# Sharp, maximal & variational restriction theory

Diogo Oliveira e Silva

**Modern Trends in Harmonic Analysis, Day 2**
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# Restriction theory

Given $1 \le p \le 2$, for which exponents $1 \le q \le \infty$ does

$$\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q \, \mathrm{d}\sigma(\omega) \lesssim \|f\|_{L^p(\mathbb{R}^d)}^q \quad \text{hold?}$$

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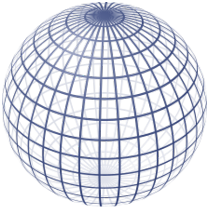
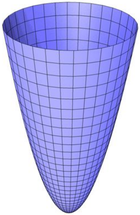
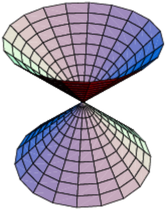
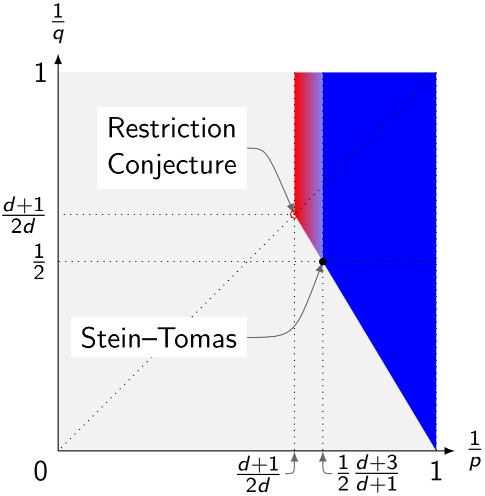






# Riesz diagram for the restriction operator



# Sharp restriction theory

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The latter inequality implies:

$$\boxed{\left| \int_{(\mathbb{S}^2)^2} f(\omega) f(\omega') g(\omega - \omega') \, \mathrm{d}\sigma(\omega) \, \mathrm{d}\sigma(\omega') \right| \lesssim \|f\|^2_{L^2(\mathbb{S}^2)} \|g\|_{L^2(\mathbb{R}^3)}}$$

# Maximal restriction theory

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## Lebesgue differentiation theorem

*If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then, for almost every $x \in \mathbb{R}^d$,*

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In fact, almost every $x \in \mathbb{R}^d$ is a **Lebesgue point** of $f \in L^1_{\text{loc}}(\mathbb{R}^d)$:

$$\lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r} |f(x-y) - f(x)| \, \mathrm{d}y = 0$$

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**Müller–Ricci–Wright (2019)**

*Let $f \in L^p(\mathbb{R}^2)$. Then:*

- *If $1 \leq p < \frac{4}{3}$, then $\lim_{\varepsilon \to 0^+}(\widehat{f} * \chi_\varepsilon)(\omega) = \mathcal{R}f(\omega)$, $\sigma$-a.e.*
- *If $1 \leq p < \frac{8}{7}$, then $\sigma$-a.e. $\omega \in \mathbb{S}^1$ is a Lebesgue point of $\widehat{f}$*

Let $\mathcal{R}f := \widehat{f}\,|_{\mathbb{S}^1}$. It is known that

$$\|\mathcal{R}f\|_{L^q(\mathbb{S}^1,\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}, \quad 1 \le p < \tfrac{4}{3},\ p' \ge 3q \qquad (1)$$

**What is the "intrinsic" pointwise relation between $\mathcal{R}f$ and $\widehat{f}$ for general $f \in L^p$?** Let $\chi \in \mathcal{S}(\mathbb{R}^2)$ be real-valued, even, $\int \chi = 1$; let $\chi_\varepsilon := \varepsilon^{-2}\chi(\varepsilon^{-1}\cdot)$. If (1) holds, then for $\sigma$-a.e. $\omega \in \mathbb{S}^1$,

$$(\widehat{f} * \chi_{\varepsilon_k})(\omega) \to \mathcal{R}f(\omega) \text{ as } k \to \infty$$

along some sequence $\varepsilon_k \to 0^+$. **What happens in general?**

### Müller–Ricci–Wright (2019)

*Let $f \in L^p(\mathbb{R}^2)$. Then:*

- *If $1 \le p < \frac{4}{3}$, then $\lim_{\varepsilon \to 0^+}(\widehat{f} * \chi_\varepsilon)(\omega) = \mathcal{R}f(\omega)$, $\sigma$-a.e.*
- *If $1 \le p < \frac{8}{7}$, then $\sigma$-a.e. $\omega \in \mathbb{S}^1$ is a Lebesgue point of $\widehat{f}$*

$L^p(\mathbb{R}^2)$–$L^q$ bounds for the two-parameter maximal function

$$\mathfrak{M}f(x) = \sup_{0<\varepsilon,\varepsilon'<1} \left| \int_{\mathbb{R}^2} \widehat{f}(x+s, \sqrt{1-x^2}+t)\chi_\varepsilon(s)\chi_{\varepsilon'}(t)\,\mathrm{d}s\,\mathrm{d}t \right|$$

# The case of $\mathbb{S}^2$

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**Maximal endpoint Stein–Tomas on $\mathbb{S}^2$ (Vitturi 2017)**

$$\Big\| \sup_{\varepsilon>0} (\widehat{f} * \chi_\varepsilon)(\omega) \Big\|_{L^2_\omega(\mathbb{S}^2)} \lesssim_\chi \|f\|_{L^{4/3}(\mathbb{R}^3)}$$

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**Corollary**

*If $f \in L^p(\mathbb{R}^3)$, $1 \leq p \leq \frac{8}{7}$, then $\sigma$-almost every point of $\mathbb{S}^2$ is a Lebesgue point of $\widehat{f}$.*

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# Variational restriction theory

Given $a\colon (0,\infty) \to \mathbb{C}$ and $1 \le \varrho < \infty$, the *$\varrho$-variation norm* of $a$ is

$$\|a\|_{V^\varrho} := \sup_{\substack{m\in\mathbb{N}\cup\{0\}\\ \varepsilon_0>\varepsilon_1>\cdots>\varepsilon_m>0}} \left( |a(\varepsilon_0)|^\varrho + \sum_{j=1}^{m} |a(\varepsilon_{j-1}) - a(\varepsilon_j)|^\varrho \right)^{\frac{1}{\varrho}}.$$

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**Variational endpoint Stein–Tomas on $\mathbb{S}^2$ (Kovač–OS 2021)**

*If $2 < \varrho < \infty$, then*

$$\Big\| (\widehat{f} * \chi_\varepsilon)(\omega) \Big\|_{L^2_\omega(\mathbb{S}^2; V^\varrho_\varepsilon)} \lesssim_{\chi,\varrho} \|f\|_{L^{4/3}(\mathbb{R}^3)} \qquad (2)$$

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- Large sets without Fourier restriction theorems

# Two ingredients

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**Variational estimates for convolution-type operators**

*If* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *and* $2 < \varrho < \infty$, *then*

$$\|(h * \varphi_\varepsilon)(x)\|_{L^2_x(\mathbb{R}^d;\widetilde{V}^\varrho_\varepsilon)} \lesssim_{d,\varphi} (\varrho - 2)^{-1} \|h\|_{L^2(\mathbb{R}^d)}$$

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**Gaussian domination (Stein 1970's & Durcik, Kovač 2012–)**

*Given $\chi \in \mathcal{S}(\mathbb{R}^3)$ real-valued and even, let $\vartheta(x) := -x \cdot (\nabla\widehat{\chi})(x)$. Let $\varphi(x) = \exp(-\pi|x|^2)$ and $\widehat{\psi}(x) := -x \cdot (\nabla\widehat{\varphi})(x)$. Then:*

$$|\vartheta(x)| \lesssim_\chi \int_1^\infty \widehat{\psi}\left(\frac{x}{\alpha}\right) \frac{\mathrm{d}\alpha}{\alpha^2}$$

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## Christ–Kiselev (2001)

*Let* $1 \leq p < q < \infty$ *and* $\beta = \frac{1}{p} - \frac{1}{q}$. *Let* $T : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ *be a bounded linear operator. Define the maximal truncations*

$$(T_\star f)(x) = \sup_{\alpha \in \mathbb{R}} |T(f\mathbf{1}_\alpha)(x)|$$

*where* $\mathbf{1}_\alpha = \mathbf{1}_{(-\infty,\alpha)}$. *Then*

$$\|T_\star f\|_{L^q(\mathbb{R})} \leq (2^\beta - 1)^{-1} \|T\|_{p\to q} \|f\|_{L^p(\mathbb{R})}$$

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- Many applications (e.g. retarded Strichartz estimates)
- Variational version: If $1 \leq p < q < \infty$ and $\varrho > p$, then

$$\|T(f\mathbf{1}_\alpha)\|_{L^q(V^\varrho_\alpha)} \lesssim_{p,q,\varrho} \|T\|_{p\to q} \|f\|_{L^p}$$

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**"Fourier restriction implies maximal and variational Fourier restriction" (Kovač 2019)**

*Under the above conditions, let $\varrho \in (p,\infty)$. Then:*

$$\Big\| \sup_{t>0} |\widehat{f} * \mu_t| \Big\|_{L^q(S,\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

$$\Bigg\| \sup_{\substack{m \in \mathbb{N}\cup\{0\} \\ t_m > \cdots > t_1 > t_0 > 0}} \Bigg( \sum_{j=1}^{m} \Big| \widehat{f} * \mu_{t_{j-1}} - \widehat{f} * \mu_{t_j} \Big|^{\varrho} \Bigg)^{\frac{1}{\varrho}} \Bigg\|_{L^q(S,\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

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**Corollary**

*If $f \in L^p(\mathbb{R}^3)$, $1 \leq p \leq \frac{8}{7}$, then $\sigma$-almost every point of $\mathbb{S}^2$ is a Lebesgue point of $\widehat{f}$.*

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## Corollary 2

*Let* $0 \leq \alpha \leq d$ *and* $1 \leq p \leq \frac{2d}{2d-\alpha}$. *There exists a compact* $E \subset \mathbb{R}^d$ *such that* $\dim_H(E) = \alpha$ *and* $p_{\text{res}}(E) = p$.

# Thank you