From non-abelian superfluids to soft pion production

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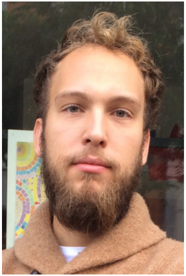
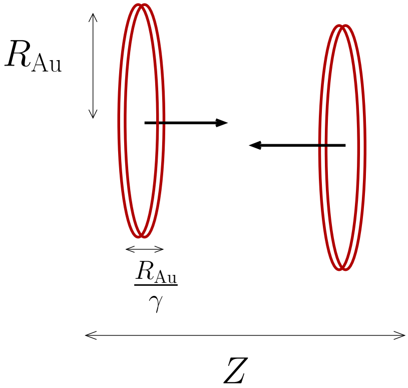
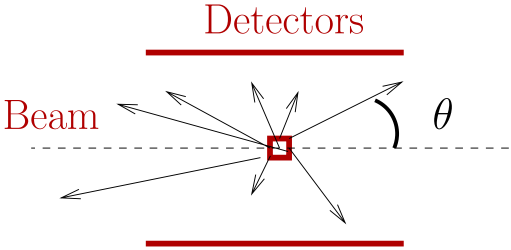
- Eduardo Grossi, Alex Soloviev, DT, Fanglida Yan: PRD, arxiv:arXiv:2005.02885
- Eduardo Grossi, Alex Soloviev, DT, Fanglida Yan: in progress

Splat: a heavy ion collision is born!





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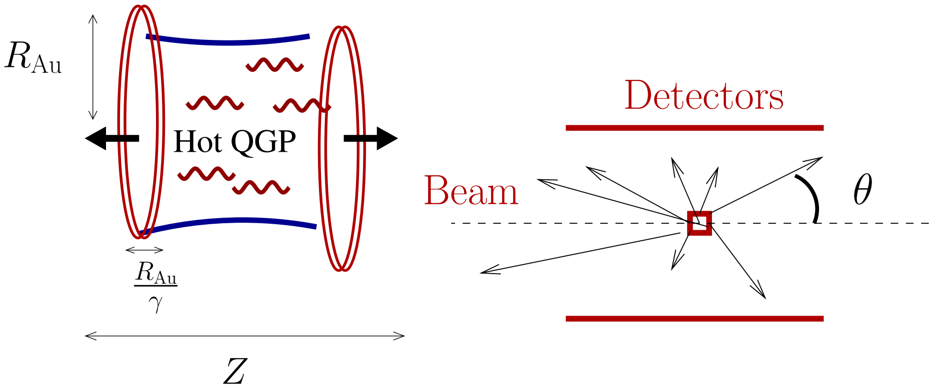

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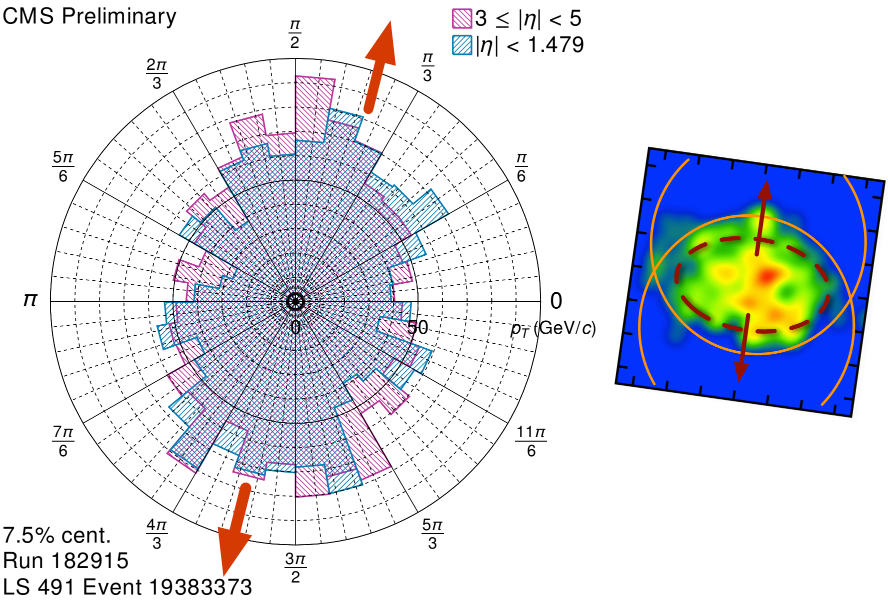

CMS Preliminary
3 ≤ |η| < 5
|η| < 1.479
π/2
π/3
2π/3
π/6
5π/6
π
0
50
0
p_T (GeV/c)
7π/6
11π/6
4π/3
5π/3
3π/2
7.5% cent.
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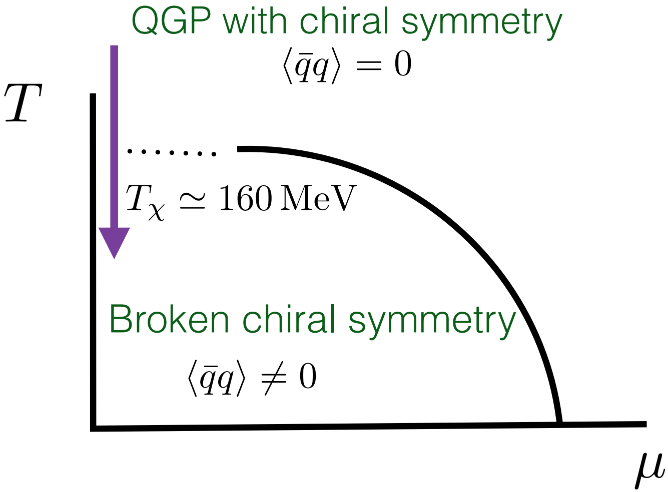
Trajectory of a heavy ion collision in the phase diagram

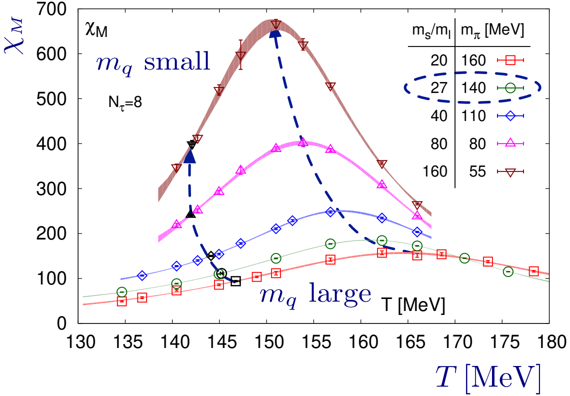


Chiral symmetry plays no role in the “Standard Model” of heavy ions ...

Chiral symmetry and $O(4)$ scaling in lattice QCD:

Hot QCD, PRL 2019

$$\chi_M \propto \frac{\partial \langle \bar{q}q \rangle}{\partial m_q} = \frac{\partial^2 \log Z}{\partial m_q^2}$$



O(4) Scaling predictions

$$\chi_M = m_q^{1/\delta - 1} f_\chi(z)$$

$$z \equiv t m_q^{-1/\beta\delta}$$

The QCD lattice knows about the $O(4) = SU_L(2) \times SU_R(2)$ critical point!

QCD and the Chiral limit and Broken Symmetry:

Son hep-ph/9912267; Son and Stephanov hep-ph/020422

1. The approximately conserved quantities $\hat{J}^\mu_a = \bar{\psi}\gamma^5\gamma^\mu\tau^a\psi$

$$\underbrace{T^{\mu\nu}}_{\text{stress}} \qquad \underbrace{J^\mu_B}_{\text{Baryon number}} \qquad \underbrace{J^\mu_a}_{\text{isovector}} \quad \textit{and} \quad \underbrace{\hat{J}^\mu_a}_{\text{iso-axial vector}}$$

2. There is the phase of the chiral condensate and pion field: $\varphi^a = \pi^a/F$

$$U = \text{Phase of } \langle \bar{q}q \rangle \equiv e^{i\tau^a\varphi^a}$$

3. The pion φ^a is like T, $\vec{u}$, μ_I, *and* $\hat{\mu}$ in the constitutive relations
4. Include a mass term so the Goldstone fields decay at large distances

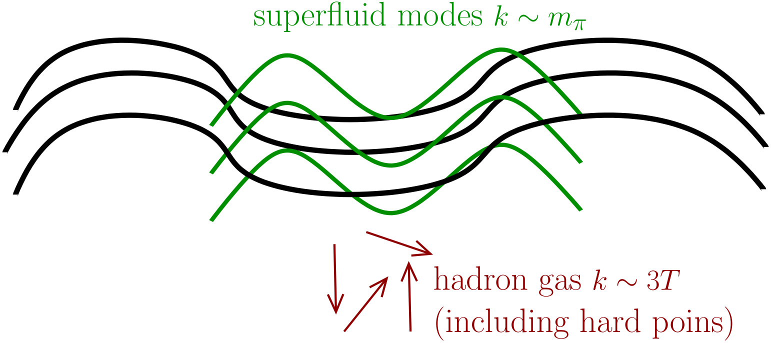
Need to write down a theory of superfluid hydro (Son '99)

Equilibrium picture:

- Work in the regime

$$k \ll m_\pi \ll \pi T \sim \pi \Lambda_{QCD}$$

equilbriated hydro modes $k \ll m_\pi$



How do these superfluid modes contribute to pressure and viscosities, and the diffusion rate of isovector charge ?

Expansion of transport coefficients in powers of the quark mass:

We will show that the normal transport parameters have the expansions:

$$\underbrace{\zeta}_{\text{bulk viscosity}} = \zeta^{(0)} - \underbrace{m_\pi \zeta^{(1)}}_{\text{non-analytic}} + \dots$$

$$\underbrace{\eta}_{\text{shear viscosity}} = \eta^{(0)} - \underbrace{m_\pi \eta^{(1)}}_{\text{non-analytic}} + \dots$$

$$\underbrace{\sigma_I}_{\text{isovect conductivity}} = \underbrace{\frac{1}{m_\pi}\sigma_I^{(-1)}}_{\text{divergent!}} + \sigma_I^{(0)} + \dots$$

The expansion is non-analytic in quark mass

$$m_\pi \propto \sqrt{m_q}$$

and the non-analyticity comes from superfluid hydrodynamic loops

The pressure from soft modes:

- Use 3D dimensionally reduced chiral perturbation theory:

$$Z_{QCD} = \underbrace{e^{\beta p_0(T,\hat{\mu})V}}_{\text{from hard modes } p\sim T} \times \underbrace{\int^{\Lambda}[D\varphi]\exp\left(-\beta\int d^3\boldsymbol{x}\mathcal{L}_{\rm eff}\right)}_{\text{from soft modes } p\sim m_\pi T}$$

where using $U = e^{i\varphi_a\tau_a}$

$$\mathcal{L}_{\rm eff} \simeq \frac{f^2(T)}{4}\,{\rm Tr}\,\vec{\nabla}U\cdot\vec{\nabla}U^\dagger + \frac{f^2m^2(T)}{2}{\rm Re\,Tr}\,U \quad \Rightarrow \quad \frac{f^2}{2}(\nabla\varphi_a)^2 + \frac{f^2m^2}{2}\varphi_a^2$$

This leads to a correction to the QCD pressure:

$$p(T,\hat{\mu}) = \underbrace{p_0(T) + \tfrac{1}{2}\hat{\chi}\hat{\mu}^2}_{\text{analytic in } m_q} + \underbrace{\frac{Tm^3}{4\pi}}_{\propto m_q^{3/2}} + \mathcal{O}(m^4)$$

The effective Lagrangian sums all non-analytic terms in the quark mass!

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The parameters have universal dependence near the $O(4)$ critical point:

$$f^2m^2 \propto m_q\left\langle\bar{\psi}\psi\right\rangle \propto m_q t^\beta \qquad\qquad t \equiv (T-T_c)/T_c$$

$$f^2 \propto t^{\nu(d-2)}$$

Can compute $f^2(T)$ and $m^2(T)$ the real world lattice QCD with precision!

Stress and Current for Superfluids:

Son; Jensen et al 1203.3556, Bhattacharya et al

The pressure in the presence of the phase is p_φ

$$p_\varphi(T, \nabla\varphi, \varphi^2) = p_0(T) + \frac{1}{2}\hat{\chi}\hat{\mu}^2 - \frac{f^2}{2}\left((\nabla\varphi)^2 + m^2\varphi^2\right)$$

- Derive the ideal stress and current from pressure $W = \int d^4x \sqrt{g}\, p_\varphi$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}} = (e_\varphi + p_\varphi)u^\mu u^\nu + \eta^{\mu\nu}p_\varphi + \underbrace{f^2\partial^\mu\varphi\partial^\nu\varphi}_{\text{super fluid stress}}$$

$$\hat{J}^\mu_a = \frac{1}{\sqrt{-g}}\frac{\partial W}{\partial A_\mu} = \underbrace{\hat{n}_a u^\mu}_{\text{normal fluid}} + \underbrace{f^2\partial^\mu\varphi_a}_{\text{super fluid current}}$$

- In an extension of the formalism coming from $[\bar{q}_R q_L, H - \mu N] = 0$

$$\underbrace{-u^\mu\partial_\mu\varphi_a = \hat{\mu}_a}_{\text{Josephson constraint}} \quad \text{and} \quad \underbrace{\partial_\mu \hat{J}^\mu_a = f^2 m^2 \varphi_a}_{\text{PCAC}}$$

Dissipative corrections to EOM

Son, Stephanov hep-ph/020422 + a bit by us

$$\underbrace{\partial_t \hat{J}^0 + \nabla\cdot\hat{\boldsymbol{J}} = f^2 m^2 \varphi}_{\text{PCAC}} \qquad \text{and} \qquad \underbrace{-\partial_t \varphi_a = \hat{\mu}_a}_{\text{Josephn's constraint}}$$

- Then expand the current in gradients

$$\hat{\boldsymbol{J}} = \underbrace{f\nabla\varphi}_{\text{ideal current}} - \underbrace{\sigma_A\,\nabla\hat{\mu}}_{\text{axial conductivity}} + \underbrace{\vec{\xi}_J}_{\text{noise}}$$

and the josephson constraint

$$-\partial_t\varphi = \underbrace{\hat{\mu}}_{\text{ideal fluid}} + \underbrace{-\kappa_2\,\nabla^2\varphi + \kappa_1\,m^2\varphi}_{\text{visc correction}} + \underbrace{\xi_S}_{\text{noise}}$$

- To reach the equilibrium fluctuations we must have:

$$\left\langle \xi_J^i \xi_J^j \right\rangle = 2T\sigma_A\,\delta^{ij}\delta^4(x-x') \qquad \langle \xi_S \xi_S \rangle = 2T\zeta^{(2)}\,\delta^4(x-x')$$

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Long wavelength pion (superfluid) modes:

Son, Stephanov hep-ph/020422 + a bit by us

- Linearizing the equation of motion $\varphi = Ce^{-i\omega t + i\boldsymbol{q}\cdot\boldsymbol{x}}$ one finds

$$\varphi(t, \boldsymbol{q}) = Ce^{-(\Gamma/2)t} e^{-i\omega_q t}$$ $\Leftarrow$ This is second sound!

- The quasi-particle energy is:

$$\omega_q^2 \equiv v_0^2(q^2 + m^2) \qquad v_0^2(T) \equiv \frac{f^2}{\hat{\chi}}$$ $\Leftarrow$ pion velocity

- The damping rate is set by two diffusion coefficients, D_A and D_m:

$$\Gamma \equiv D_A q^2 + D_m m^2$$

$$D_A = \sigma_A + (\hat{\chi} v_0)^2 \zeta^{(2)}$$ $\Leftarrow$ Axial charge diffusion coefficient

$$D_m = (\hat{\chi} v_0)^2 \zeta^{(2)}$$ $\Leftarrow$ Axial damping coefficient

Interpretation of the pion velocity

Son, Stephanov hep-ph/020422

- The ideal massless pion equation of motion: $(f_t^2 \equiv \hat{\chi}$ and $f_s^2 \equiv f^2)$

$$\underbrace{-\partial_t(\hat{\chi}\partial_t\varphi)}_{\partial_t(\text{axial-chrg})} + \underbrace{\partial_x(f^2\partial_x\varphi)}_{\partial_x(\text{super-current})} = 0$$

The conserved charge is

$$J^0 = \underbrace{\hat{\chi}\partial^t\varphi}_{\text{total axial-chrg}} = \underbrace{f^2\partial^t\varphi}_{\text{super component}} + \underbrace{\Delta\hat{\chi}\,\partial^t\varphi}_{\text{normal component}}$$

So the pion velocity has a simple interpretation

$$v_0^2 \equiv \frac{f^2}{\hat{\chi}} = \frac{f^2}{f^2 + \Delta\hat{\chi}} = \frac{\text{super}}{\text{super} + \text{normal}}$$

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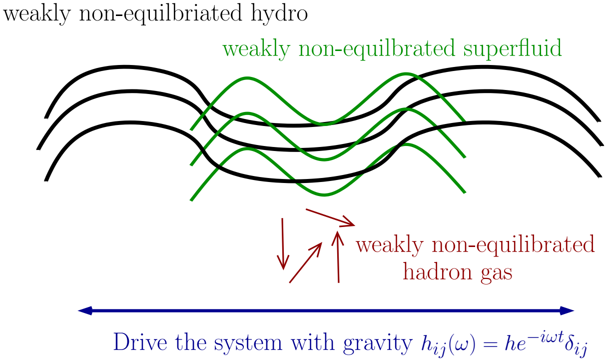
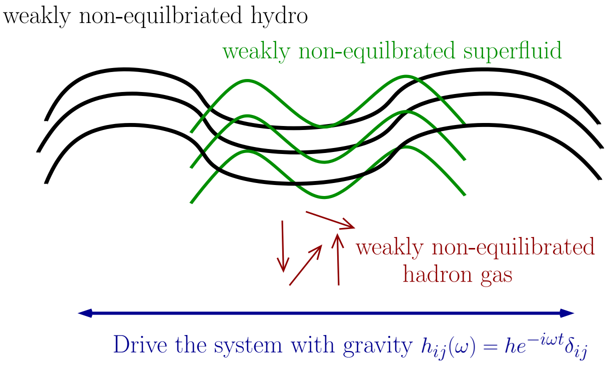
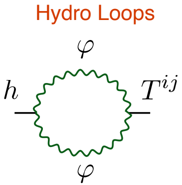


Superfluid fluctns and shorter are *absorbed* into the transport coefficients

$$\underbrace{\tfrac{1}{3}\left\langle \delta T^{i}_{\ i} \right\rangle}_{\text{non-equil. stress}} = +i\tfrac{3}{2}\omega h\zeta = -\zeta \underbrace{\nabla \cdot u}_{\text{cov-d}}$$





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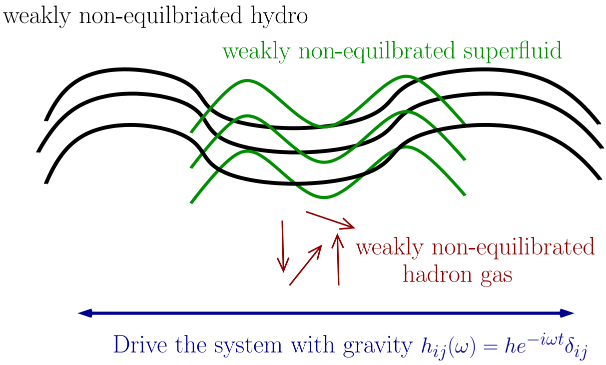

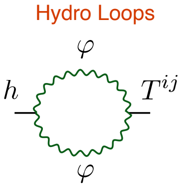

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Let's evolve the phase-space density of the stochastic superfluid with (hydro) kinetics

Developing hydro-kinetics – linearized hydro in a uniform system

1. Evolve fields of linearized super fluid hydro:

$$\phi_a(\boldsymbol{k}) \equiv \left(\varphi(\boldsymbol{k}), \hat{J}^0\right)$$

2. The stochastic EOM are matrix versions of Brownian motion:

$$\frac{d\phi_a(\boldsymbol{k})}{dt} = \underbrace{\mathcal{L}_{ab}(\boldsymbol{k})}_{\text{ideal}}\,\phi_b(\boldsymbol{k}) + \underbrace{\mathcal{D}_{ab}\phi_b}_{\text{damping}} + \underbrace{\xi_a}_{\text{noise}}$$

3. Break up the equations into eigen modes of $\mathcal{L}_{ab}$, and analyze:

   - The eigenmodes of the superfluid equations are the propagating pions:

$$\hat{\phi}_{\pm} = \frac{\hat{J}^0 \pm i\omega_k\,\hat{\chi}\,\varphi}{\sqrt{2}} \qquad \text{with} \qquad \omega_{k,\pm} = \underbrace{\pm(v_0^2\,(k^2 + m^2))^{1/2}}_{\text{eigen-vals}}$$

The hydro-kinetic equations without expansion

1. Find the evolution of the phase-space density of 2nd sound modes:

$$\hat{W}_{++}(t,\boldsymbol{x},\boldsymbol{q}) \equiv \int d\boldsymbol{y}\, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}} \left\langle \hat{\phi}^*_+(t,\boldsymbol{x}+\boldsymbol{y}/2)\hat{\phi}_+(t,\boldsymbol{x}-\boldsymbol{y}/2) \right\rangle$$

   The phase space density is $W_{++} \equiv \chi\omega_q f_\pi(t,\boldsymbol{x},\boldsymbol{q})$.

2. The phase-space distribution evolution follows the Boltzmann eqn:

$$\left(\partial_t + \frac{\partial\omega_q}{\partial\boldsymbol{q}}\cdot\frac{\partial f_\pi}{\partial\boldsymbol{x}} - \frac{\partial\omega_q}{\partial\boldsymbol{x}}\cdot\frac{\partial f_\pi}{\partial\boldsymbol{q}}\right) = \underbrace{-(D_A q^2 + D_m m^2)\left[f_\pi - \frac{T}{\omega_q}\right]}_{\text{damping to equilibrium}}$$

3. The particles stream with effective 4D Hamiltonian — An,Basar,Yee,Stephanov

$$\mathcal{H} = \tfrac{1}{2}G^{\mu\nu}q_\mu q_\nu + \tfrac{1}{2}f^2m^2 \qquad \underbrace{G^{\mu\nu} = -u^\mu u^\nu + v_0^2\Delta^{\mu\nu}}_{\text{fluid metric}}$$

   i.e. $\dot{q}_\mu = -\partial\mathcal{H}/\partial x^\mu$ etc. with particles onshell $\mathcal{H}=0$.

Final kinetic theory expression for bulk viscosity

1. Expression

$$\zeta = \zeta^{(0)}(\Lambda) + \int^{\Lambda} \frac{d^3q}{(2\pi)^3} \frac{T}{D_A\, q^2 + D_m\, m^2} \left[\frac{\boldsymbol{q}}{\omega_q} \cdot \frac{\partial \omega_q}{\partial \boldsymbol{q}} - \frac{c_s^2}{\omega_q} \frac{\partial(\beta\omega_q)}{\partial \beta} \right]^2$$

2. Definitions characterizing the dispersion curve, $m_p^2 \equiv v_0^2 m^2$

$$\underbrace{\tilde{v}_0^2 = v_0^2 - T^2 \frac{\partial v_0^2}{\partial T^2}}_{\text{thermal velocity (euclidean!)}} \qquad \text{and} \qquad \underbrace{\tilde{m}_p^2 = m_p^2 - T^2 \frac{\partial m_p^2}{\partial T^2}}_{\text{thermal mass (euclidean!)}}$$

3. Find, with $r \equiv \sqrt{D_m/D_A}$, the first correction to the chiral limit:

$$\zeta = \zeta^{(0)} + \frac{3Tm}{8\pi D_A} \left[\left(\frac{c_s^2}{1+r} \frac{\tilde{m}_p^2}{m_p^2} - \frac{1+2r}{1+r} \left(\tfrac{1}{3} - c_s^2 \frac{\tilde{v}_0^2}{v_0^2} \right) \right)^2 - (4+2r) \left(\tfrac{1}{3} - c_s^2 \frac{\tilde{v}_0^2}{v_0^2} \right)^2 \right]$$

Final formulas:

(in terms of superfluid parameters D_A and $r = \sqrt{D_m/D_A}$)

$r = \sqrt{D_m/D_A}$ is an order one number:

$r^2 = 3/4$ in χPT.

$$\zeta \simeq \zeta^{(0)} + \frac{3Tm}{8\pi D_A}\left(\left(\frac{c_s^2}{1+r}\frac{\tilde{m}_p^2}{m_p^2} - \frac{1+2r}{1+r}\left(\frac{1}{3} - c_s^2\frac{\tilde{v}_0^2}{v_0^2}\right)\right)^2 - (4+2r)\left(\frac{1}{3} - c_s^2\frac{\tilde{v}_0^2}{v_0^2}\right)^2\right)$$

$$\eta = \eta^{(0)} - \frac{Tm}{40\pi D_A}\left[\frac{2r^3+4r^2+6r+3}{(1+r)^2}\right]$$

$$\sigma_I = \frac{T}{12\pi m D_A}\left[\frac{1+2r}{(1+r)^2}\right] + \sigma_I^{(0)}$$

These show how chiral phase flucts modify transport coeffs: $D_A \equiv \lambda_A/\hat{\chi}$

What happens near (but not too near) the $O(4)$ critical point ?

$$\underbrace{\Phi \equiv \langle \bar{q}_R q_L \rangle}_{\text{order parameter}} = \underbrace{\sigma}_{\text{amplitude}} \times \underbrace{U}_{\text{phase}}$$

The final formulas near T_c:

Take $r \simeq 0$ for simplicity:

$$\zeta \simeq \zeta^{(0)} + \frac{3Tc_s^4}{8\pi D_A}\, m \left(\left(\frac{\tilde{m}_p^2}{m_p^2} - \frac{\tilde{v}_0^2}{v_0^2} \right)^2 - 4 \left(\frac{\tilde{v}_0^2}{v_0^2} \right)^2 \right)$$

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Given by static O(4) universality

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- Near T_c the pion relaxation rate for $q \sim \xi^{-1}$ tracks the order param:

$$\Gamma_\pi \equiv D_A q^2 \sim \text{order parameter damping rate}\ ,$$

which is given by dynamic universality $\Gamma_\pi \sim \xi^{-z}$ where $z = d/2$. So:

with $\xi \sim t^{-\nu}$ find: $D_A \sim t^{-\nu(2-d/2)}$

Final formulas near T_c:

Final expectations as we approach T_c from below:

$$\zeta = \zeta^{(0)} + \underbrace{-C_\zeta \sqrt{m_q}\, t^{\beta/2} \left(\frac{0.43 c_{s0}}{t}\right)^4}_{\text{decreases sharply } |t|^{-1.81}}$$

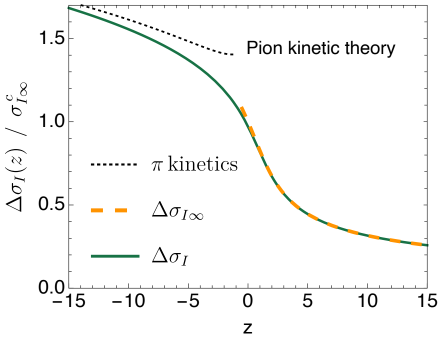
$$\eta = \eta^{(0)} + \underbrace{-C_\eta \sqrt{m_q}\, t^{\beta/2}}_{\text{grows slowly } -|t|^{0.19}}$$

The soft pion contribution to the charge diffusion *dominates* the result:

$$\sigma_I = \underbrace{\frac{C_\sigma}{\sqrt{m_q}} t^{\nu-\beta/2}}_{\text{falls as } |t|^{0.55}}$$

Outlook I (see the next talk): the conductivity through T_c

$$\sigma = \sigma_{\rm reg} + \underbrace{\Delta\sigma_I}_{\text{crit. part}}$$



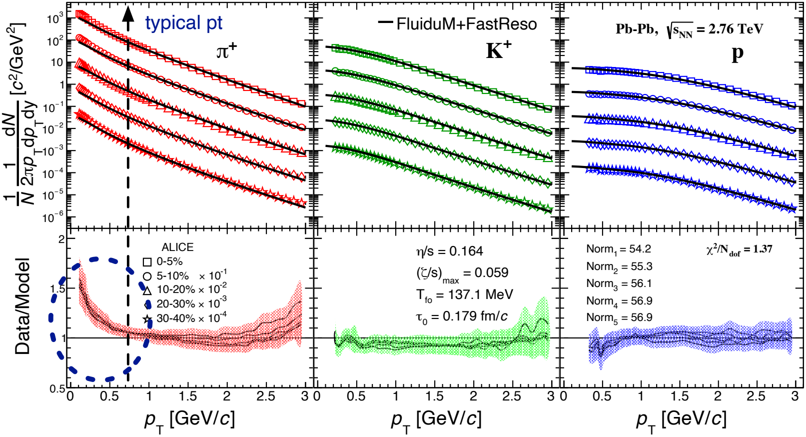
$$\sigma^c_{I\infty} \equiv \underbrace{\frac{T}{16\pi D_m m_c}}_{\text{constant}}$$

Estimate of the absolute magnitude for $\Delta\sigma_I/\sigma^c_{I\infty} = 1$:

$$\Delta D_I = \left(\frac{\Delta\sigma}{\chi}\right) = \frac{0.73}{2\pi T} \times \left(\frac{1}{m_c/T}\right)\left(\frac{3}{2\pi T D_m}\right)\left(\frac{0.36}{\chi/T^2}\right)$$

Outlook II: evidence for the chiral crossover in the heavy ion data?

A recent ordinary hydro fit from Devetak et al 1909.10485

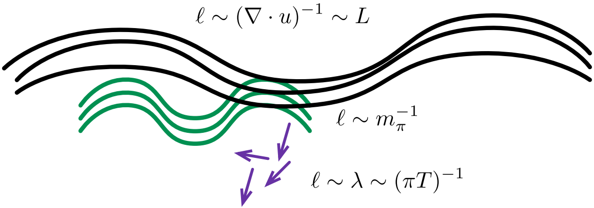


Because the pions are the Goldstones expect an enhancement at low p_T:

$$n(\omega_q) = \frac{1}{e^{vq/T} - 1} \simeq \frac{T}{vq} \Rightarrow \infty,$$

Since at T_c the velocity $\Rightarrow 0$!

Summary



1. Wrote down the appropriate $SU_L(2) \times SU_R(2)$ superfluid theory

   Included non-linear forms, viscous and mass corrections, noise etc.

2. Determined a kinetic equation for soft pions from the hydro EOM

   This can be used for the real world!

3. Determined how the ordinary transport parameters depend on m_π

   Solved the driven kinetic equation, or integrating out the hydro loops

4. Should develop the $O(4)$ scaling theory in hydro with $\Sigma = \sigma e^{i\varphi}$ field:

See the next talk!

The Josephson constraint:

$\langle \bar{q}_R q_L \rangle$ is stable

- The phase U is related to $\hat{\mu}$, since $\Sigma \equiv \bar{q}_R q_L = \sigma U$ is stationary:

$$[\Sigma,\ H - \mu_L \cdot Q_L - \mu_R \cdot Q_R] = 0$$

using the transformation properties e.g. $[\Sigma, Q_L^a] = -it^a \Sigma$, we find:

$$\underbrace{i\partial_t U U^\dagger}_{\text{(minus) deriv of phase}} \quad = \quad \underbrace{\mu_L - U \mu_R U^\dagger}_{\text{the axial chem } \hat{\mu}}$$

- In linearized form:

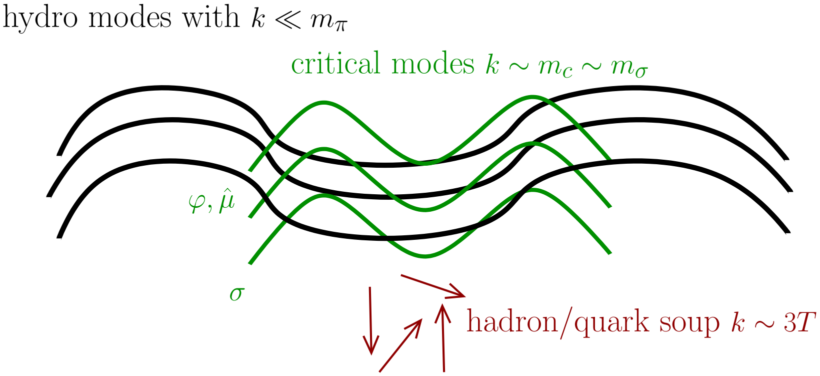
$$\underbrace{-\partial_t \varphi = \hat{\mu}}_{\text{Josephson constraint}}$$

Dynamics near the critical point

- Work in the regime:

$m_c \equiv$ pion mass on critical line

$$k \ll m_c \sim m_\sigma \ll \pi T_C \sim \pi \Lambda_{QCD}$$



How do the critical modes contribute to pressure and viscosities, and the diffusion rate of isovector charge ?

The pressure from the critical modes:

- The eff. Lagrangian near the critical point is a Landau-Ginzburg type

$$Z_{QCD} = \underbrace{e^{\beta p_0(T,\hat{\mu})V}}_{\text{from hard modes } p\sim T} \times \underbrace{\int^{\Lambda} [D\varphi] \exp\left(-\beta \int d^3\boldsymbol{x}\mathcal{L}_{\rm eff}\right)}_{\text{from soft modes } p\sim m_c}$$

where using $\Sigma = \sigma e^{i\varphi_a \tau_a}$, where σ and φ now fluctuate:

$$p_\Sigma = p_0(T) + \frac{\chi_0}{2}\mathrm{tr}(\mu_L^2 + \mu_R^2) - \left(\frac{\mathrm{tr}}{4}\partial_i\Sigma\partial_i\Sigma^\dagger + \frac{m_0^2(T)}{2}\sigma^2 + \frac{\lambda}{4}\sigma^4\right) + \frac{H}{4}\mathrm{tr}(\Sigma + \Sigma^\dagger)$$

with $m_0 \propto (T - T_c)/T_c$.

We will work with a mean field approximation

$$\Sigma \simeq \underbrace{\bar{\sigma}}_{\text{mean field}} + \underbrace{\delta\sigma + i\bar{\sigma}\vec{\varphi}\cdot\vec{\tau}}_{\text{flucts}}$$

Mean field approximation

- The mean order parameter, or EOS, takes takes the scaling form

$$\bar{\sigma} = h^{1/3} f_G(z) \qquad z = t h^{-2/3}$$

where $h = H/\lambda$ and $t = m_0^2(T)/\lambda$ with $f_G(0) = 1$.

- The action for the quadratic fluctuations takes the forms

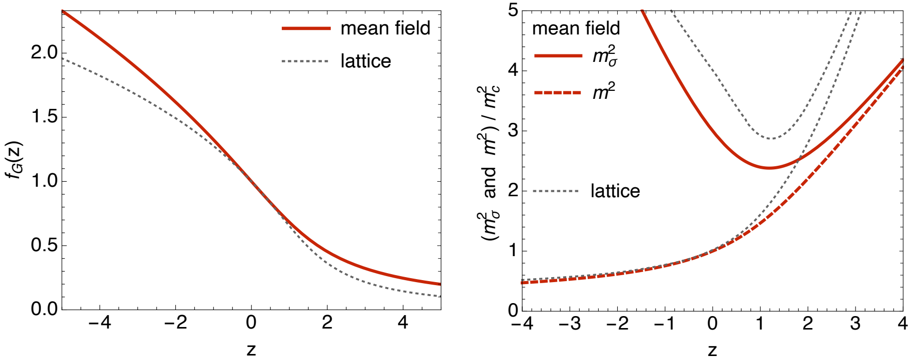
$$p_\Sigma = p_0(T) + \frac{1}{2}\chi_0 \mathrm{tr}(\mu_L^2 + \mu_R^2) - \frac{1}{2}\left(\nabla\delta\sigma \cdot \nabla\delta\sigma + m_\sigma^2 \delta\sigma^2\right) - \frac{1}{2}\chi_0 v^2(T)\left(\nabla\varphi^a \cdot \nabla\varphi^a + m^2 \varphi^a \varphi^a\right)$$

Relating the pion $m^2(z)$ and sigma screening masses to the EOS, e.g.

$$\underbrace{m^2(z)}_{\text{pion mass}} = \frac{H}{\bar{\sigma}(z)} = \frac{m_c^2}{f_G(z)}$$

Screening masses and magnetic EOS compared to lattice

Lattice: Engels, Vogt 0911.1939. Engels, Karsch 1105.0584



General trends are reproduced by mean field analysis

Stochastic hydrodynamic equations

Wilczek-Rajagopal hep-ph/9210253

Given the pressure, we go find the hydro equations for $\Sigma = \sigma e^{i\varphi_a \tau_a}$

- For example: the equation of the phase is, with $\vec{L} \equiv -i\nabla U U^\dagger$

$$\underbrace{(\tfrac{-i}{2}\partial_t U U^\dagger)}_{\text{phase deriv}} = \underbrace{-\tfrac{1}{2}(\mu_L - U\mu_R U^\dagger)}_{\text{joseph constraint}} + \underbrace{\frac{D_m}{\sigma^2}\left[\nabla\cdot(\tfrac{\sigma^2}{2}\vec{L}) + \frac{H\sigma}{4} i(U - U^\dagger)\right]}_{\text{viscous correction}} + \xi$$

And are coupled to the partially conserved currents, e.g.

$$\partial_\mu J^\mu_L = -i\frac{H}{8}\left(\Sigma - \Sigma^\dagger\right) \qquad \text{with} \qquad \vec{J}_L = \tfrac{\sigma^2}{4}\vec{L} + \underbrace{\lambda_0\vec{\nabla}\mu_L}_{\text{diffusion}} + \underbrace{\xi}_{\text{noise}}$$

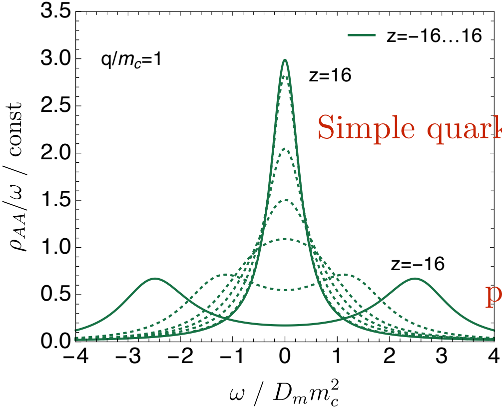
and the pion contribution to the current

$$\sigma^2\vec{L} = \frac{i}{2}\underbrace{(\Sigma\vec{\nabla}\Sigma^\dagger - \vec{\nabla}\Sigma\Sigma^\dagger)}_{\text{pion current}}$$

- And a similar equation for σ, e.g. $\partial_t\sigma = D_m\left(\delta S/\delta\sigma\right) + \xi$.

Iso-axial charge-charge correlator mixes with pions pions:

$$\frac{\rho_{AA}(\omega,\boldsymbol{q})}{\omega} = \frac{1}{T}\int_{-\infty}^{\infty} d^4x\, e^{-i\omega t + i\boldsymbol{q}\cdot\boldsymbol{x}} \left\langle \hat{J}^0_a(x)\hat{J}^0_a(0)\right\rangle$$

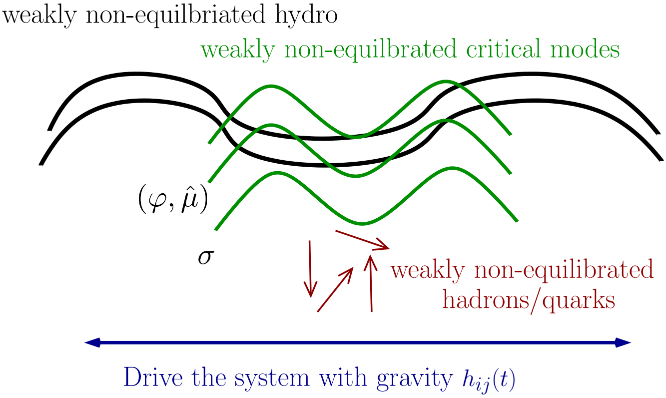


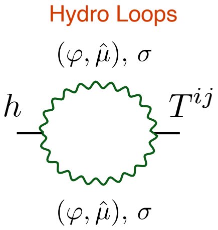
Simple quark diffusion for $T \gg T_c$

pions for $T \ll T_c$

Can see the transition from QGP to propagating pions from the EOM.

Hydro loops:

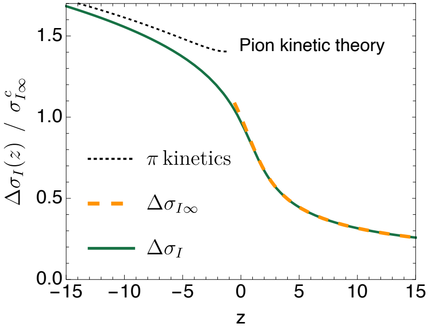




The retarded propagators are given by the linearized stochastic EOM.
Integrate out $(\varphi, \hat{\mu})$, σ to find its influence on hydro

Shift in the diffusion coefficient from critical modes:

$$\sigma = \sigma_{\rm reg} + \underbrace{\Delta\sigma_I}_{\text{crit. part}}$$

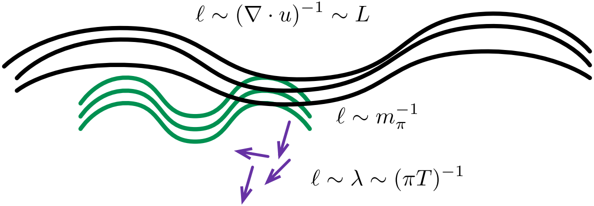


$$\sigma^c_{I\infty} \equiv \underbrace{\frac{T}{16\pi D_m m_c}}_{\text{constant}}$$

Estimate of the absolute magnitude for $\Delta\sigma_I/\mathrm{const} = 1$:

$$\Delta D_I = \left(\frac{\Delta\sigma}{\chi}\right) = \frac{0.73}{2\pi T} \times \left(\frac{1}{m_c/T}\right)\left(\frac{3}{2\pi T D_m}\right)\left(\frac{0.36}{\chi/T^2}\right)$$

Summary (Part II)



1. Wrote down the $SU_L(2) \times SU_R(2)$ critical hydro theory theory
   Included non-linear forms, viscous and mass corrections, noise etc.

2. Showed how pions are formed through the phase transition, and match onto a specific kinetic theory.

3. Determined how the ordinary transport parameters depend near T_c
   Worked in a mean field approximation, but it can be improved

There are exciting hints of the chiral crossover in the data!
Thank you and stay safe!

Backup

The Josephson constraint:

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- In linearized form:

$$\underbrace{-\partial_t \varphi = \hat{\mu}}_{\text{Josephson constraint}}$$

Extra: Superfluid kinetics of chiral perturbation theory

Kinetic coefficients of χPT near the chiral limit:

- When the temperature is low then χPT is valid

$$m_\pi^2 \ll (\pi T)^2 \ll (\pi \Lambda)^2 \quad \Leftarrow \text{Theorist's world!}$$

The microscopic Lagrangian is

$$\mathcal{L} = \frac{F^2}{4} \,\mathrm{Tr}\left[\partial_\mu \Sigma \partial^\mu \Sigma\right] + \ldots$$

Note:

$$\underbrace{\Sigma}_{\text{micro-field}} \neq \underbrace{U}_{\text{long wavelength field}}$$

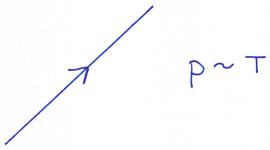
- The 0th order bulk $\zeta^{(0)}$ viscosity vanishes (in this limit) and the ζ is

$$\zeta = \frac{3Tm}{8\pi}\left(\frac{c_s^2}{\sqrt{D_A} + \sqrt{D_m}}\right)^2 \qquad \Leftarrow \text{I'm cute!}$$

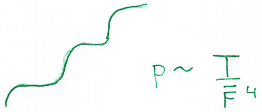
Let's go compute the chiral limit parameters D_A and D_m in χPT ...

Modes in χPT: Hard, Soft, and Hydrodynamic:

- Hard pions – not superfluid like:

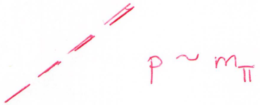


- Soft pions – wavelength comparable to spacing between collisions



$$\overline{F} \equiv \frac{F}{T} \gg 1$$

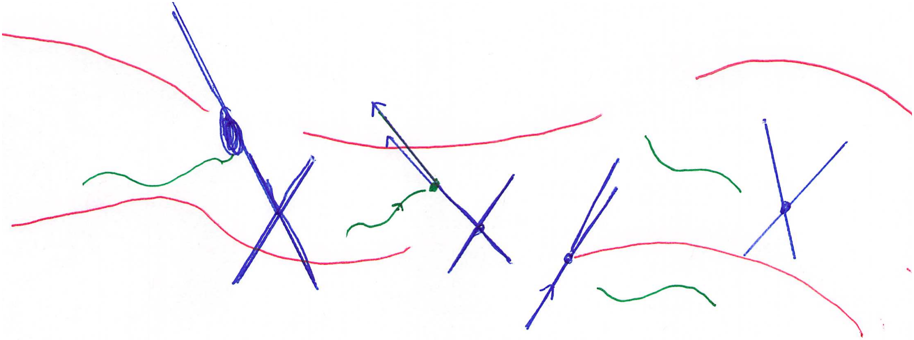
- Ultra-soft or hydrodynamic pions:



$$m_\pi \ll \frac{T}{\bar{F}^4}$$

Our goal is "integrate out" the hard and soft pions, to leave a hydrodynamic theory of pions

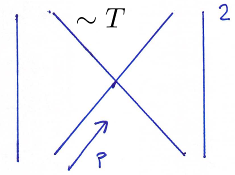
“Artists” conception



Hard Modes (Easy):

- Scattering rate: (Shuryak; Goity, Leutwyler)

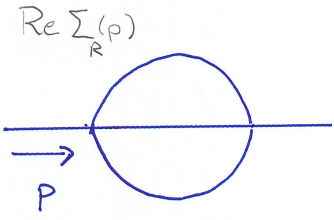
$$\frac{\mathrm{Im}\Sigma(p)}{E_p} \sim$$



The mean scattering rate is:

$$\bar{\gamma} = 2.333 \frac{T}{27\bar{F}^4}$$

- Dispersion curve: (Pisarski; Schenk; Toublan)



Parametrized by $v(p)$:

$$E_{\boldsymbol{p}} = v(p)p \qquad v_0 \equiv v(0)$$

which is close to unity

$$v(p) - v_0 \propto \frac{1}{\bar{F}^4}$$

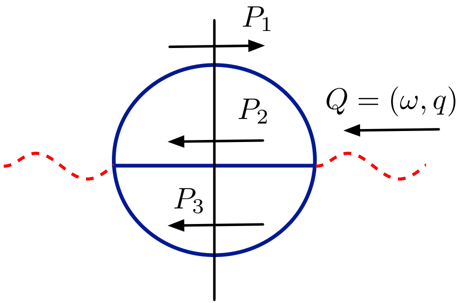
The axial charge diffusion coefficient:

- The equation of motion in the chiral limit:

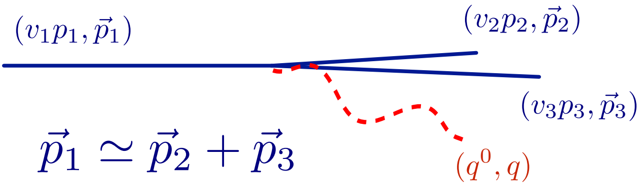
$$\underbrace{-\partial_t(\hat{\chi}\partial_t\varphi) + \partial_x(f^2\partial_x\varphi)}_{\text{ideal superfluid}} = \underbrace{\hat{\chi} D_A \nabla^2 \partial_t \varphi}_{\text{dissipation}}$$

- Comparison with the eqn. of motion, $(Q^2 - \Sigma_R(Q))\pi(Q) = 0$, shows

$$\lim_{q\to 0} \; \mathrm{Im}\Sigma_R(\omega, q)\Big|_{\omega = v_0 q} = D_A q^2 \omega$$



Problem: The energy difference and the collisional width



1. The energy difference is the same order as the scattering rate:

$$\delta E_{123} = E_1 - E_2 - E_3 - \underbrace{q^0}_{\text{usoft, ignore-me}}$$

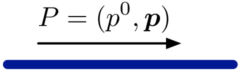
$$= (v_1 - v_2) p_2 + (v_1 - v_3) p_3 \sim \frac{T}{\bar{F}^4}$$

So the collisional width can not be ignored!

2. $1 \leftrightarrow 3$ processes are totally OK, the process is effectively $1 \leftrightarrow 2$

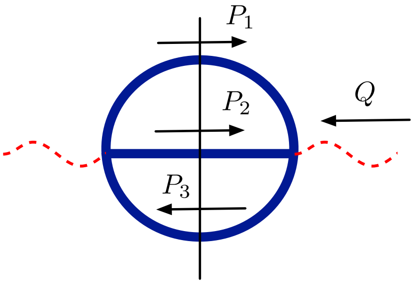
Including the width

1. Thick hard lines include the collisional width

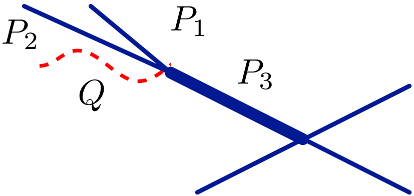


$$= \frac{1}{-p^0 + E_{\boldsymbol{p}} + \frac{i}{2}\Gamma_{\boldsymbol{p}}}$$

2. The self energy then includes $2 \leftrightarrow 2$ and $3 \leftrightarrow 3$ and $2 \leftrightarrow 4$ processes



For example:



Summary:

- Find the axial charge diffusion rate and damping rate:

$$TD_A = \frac{T^4}{9\pi F^4} \log\left(\frac{2.35T}{\bar{\gamma}}\right) \qquad r = \sqrt{\frac{D_m}{D_A}} = \sqrt{\frac{3}{4}}$$

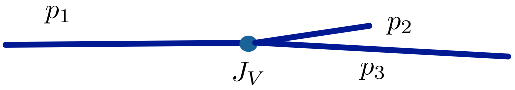
- Find that axial diffusion coefficient is of order the scattering rate:
  - But, the "chiral log" is screened by the mean free path
- The leading order bulk viscosity and conductivity are then determined:

$$\zeta = \frac{3Tm_\pi c_s^4}{8\pi D_A} \left[\frac{1}{(1+r)^2}\right]$$

$$\sigma_I = \frac{T}{12\pi m_\pi D_A} \left[\frac{1+2r}{(1+r)^2}\right]$$

Backup

Imaginary part of self energy:



$$\delta E_{123} = E_1 - E_2 - E_3$$
$$\Gamma_{123} = \Gamma_1 + \Gamma_2 + \Gamma_3$$

- Find a relatively compact formula:

$$\mathrm{Im}\Sigma_R \propto \frac{\omega q^2}{F^4} \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} p_1^2 n_1 (1+n_2)(1+n_3)$$
$$\times 2\pi\delta(p_1 - p_2 - p_3)\left(\frac{1}{2} + \underbrace{\tan^{-1}(2\delta E_{123}/\Gamma_{123})}_{\text{width and coherence time}}\right)$$

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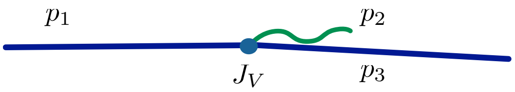

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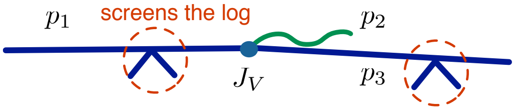
But this diverges when for example p_2 gets soft (the green line):

$$TD_A = \frac{T^4}{9\pi F^4}\left(\underbrace{\log\left(\frac{T}{\Lambda}\right)}_{\text{soft-divergence}} + 0.372\right)$$

Imaginary part of self energy:



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But this diverges when for example p_2 gets soft (the green line):

$$TD_A = \frac{T^4}{9\pi F^4} \left(\underbrace{\log\left(\frac{T}{\Lambda}\right)}_{\text{screened w. collisions}} + 0.372 \right)$$

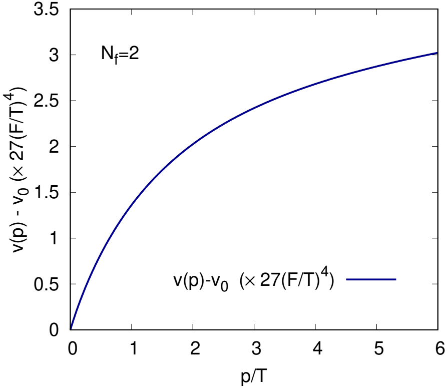
The dispersion curve:

(Schenk; Toublan)

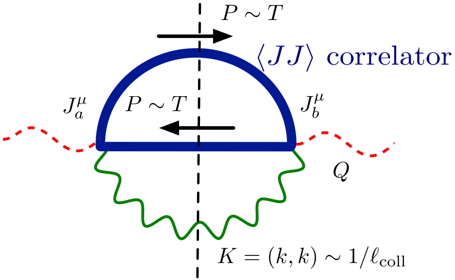
The velocity squared is

$$v_0^2 = 1 - \frac{T^4}{27F^4}\log\left(\frac{\Lambda}{T}\right) , \qquad \underbrace{\Lambda \simeq 1.8\,\mathrm{GeV}}_{\text{depends on Leutwyler-coefficients}}$$

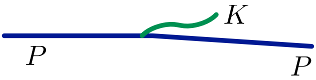
We will need only the difference $v(p) - v_0$ and width γ_p:



Analysis of the infrared divergence:



The JJ correlator
is given by
free kinetic theory

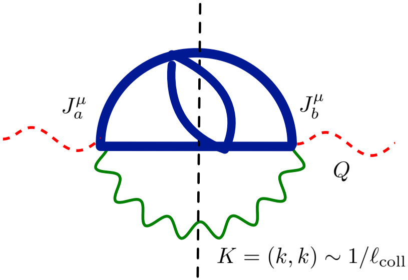


- The graph is given by the symmetrized $G^{JJ}_{\mu\nu}(X, Y) = \langle \{ J_\mu(X), J_\nu(Y) \} \rangle$ correlator on the light cone:

$$\mathrm{Im}\Sigma_R(\omega, q) \propto \frac{\omega q^2}{F^4} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \left[2G_L^{JJ}(k^0, k^0) + G_T^{JJ}(k^0, k^0) \right]_{HTL}$$

For small K, we should add collisions to the free kinetic theory

Analysis of the infrared divergence:



The JJ correlator
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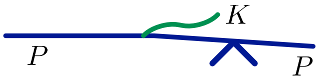
kinetics with collisions



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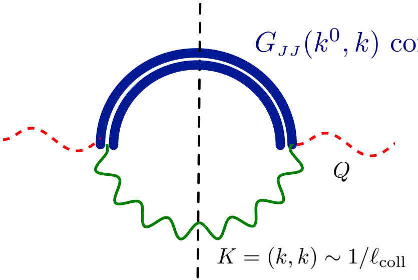

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For small K, we should add collisions to the free kinetic theory

Thermal light-cone sum rules ala Caron-Huot:

$$\mathrm{Im}\Sigma_R(\omega, q) \propto \frac{\omega q^2}{F^4} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \left[2G_L^{JJ}(k^0, k^0) + G_T^{JJ}(k^0, k^0)\right]$$

- Use the Fluctuation Dissipation Theorem:

$$G_{\mathrm{sym}}^{JJ}(K) = \frac{T}{k^0} \left(G_R^{JJ}(K) - G_A^{JJ}(K)\right)$$

- Then the contour can be deformed to the arc at infinity, where the HTL expression for $G_R^{JJ}(K)$ (with the width) applies.
- The result is:

$$T\mathrm{Im}\Sigma_R = \omega q^2 \frac{T^4}{9\pi F^4} \Big(\underbrace{\log\left(\frac{\Lambda}{\bar{\gamma}}\right)}_{\text{IR-cutoff by width}} + 0.4816 \Big)$$

Adding the hard and soft contribution contributions gives a finite result

Attempt at a physical explanation of sum rule: