Dimension drop conjecture in homogeneous dynamics

Dmitry Kleinbock,
joint with Shahriar Mirzadeh

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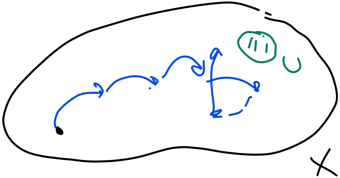
Abstract

Happy Birthday, Professor Dani!

Basic set-up



- X a metric space
- μ a probability measure on X of full support
- F an infinite set of self-maps $X \to X$
- U a non-empty subset of X

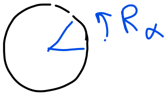
Define the set

$$E(F,U) := \{x \in X : \overline{Fx} \cap U = \varnothing\}$$

of points in X whose F-trajectory stays away from U.

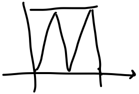
If F is a group or semigroup of μ-preserving transformations acting ergodically on (X,μ), it follows that $\mu\big(E(F,U)\big) = 0$.

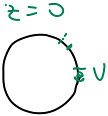
Example 1. $X = \mathbb{S}^1$, $\mu =$ Lebesgue,
$F = \langle$rotation by $\alpha \notin \mathbb{Q}\rangle$.



Then for any $z \in X$, $E(F, \{z\}) = \varnothing$.

Example 2. $X = \mathbb{S}^1$, $\mu =$ Lebesgue,
$F = \langle$multiplication by 2 mod 1$\rangle$.





Then for any $z \in X$, $E(F, \{z\})$, although still null, is quite big (full Hausdorff dimension). Equivalently,

$$\dim E\big(F, B(z,r)\big) \to 1 \text{ as } r \to 0.$$

In the latter set-up it can be proved that

- $\dim E\big(F, B(z,r)\big) < 1$ whenever $r > 0$, and, moreover,
- $\operatorname{codim} E\big(F, B(z,r)\big) = 1 - \dim E\big(F, B(z,r)\big) \sim \text{const}_z \cdot \mu\big(B(z,r)\big)$

[Bunimovich–Yurchenko, Ferguson–Pollicott]

In this work we consider dynamical systems on homogeneous spaces:

- G a Lie group
- Γ a lattice in G
- $X = G/\Gamma$
- μ the G-invariant probability measure on X
- $F \subset G$ a semigroup acting on X by left translations

The two examples above give rise to two special cases:
unipotent and partially hyperbolic flows.

Ratner's Theorems,
Dani–Margulis linearlization:
$\{x \in X : Fx \text{ is not dense}\}$
is contained in a countable union
of proper submanifolds of X

"$E(F,z)$ is small"

exponential expansion
of unstable leaves:
full Hausdorff
dimension
of $E(F,\{z\})$

The study of exceptional orbits of partially hyperbolic homogeneous flows was initiated by Dani in the 1980s. Let us state one of Dani's observations from his 1985 paper: for an $m \times n$ matrix A denote

$$u_A := \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} \in G = \mathrm{SL}_{m+n}(\mathbb{R});$$

then the trajectory $\{g_t u_A \mathbb{Z}^{m+n} : t \geq 0\}$, where

$f = g_{\mathbb{R}_+}$

$$g_t = \mathrm{diag}(\underbrace{e^{t/m}, \ldots, e^{t/m}}_{m \text{ times}}, \underbrace{e^{-t/n}, \ldots, e^{-t/n}}_{n \text{ times}}),$$

is bounded in the space of unimodular lattices in $\mathbb{R}^{m+n}$, i.e.

$$u_A \mathbb{Z}^{m+n} \in E(g_{\mathbb{R}_+}, \infty)$$

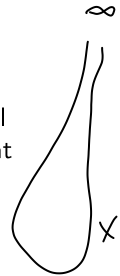
$X = \mathrm{SL}_{m+n}(\mathbb{R}) / \mathrm{SL}_{m+n}(\mathbb{Z})$

$\Updownarrow$ (Dani Correspondence)

A is badly approximable, that is, $A \in \mathbf{BA}_{m,n} :=$

$$\left\{ A : \exists\, c > 0 \text{ s.t. } \|A\mathbf{q} + \mathbf{p}\|^m \|\mathbf{q}\|^n \geq c \quad \forall\, \mathbf{p} \in \mathbb{Z}^m,\ \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\} \right\}.$$

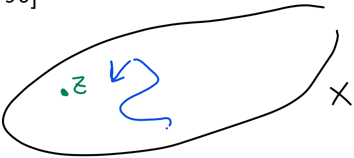
[Schmidt '69]: $\mathbf{BA}_{m,n}$ is a winning set,
in particular it has full Hausdorff dimension.



This, and the fact that $\{u_A\}$ is the expanding horospherical subgroup of G relative to g_1, enabled Dani to conclude that

$$E(g_{\mathbb{R}_+}, \infty) \text{ has full Hausdorff dimension.}$$

In a follow-up paper Dani modified Schmidt's argument to prove a similar statement for quotients of rank 1 Lie groups. For arbitrary partially hyperbolic flows on higher rank spaces this was conjectured in [Margulis '90] and settled in [K–Margulis '96].



Another related result:

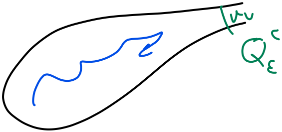
$\dim E(g_{\mathbb{R}_+}, \{z\})$ has full dimension $\forall\, z \in X = G/\Gamma$ [K '98].

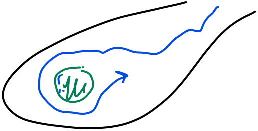
In other words, the aforementioned results state that

$$\lim_{\varepsilon\to 0} \dim E(g_{\mathbb{R}_+}, Q_\varepsilon^c) = \dim X,$$

where $\{Q_\varepsilon\}$ is a family of compact sets exhausting X, and

$$\lim_{\varepsilon\to 0} \dim E\big(g_{\mathbb{R}_+}, B(z,\varepsilon)\big) = \dim X.$$





But what about $E(g_{\mathbb{R}_+}, Q_\varepsilon^c)$ and $E\big(g_{\mathbb{R}_+}, B(z,\varepsilon)\big)$ for a fixed $\varepsilon > 0$?

Let us pose the following natural question (from Maryam Mirzakhani):

- if the F-action is ergodic and U is open and nonempty, does $E(F, U)$ necessarily have less than full dimension?

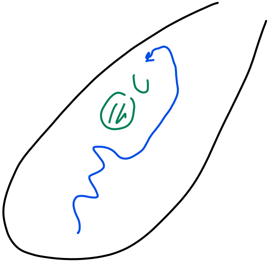
In fact it is reasonable to conjecture that the answer is 'yes'; in other words, that the following holds:

Dimension Drop Conjecture.

$F \subset G$ a subsemigroup, $U \subset X$ open

$$\Downarrow$$

either $E(F, U)$ has positive measure, or $\operatorname{codim} E\big(F, U\big) > 0$.

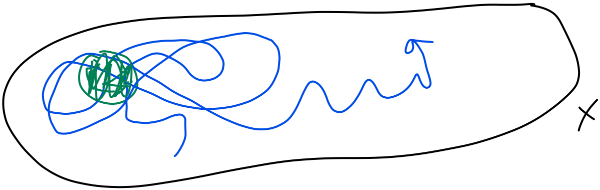


(In other words, we cannot have a proper closed F-invariant subset of full Hausdorff dimension.)

Remark. If $F = \{g_t : t \in \mathbb{R}_+\}$ or $\{g_t : t \in \mathbb{Z}_+\}$ (a one-parameter semigroup), we will also consider

$$\widetilde{E}(F, U) := \{x \in X : \exists\, N \in \mathbb{N} \text{ such that } g_t x \notin U \ \forall\, t \geq N\}$$
$$= \bigcup_{N \in \mathbb{N}} E(g_{\{t \geq N\}}, U)$$

eventually escape U



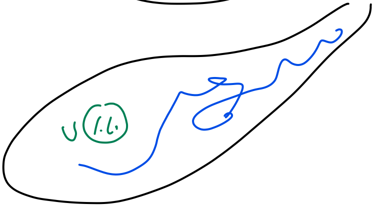
Clearly DDC as stated above implies the corresponding statement for the (bigger) set $\widetilde{E}(F, U)$.

Note: this conjecture is interesting only if

- F is partially hyperbolic (in view of Ratner etc.)
- X is not compact, that is, $\Gamma \subset G$ is non-uniform, by an argument due to Einsiedler–Lindenstrauss, see [K–Weiss '13]

(When X is compact, can use
Hausdorff dimension $\longleftrightarrow$ entropy,
variational principle,
uniqueness of measure of max entropy)

But when X is not compact,
the situation is more complicated
due to a (theoretical) possibility
of the 'escape of mass'.

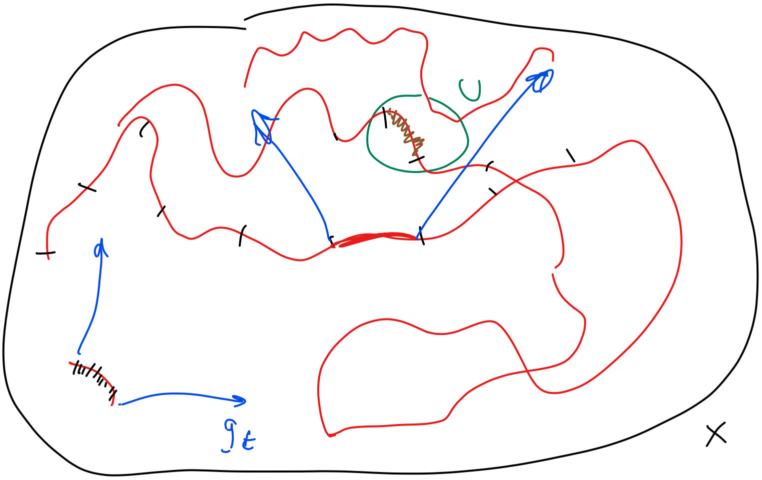


[Einsiedler–Kadyrov–Pohl '15]: the case $\mathrm{rank}_{\mathbb{R}} G = 1$

Another question: if the dimension is less than full, maybe the codimension of $E(F, U)$ can be explicitly estimated?

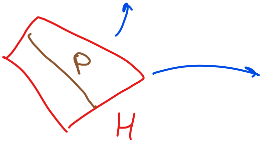
This is interesting even when X is compact.

Here is a rather simple idea how to do it,
first tested in [Broderick–K '15], then in [K–Mirzadeh '20].



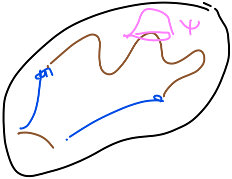
Details: let

- $F = \{g_t : t \geq 0\}$ be an Ad-diagonalizable one-parameter subsemigroup of G;
- $H := \{g \in G : \text{dist}(g_t g g_{-t}, e) \to 0 \ \ as \ \ t \to -\infty\}$ the unstable horospherical subgroup with respect to F;



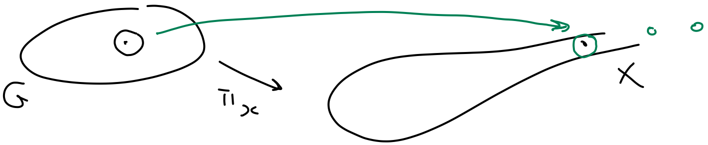
- P a connected subgroup of H normalized by F;
- ν the Haar measure on P normalized so that $\nu\big(B^P(r)\big) = 1$;
- if f a function on P and $t \geq 0$, define the integral operator $\mathcal{I}_{f,t}$ acting on functions ψ on X via

$$(\mathcal{I}_{f,t}\psi)(x) := \int_P f(h)\psi(g_t h x)\, d\nu(h)\,.$$



For $x \in X$ denote by π_x the map $G \to X$ given by $\pi_x(g) := gx$, and by $r_0(x)$ the injectivity radius of x:

$$r_0(x) := \sup\{r > 0 : \pi_x \text{ is injective on } B(r)\}.$$



If Q is a subset of X, let us denote by $r_0(Q)$ the injectivity radius of Q:

$$r_0(Q) := \inf_{x \in Q} r_0(x) = \sup\{r > 0 : \pi_x \text{ is injective on } B(r) \ \ \forall\, x \in Q\};$$

(positive if and only if Q is bounded).

Definition. Say that a subgroup P of G has

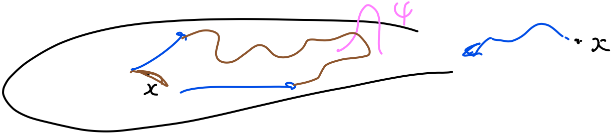
Effective Equidistribution Property (EEP) w.r.t. the flow (X, F)

if there exists constants $a, b, \lambda > 0$ and $\ell \in \mathbb{N}$ such that for any $x \in X$ and $t > 0$ with

$$t \geq a + b \log \frac{1}{r_0(x)},$$

any $f \in C^\infty(P)$ with $\operatorname{supp} f \subset B^P(1)$ and any $\psi \in C_2^\infty(X)$ it holds that

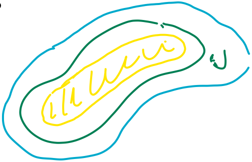
$$\left| (\mathcal{I}_{f,t}\psi)(x) - \int_P f \, d\nu \int_X \psi \, d\mu \right| \ll \max\left(\|\psi\|_{C^1}, \|\psi\|_{\ell,2} \right) \|f\|_{C^\ell} e^{-\lambda t}.$$



Fact: exponential mixing $\Longrightarrow$ (EEP) for $P = H$ [K–Margulis '96, '12].

Some more notation: for a subset U of X and $r > 0$ denote by $\sigma_r U$ the inner r-core of U, defined as

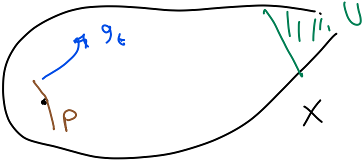
$$\sigma_r U := \{x \in X : \mathrm{dist}(x, U^c) > r\},$$



and by $\partial_r U$ the r-neighborhood of U by

$$\partial_r U := \{x \in X : \mathrm{dist}(x, U) < r\}.$$

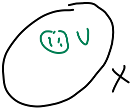
Theorem 1 [K–Mirzadeh '20]**.**
Let P have property (EEP) with respect to the flow (X, F). Then $\exists\ \rho > 0$ such that $\forall\, x \in X$, $\forall\, U \subset X$ such that U^c is compact and $\forall\, 0 < r < \min\big(r_0(\partial_{1/2} U^c), \rho\big)$ one has



no r

an extra δ

$$\mathrm{codim}\left\{g \in P : gx \in \widetilde{E}(F, U)\right\} \gg \frac{\mu(\sigma_r U)}{\log \frac{1}{r} + \log \frac{1}{\mu(\sigma_r U)}}.$$

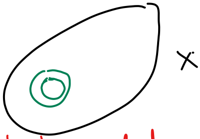


Corollaries:

- If X is compact and $U = B(x, r)$, then

$$\operatorname{codim}\left\{g \in P : gx \in \widetilde{E}(F, U)\right\} \gg \frac{r^{\dim X}}{\log(1/r)};$$

not needed



- Same estimate for the codimension of $\widetilde{E}(F, U)$ in X;

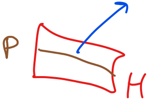
- For any $c > 0$ set

$\mathbf{i} = (i_1, \dots, i_m)$, $\mathbf{j} = (j_1, \dots, j_n)$, $\sum i_k = \sum j_l = 1$

$g_t = \operatorname{diag}(e^{i_1 t}, \dots, e^{i_m t}, e^{-j_1 t}, \dots, e^{-j_n t})$

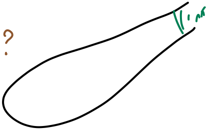
$$\mathbf{BA}_{\mathbf{i},\mathbf{j}}(c) := \left\{ A \in M_{m,n} : \inf_{\mathbf{p} \in \mathbb{Z}^m,\ \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \|A\mathbf{q} + \mathbf{p}\|_{\mathbf{i}} \, \|\mathbf{q}\|_{\mathbf{j}} \geq c \right\};$$

then $\exists\, c_0 > 0$ such that for any $\mathbf{i}, \mathbf{j}$ and any $0 < c < c_0$ one has

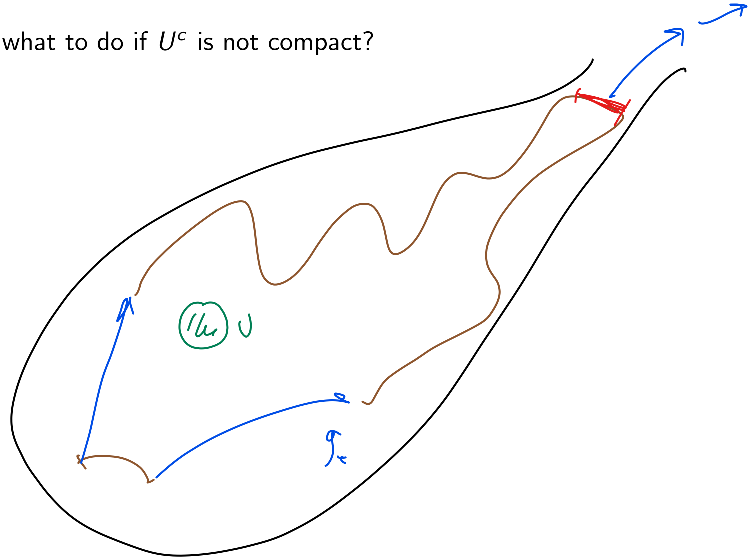


$$\operatorname{codim} \mathbf{BA}_{\mathbf{i},\mathbf{j}}(c) \gg \frac{c}{\log \frac{1}{c}}.$$

?



But what to do if U^c is not compact?



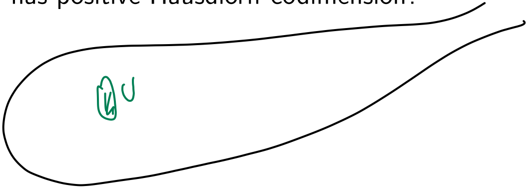
Example: fix $m, n \in \mathbb{N}$, let

$$G = \mathrm{SL}_{m+n}(\mathbb{R}),\ \Gamma = \mathrm{SL}_{m+n}(\mathbb{Z}),\ X = G/\Gamma,$$

and

$$g_t = \mathrm{diag}(\underbrace{e^{t/m}, \ldots, e^{t/m}}_{m \text{ times}}, \underbrace{e^{-t/n}, \ldots, e^{-t/n}}_{n \text{ times}}),$$

Can one prove that for any non-empty open $U \subset X$, the set $\widetilde{E}(F, U)$ has positive Hausdiorff codimension?



It is probably not hard to guess that an affirmative answer would have some consequences for Diophantine approximation.

(Remember, we are at the Dani birthday conference!)

And indeed, given $c \le 1$, say that

$$A \in M_{m\times n}(\mathbb{R}) \text{ is } c\text{-Dirichlet improvable}$$

if for all sufficiently large $N \in \mathbb{N}$

$$\text{there exists } \mathbf{p} \in \mathbb{Z}^m \text{ and } \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\} \text{ such that}$$
$$\|A\mathbf{q} - \mathbf{p}\|_\infty < cN^{-n/m} \text{ and } 0 < \|\mathbf{q}\|_\infty < N.$$

We let $\mathbf{DI}_{m,n}(c)$ be the set of c-Dirichlet improvable $m \times n$ matrices.

Dirichlet's theorem implies that $\mathbf{DI}_{m,n}(1) = M_{m\times n}(\mathbb{R})$.

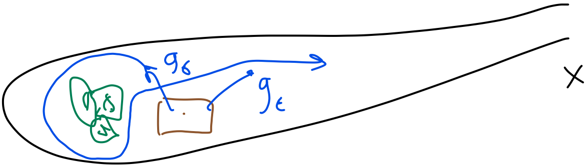
[Davenport and Schmidt, '69] proved that

- $\text{Leb}\big(\mathbf{DI}_{m,n}(c)\big) = 0$ for any $c < 1$;
- $\bigcup_{c<1} \mathbf{DI}_{m,n}(c) \supset \mathbf{BA}_{m,n} \implies \dim \mathbf{DI}_{m,n}(c) \to mn$ as $c \to 1$.

But can we have $\dim \mathbf{DI}_{m,n}(c) = mn$ for some $c < 1$?

Proposition [K–Weiss '08], based on [Dani '85]**.**
For any $c < 1$ there is a non-empty open subset U_c of X
(a neighborhood of the critical locus of the supremum norm)
such that

$$A \in \mathbf{DI}_{m,n}(c) \iff u_A\mathbb{Z}^{m+n} \in \widetilde{E}(F, U_c).$$



Thus an affirmative solution to DDC for this case [K–Mirzadeh '20], powered by measure estimates from [K–Strömbergsson–Yu '22], implies

Corollary. For any $m, n \in \mathbb{N}$ there exist explicit constants $a = a_{m,n}$, $b = b_{m,n}$ such that

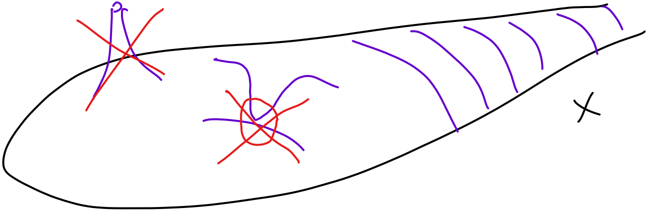
$$\operatorname{codim} \mathbf{DI}_{m,n}(c) \gg (1-c)^a \log^b\left(\frac{1}{1-c}\right)$$

So how does the proof go in **the non-compact case**?
We need some more terminology.

Say that a non-negative continuous function u on X is a height function if it is

- proper, that is $u(x) \to \infty$ if and only if $x \to \infty$ in X, and
- regular, that is
  $\exists$ a non-empty neighborhood B of $e \in G$ and $C > 0$ (equivalently, $\forall$ bounded $B \subset G$ there exists $C > 0$) such that

$$u(hx) \leq Cu(x) \text{ for every } h \in B \text{ and all } x \in X.$$



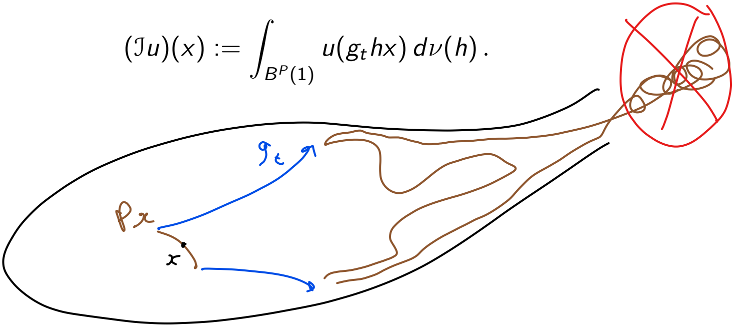
Also let us say that u satisfies the (c, d)-Margulis inequality with respect to an operator $\mathcal{I} : C(X) \to C(X)$ if for all $x \in X$ one has

$$(\mathcal{I}u)(x) \leq cu(x) + d.$$

Functions u satisfying the (c, d)-Margulis inequality for some $c < 1$ and $d \in \mathbb{R}$ are often called Margulis functions (with respect to $\mathcal{I}$).

We wil consider $\mathcal{I} = \mathcal{I}_{f,t}$ where $f = 1_{B^P(1)}$, that is,

$$(\mathcal{I}u)(x) := \int_{B^P(1)} u(g_t hx)\, d\nu(h)\,.$$



Definition. Say that a subgroup P of G has

Effective Non-Divergence Property (ENDP) w.r.t. the flow (X, F)

if there exists $0 < c_0 < 1$ and $t_0 > 0$ such that
for any $t \geq t_0$ one can find $d_t > 0$ and a height function u_t
satisfying the (c_0, d_t)-Margulis inequality with respect to $\mathcal{I}_{B^P(1),t}$.

In other words, we have

$$\int_{B^P(1)} u_t(g_t hx)\, d\nu(h) \leq c_0 u_t(x) + d_t.$$

The prototypical example: [Eskin–Margulis–Mozes '98].

More examples: [Kadyrov–K–Lindenstrauss–Margulis '17], [Guan–Shi '20], [K–Mirzadeh '20], [Rodriguez Hertz–Wang '21].

Theorem 2 [K–Mirzadeh '22]**.**
Suppose P has properties (EEP) and (ENDP) w.r.t. (X, F).
Then for any open $U \subset X$ one has

$$\inf_{x \in X} \operatorname{codim} \left\{ g \in P : gx \in \widetilde{E}(F, U) \right\} > 0.$$

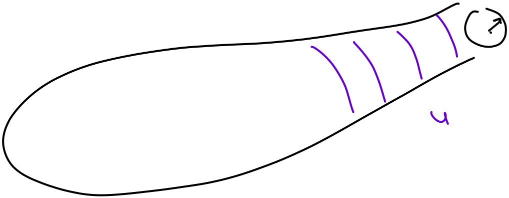
$\Rightarrow$ DDC along P

To make it effective, need an additional assumption on the height functions:

Definition. Say that $P \subset G$ has property $(\mathrm{ENDP}+)$ w.r.t. (X, F) if $\exists\, \alpha > 0$ such that the functions u_t in addition satisfy

$$u_t(x) \gg r_0(x)^{-\alpha} \qquad \forall\, x \in X \ \ \forall\, t \geq t_0.$$

That is, they grow uniformly fast enough.
(This holds in all the examples we care about.)

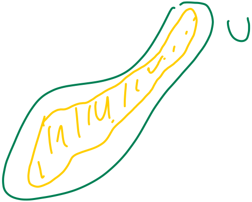


Theorem 3 [K–Mirzadeh '22]**.**
Suppose P has properties (EEP) and (ENDP+) w.r.t. (X, F).
Then $\exists\ \rho > 0$ such that for any non-empty open $U \subset X$ one has

$$\inf_{x\in X} \operatorname{codim} \left\{ g \in P : gx \in \widetilde{E}(F, U) \right\} \gg \frac{\mu(U)}{\log \frac{1}{\min(\mu(U), \theta_U, \rho)}},$$

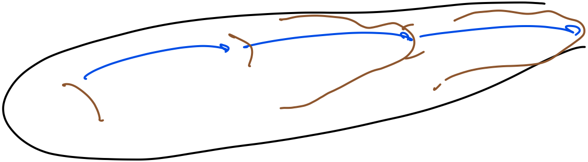
where

$$\theta_U := \sup \left\{ r > 0 : \mu(\sigma_{4r} U) \geq \frac{1}{2}\mu(U) \right\}.$$



Example: if $U = B(z, r)$, then
both $\mu(U)$ and θ_U are polynomial in r,
so we are getting $\frac{r^{\dim X}}{\log(1/r)}$ again.

How to prove all that: one combines Theorem 1 (EEP)
with a method from [Kadyrov–K–Lindenstrauss–Margulis '17]
where the goal was to control trajectories divergent on average:

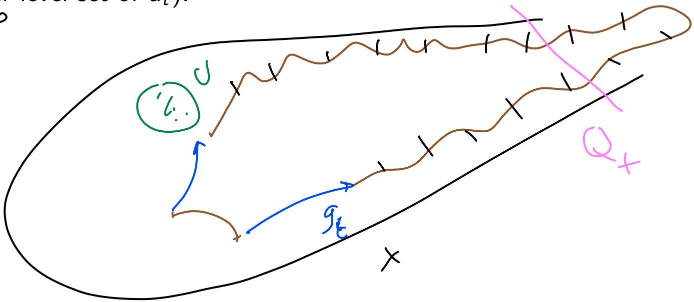


Roughly speaking, if the trajectory Fx diverges on average, then for a fixed t and large N most of $\{g_{it}x : i = 1, \dots, N\}$ is far away

$$\Downarrow \quad \text{(Margulis inequality + regularity of } u\text{)}$$

the set of those x in a piece of a P-orbit can be effectively covered by a relatively small number of balls.

The same approach is applied to the problem of escaping U: one fixes $t \geq t_0$ and chooses a big compact subset Q_t (a ~~sup~~er-level set of u_t).
sub



Then at each stage of the induction:
if $g_{it}x$ lands in Q_t, apply (EEP);
if not, apply the Margulis inequality.

Thank you for your attention!

My collaborators:

Shahriar Mirzadeh 

Andreas Strömbergsson 

Shucheng Yu 

And many more productive years ahead!