Global Maximizers for Spherical Restriction

Diogo Oliveira e Silva

Modern Trends in Harmonic Analysis, Day 3
ICTS, 28 June 2023

Three problems

Geometry

Given $d \geq 2$ and $0 < k < d$, what is the maximal volume of the intersection of the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ with a k-dimensional subspace of $\mathbb{R}^d$?

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Given $d, n \geq 2$, what is the probability distribution of an n-step uniform random walk in $\mathbb{R}^d$?

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Algebra

Given $d \geq 2$, what is the minimal codimension of a proper subalgebra of the special orthogonal Lie algebra $\mathfrak{so}(d)$?

Restriction theory

Given $1 \leq p \leq 2$, for which exponents $1 \leq q \leq \infty$ does

$$\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q \, \mathrm{d}\sigma(\omega) \lesssim \|f\|_{L^p(\mathbb{R}^d)}^q \quad \text{hold?}$$

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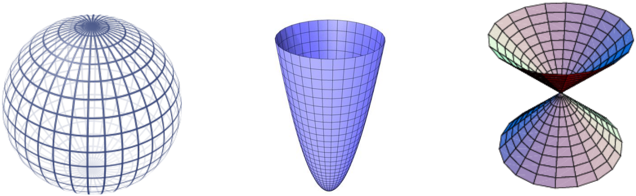
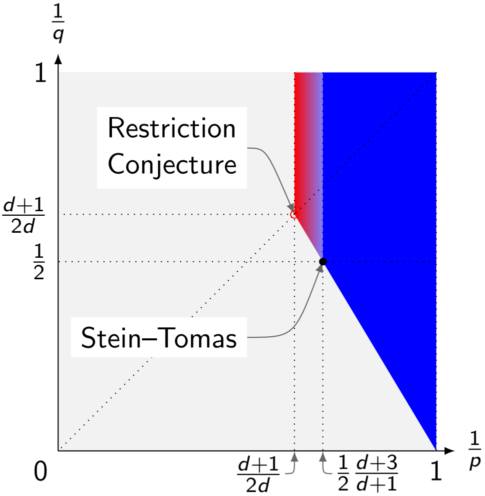


Riesz diagram for the restriction operator to $\mathbb{S}^{d-1}$



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Let $d \in \{3,4,5,6,7\}$ and $q \geq 6$ be an even integer. Then constant functions are the unique real-valued maximizers of $\Phi_{d,q}$.

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Special case: $(d, q) = (3, 6)$, $\Phi_q := \Phi_{3,q}$, $\mathbf{T}_q := \mathbf{T}_{3,q}$

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(uses evenness of $q = 6$ in a crucial way)

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Therefore:

$$\mathbf{T}_6^6 = \max_{\substack{f \in C^\infty(\mathbb{S}^2)\setminus\{\mathbf{0}\} \\ f_\star = f \geq 0}} \Phi_6(f)$$

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- $K_f(\xi) = \overline{K_f(-\xi)}$, for all ξ

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Let $f : \mathbb{S}^2 \to \mathbb{R}$ be non-constant, such that $f_\star = f \in C^1(\mathbb{S}^2)$ and $\|f\|_{L^2} = 1$. Assume $T_f(f) = \lambda f$.

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Key: The codimension of a proper, nontrivial subalgebra of $\mathfrak{so}(3)$ is equal to 2. Think $SO(2) \subsetneq SO(3)$.

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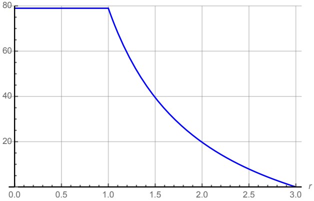

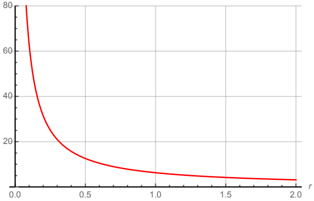

Left: Plot of $r \mapsto (\sigma * \sigma * \sigma)(r)$ for $0 \le r \le 3$
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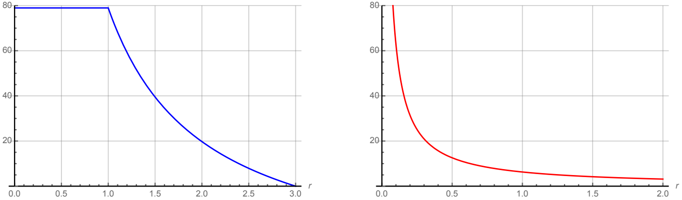

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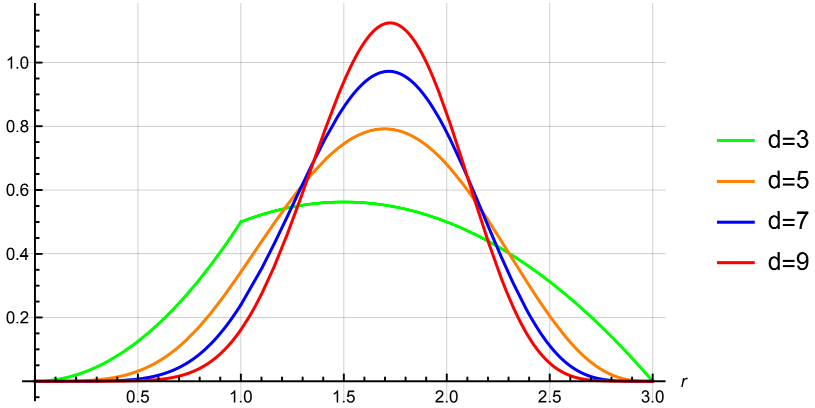
3-step uniform random walks in $\mathbb{R}^d$



Plot of the function $r \mapsto r^{d-1}(\sigma * \sigma * \sigma)(r)$ for $0 \leq r \leq 3$, when $d \in \{3, 5, 7, 9\}$.

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$$\int_0^\infty |J_\nu(r)|^q \, r^{d-1-q\nu} \, \mathrm{d}r \leq \frac{\Gamma(\frac{d}{2})^2}{2^{2-d}} \frac{q+2d-\delta_{d,4}}{q+1} \int_0^\infty |J_\nu(r)|^{q+2} \, r^{d-1-(q+2)\nu} \, \mathrm{d}r$$

$$E(d,q) := \frac{\Gamma(\frac{d}{2})^2}{2^{2-d}} \frac{\int_0^\infty |J_\nu(r)|^{q+2}\, r^{d-1-(q+2)\nu}\,\mathrm{d}r}{\int_0^\infty |J_\nu(r)|^{q}\, r^{d-1-q\nu}\,\mathrm{d}r} - \frac{q+1}{q+2d-\delta_{d,4}} \overset{?}{\geq} 0$$

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$d \backslash q$	4	6	8	10	12	14	16	18	20
2	*	0.57	5.16	5.32	4.95	4.55	4.19	3.88	3.61
3	0.00	8.33	8.63	8.29	7.85	7.40	6.97	6.57	6.20
4	0.82	4.83	5.67	5.93	5.94	5.83	5.66	5.45	5.24
5	3.98	7.68	9.13	9.77	9.97	9.94	9.77	9.53	9.25
6	2.26	6.16	8.24	9.38	9.97	10.22	10.27	10.18	10.03
7	0.36	4.46	7.03	8.62	9.57	10.10	10.38	10.47	10.45
8	−1.42	2.75	5.66	7.62	8.89	9.70	10.20	10.47	10.60
9	−2.98	1.11	4.25	6.49	8.04	9.10	9.81	10.26	10.54
10	−4.31	−0.39	2.85	5.31	7.09	8.36	9.27	9.89	10.31
11	−5.41	−1.76	1.51	4.11	6.08	7.54	8.62	9.40	9.96

Values of $100 \times E(d,q)$, obtained through numerical evaluation of Bessel integrals and truncated to two decimal places.

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Theorem (OS–Quilodrán 2021)

Let $d \geq 2$ and $q \geq 2\frac{d+1}{d-1}$ be an even integer. Then each complex-valued maximizer of $\Phi_{d,q}$ is of the form

$$ce^{i\xi\cdot\omega}F(\omega)$$

for some $\xi \in \mathbb{R}^d$, some $c \in \mathbb{C} \setminus \{0\}$, and some nonnegative, antipodally symmetric maximizer F of $\Phi_{d,q}$.

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Corollary (OS–Quilodrán)

Let $d \in \{3, 4, 5, 6, 7\}$ and $q \geq 4$ be an even integer. Then all complex-valued maximizers of $\Phi_{d,q}$ are given by

$$f(\omega) = ce^{i\xi\cdot\omega}$$

for some $\xi \in \mathbb{R}^d$ and $c \in \mathbb{C} \setminus \{0\}$.

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6. Conversely, are all real-valued maximizers of $\Phi_{d,q}$ if $q = 2\frac{d+1}{d-1}$ constant?

Thank you