Holomorphic curves and the ADHM vortex equations

Aleksander Doan

University College London & Trinity College, Cambridge

Vortex Moduli, International Centre for Theoretical Sciences, 15 Feb 2023

this talk is based on joint work with
Thomas Walpuski (Humboldt-Universität zu Berlin)

Curve counting

Old problem in algebraic geometry

Count holomorphic curves in a complex projective manifold X
(given genus, degree/homology class, additional constraints...)

Examples

Two points in $\mathbb{CP}^n$ determine a line.

There are 27 lines contained in a cubic surface.

There are 2875 lines contained in a quintic threefold...

The moduli space of curves of genus g in a homology class $A \in H_2(X, \mathbb{Z})$ has virtual dimension

$$\mathrm{vdim} = (\dim_{\mathbb{C}} X - 3)(1 - g) + \langle c_1(X), A \rangle$$

When $\mathrm{vdim} = 0$, we can try to count the curves.

Important case: Calabi–Yau threefolds

$$\dim_{\mathbb{C}} X = 3 \text{ and } c_1(X) = 0$$

Two problems:

- Transversality: moduli space can have incorrect dimension
- Compactness: curves can degenerate

Pseudo-holomorphic maps

More generally, let

(X, ω) – symplectic manifold, J – almost complex structure inducing a Riemannian metric

$$g(v, w) = \omega(v, Jw).$$

Gromov–Witten theory studies moduli spaces $\mathcal{M}_{g,A}(X, J)$ of pseudo-holomorphic maps of genus g and homology class A:

$$u : \Sigma \to X$$
$$\mathrm{d}u \circ j = J \circ \mathrm{d}u$$

Such maps are harmonic.

One defines a compactification

$$\mathcal{M}_{g,A}(X,J) \subset \overline{\mathcal{M}}_{g,A}(X,J)$$

by allowing the domains of maps to degenerate. When $\mathrm{vdim} = 0$, this leads to the Gromov–Witten invariants

$$\mathrm{GW}_{g,A}(X) \in \mathbb{Q}.$$

Most maps in $\overline{\mathcal{M}}_{g,A}(X,J)$ are not embeddings, for example

- Multiple covers: $\tilde{\Sigma} \to \Sigma \to X$,
- Ghosts: Maps constant on some components of Σ

Question

Are there symplectic invariants which count embedded pseudo-holomorphic curves?

(e.g. similar to the invariant defined by Taubes in dimension four)

Theorem (with T. Walpuski)

X – compact symplectic manifold with

$$\dim_{\mathbb{R}} X = 6 \quad \text{and} \quad c_1(X) = 0.$$

For a generic J there are finitely many embedded pseudo-holomorphic curves in every homology class $A \in H_2(X, \mathbb{Z})$.

If A is a primitive class, then the signed count $n_{g,A}$ of curves of genus g and homology class A is independent of J, and defines a symplectic invariant of X.

(There is a generalization to arbitrary symplectic six-manifolds.)

Idea of proof

- For a generic J, the moduli space of curves is discrete. Moreover, they are all embedded and pairwise disjoint.
- By contradiction, assume there is a sequence of curves.
- By Federer's Compactness Theorem and work of DeLellis–Spadaro–Spaloar there is a limit current

$$\sum_{i=1}^{n} m_i \delta_{C_i}, \quad m_i \in \mathbb{N}.$$

For a generic J we must have $n = 1$ and the limit is $m\delta_C$.

- Idea of Taubes:
  Rescale the sequence in the normal direction to C. Take limit to get another curve $\tilde{C}$ in the normal bundle of C.
- $\tilde{C}$ is the graph of a multi-valued pseudo-holomorphic section of the normal bundle of C.
- Existence of such sections is a non-generic phenomenon. This is the content of the super-rigidity conjecture proved by Wendl.

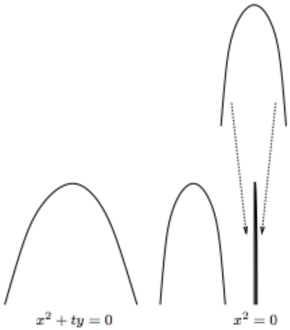
This proves that there are finitely many curves for a generic J.

Why is their count independent of J if A is primitive?

The proof fails when J varies in a family $(J_t)_{t\in[0,1]}$ because multi-valued pseudo-holomorphic sections can appear.

This is related to multiple covers.

Suppose that $A = mB$. A sequence of pseudo-holomorphic curves in the class A can collapse to an m–fold branched cover of a curve in the homology class B. The m–valued section of the normal bundle remembers the infinitesimal direction of this collapse.



This is why the naive count of curves $n_{g,A}$ is not independent of J for a general A.

If A is primitive, there are no multiple covers. You still have to worry about degenerations to ghosts but with some work you can rule those out too using gluing theory for pseudo-holomorphic maps.

$\implies n_{g,A}$ is independent of J if A is primitive. $\square$

Digression: Gopakumar–Vafa invariants

Our result is closely related to

The Gopakumar–Vafa conjecture

(a) the Gromov–Witten invariants $\mathrm{GW}_{g,A}(X)$ can be expressed by an explicit formula in terms of integer invariants $\mathrm{BPS}_{g,A}(X) \in \mathbb{Z}$

(b) these integer invariants satisfy

$$\mathrm{BPS}_{g,A}(X) = 0 \quad \text{for } g \gg 1.$$

Part (a) was proved by Ionel–Parker in 2018.

Zinger proved that for a primitive class $A \in H_2(X, \mathbb{Z})$,

$$\mathrm{BPS}_{g,A}(X) = n_{g,A}(X)$$

where the right-hand side is our "naive count". Therefore, our finiteness result proves (b) when A is a primitive homology class.

Theorem (with E. Ionel and T. Walpuski)

Part (*b*) of the Gopakumar–Vafa conjecture holds.

For A non-primitive

$$\mathrm{BPS}_{g,A}(X) \neq n_{g,A}(X)$$

but we can use tools from geometric measure theory, as we did to prove the earlier theorem, to conclude finiteness of $\mathrm{BPS}_{g,A}(X)$.

In particular, these two theorems imply that

$$\mathrm{BPS}_{g,A}(X) = \sum_{d} \sum_{[C]=A/d} w_{g,d}(C, J)$$

for some integer weights $w_{g,d}$ depending on J.

What is the geometric meaning of these weights?

cf. recent work of Bai–Swamanathan

End of digression.

Relation to gauge theory

We want to correct the naive count to get an invariant.

Let $A = 2B$ with B primitive. Look for invariant of the form

$$PT_A(X, J) = \sum_{[C]=A} w_1(C, J) + \sum_{[C]=B} w_2(C, J)$$

w_1, w_2 = integer weights depending on J

Let (J_t) be a family such that as $\tilde{C} \to 2C$ as $t \to 1/2$.

$$[\tilde{C}] = A, \qquad [C] = B.$$

We want that

$$w_1(\tilde{C}, J_0) + w_2(C, J_0) = w_2(C, J_1)$$

How to find such weights?

The degeneration $\tilde{C} \to 2C$ happens when there exists a two-valued J–holomorphic section of the normal bundle:

$$C \to \mathrm{Sym}^2 N_{C/X}.$$

Therefore, we want $w_2(C, J)$ to change precisely whenever such a section exists.

A similar problem was studied in G_2 geometry by Haydys and Walpuski, following ideas of Donaldson, Thomas, and Segal.

They proposed to count calibrated 3–manifolds inside 7–dimensional G_2 manifolds with weights given by counting solutions to non-abelian Seiberg–Witten equations.

By consider the equations over $C \times \mathbb{R}$ we get equations on C.

In conclusion, we want to define $w_2(C, J)$ by counting solutions to non-abelian vortex equations on C depending on J:

$$\begin{cases} \bar{\partial}_{J,A}\xi = 0, \quad \bar{\partial}_A\alpha = 0, \quad \bar{\partial}_A\beta = 0 \\ [\xi \wedge \xi] + \alpha \cdot \beta = 0 \\ i * F_A + [\xi \wedge \xi^*] + \alpha\alpha^* - \beta^*\beta = 0 \end{cases}$$

ξ is a section of $N_{C/X} \otimes \mathrm{End}\, E$ for a rank two bundle $E \to C$

A is a $\mathrm{U}(2)$ connection on E

(They are called the ADHM vortex equations because they are closely related to the ADHM construction of instantons on $\mathbb{R}^4$.)

The only way for the count of solutions to change is when there is a sequence of solutions with $\|\xi\|_{L^2} \to \infty$ (and A, α, β bounded). After rescaling, this leads to

$$\xi \in \Gamma(N_{C/X} \otimes \mathrm{End} E)$$
$$[\xi \wedge \xi] = 0,$$
$$[\xi \wedge \xi^*] = 0.$$

Here $E \to C$ is a rank two bundle. By Hitchin's spectral curve construction, such data is equivalent to a section $C \to \mathrm{Sym}^2 N_{C/X}$.

Conjecture (work in progress with T. Walpuski)

1. For generic J, we can define

$w_k(C, J) =$ count of ADHM vortices with structure group $\mathrm{U}(k)$

2. The sum

$$PT_A(X, J) = \sum_{k \mid A} \sum_{[C]=A/k} w_k(C, J)$$

is independent of J and defines a symplectic invariant of (X, ω).

3. If (X, J) is projective, then $PT_A(X, J)$ agrees with the Pandharipande–Thomas invariant defined using sheaf theory.

To prove that $w_k(C, J)$ is well-defined we need to study the compactness problem for these equations (following Taubes, Haydys–Walpuski, Walpuski–Zhang).

The main issue is the appearance of singular sets in the limit $\|\xi\|_{L^2} \to \infty$. Such a singular set corresponds to the branching locus of a section $C \to \mathrm{Sym}^k N_{C/X}$.

To prove that $\mathrm{PT}_A(X, J)$ is independent of J, we need to combine this analysis with deformation and compactness theory for curves. Wendl's work on super-rigidity and methods of geometric measure theory will play an important role in this part.

Thank you for your attention!