Symmetric Stein–Tomas, and why do we care?

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Modern Trends in Harmonic Analysis, Day 4
30 June 2023

Restriction theory

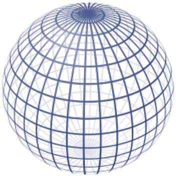
Given $1 \leq p \leq 2$, for which exponents $1 \leq q \leq \infty$ does

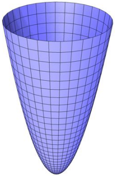
$$\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q \, \mathrm{d}\sigma(\omega) \lesssim \|f\|_{L^p(\mathbb{R}^d)}^q \quad \text{hold?}$$

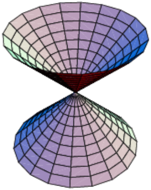
Restriction Conjecture. $1 \leq p < \frac{2d}{d+1}, \quad q \leq \frac{d-1}{d+1} p'$

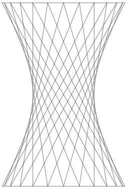
Stein–Tomas (1975). $1 \leq p \leq 2\frac{d+1}{d+3}, \quad q = 2$

Curvature plays a role: Any smooth *compact* hypersurface of *nonvanishing* Gaussian curvature will do.

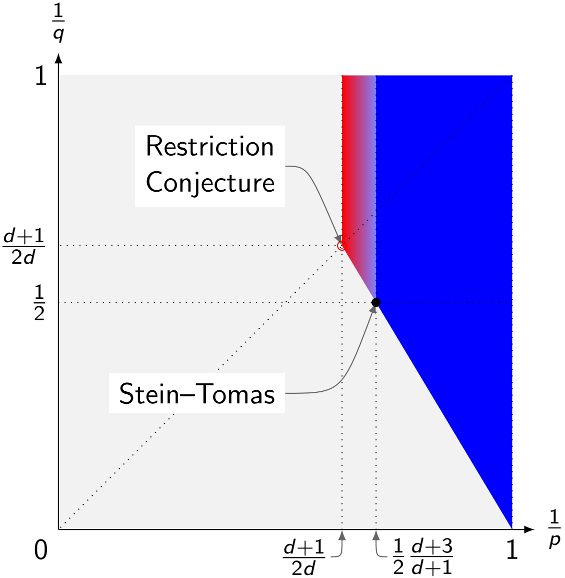








Riesz diagram for the restriction operator to $\mathbb{S}^{d-1}$



Radial case (L. Schwartz)

Let $d \geq 2$ and $1 \leq p < \frac{2d}{d+1}$. If $f \in L^p(\mathbb{R}^d)$ is radial, then $\widehat{f}$ is continuous on $\mathbb{R}^d \setminus \{0\}$.

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Proof sketch.

$$\widehat{f}(\rho) \simeq_d \rho^{\frac{2-d}{2}} \int_0^\infty f(r) J_{\frac{d-2}{2}}(\rho r) r^{\frac{d}{2}} \, \mathrm{d}r$$

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Note that $d - 1 - p'(\frac{d-1}{2}) < -1$ if and only if $p < \frac{2d}{d+1}$. □

Knapp's example on $\mathbb{S}^{d-1}$

Let $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$ and $f = \mathbf{1}_{\mathcal{C}_\delta}$, with the cap $\mathcal{C}_\delta$ given by

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$$\boxed{|\widehat{f\sigma}(x)| \gtrsim \delta^{d-1} \text{ if } x \in \mathcal{R}_\delta^\star}$$

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By letting $\delta \to 0^+$, the estimate

$$\delta^{d-1-\frac{d+1}{p'}} \lesssim \|\widehat{f\sigma}\|_{p'} \lesssim \|f\|_{L^{q'}(\mathbb{S}^{d-1})} \simeq \delta^{\frac{d-1}{q'}}$$

is only possible if $d-1-\frac{d+1}{p'} \geq \frac{d-1}{q'}$, i.e., $q \leq \frac{d-1}{d+1}p'$.

Symmetric setup

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- $m = 0$: radial functions, RC holds ($1 \leq p < \frac{2d}{d+1}$ suffices)
- $m = 1$: Knapp's example precludes any improvement over

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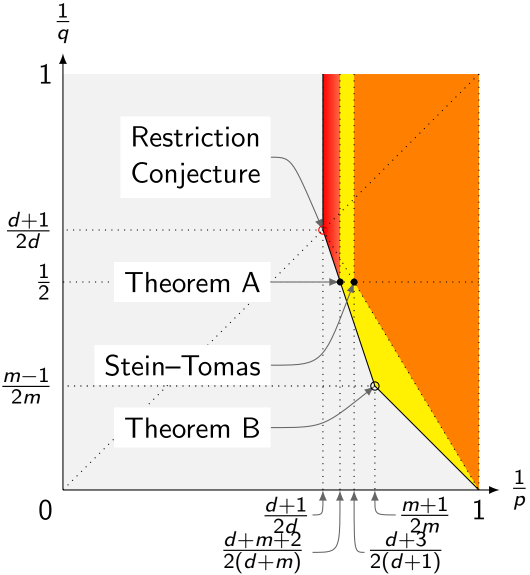

Riesz diagram for the G_k-symmetric restriction problem to $\mathbb{S}^{d-1}$

G_k-symmetric Knapp

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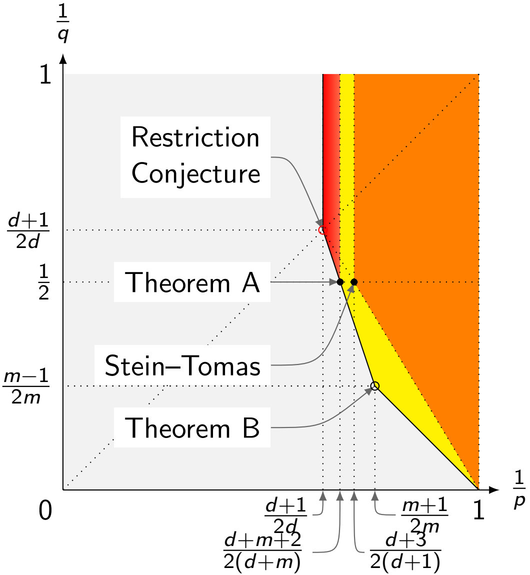
$$\frac{\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|f\|_{L^{q'}(\mathbb{S}^{d-1})}} \gtrsim \begin{cases} \delta^{\frac{d+k}{p}+\frac{d-k}{q}-d-1}, & \text{if } \frac{1}{p} < \frac{k+1}{2k} \\ \delta^{\frac{d+k}{p}+\frac{d-k}{q}-d-1}|\log(\delta)|^{\frac{1}{p'}}, & \text{if } \frac{1}{p} = \frac{k+1}{2k} \\ \delta^{\frac{d-k}{p}+\frac{d-k}{q}-d+k}, & \text{if } \frac{1}{p} > \frac{k+1}{2k} \end{cases}$$



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Theorem A (Mandel–OS 2021)

Let $d \geq 4$ *and* $2 \leq k \leq d-2$ *and* $m = (d-k) \wedge k$. *Then*

$$\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^2 \, \mathrm{d}\sigma(\omega) \lesssim_{k,p,d} \|f\|^2_{L^p(\mathbb{R}^d)}$$

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Tools: Inequalities of Pitt (weighted Hausdorff–Young) and Stein–Weiss (weighted Hardy–Littlewood–Sobolev)

Theorem B (Mandel–OS 2021)

Let $d \geq 4$ *and* $2 \leq k \leq d-2$ *and* $m = (d-k) \wedge k \neq \frac{d}{2}$. *Then*

$$\|\widehat{f}\|_{L^{\frac{2m}{m-1},\infty}(\mathbb{S}^{d-1})} \lesssim_{k,d} \|f\|_{L^{\frac{2m}{m+1},1}(\mathbb{R}^d)}$$

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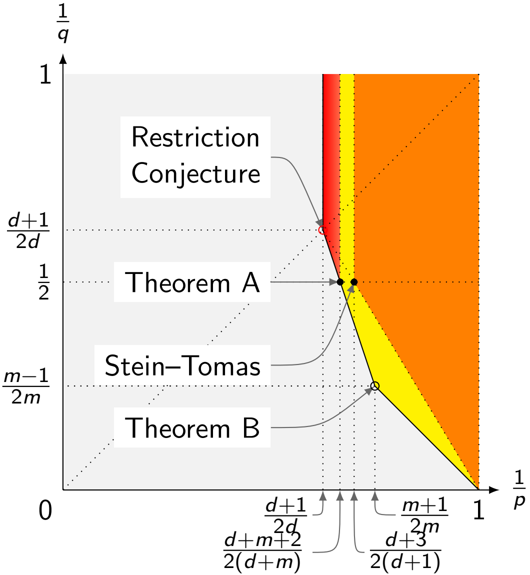
- Real interpolation for mixed Lorentz spaces (Mandel 2023)



Riesz diagram for the G_k-symmetric restriction problem to $\mathbb{S}^{d-1}$

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$$\mathbf{T}_{d,p}(k) = \sup_{0 \neq f \in L^p_k} \frac{\left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^2 \, \mathrm{d}\sigma(\omega) \right)^{\frac{1}{2}}}{\|f\|_{L^p(\mathbb{R}^d)}}$$

In particular, maximizers exist when $p = 2\frac{d+1}{d+3}$.

Maximizing sequences are precompact in $L^{2_\star}_k$

Key lemma (no vanishing)

Let $d \geq 4$ and $2 \leq k \leq d-2$. Then, for every $\varepsilon > 0$, there exists $\rho = \rho(k, d, \varepsilon) < \infty$ for which $|\widehat{f\sigma}(x)| < \varepsilon$ if $|x| > \rho$, for every G_k-symmetric $f \in L^2(\mathbb{S}^{d-1})$ such that $\|f\|_{L^2(\mathbb{S}^{d-1})} = 1$.

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Radial ground states of biharmonic NLS

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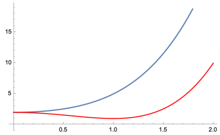

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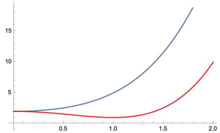

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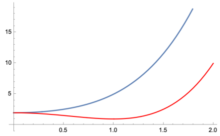

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- $a > 0$: since $g_{a,b}(r)$ is not monotonic on $\mathbb{R}_+$, Fourier rearrangement does **not** apply. By scaling, $(a, b) = (1, 1 + \varepsilon)$.

Nonradial ground states of biharmonic NLS

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Note that $2^1_\star = 2\frac{d+1}{d-1}$ is the endpoint Stein–Tomas exponent.

G_k-symmetry breaking for ground states of biharmonic NLS

$$\mathrm{R}_\varepsilon^k(p) := \inf_{0 \neq u \in H_k^2} \frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2} \qquad \mathrm{R}_\varepsilon^{\mathrm{rad}}(p) := \inf_{0 \neq u \in H_{\mathrm{rad}}^2} \frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2}$$

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- The case $k = 1$ implies Lenzmann–Weth since $2_\star^1 = 2\frac{d+1}{d-1}$.

Lower & upper bounds (Mandel–OS 2023)

Let $d \geq 2$, $1 \leq k \leq d-1$, $2 < p < 2^\star$. *As* $\varepsilon \to 0^+$,

$$\mathrm{R}^\circ_\varepsilon(p) \cong \varepsilon^{1-\frac{1\wedge\alpha_1}{2}}$$

$$\mathrm{R}^k_\varepsilon(p) \cong \varepsilon^{1-\frac{1\wedge\alpha_k}{2}}$$

$$\mathrm{R}^{\mathrm{rad}}_\varepsilon(p) \cong \begin{cases} \varepsilon^{1-\frac{1\wedge\alpha_{\mathrm{rad}}}{2}}, & \text{if } p \neq 2^{\mathrm{rad}}_\star \\ \varepsilon^{\frac{1}{2}}|\log(\varepsilon)|^{\frac{1-d}{d}}, & \text{if } p = 2^{\mathrm{rad}}_\star \end{cases}$$

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$$\alpha_k := \frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{2_\star^k}} \qquad \alpha_{\mathrm{rad}} := \frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{2_\star^{\mathrm{rad}}}}$$

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Rmk. $\alpha_k \in (0,1)$ iff $p \in (2, 2_\star^k)$ and $\alpha_{\mathrm{rad}} \in (0,1)$ iff $p \in (2, 2_\star^{\mathrm{rad}})$

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Upper bounds: test against G_k-symmetric Knapp

Lower & upper bounds (Mandel–OS 2023)

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Upper bounds: test against G_k-symmetric Knapp
Lower bounds: work hard(er)

Lower bounds: proof sketch (for $\mathrm{R}_\varepsilon^k(p)$ only)

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Decompose $u \in H_k^2$ as $u = v + w$ with $\widehat{v}(\xi) = \mathbf{1}_{||\xi|-1| \leq \frac{1}{2}} \widehat{u}(\xi)$.

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Lower bounds: proof sketch (for $\mathrm{R}_\varepsilon^k(p)$ only)

Decompose $u \in H^2_k$ as $u = v + w$ with $\widehat{v}(\xi) = \mathbf{1}_{||\xi|-1|\leq\frac{1}{2}}\widehat{u}(\xi)$.
Since $\|u\|_p \leq \|v\|_p + \|w\|_p$ and $\mathrm{q}_\varepsilon(u) = \mathrm{q}_\varepsilon(v) + \mathrm{q}_\varepsilon(w)$,

$$\frac{\mathrm{q}_\varepsilon(u)}{\|u\|_p^2} \geq \frac{\mathrm{q}_\varepsilon(v)+\mathrm{q}_\varepsilon(w)}{(\|v\|_p+\|w\|_p)^2} \geq \frac{1}{2}\frac{\mathrm{q}_\varepsilon(v)+\mathrm{q}_\varepsilon(w)}{\|v\|_p^2+\|w\|_p^2} \geq \frac{1}{2}\left(\frac{\mathrm{q}_\varepsilon(v)}{\|v\|_p^2} \wedge \frac{\mathrm{q}_\varepsilon(w)}{\|w\|_p^2}\right)$$

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Sobolev embedding takes care of w.

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$$\|v\|_p^2 \cong \left\| \int_{\mathbb{R}^d} e^{i\langle \xi, \cdot \rangle} \widehat{v}(\xi)\, \mathrm{d}\xi \right\|_p^2 \leq \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\| \int_{\mathbb{S}^{d-1}} e^{i\langle \omega, r\cdot \rangle} \widehat{v}(r\omega)\, \mathrm{d}\sigma(\omega) \right\|_p \mathrm{d}r \right)^2$$

$$\cong \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \left\| \int_{\mathbb{S}^{d-1}} e^{i\langle \omega, \cdot \rangle} \widehat{v}(r\omega)\, \mathrm{d}\sigma(\omega) \right\|_p \mathrm{d}r \right)^2 \lesssim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|\widehat{v}(r\cdot)\|_{L^2(\mathbb{S}^{d-1})}\, \mathrm{d}r \right)^2$$

$$\lesssim \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathrm{d}r}{g_\varepsilon(r)} \right) \left(\int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 g_\varepsilon(|\xi|)\, \mathrm{d}\xi \right) \cong \varepsilon^{-\frac{1}{2}} \mathrm{q}_\varepsilon(v)$$

Decompose $u \in H^2_k$ as $u = v + w$ with $\widehat{v}(\xi) = \mathbf{1}_{||\xi|-1|\le\frac12}\widehat{u}(\xi)$.
Since $\|u\|_p \le \|v\|_p + \|w\|_p$ and $q_\varepsilon(u) = q_\varepsilon(v) + q_\varepsilon(w)$,

$$\frac{q_\varepsilon(u)}{\|u\|_p^2} \ge \frac{q_\varepsilon(v)+q_\varepsilon(w)}{(\|v\|_p+\|w\|_p)^2} \ge \frac12 \frac{q_\varepsilon(v)+q_\varepsilon(w)}{\|v\|_p^2+\|w\|_p^2} \ge \frac12 \left(\frac{q_\varepsilon(v)}{\|v\|_p^2} \wedge \frac{q_\varepsilon(w)}{\|w\|_p^2}\right)$$

Sobolev embedding takes care of w. For v, **estimate** for $p \ge 2^k_\star$:

$$\|v\|_p^2 \cong \left\|\int_{\mathbb{R}^d} e^{i\langle\xi,\cdot\rangle}\widehat{v}(\xi)\,\mathrm{d}\xi\right\|_p^2 \le \left(\int_{\frac12}^{\frac32}\left\|\int_{\mathbb{S}^{d-1}} e^{i\langle\omega,r\cdot\rangle}\widehat{v}(r\omega)\,\mathrm{d}\sigma(\omega)\right\|_p \mathrm{d}r\right)^2$$

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$$\lesssim \left(\int_{\frac12}^{\frac32}\frac{\mathrm{d}r}{g_\varepsilon(r)}\right)\left(\int_{\mathbb{R}^d}|\widehat{v}(\xi)|^2 g_\varepsilon(|\xi|)\,\mathrm{d}\xi\right) \cong \varepsilon^{-\frac12} q_\varepsilon(v)$$

If $p \in (2, 2^k_\star)$, interpolate with $\|v\|_2^2 \lesssim \frac{1}{\varepsilon}\int|\widehat{v}|^2 g_\varepsilon = \varepsilon^{-1}q_\varepsilon(v)$.

Asymptotic behaviour of ground states as $\varepsilon \to 0^+$

Let $d \geq 2$, $1 \leq k \leq d-1$, $A_{\varepsilon,\delta} := \{\xi \in \mathbb{R}^d : ||\xi| - 1| \in I_{\varepsilon,\delta}\}$ with $I_{\varepsilon,\delta} := [\delta\sqrt{\varepsilon}, \delta^{-1}\sqrt{\varepsilon}]$. Decompose $u_\varepsilon = v_\varepsilon + w_\varepsilon$ where $\widehat{v}_\varepsilon = \mathbf{1}_{A_{\varepsilon,\delta}}\widehat{u}_\varepsilon$.

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Concentration (Mandel–OS 2023)

Let $2 < p < 2^\star$. Let $* \in \{\circ, k, \mathrm{rad}\}$. If $u_\varepsilon = v_\varepsilon + w_\varepsilon$ is a minimizer for $\mathrm{R}^*_\varepsilon(p)$ for a given positive null sequence $(\delta_\varepsilon)_{\varepsilon>0}$, then

$$\lim_{\varepsilon \to 0^+} \frac{\|w_\varepsilon\|_p}{\|v_\varepsilon\|_p} = \lim_{\varepsilon \to 0^+} \frac{\mathrm{q}_\varepsilon(w_\varepsilon)}{\mathrm{q}_\varepsilon(v_\varepsilon)} = 0$$

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Roughness (Mandel–OS 2023)

Let $2 < p < 2^k_\star$. Let $* \in \{\circ, k\}$. If u_ε is a minimizer for $\mathrm{R}^*_\varepsilon(p)$, then for every $t > 0$ and every positive null sequence $(\delta_\varepsilon)_{\varepsilon>0}$,

$$\lim_{\varepsilon\to0^+} \sup_{|r-1|\in I_{\varepsilon,\delta_\varepsilon}} \frac{\|\widehat{u}_\varepsilon(r\,\cdot)\|_{H^t(\mathbb{S}^{d-1})}}{\|\widehat{u}_\varepsilon(r\,\cdot)\|_{L^2(\mathbb{S}^{d-1})}} = \infty$$

Thank you very much

Also to my collaborators:

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