Loop Equation in Decaying Turbuence

Alexander Migdal[1]

[1]Department of Physics, New York University Abu Dhabi,
PO Box 129188, Saadiyat Island, Abu Dhabi, United Arab Emirates,

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Abstract

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We investigate the exact solution of the loop equation for decaying turbulence. This equation (the Euler ensemble of rational numbers coupled with the Ising spin chain), is also equivalent to a quantum partition function of N Fermi particles on a ring. Combining the number theory methods with the supercomputer simulation of this 1D system at $N = 200,000,000$, we compute various quantities in the statistical limit $N \to \infty$, including the energy decay $(1/t)$ and energy spectrum of decaying turbulence. This spectrum is **discrete**, but it approximates power laws on average. These quantum effects modify the paradigm of multifractal scaling.

Introduction

The turbulence problem looks deceptively simple: find the limit of the solution of the Navier-Stokes equations when viscosity goes to zero at a fixed energy dissipation rate.

$$\partial_t \vec{v} = -\nu \vec{\nabla} \times \vec{\omega} + \vec{v} \times \vec{\omega} - \vec{\nabla} \left(p + \frac{\vec{v}^2}{2} \right); \tag{1}$$

$$\vec{\nabla} \cdot \vec{v} = 0; \tag{2}$$

In this limit, the Navier-Stokes equation tends to the Euler equation everywhere except some singular regions: Vortex sheets and lines, where large velocity gradients could compensate the factor of ν.

These regions are randomly distributed in space, making velocity and vorticity stochastic variables at every point, with local vorticity values divergent in the turbulent limit.

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Hopf Functional

Hopf outlined the general approach to the turbulence problem using the functional

$$Z[\vec{J}] = \left\langle \exp\left(\int_{\vec{r}\in\mathbb{R}_d} \vec{J}(\vec{r}) \cdot \vec{v}(\vec{r}) \right) \right\rangle \tag{3}$$

This functional generates the correlation functions of the velocity field and satisfies the functional differential equation of the form

$$\partial_t Z[\vec{J}] = \hat{H}\left[\vec{J}, \frac{\delta}{\delta \vec{J}}\right] Z[\vec{J}] \tag{4}$$

The turbulence corresponds to a degenerate fixed point of the Navier-Stokes dynamics for Z, in the same way as the Gibbs distribution is a degenerate fixed point $Z = \delta(E - H(\vec{p}, \vec{q}))$ of Newton's dynamics (independent of the position at the energy surface $H(\vec{p}, \vec{q})$).

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The decaying turbulence is a degenerate fixed trajectory, slowly approaching the stable fixed point at zero velocity due to friction forces represented by viscosity.

The Hopf equation for the Navier-Stokes dynamics is compatible with such a trajectory, but it is too general and too complex to compute anything.

Its complexity is equivalent to the non-Gaussian functional integral, misplacing turbulence in the same category as critical phenomena in statistical physics.

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Loop Average and dimension reduction

The loop average is a particular case of the Hopf functional with the source $\vec{J}(\vec{r})$ concentrated on a fixed loop in space

$$\vec{J}_C(\vec{r}) = \frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta)\delta\left(\vec{r} - \vec{C}(\theta)\right) \tag{5}$$

The loop average is defined as

$$\Psi[\gamma, C] = \left\langle \exp\left(\int_{\vec{r}\in\mathbb{R}_d} \vec{J}_C(\vec{r}) \cdot \vec{v}(\vec{r})\right)\right\rangle = \left\langle \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle; \tag{6}$$

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Loop equation as dimension reduction

We derived a closed functional equation for the loop average in incompressible Navier-Stokes equation**M93, M23PR**

$$\imath\nu\partial_t\Psi[\gamma, C] = \left\langle \gamma \oint d\vec{C}(\theta) \cdot \left(-\nu\vec{\nabla} \times \vec{\omega} + \vec{v} \times \vec{\omega}\right) \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta\vec{C}(.)}\right]\Psi[\gamma, C] \quad (8)$$

The operator $\vec{L}\left[\frac{\delta}{\delta\vec{C}(.)}\right]$ only depends on the functional derivative, but does not depend on the coordinate $\vec{C}(.)$ in loop space.

This independence (translation invariance) is the key to the solution.

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A plane wave in loop space solves this Schrödinger equation

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta) \cdot \vec{P}(t,\theta)\right)\right\rangle; \tag{9}$$

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The WKB approximation

The simplest thing to do with the Schrödingen equation beyond perturbation theory is the WBK approximation.

In the case of the loop equation, the WKB approximation produced the Area law **M93, M23PR**

$$\Psi[\gamma, C] \to f(\gamma, A_{min}[C]); \tag{12}$$

$$P[\Gamma, C] = \int_{-\infty}^{\infty} d\gamma \Psi[\gamma, C] \exp\left(-\imath \gamma \Gamma / \nu\right) \to \frac{\exp\left(a - b|\Gamma| / \sqrt{A_{min}[C]}\right)}{\sqrt{|\Gamma|}} \tag{13}$$

This Area law was verified in remarkable DNS by Sreeni and Kartik Iyer in 2019, 2020 in ten decades of PDF tail. The scaling law $\sqrt{A_{min}[C]}$ was recently confirmed by Kartik Iyer.

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Exact solution

An analytic solution was found a few months ago.

Surprisingly, its parameters are random integers.

$$\vec{P}(t,\theta) = \sqrt{\frac{\nu}{2(t+t_0)}}\hat{\Omega} \cdot \frac{\vec{F}(\theta)}{\gamma};\ \hat{\Omega} \in O(3); \tag{14}$$

$$\vec{F}_k = \frac{\left\{\cos(\alpha_k), \sin(\alpha_k), i \cos\left(\frac{\beta}{2}\right)\right\}}{2 \sin\left(\frac{\beta}{2}\right)}; \tag{15}$$

$$\theta_k = \frac{2\pi k}{N};\ \beta = \frac{2\pi p}{q};\ N \to \infty; \tag{16}$$

$$\alpha_{k+1} = \alpha_k + \sigma_k \beta;\ \sigma_k = \pm 1,\ \beta \sum \sigma_k = 2\pi p r; \tag{17}$$

The parameters $\hat{\Omega}, N, p, q, r, \sigma_0 \dots \sigma_{N-1}$ are random, making this solution for $\vec{F}(\theta)$ a fixed manifold rather than a fixed point.

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Euler ensemble

This is not a toy model like the 1D Burgers equation but an exact stochastic solution of the 3D Navier-Stokes equation.

We treat it as a quantum statistical system with a chemical potential $\mu \to 0$ (the Euler ensemble). The partition function is calculable

$$Z(\mu) = \sum_N e^{-\mu N} \sum_{2<q<N} \varphi(q) \sum_{\substack{r \\ 2|(N-qr)}} 2^{-N} \binom{N}{(N+qr)/2}; \quad (18)$$

$$\varphi(q) = q \prod_{p|q} \left(1 - \frac{1}{p}\right); \quad (19)$$

$$Z(\mu) \to \frac{9}{4\sqrt{2}\pi^2 \mu^{5/2}}; \quad (20)$$

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Enstrophy (exact formula)

The enstrophy

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Local Vorticity moments $(\mu \to 0)$

Anomalous dissipation

$$\partial_t E = -\nu \left\langle \vec{\omega}(\vec{0})^2 \right\rangle = -\frac{\mathcal{B}\nu}{\mu^2 (t + t_0)^2}; \tag{25}$$

$$\mathcal{B} = \frac{35\pi^2}{13824\zeta(3)}; \tag{26}$$

Vorticity moments

$$\left\langle \omega(\vec{0})^{2n} \right\rangle \to \frac{\Xi_n}{(t + t_0)^{2n} \mu^{3n-1}} \quad \text{if } n > 0; \tag{27}$$

$$\Xi_n = \frac{\pi^{\frac{3}{2}-2n} 2^{3-5n} \zeta(2n) \Gamma\left(3n + \frac{3}{2}\right)}{9(2n+1)\zeta(2n+1)\Gamma(n+1)} \tag{28}$$

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Markov process

Our system is less than one-dimensional.

This is a periodic Markov process with just two integer variables.

$$n_+ = N_+;\ n_- = N_-; \tag{29}$$

$$\mathbb{P}(n_+ \Rightarrow n_+ - 1) = \frac{n_+}{n_+ + n_-};\ \sigma = +1; \tag{30}$$

$$\mathbb{P}(n_- \Rightarrow n_- - 1) = \frac{n_-}{n_+ + n_-};\ \sigma = -1; \tag{31}$$

This Markov property leads to $O(N^0)$ memory in simulation, which allows simulations of astronomically large systems. Maxim Bulatov and I simulated $N = 2 * 10^8$ at NYU AD GPU cluster with $T \sim 2 * 10^8$ random samples of the Euler ensemble.

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Conclusions

Our solution for the loop average in decaying turbulence shows nonperturbative effects, which are missing in the weak turbulence ($1/\nu$ expansion), particularly the quantization of the solution's parameters.

These quantum effects follow from the mathematical equivalence of the Navier-Stokes statistics to the quantum mechanics in loop space.

The quantum statistical system, corresponding to the solution of the Schrödinger equation in loop space, can be regarded as a **dual** to the turbulent velocity field theory; in the same sense, quantum gravity is dual to the gauge theory in ADS/CFT correspondence: correlation functions coincide, though dynamical variables differ.

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Compared to the other critical phenomena, this theory is quite simple: It is not a field theory but a quantum statistical system isomorphic to a periodic Markov process with two integer variables.

The spectrum of anomalous dimensions presents a formidable mathematical problem in conformal bootstrap. Still, the spectral equation in this theory is explicitly calculated at any finite N.

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The solution is conceptually simple but technically involved. It reveals unexpected relations between turbulence and number theory.

In particular, the anomalous energy dissipation constant $\mathcal{B}$ is related to the prime factorization of large integers.

The analytic computation of the decay spectrum from the characteristic equation (33) in the local limit $N \to \infty$ remains a problem for the number theory.

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