Experimental and numerical explorations around the 4th Millenium Problem: *(Navier–Stokes existence and smoothness)*

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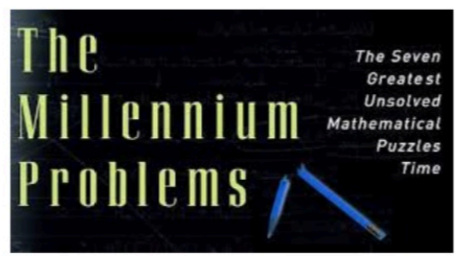
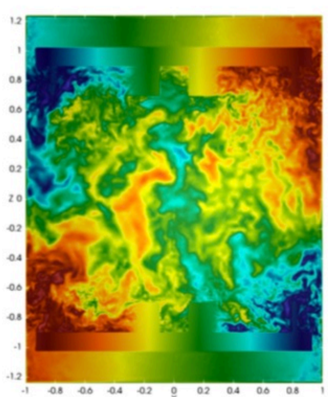
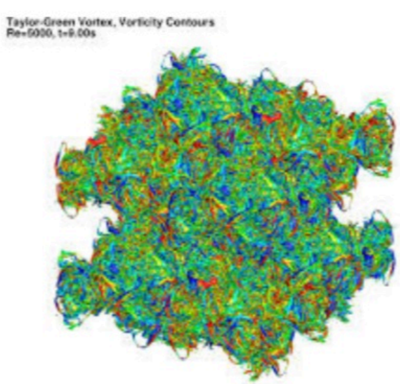
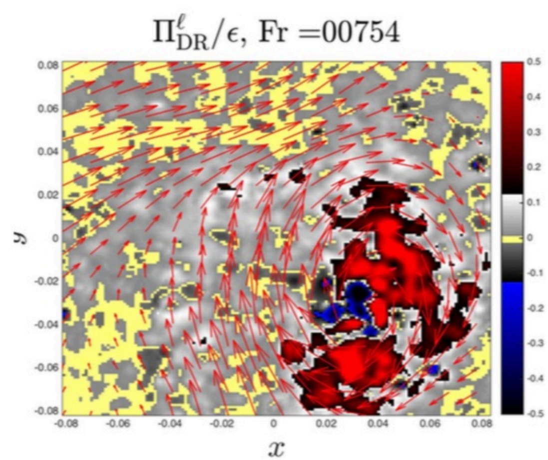
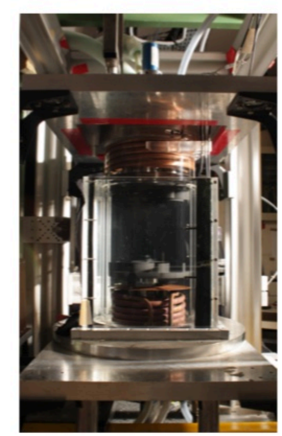








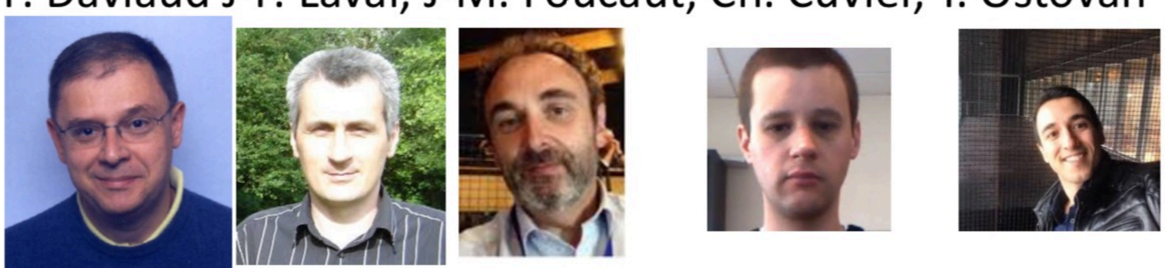


Work done with

PhD's: D. Kuzzay, P. Debue, D. Geneste, H. Faller, F. Nguyen, T. Chaabo

Post-Docs: D. Faranda, E-W. Saw, V. Shukla, V. Valori, A. Cheminet

Team EXPLOIT: F. Daviaud J-P. Laval, J-M. Foucaut, Ch. Cuvier, Y. Ostovan



C. Nore

Some Mathematical aspects of Navier-Stokes equations

$$\vec{\nabla} \bullet \vec{u} = \vec{0}$$

$$\partial_t \vec{u} + (\vec{u} \bullet \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} p + \nu \Delta \vec{u}$$

$$u(0,.) = u_0$$

Millenium Problem:
Well posedness of the Cauchy problem for finite energy solutions: existence, uniqness, regularity

2D: yes *(Ladyzhenskaya, 1958)*
3D: existence of global weak solutions. *(Leray, 1934)* but uniqness and regularity=open for

$u \in L^p$(0;T;$L^q(\mathbb{R}^3)$) with $\frac{2}{p} + \frac{3}{q} > 1$ (Serrin's criterium)

Interesting questions:
If there is a blow-up solution, what is its shape?
Is there loss of unicity after blow-up?

Some Mathematical aspects of Navier-Stokes equations

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$$\partial_t \vec{u} + (\vec{u} \bullet \vec{\nabla}) \vec{u} = -\frac{1}{\rho} \vec{\nabla} p + \nu \Delta \vec{u}$$

Symmetries

Tme-translation $t \to t + h$

Space translation $\vec{x} \to \vec{x} + \vec{h}$

Space-reversal $(\vec{x}, \vec{u}) \to (-\vec{x}, -\vec{u})$

Galilean invariance $(\vec{x}, \vec{u}) \to (\vec{x} + \vec{U} t, \vec{u} + \vec{U})$

Scaling $(t, \vec{x}, \vec{u}) \to (\lambda^2 t, \lambda \vec{x}, \lambda^{-1} \vec{u}) \quad \nu \neq 0$

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Are they self-similar solutions? (cf Leray)

Some Mathematical aspects of Navier-Stokes equations

Self-similar Solutions

$$u_0(x) = \frac{\varphi(\hat{x})}{\|x\|}$$

$$u(t,x) = \frac{1}{\sqrt{2\kappa t}} U\left(\frac{x}{\sqrt{2\kappa t}}\right)$$

Forward self-similar if κ $> \mathbf{0}$

Existence of FSS solution for $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ (Jia&Sverak, 2013)

Non-uniqness of FSS solution for $u_0 \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ (Guillod &Sverak, 2017)

Backward self-similar if κ $< \mathbf{0}$

No non-zero BSS solution (Tsai, 1998)

>>>>>> Generalized Backward self-similar if κ $< \mathbf{0}$ (Guillod & Wittwer, 2015)

$$u(t,x) = \frac{e^{-1/2 \log(2\kappa t)L}}{\sqrt{2\kappa t}} U\left(\frac{e^{1/2 \log(2\kappa t)L} x}{\sqrt{2\kappa t}}\right) \qquad L \in SO(3)$$

Some Mathematical aspects of Navier-Stokes equations

Conjecture (Guillod-Sverak, 2018)
Existence of generalized backward self-similar blow-up followed by loss of unicity

Proved for Complex Ginzburg-Landau system (3D)

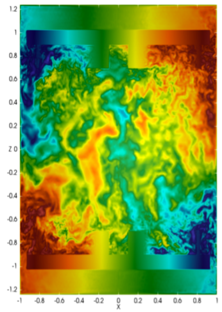
$$\partial_t u - i(\Delta u + u|u|^2) = \nu \Delta u$$

Same energy conservation as NS
Same scale invariance as NS
Invariance under U(1) instead of SO(3)

>>>>>> For NS: $$u(t,x) = \frac{e^{-1/2\log(2\kappa t)L}}{\sqrt{2\kappa t}} U\left(\frac{e^{1/2\log(2\kappa t)L}x}{\sqrt{2\kappa t}}\right)$$

Can we guide the intuition of mathematicians by suitable in silico or in fluido experiments to find the releveant shape of U?

Exploring issues in NS: Numerics vs experiments



$$\vec{\nabla} \bullet \vec{u} = \vec{0}$$

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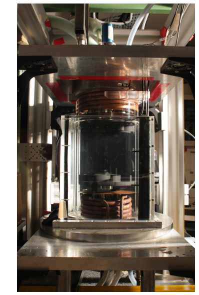


Numerics

Equation is exact
All variables available
Easy to vary nu and u0
Periodic BC easy

Non periodic BC hard
Discretization of equations
Limitation in number of grid points
- No limit l small
- Limit t small hard

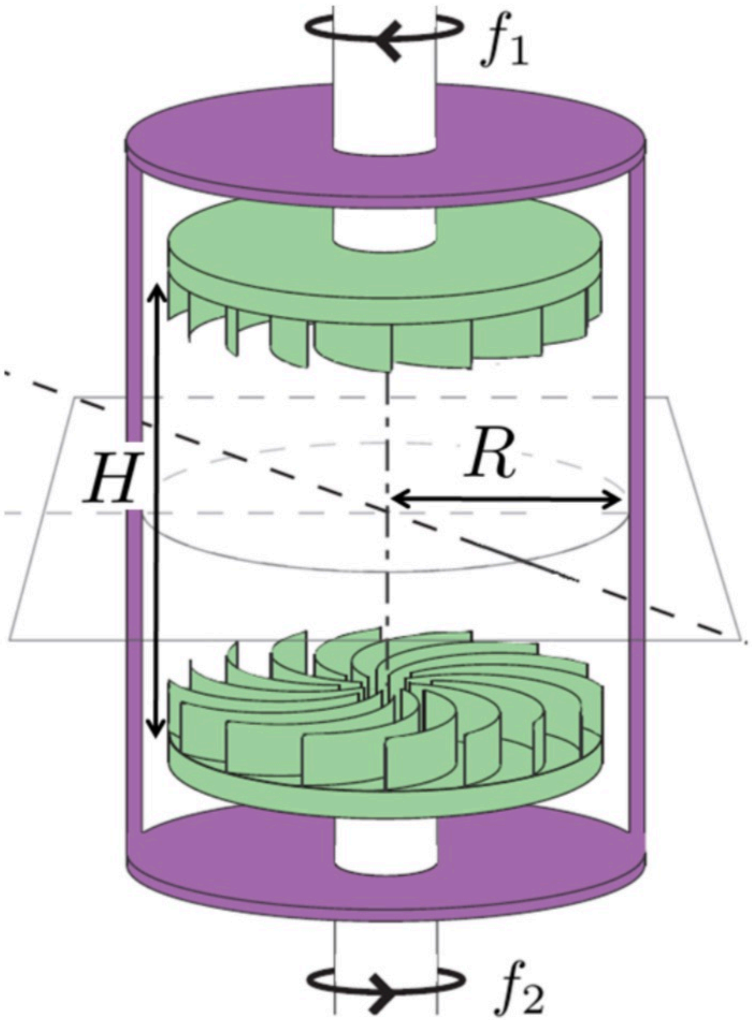
Lack of statistics

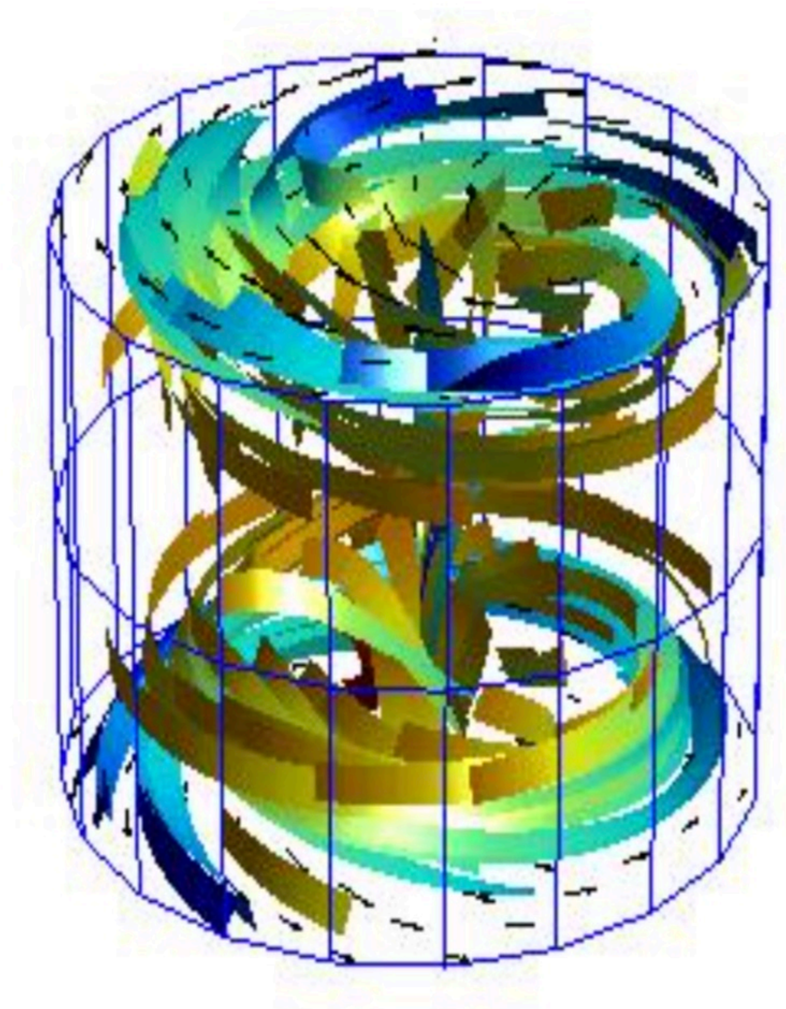
Experiments

Limit l small and t small are automatic
Huge statistics easy
Easy to implement almost any BC

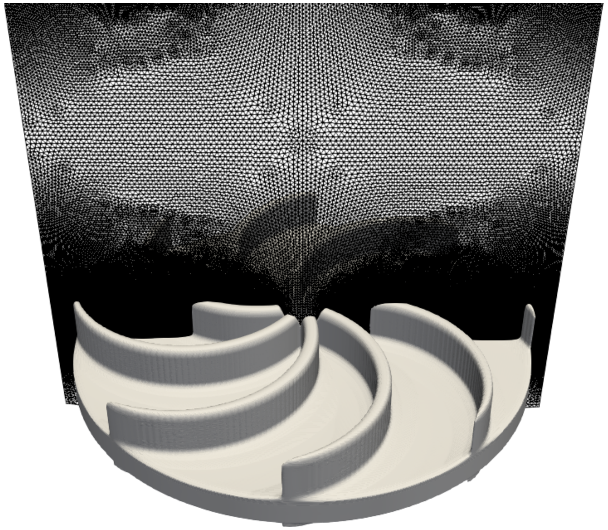
Periodic 3D BC impossible
Difficulty of measurements
Resolution (time and scale) issue
More challenging to vary nu and u0
Issue with compressibility if blow-up

The von Karman flow





Numerical simulation of von Karman flow: SFEMaNS



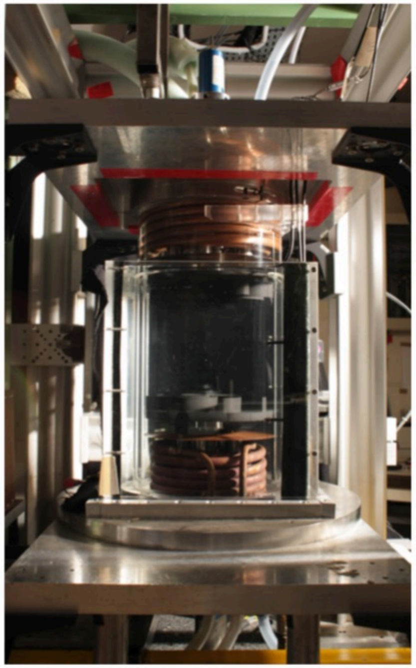
spatial resolution in r, z	$\left(\frac{1}{200} \to \frac{1}{600}\right) L \sim (1 \to 0.4)\,\eta$
number of fourier modes in θ	512
number of procs	2048
$\omega \mathrm{d}t$	$1,5 \times 10^{-4}$
Re	6×10^{3}

Spectra/Finite Elements for Maxwell and Navier-Stokes

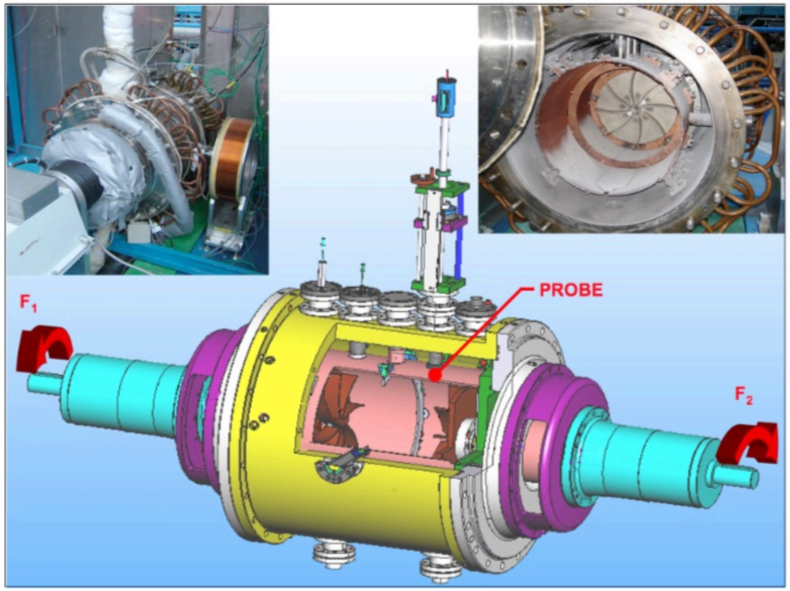
Cylindrical geometry
Radial and vertical: Finite elements
Azimuthal Spectral: (Fourier)
Penalty method for moving boundaries (impellers)

DNS and LES using entropy viscosity (Guermond et al, 2011)

Changing viscosity in the lab



VKE

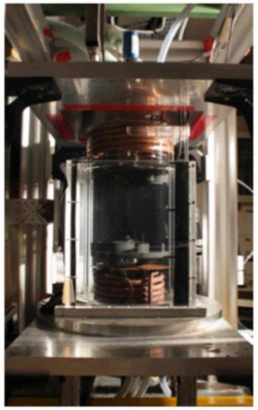


VKS=VKEx2

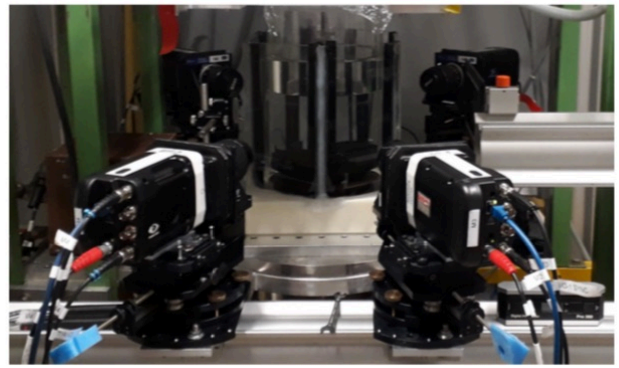
SHREK=VKEx4

SetUp	Fluid	P(bars)	T(K)	Re $(1/\nu)$
SHREK	Helium	1.1	2.62 or 2	10^8 or ∞
SHREK	Nitrogen	1.1	284	10^5
VKS	Sodium		410	10^7
VKE VKE	Water Glycerol	1.8 1.8	300 300	10^5 10^2

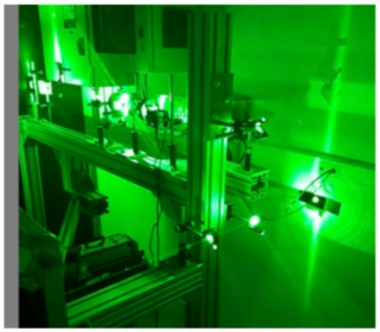
Measuring velocities in the lab



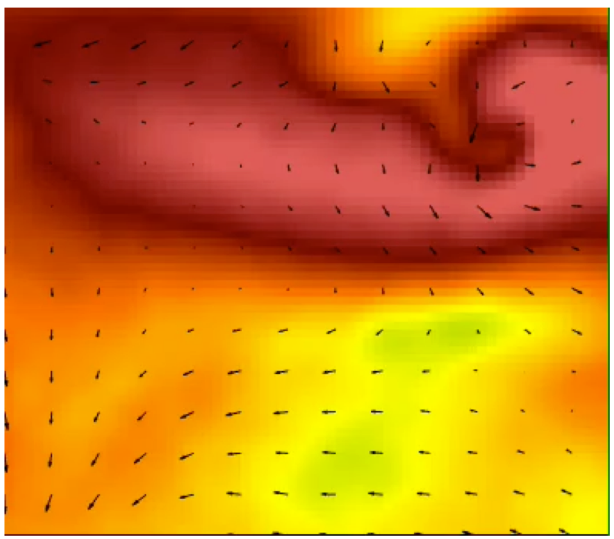
4 fast cameras



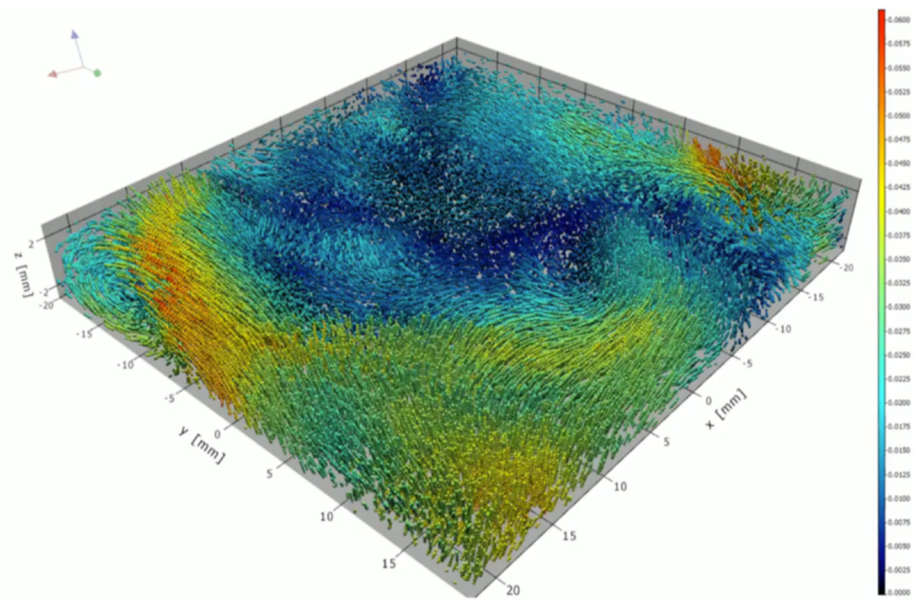
Laser



Eulerian velocity measurement



Lagrangian velocity measurements

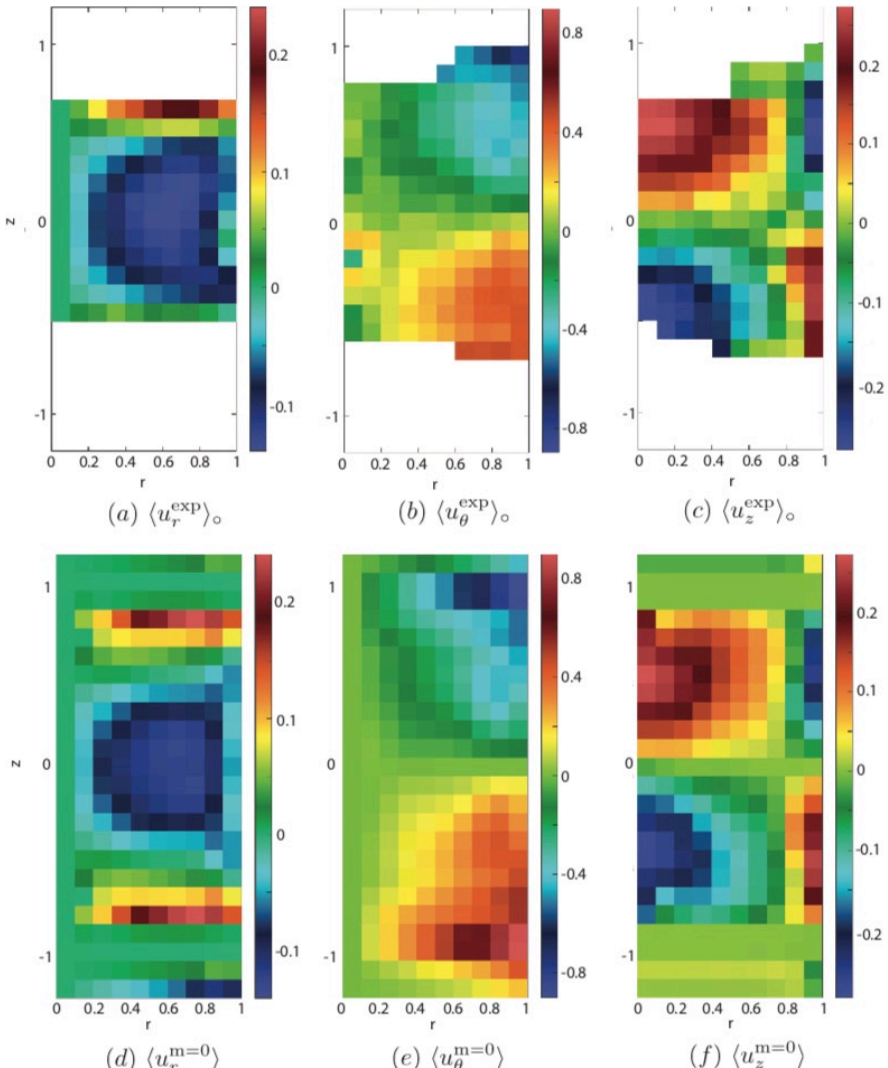


ANR EXPLOIT: T. Chabo, A. Cheminet,C. Cuvier, F. Daviaud, P. Debue, B. Dubrulle, J-M. Foucault, D. Geneste, J-P. Laval, Y. Ostovan, V. Padilla, V. Valori, C. Wiertel

Numerics vs lab: mean velocity at Re=10^3

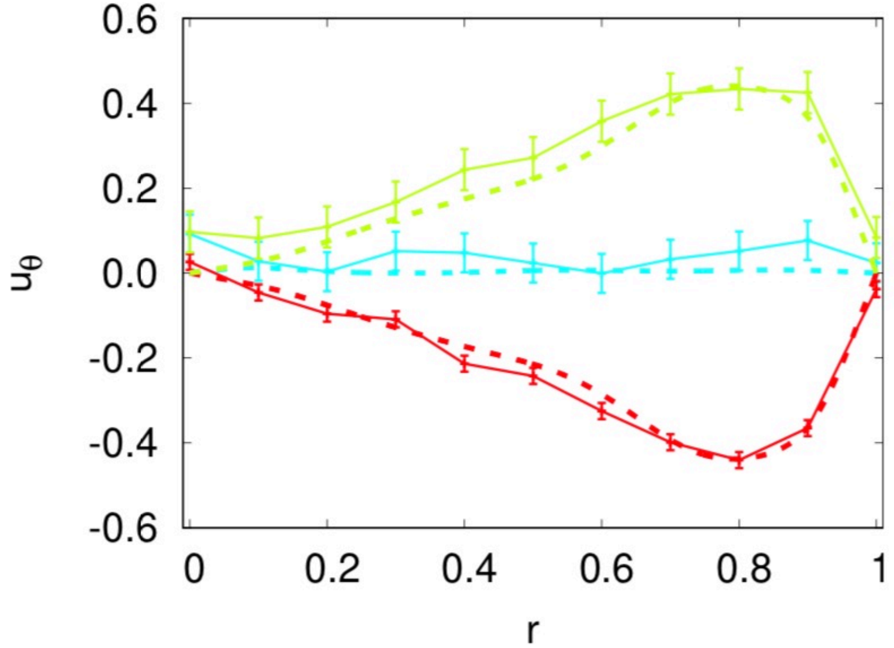
Lab

DNS

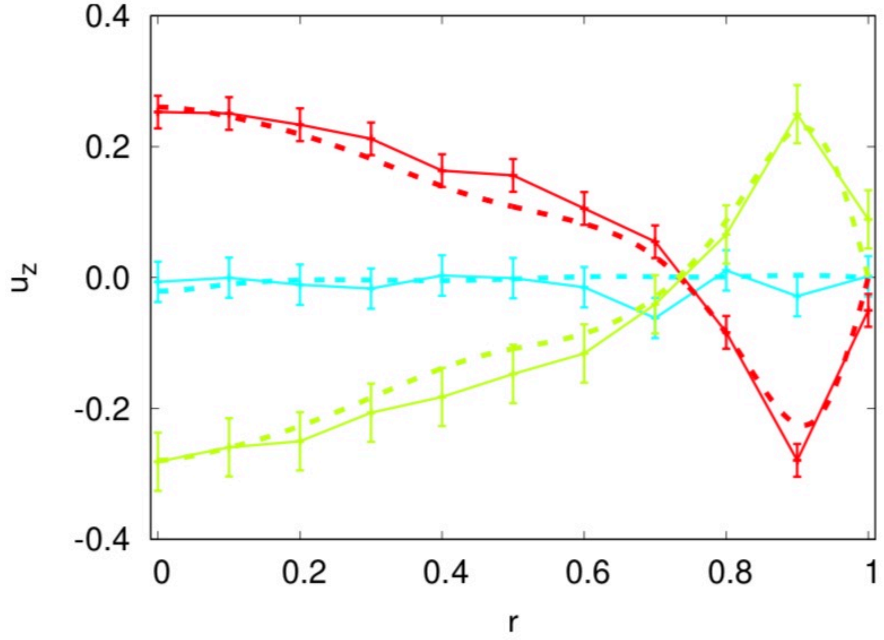


Cappanera et al, Computers&Fluids, 2020

Numerics vs lab: velocity profiles at Re=10^3



Azimuthal velocity profile

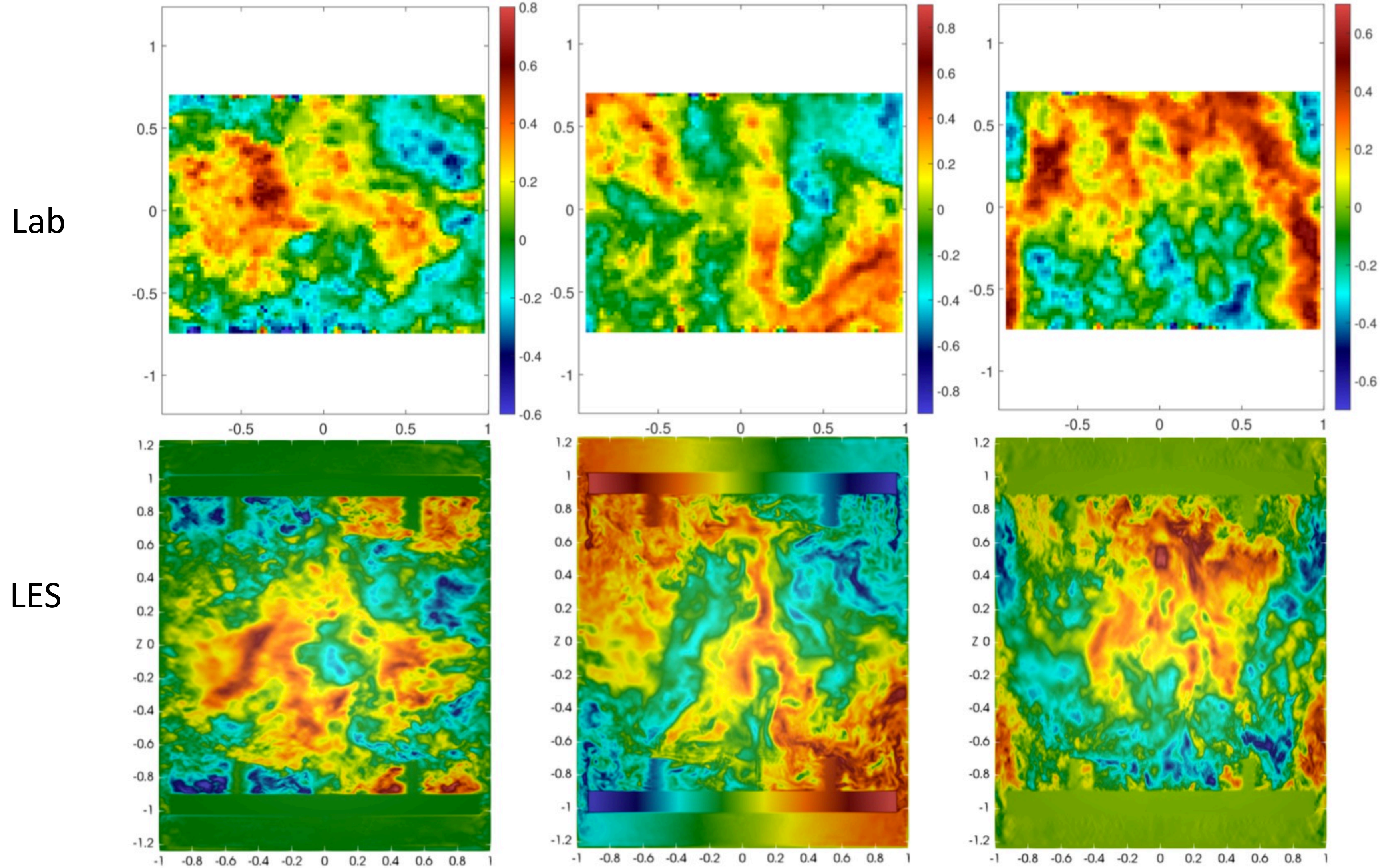


Radial velocity profile

At z=+- 0.4 and 0

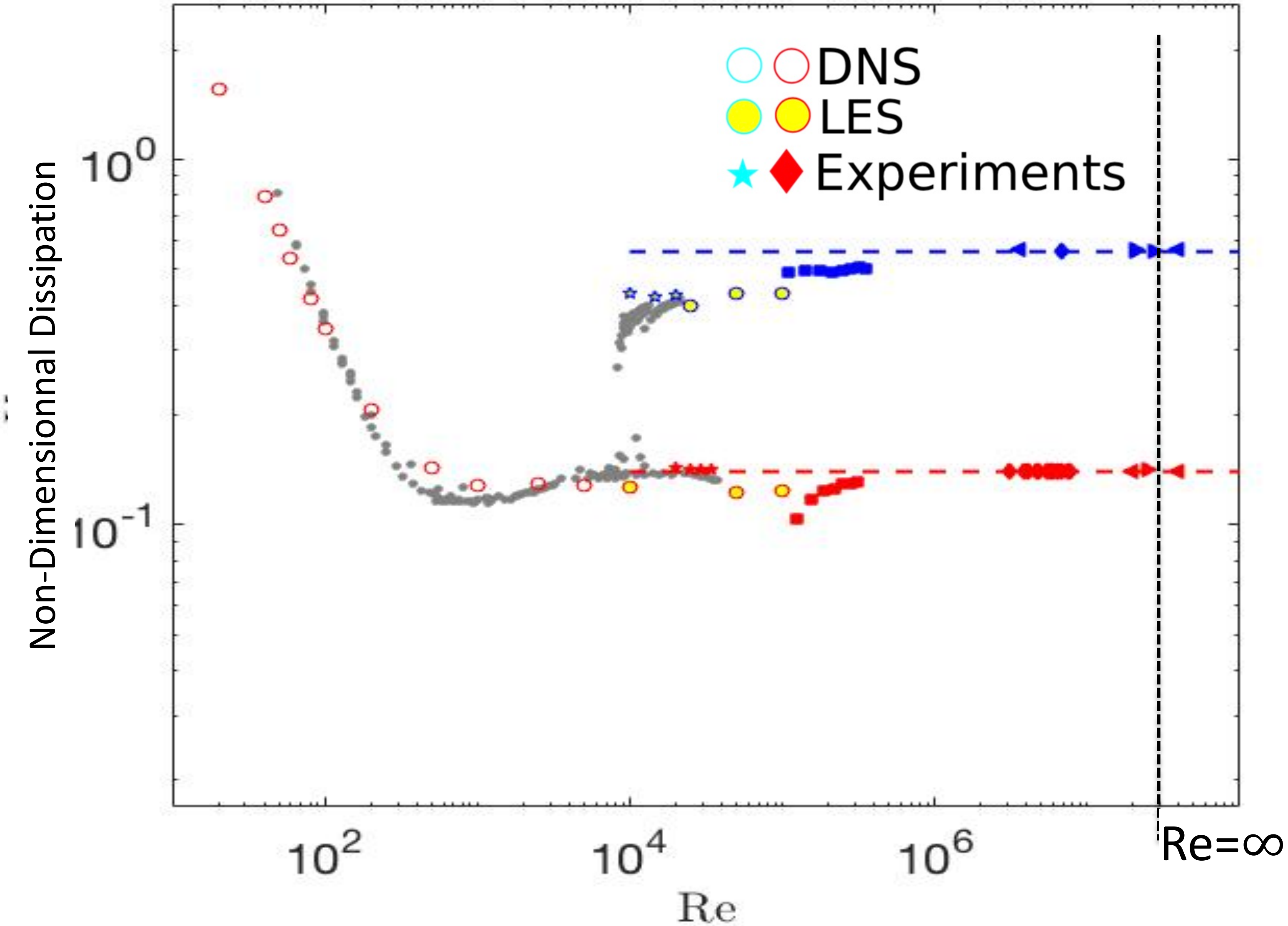
Cappanera et al, Computers&Fluids, 2020

Numerics vs lab: instantaneous velocity at Re=10^5



Cappanera et al, Computers&Fluids, 2020

Numerics vs lab: energy dissipation



Cappanera et al, Computers&Fluids, 2020

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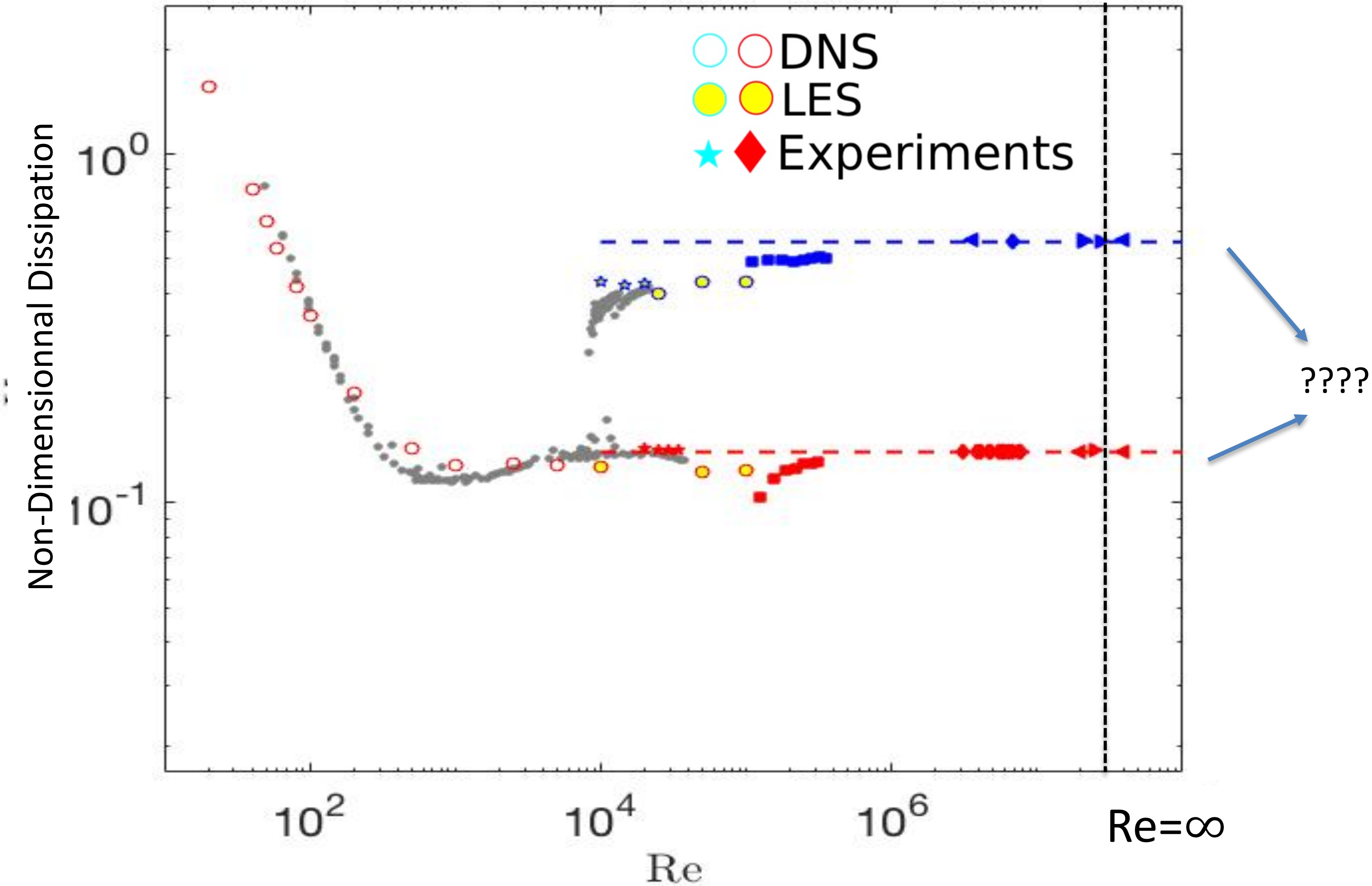
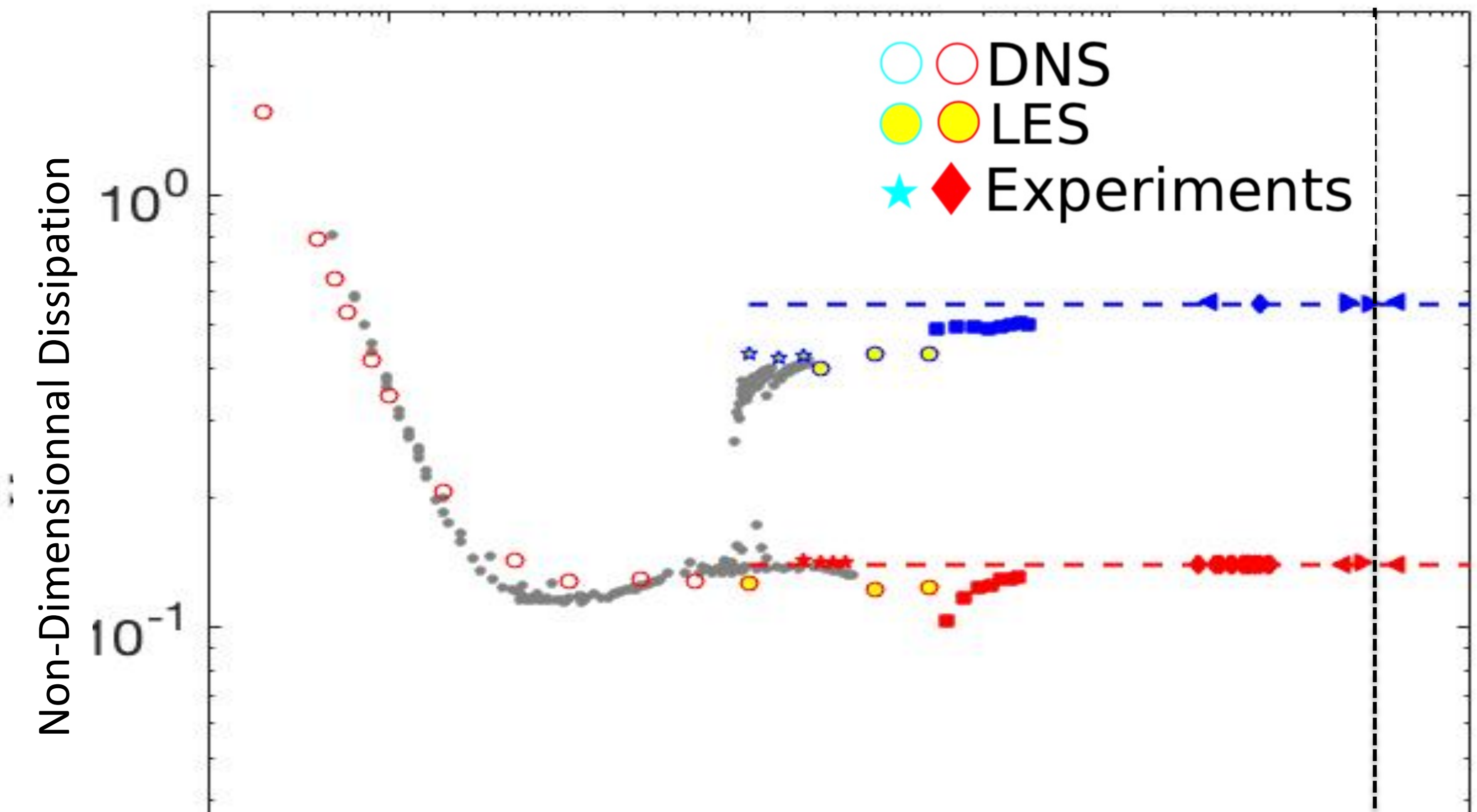
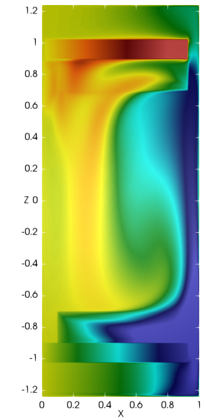
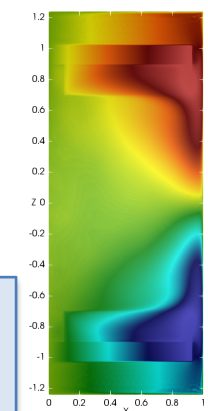


Numerics vs lab: non-uniqueness

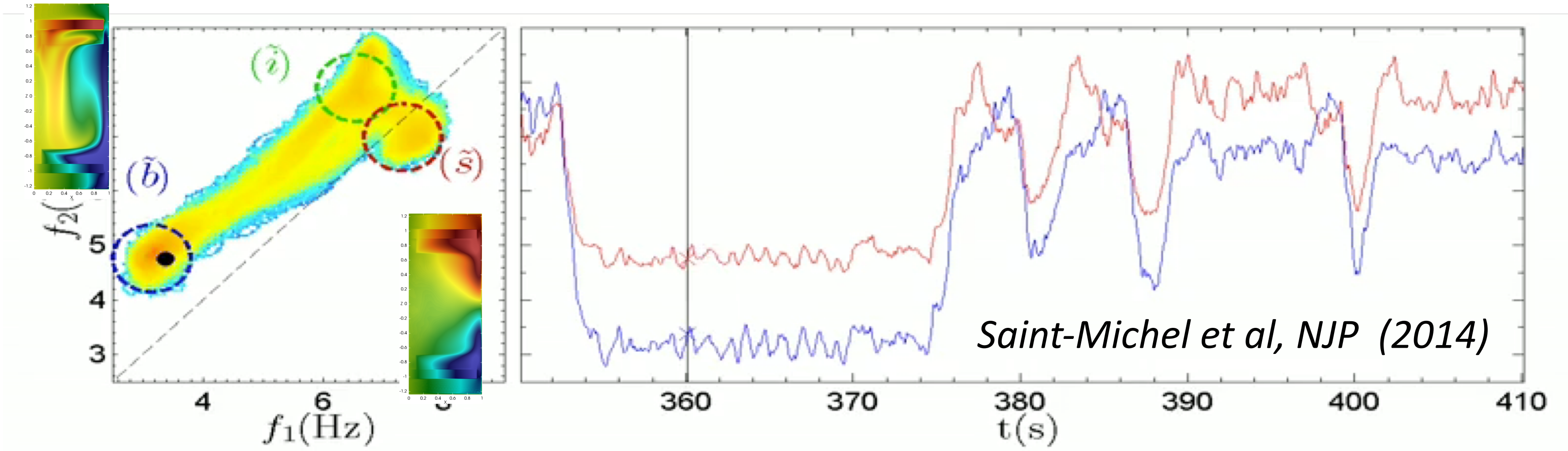






Multiple solutions: non-uniqueness!!!!

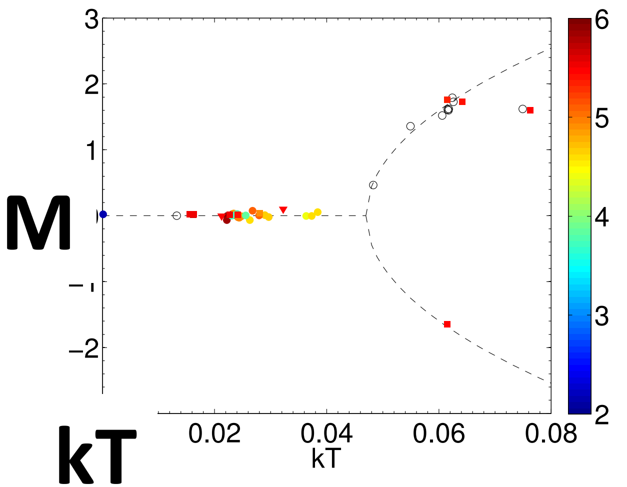
Non-uniqueness: phase transition, stochastic attractor at large scales

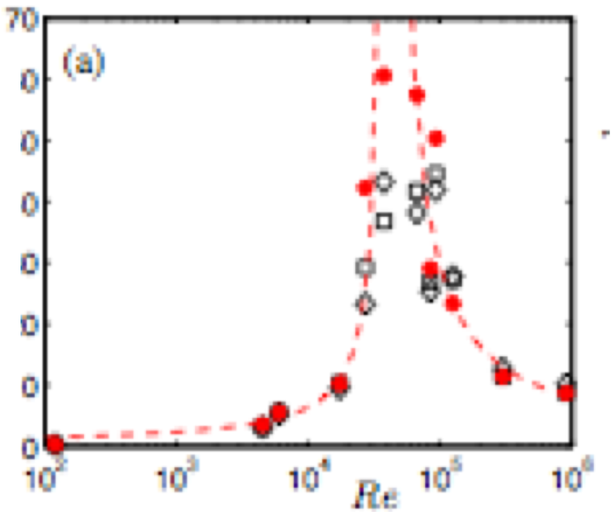


Saint-Michel et al, NJP (2014)

Statistical theory à la Miller-Robert-Sommeria (inviscid)

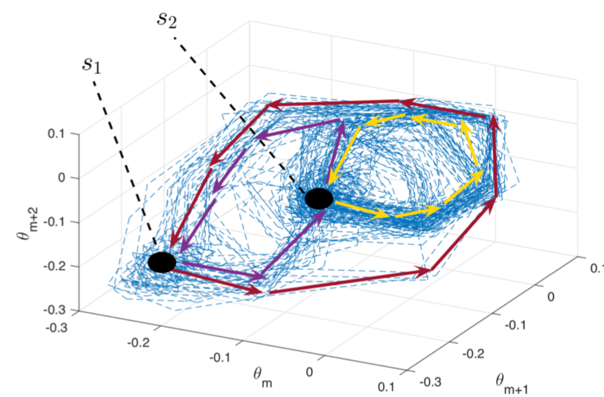
Log(Re)





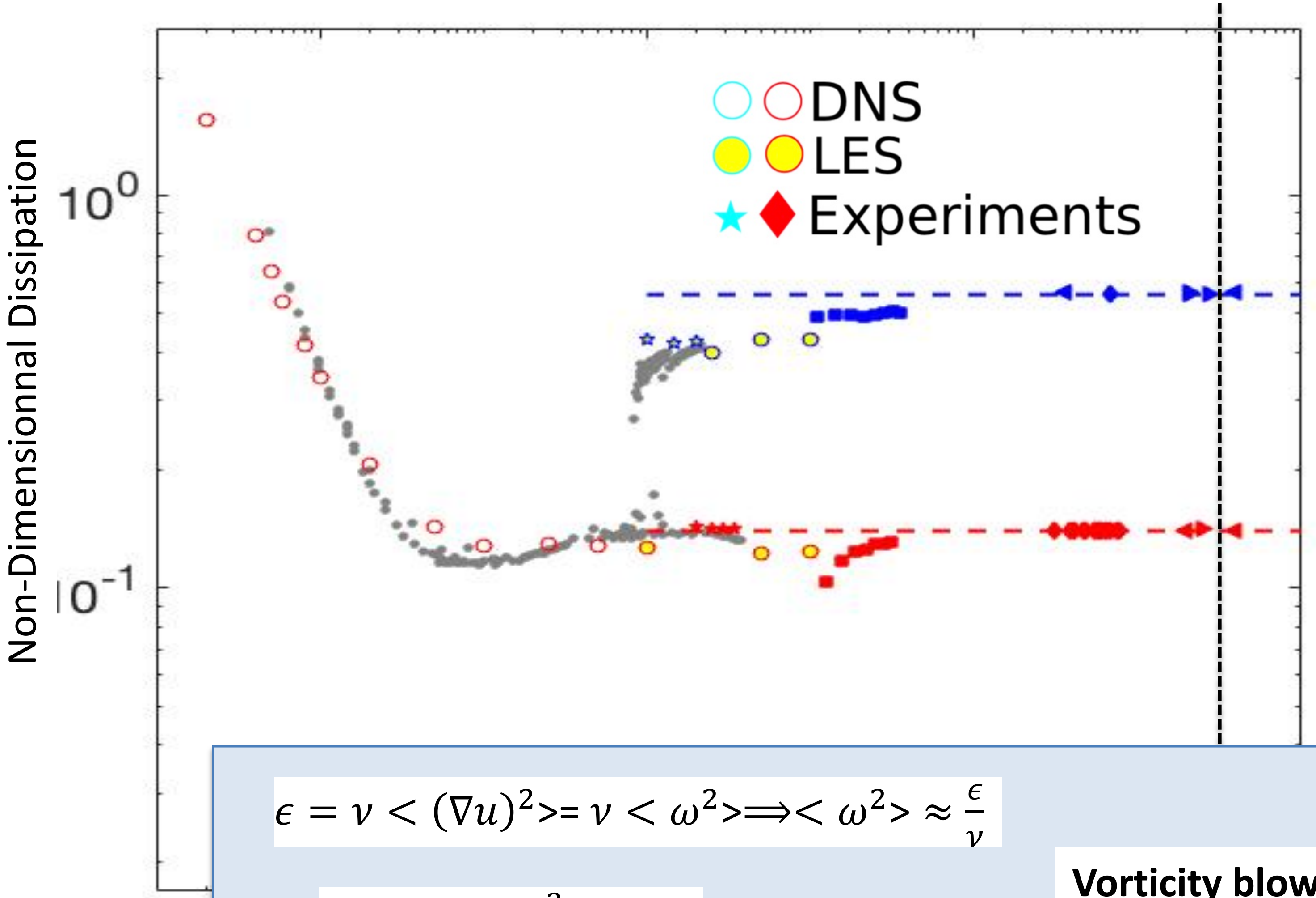
Thalabard et al, NJP (2015)

Stochastic attractor



Faranda et al, PRL (2017)

Numerics vs lab: blow-up?

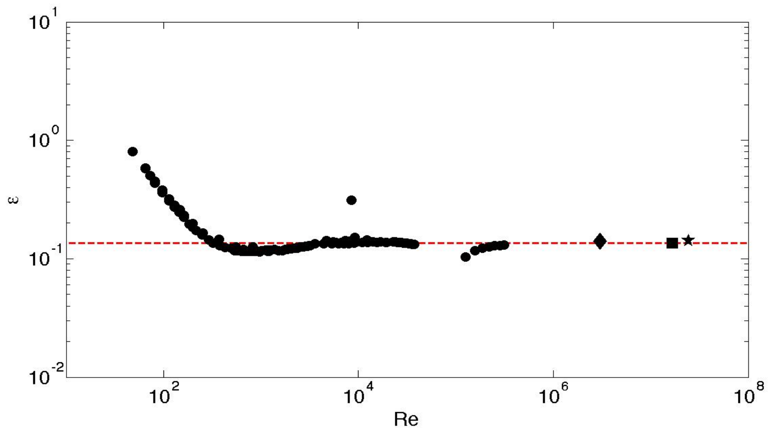


$$\epsilon = \nu < (\nabla u)^2 >= \nu < \omega^2 > \Longrightarrow < \omega^2 > \approx \frac{\epsilon}{\nu}$$

$$lim_{\nu \to 0} < \omega^2 >= \infty$$

Vorticity blow-up!!!

Local energy balance for weak solutions



"...in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available."

L. Onsager, 1949

See Eyink&Sreenivasan (2006)

$$\frac{1}{2}\partial_t \mathbf{u}^2 + \mathrm{div}\left(\mathbf{u}\left(\frac{1}{2}\mathbf{u}^2 + p\right) - \nu\nabla\mathbf{u}\right) = D(u) - \nu(\nabla\mathbf{u})^2$$

Duchon&Robert. Nonlinearity (2000),

Regular Test function of width ℓ

Inertial dissipation:

$$D(u) = \lim_{\ell\to 0}\frac{1}{4}\int_{r\leq\ell} d^3r\ \nabla\phi_\ell(r)\cdot\delta u_r|\delta u_r|^2$$

$\delta u(\ell) \sim \ell^h$ In the limit of $\ell \approx 0$

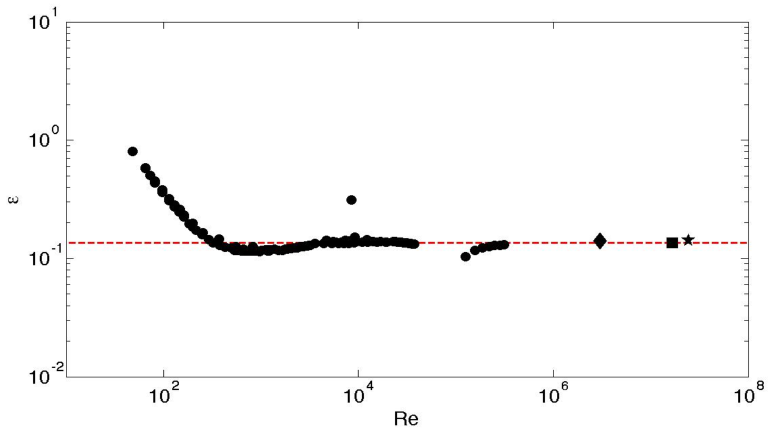
$$D(u)[x] \propto \lim_{\ell\to 0} \ell^{3h-1}$$

δu=u(x+r)-u(x) Velocity increment

If *h > 1/3* → Euler equation conserves energy,
Dissipation in Navier-Stokes by viscosity.
(Eyink 1994, Constantin et al, 1994)

If *h ≤ 1/3* → Dissipation through irregularities (singularities)
Without viscosity !
(Isett, 2018)

Local energy balance vs regularity



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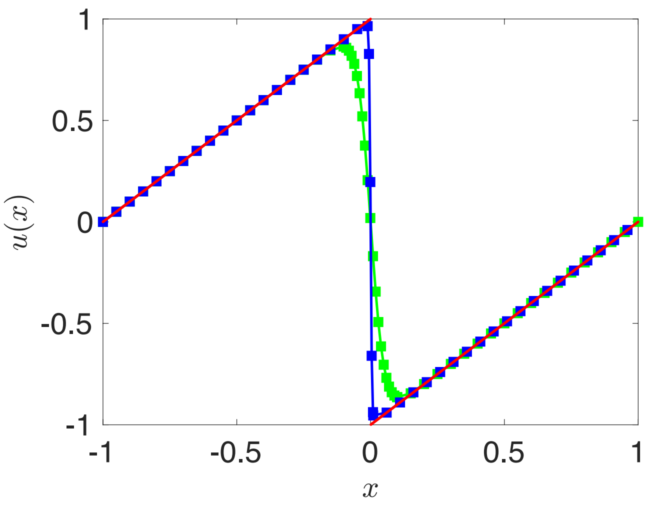
Scalar regularity indicator D_ℓ^I

Corresponds to local energy transfer
+= towards small scale
-= towards large scale

B. Dubrulle, JFM perspectives, 2019

Local energy balance vs regularity



Dubrulle 2019 JFM Perspectives

$$\partial_t u + u\partial_x u = \nu\, \partial_{xx} u$$

Develops shocks=singularity in the inviscid limit

$$\frac{1}{2}\partial_t \mathbf{u}^2 + \mathrm{div}\left(\mathbf{u}\left(\frac{1}{2}\mathbf{u}^2 + p\right) - \nu\nabla\mathbf{u}\right) = D(u) - \nu(\nabla\mathbf{u})^2$$

Duchon&Robert. Nonlinearity (2000),

Inertial dissipation= singularity/large gradient detector!

$$D(u) = \lim_{\ell\to 0}\frac{1}{4}\int_{r\leq\ell} d^3r\ \nabla\phi_\ell(r)\cdot\delta u_r|\delta u_r|^2$$

δu=u(x+r)-u(x) Velocity increment

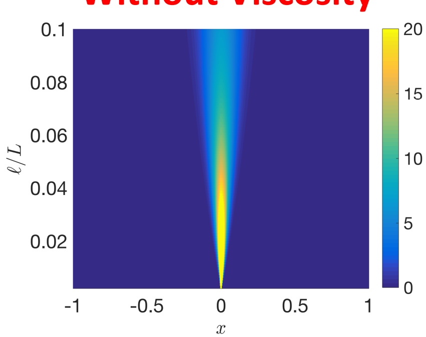
$\delta u = \Delta,\quad x = 0$

h=0; $D(u) \neq 0$

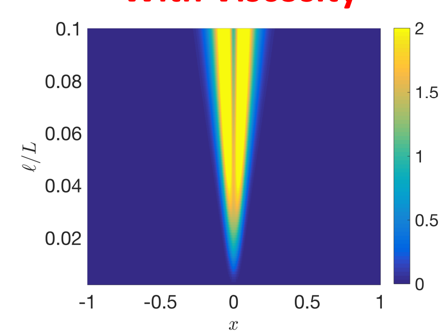
$\delta u = \ell,\quad x \neq 0$

h=1; $D(u) = 0$

Without Viscosity



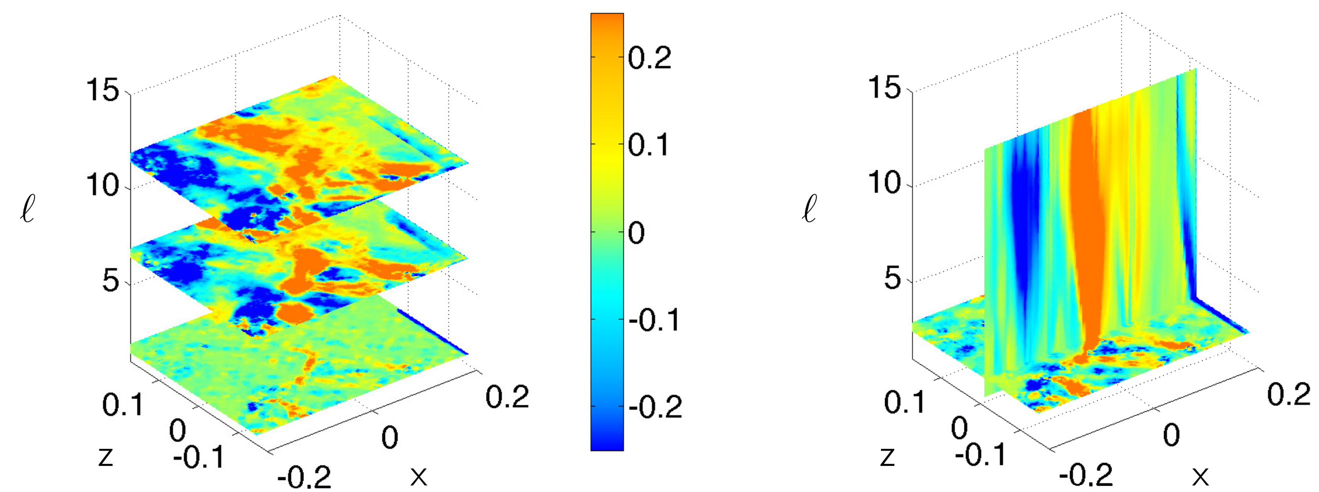
With Viscosity



Finding the irregularities in von Karman 3D flow Experiments

$$D_\ell(\mathbf{u}) = \frac{1}{4}\int_{\mathcal{V}} d^3r\, (\nabla G_\ell)(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})\, |\delta\mathbf{u}(\mathbf{r})|^2, \qquad G_\ell(r) = \frac{1}{N}\exp(-1/(1-(r/2\ell)^2)),$$

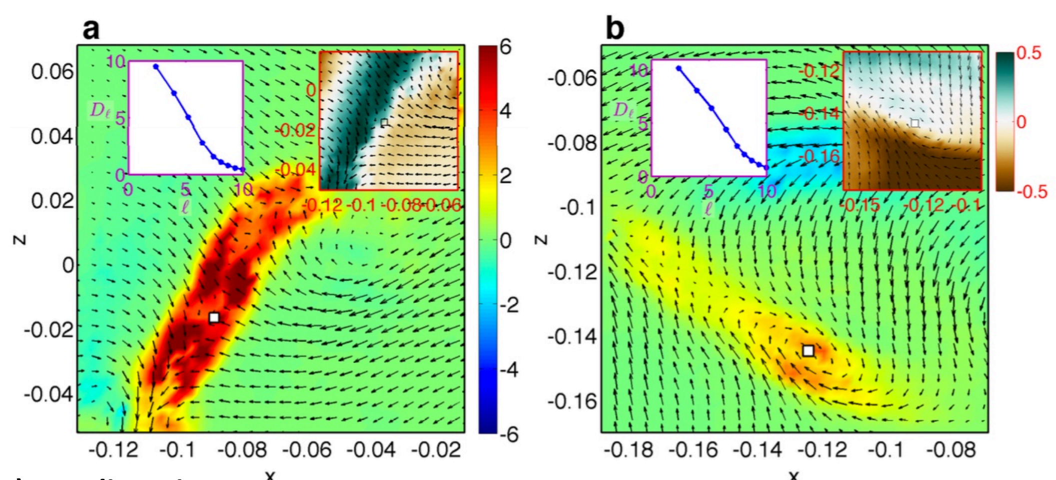
We observe the same
Kind of structure than in
Burgers.
We can map areas with
Strong local transferts at the
Kolmogorov scale!
We can study these events!



$$\delta u(\ell) \sim \ell^h$$

$$D(u)[x] \propto \lim_{\ell\to 0} \ell^{3h-1}$$

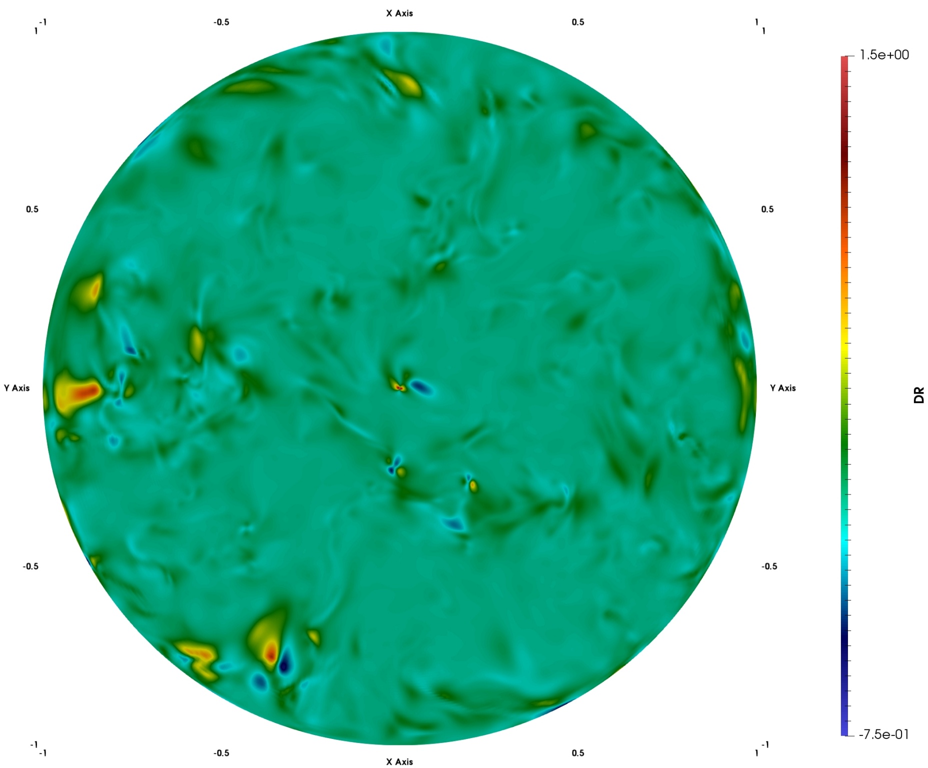
Low DR Tracks regions with h>1/3
High DR: Tracks regions with h<1/3



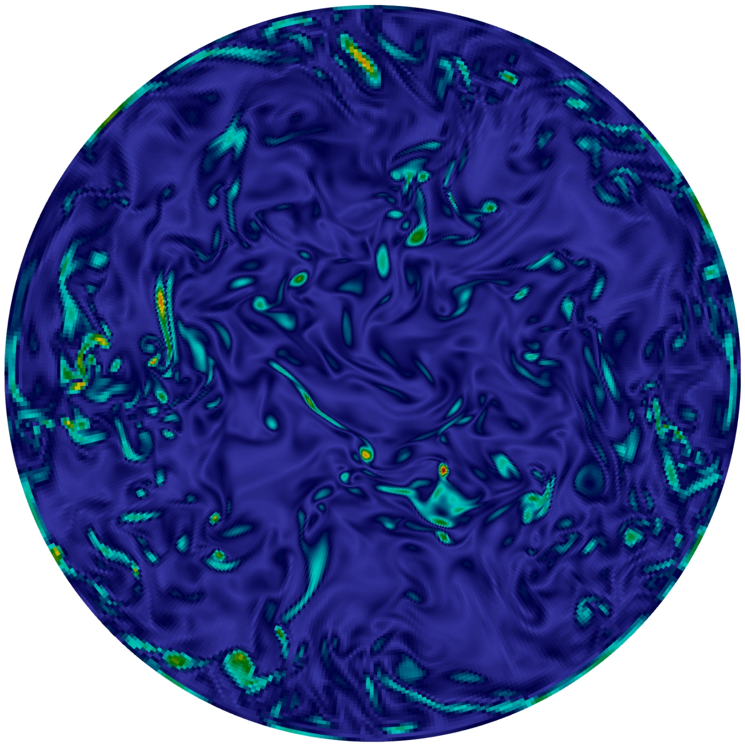
Saw et al. (2016), Nature-Comm. 7 – Kuzzay D. et al. (2017), *Nonlinearity.*

Finding the irregularities in von Karman 3D flow DNS

$$D_\ell(\mathbf{u}) = \frac{1}{4}\int_{\mathcal{V}} d^3r \; (\boldsymbol{\nabla} G_\ell)(\mathbf{r}) \cdot \delta\mathbf{u}(\mathbf{r}) \; |\delta\mathbf{u}(\mathbf{r})|^2, \qquad G_\ell(r) = \frac{1}{N}\exp(-1/(1-(r/2\ell)^2)),$$



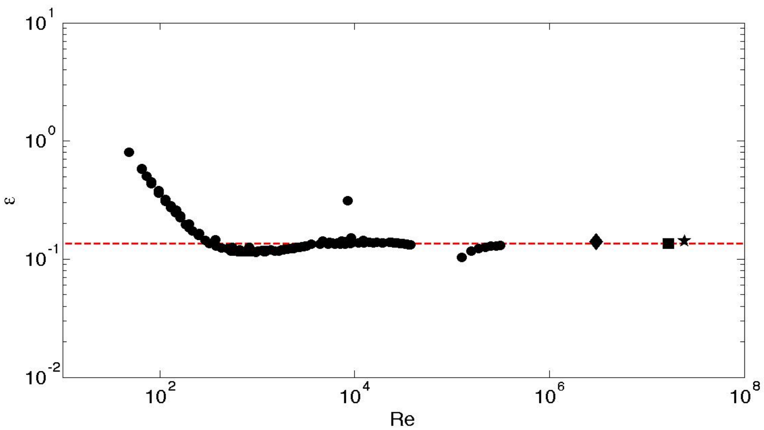
D_ℓ



ω^2

Faller et al. (2020), JFM (2021)

How frequent are singularities, if any?



$$\frac{1}{2}\partial_t \mathbf{u}^2 + \mathrm{div}\left(\mathbf{u}\left(\frac{1}{2}\mathbf{u}^2 + p\right) - \nu\nabla\mathbf{u}\right) = D(u) - \nu\ \nabla\mathbf{u})^2$$

Constraints on the singularity using dissipation

$$\left\langle D_\ell^I \right\rangle \approx \int d\mu(h,\ell)\,\ell^{3h-1} \approx \ell^{3h-1+C(h)}$$

$$\langle D(u)\rangle = \varepsilon = \ell^0 \Longrightarrow C(h) = 1 - 3h$$

Burgers

$$\partial_t u + u\partial_x u = 0$$

$h = 0 \Rightarrow C = 1$

Isolated point in space: shocks

Navier-Stokes

$$\partial_t \vec{u} + (\vec{u}\bullet\vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} p + \nu\Delta\vec{u}$$

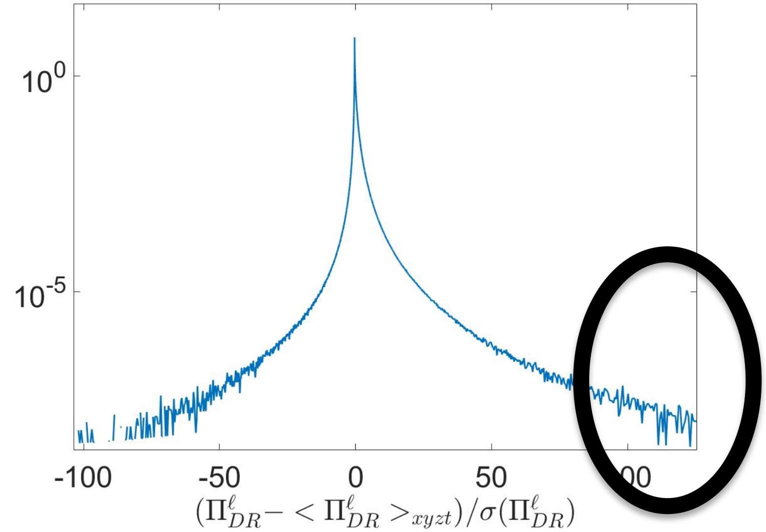
$h = -1 \Rightarrow C = 4$

Isolated points in space-time, consistent with Cafarelli's theorem

Very rare!!!!!-> need a statistical study to find them

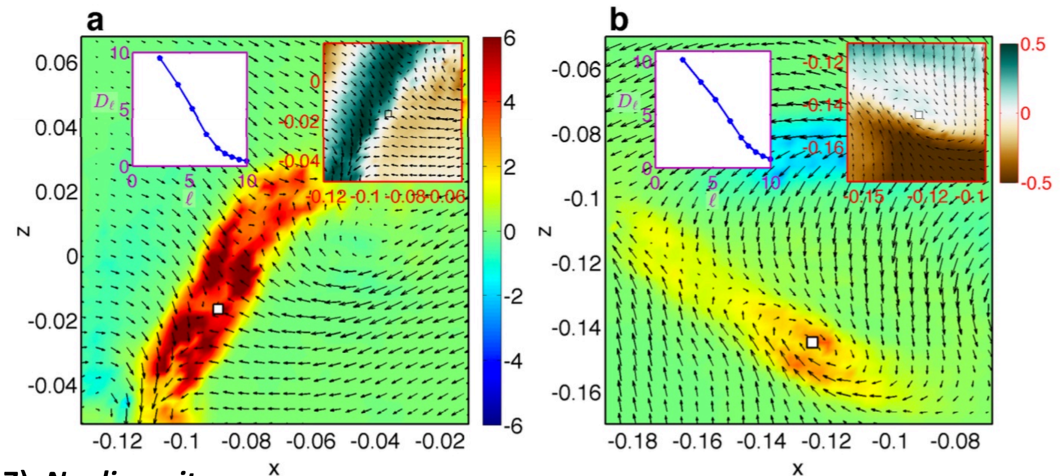
Finding the irregularities in von Karman 3D flow

$$D_\ell(\mathbf{u}) = \frac{1}{4}\int_{\mathcal{V}} d^3r\ (\boldsymbol{\nabla} G_\ell)(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})\ |\delta\mathbf{u}(\mathbf{r})|^2, \qquad G_\ell(r) = \frac{1}{N}\exp(-1/(1-(r/2\ell)^2)),$$



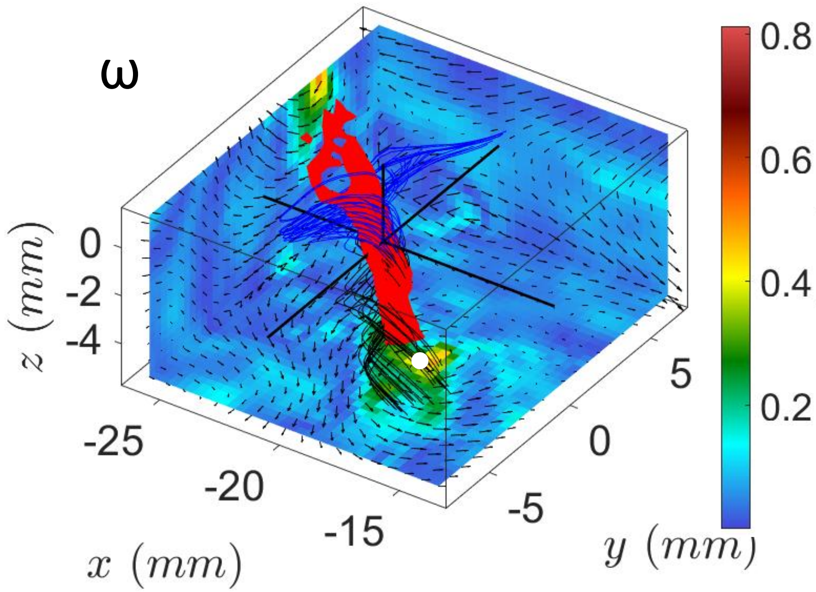
We look for the rarest,
Strongest events

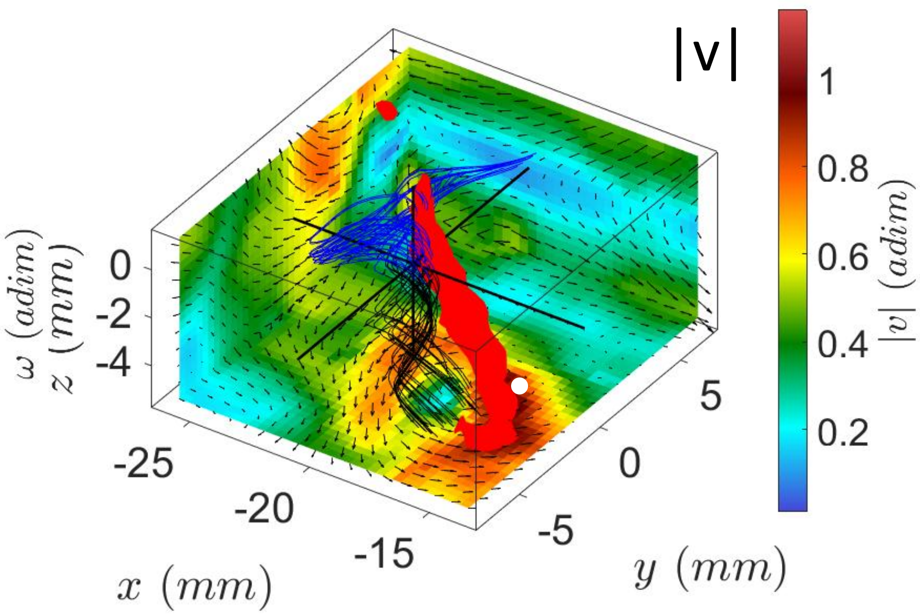
We study velocity fields at places where DR > 160 std

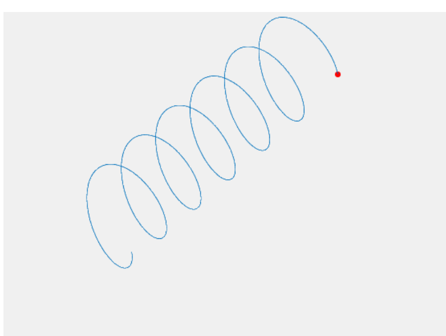


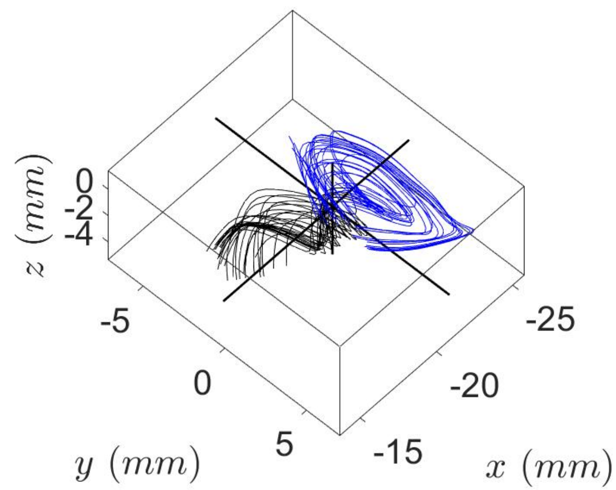
Saw et al. (2016), Nature-Comm. 7 – Kuzzay D. et al. (2017), *Nonlinearity.*

Screw vortices: experiments



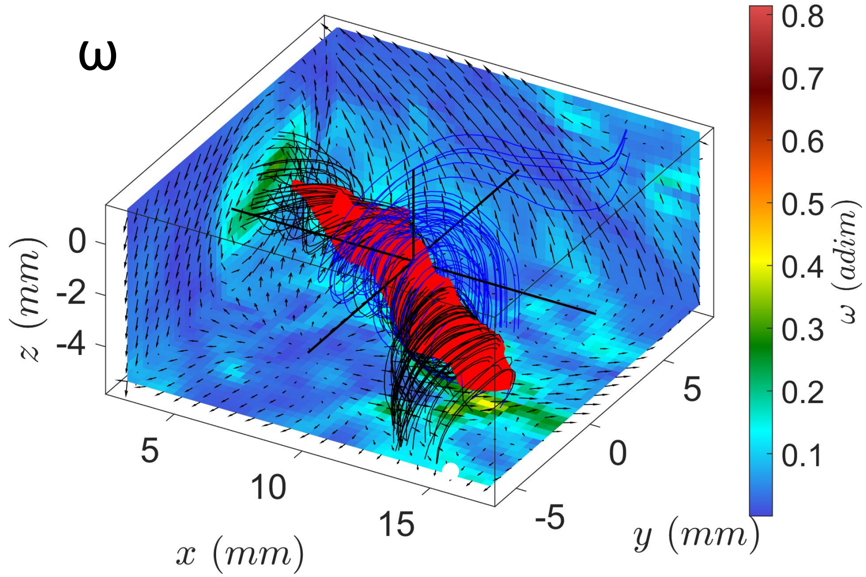


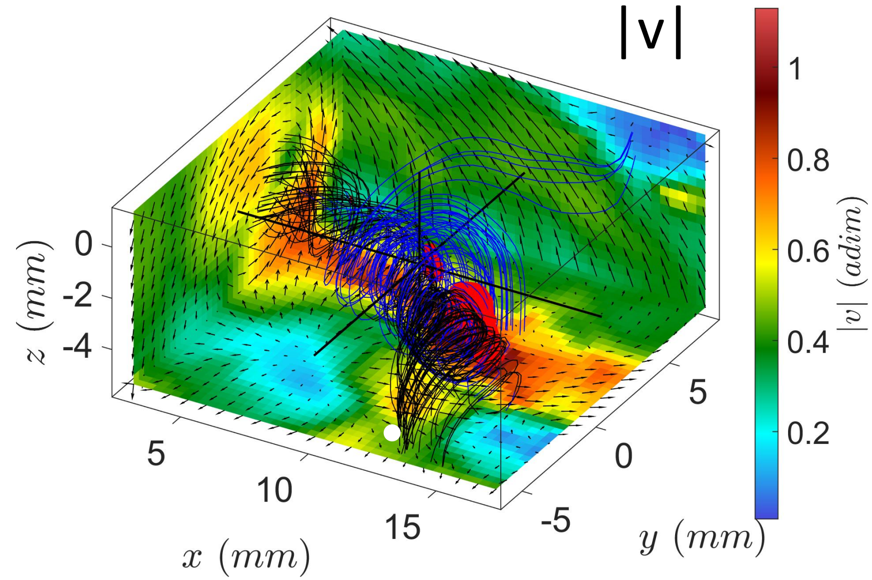


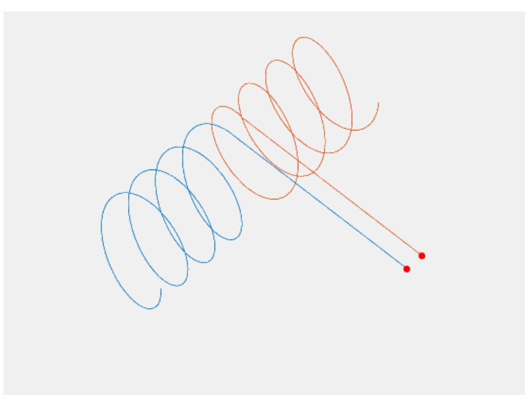


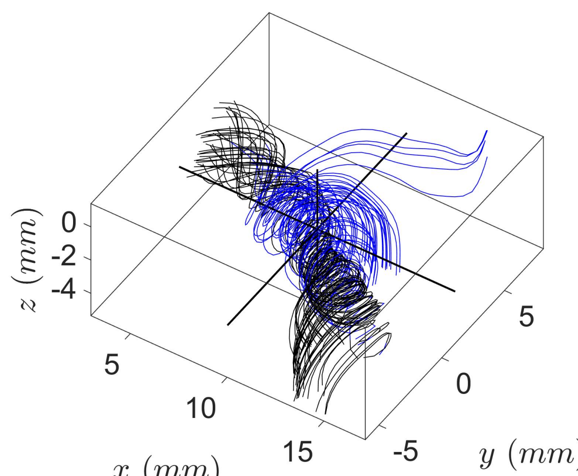
Debue PhD Thesis (2019); ;
Debue et al, JFM (2020)

Roll vortices : experiments



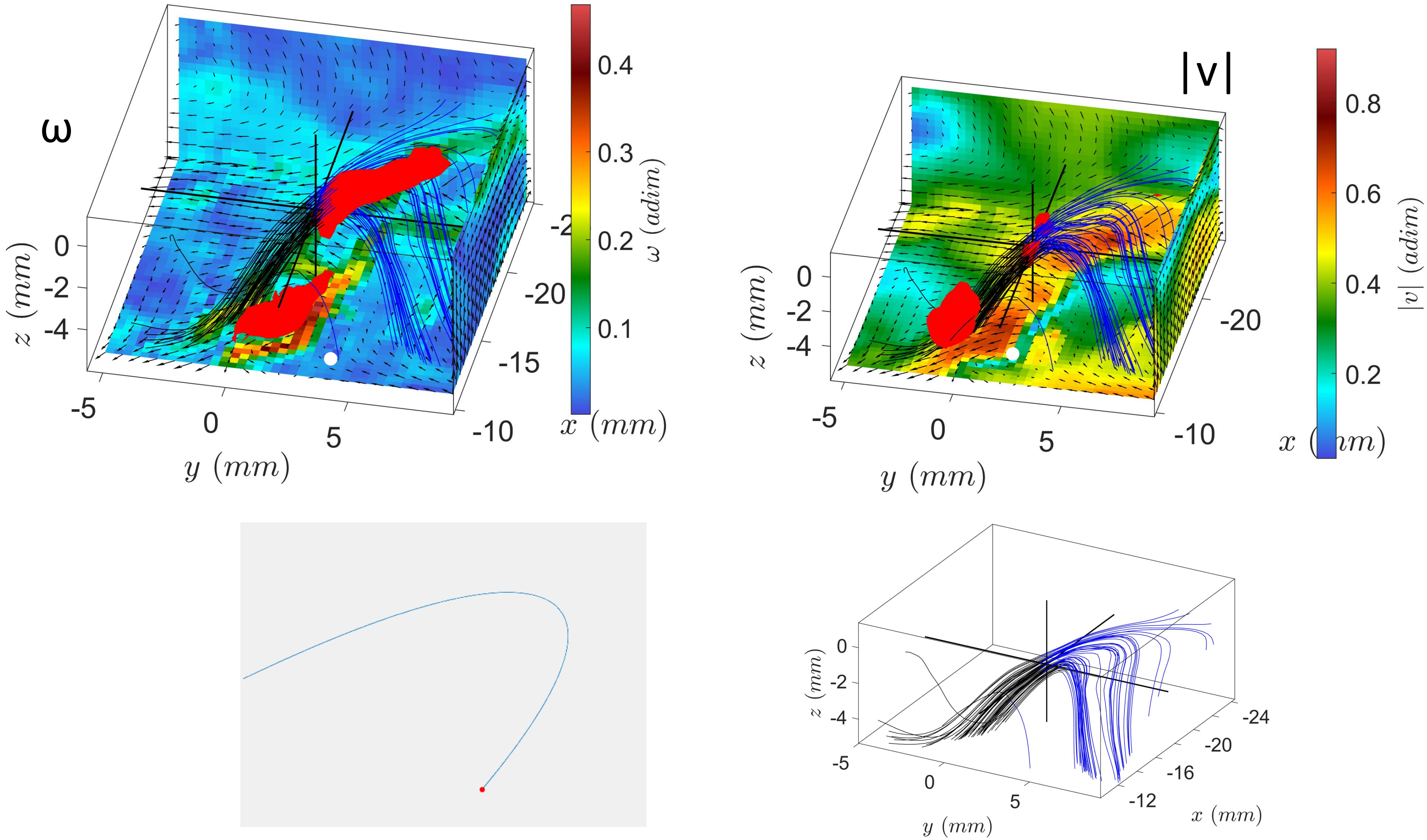






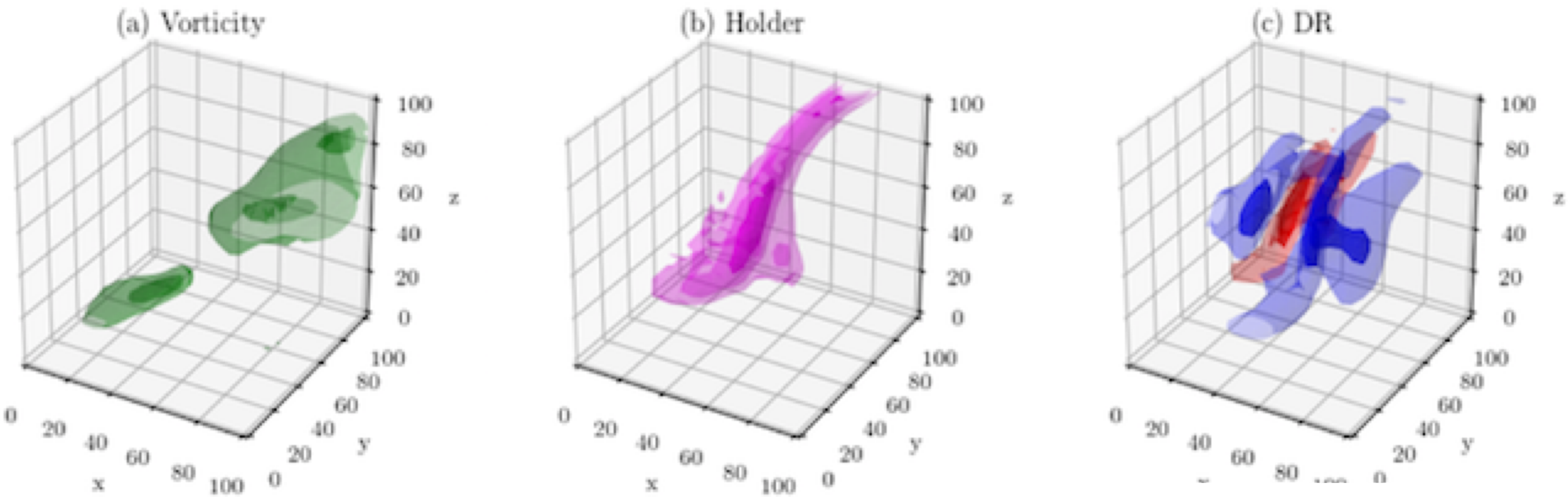
Debue PhD Thesis (2019); ; Debue et al, JFM (2020)

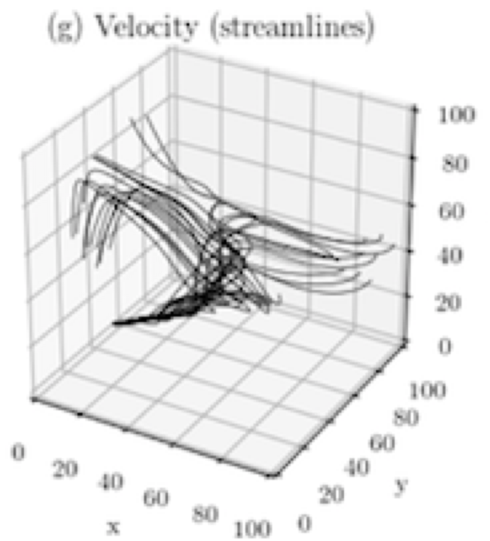
U-turns : experiments



Debue PhD Thesis (2019); ; Debue et al, JFM (2020)

Screw vortices : numerics

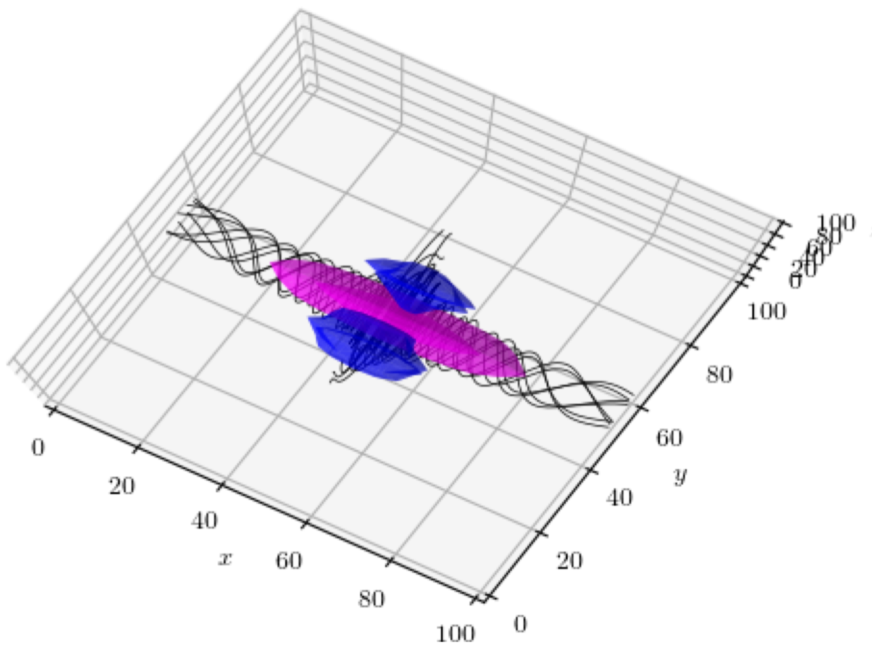




Nguyen PhD Thesis (2020); Nguyen et al, PRE (2020)

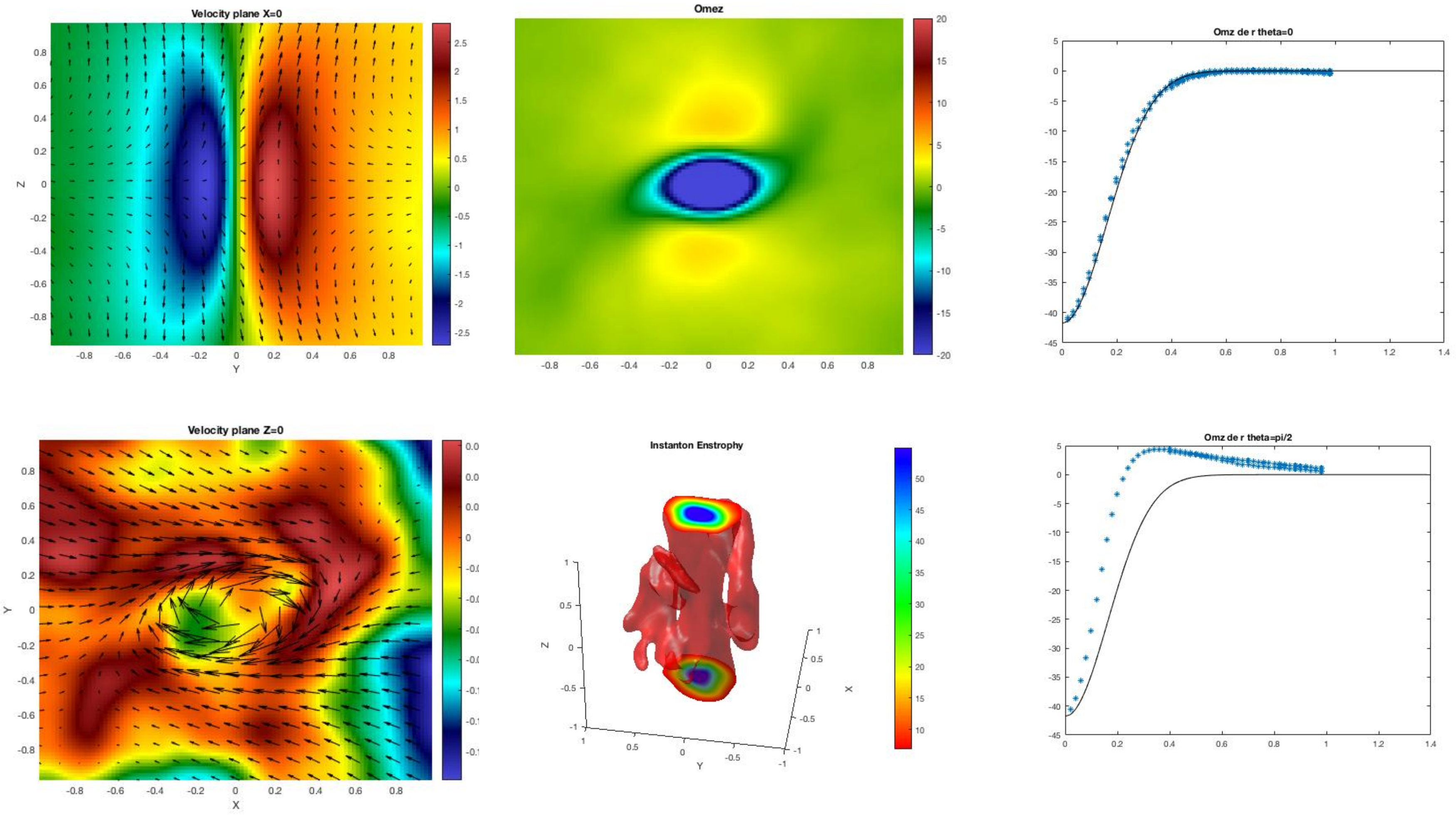
A « typical event » : numerics

Average over the 200 events with lower holder, after reorientation of event along same axis.



Nguyen PhD Thesis (2020); Nguyen et al, PRE (2020)

A « typical event » compared with Burgers vortex

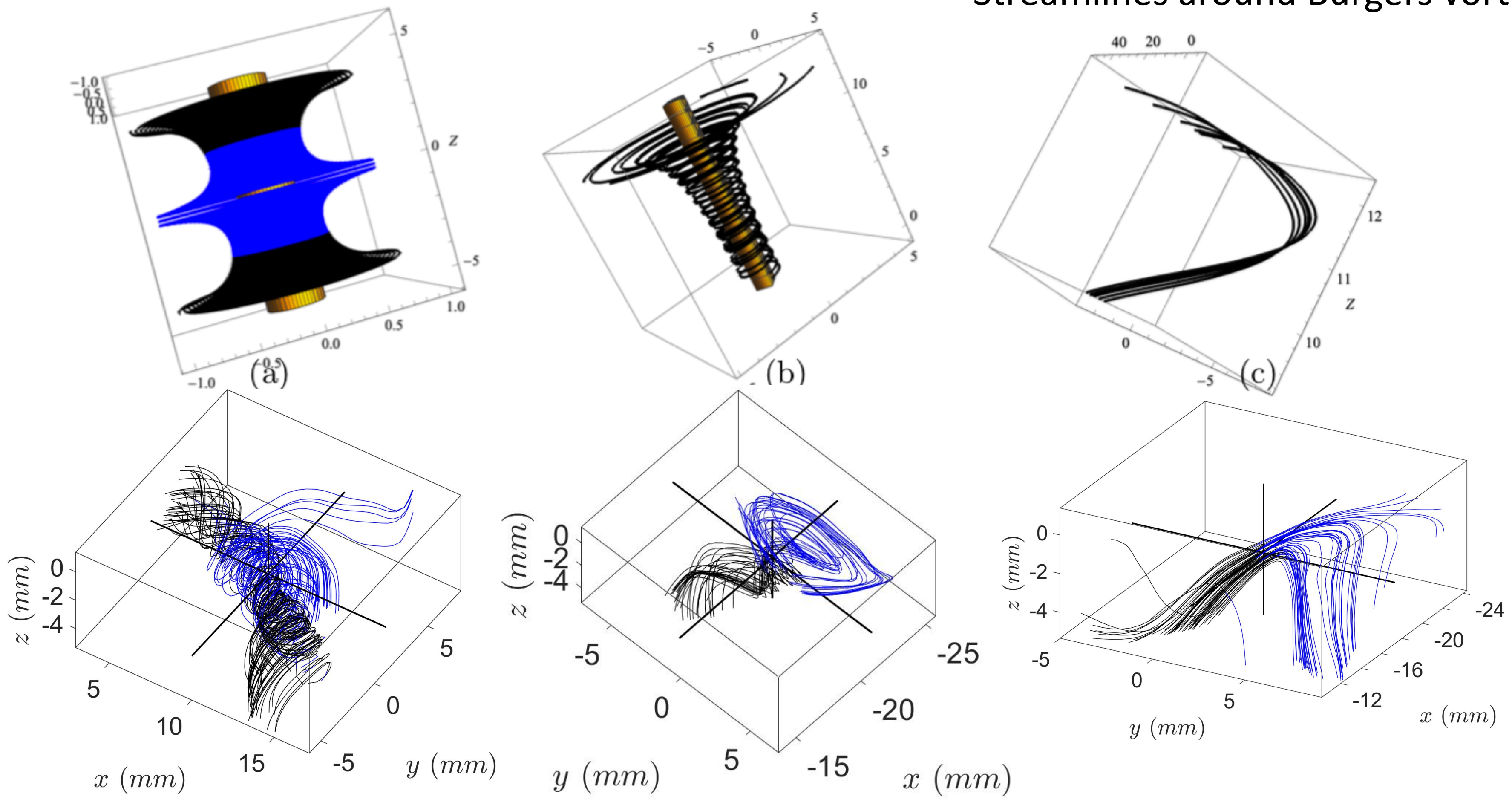


Comparison with Burgers vortex

Nguyen PhD Thesis (2020); Nguyen et al, PRE (2020)

Are we seeing Burgers vortices?

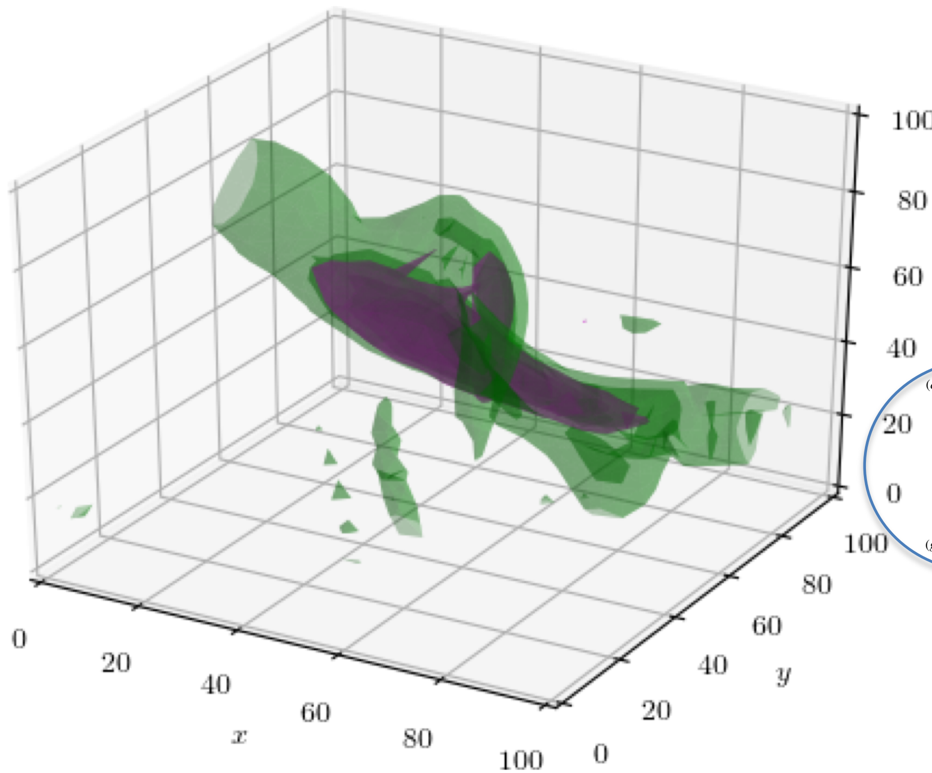
Streamlines around Burgers vortex

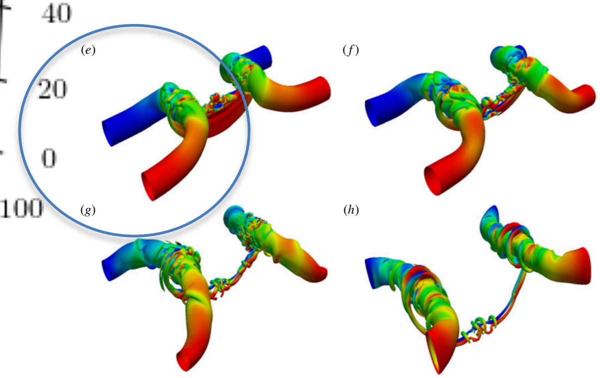


Moffatt, Focus on Fluids (2020)

Link with reconnection? Numerics

Iso-surface of Holder exponent



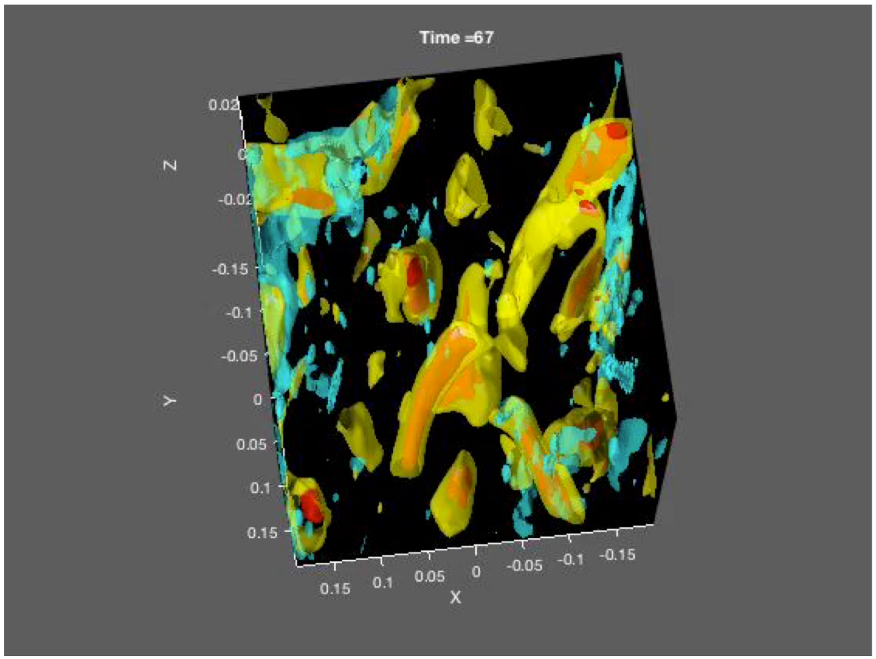


Nguyen PhD Thesis (2020); Nguyen et al, PRE (2020)

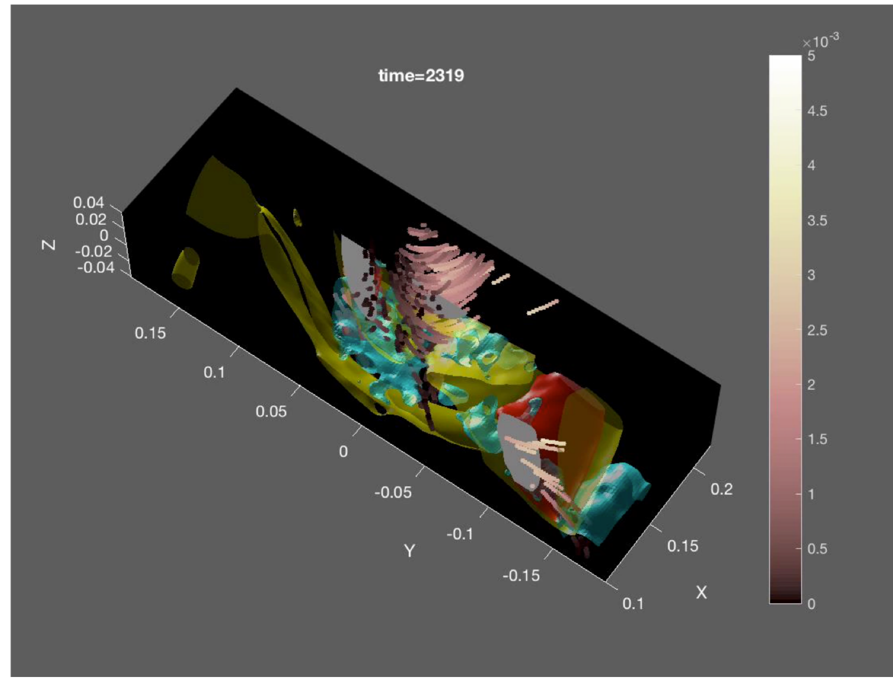
Link with reconnection? Numerics

Nguyen PhD Thesis (2020); Nguyen et al, PRE (2020)

Link with reconnection? Experiment



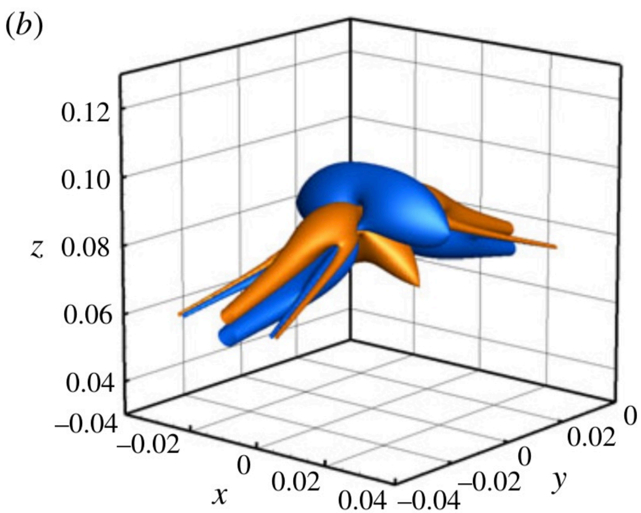
Eulerian

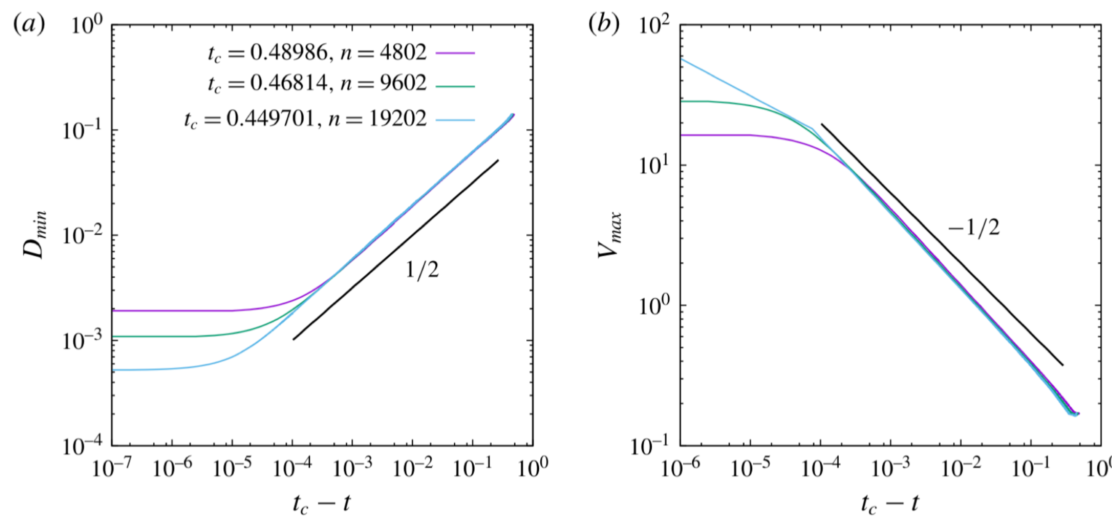


Eulerian+Lagrangian

Geneste et al, in preparation (2021)

Reconnection and Blow-up





Biot-Savart self-similar evolution
Blow-up?

Kimura&Moffatt JFM (2017), (2020)

Reconnection and Blow-up: Numerics

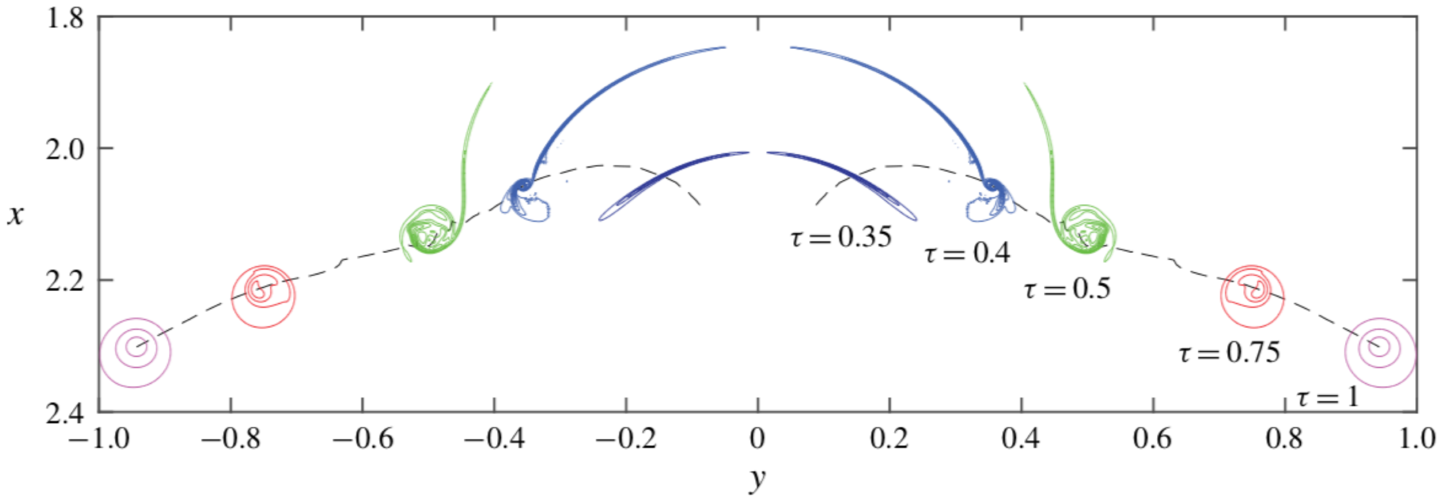


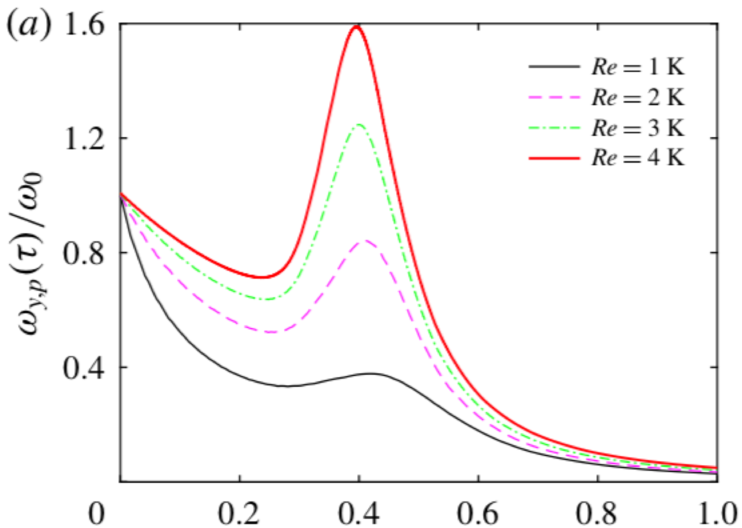
FIGURE 4. Evolution of the bridge vortex core (represented by the vorticity iso-contours $\omega_z = [0.05 : 0.1 : 2]\omega_0$) on the S_c plane for $Re = 4000$. The dashed line (- - - -) denotes the vortex core trajectory; see the supplementary movie for the full-time evolution of the vortex core for different Re cases.

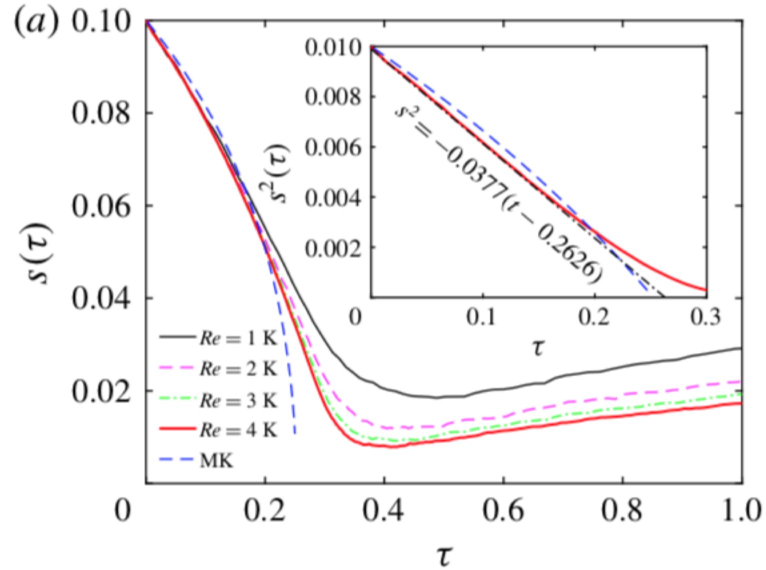
DNS simulation of rings reconnection

No blow-up due to vortex core deformation, but…

Yao&Hussain JFM (2020)

Reconnection and Blow-up: Numerics



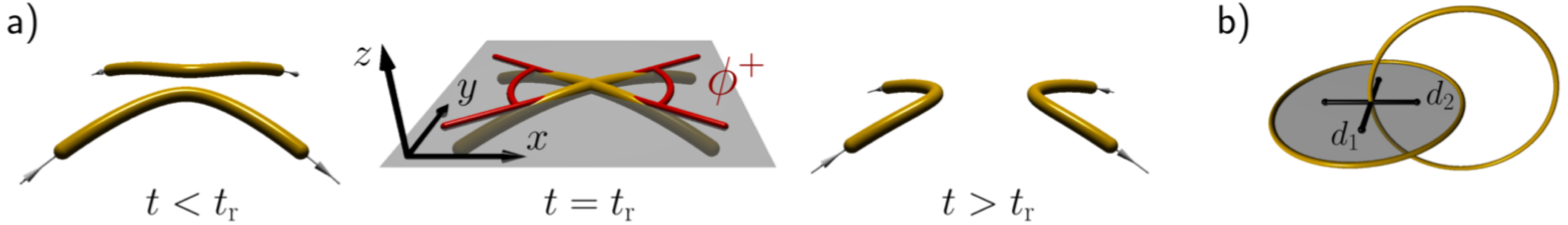


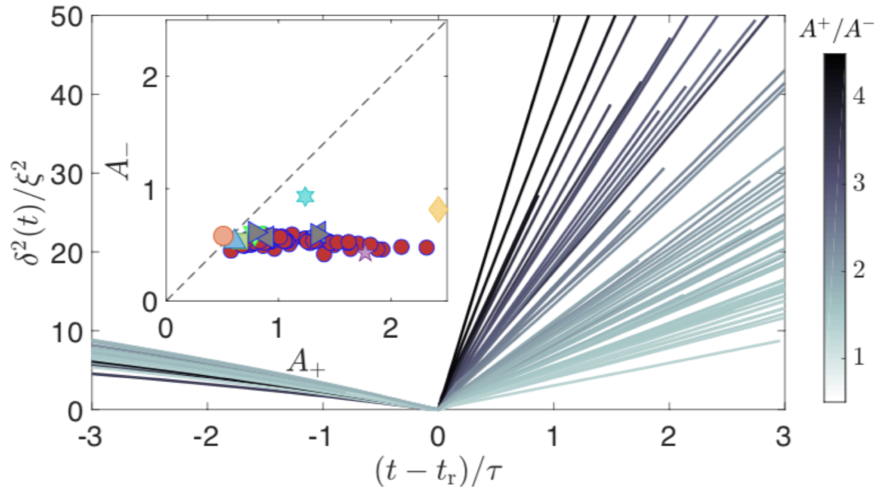
DNS simulation of rings reconnection

No blow-up due to vortex core deformation, but… self-similar behaviour prereconnection

Yao&Hussain JFM (2020)

Reconnection and Blow-up: Gross-Pitaievsky

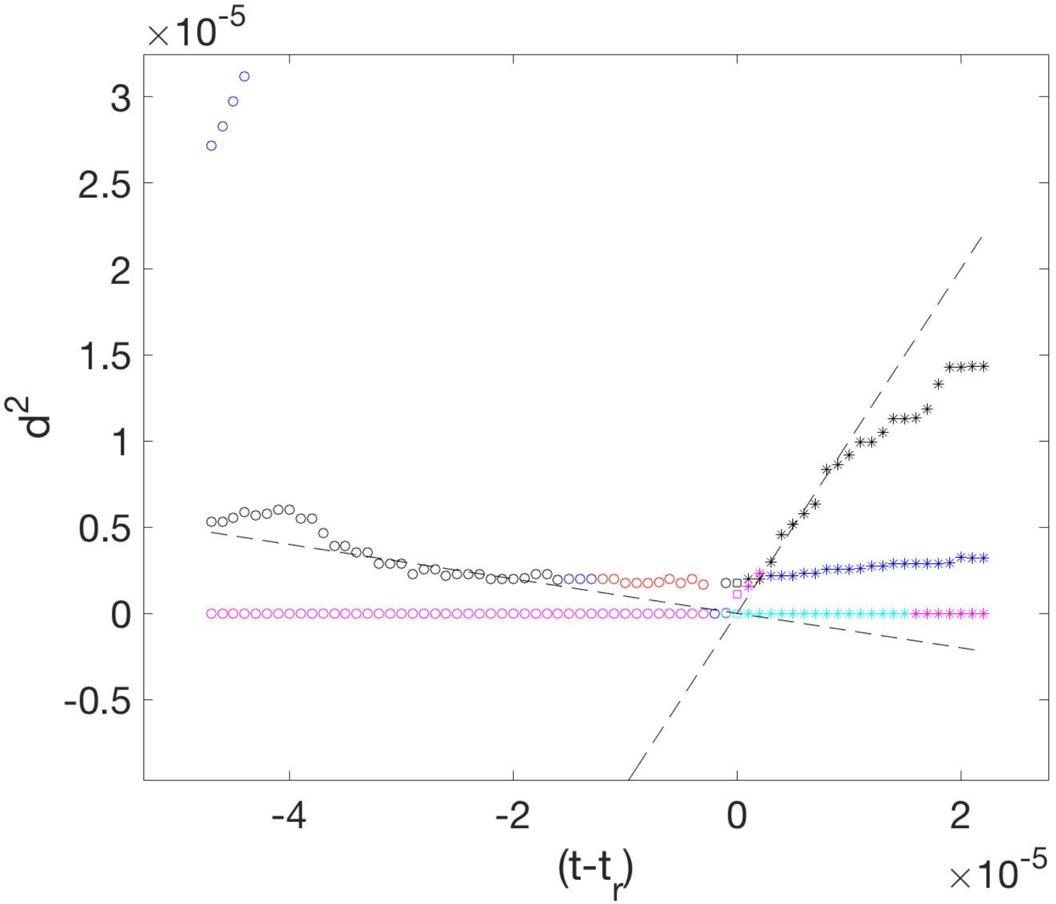




Self-similar behaviour behaviour pre and after reconnection
Irreversibility, spontaneous stochasticity

Villois et al preprint (2020)

Link with reconnection? Experiment



Geneste et al, in preparation (2020)

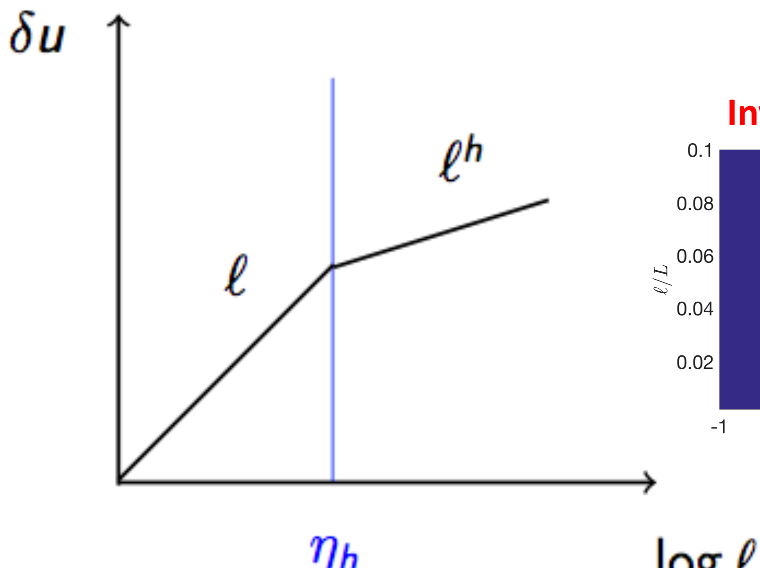
Have we found a singularity of NS?

$$\partial_t E^\ell + \partial_j J_j^\ell = -\frac{1}{4}\int \nabla\phi^\ell(\xi)\cdot\delta\mathbf{u}(\delta u)^2 d\xi + \nu\partial^2 E^\ell$$
$$\equiv -D_\ell^I - D_\ell^\nu,$$

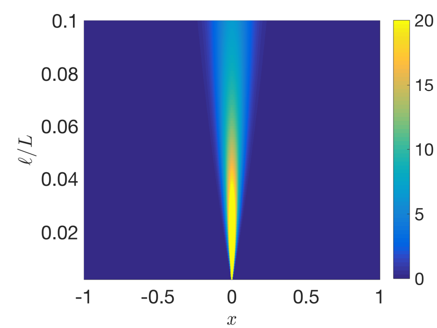
$$\delta u \approx \ell^h \Rightarrow D_\ell^I \approx \ell^{3h-1} \; and \; D_\ell^\nu \approx \nu\ell^{2h-2}$$

$$\frac{D_\ell^I}{D_\ell^\nu} \approx \frac{\ell^{3h-1}}{\nu\ell^{2h-2}} \approx \frac{\ell^{h+1}}{\nu} \Rightarrow \eta_h \approx \mathrm{Re}^{-1/(1+h)}$$

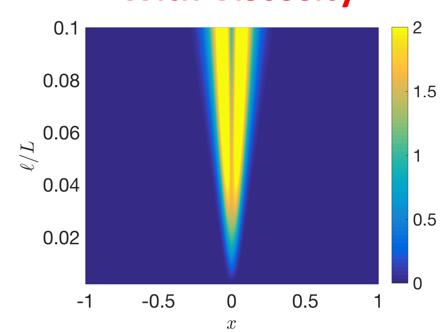
$\eta_{-1}=0$



Inviscid limit



With Viscosity



Quasi-singularity if h>-1

Kuzzay et al et al. (2017), Nonlinearity

Have we found a singularity of NS?

$$\partial_t E^\ell + \partial_j J_j^\ell = -\frac{1}{4}\int \nabla\phi^\ell(\xi)\cdot\delta\mathbf{u}(\delta u)^2 d\xi + \nu\partial^2 E^\ell$$
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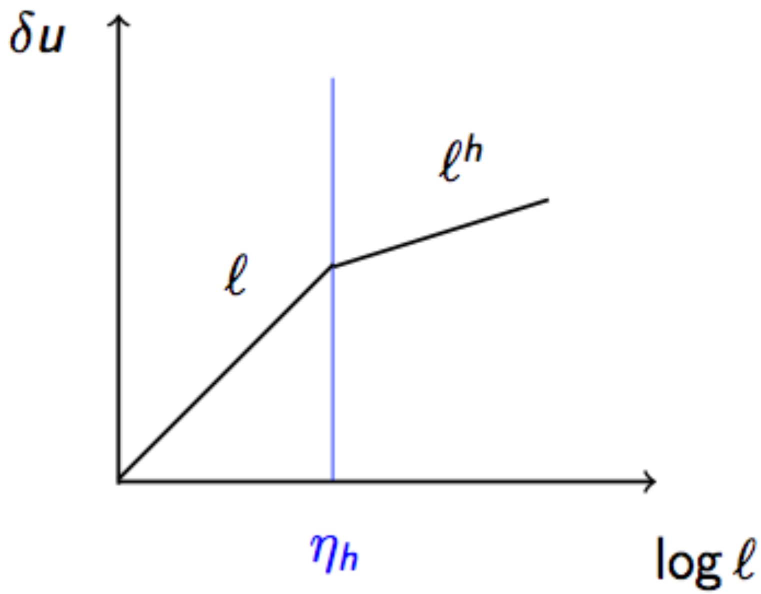
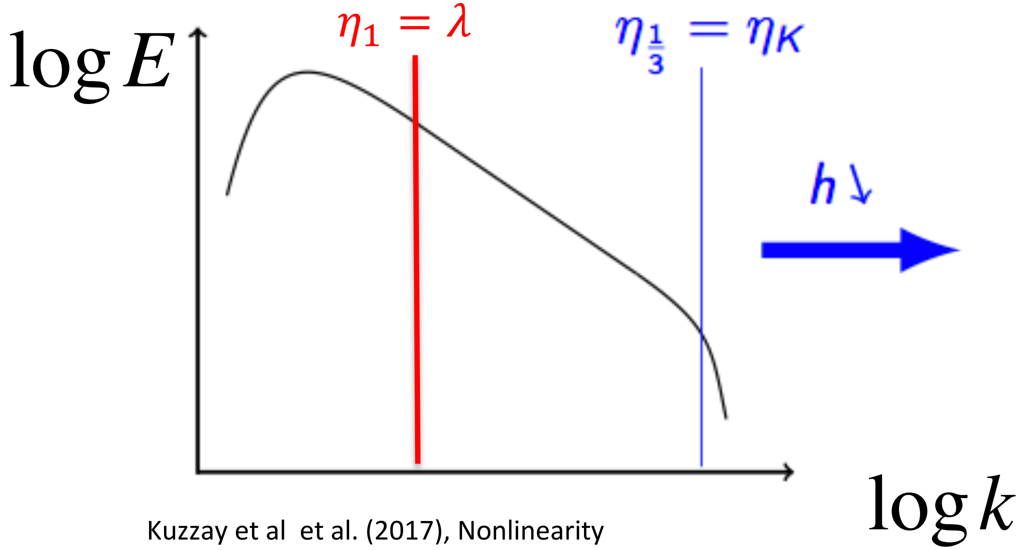
$\eta_{-1}=0$





Kuzzay et al et al. (2017), Nonlinearity

Exploring smaller scales with GVK experiment

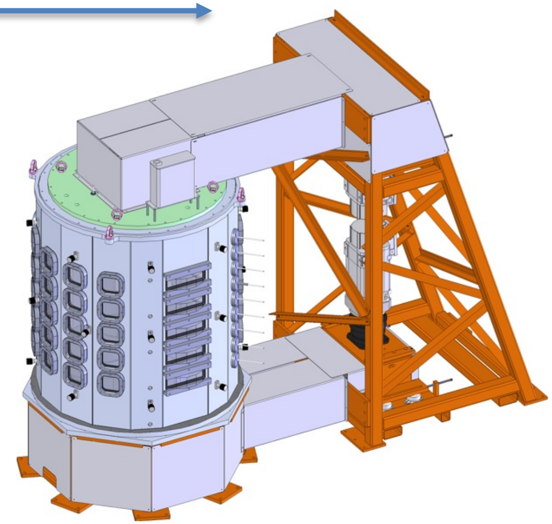
20 cm

Present experiment R=10 cm

Re=10^6
η=0,01 mm

Re=6×10^3
η=0,4 mm
$\Delta x \sim \eta$

1 m



GVK experiment R=50 cm

Re=10^6
η=0,05 mm

Re=6×10^3
η=2 mm
$\Delta x \sim \eta/5$

Possibility to explore sub-Kolmogorov scales
Detection of stronger velocity gradients and quasi-singularities

Perspectives

- Spontaneous stochasticity (ongoing work)
- Look for self-similarity (time resolved meas.)
- Go towards smaller scale (GVK)
- Strengthen link with reconnection
- ?????

- Inputs from mathematicians=welcome!