Sharp restriction theory: highlights and future directions

Diogo Oliveira e Silva

Modern Trends in Harmonic Analysis, Day 1
ICTS, 26 June 2023

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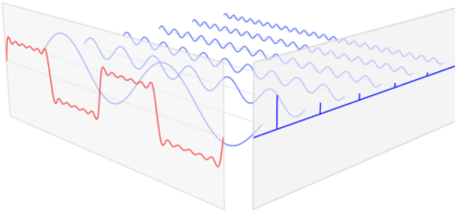

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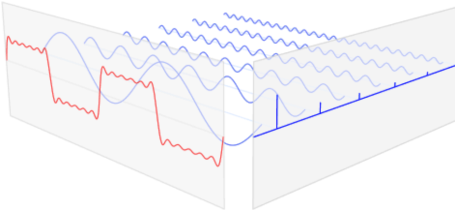

- This defines a contraction from L^1 to L^∞
- It extends to a unitary operator on L^2
- It extends to contraction from L^p to $L^{p'}$, if $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$

Sharp Hausdorff–Young: Beckner (1975)

If $d \geq 1$ *and* $1 \leq p \leq 2$, *then*

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \mathbf{B}_p^d \|f\|_{L^p(\mathbb{R}^d)}$$

where $\mathbf{B}_p = p^{\frac{1}{2p}} (p')^{-\frac{1}{2p'}}$. *A maximizer is* $f = \exp(-|\cdot|^2)$.

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Sharp Hardy–Littlewood–Sobolev: Lieb (1983)

If $d \geq 1$, $0 < \lambda < d$ and $p = \frac{2d}{2d-\lambda}$, then

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)|x-y|^{-\lambda} g(y)\,\mathrm{d}x\,\mathrm{d}y \right| \leq \mathbf{L}_{\lambda,d} \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}$$

with $\mathbf{L}_{\lambda,d} = \pi^{\frac{\lambda}{2}} \frac{\Gamma(\frac{d-\lambda}{2})}{\Gamma(d-\frac{\lambda}{2})} \left(\frac{\Gamma(\frac{d}{2})}{\Gamma(d)}\right)^{\frac{\lambda}{d}-1}$. A maximizer is $f = (1+|\cdot|^2)^{\frac{\lambda}{2}-d}$.

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$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \left(\mathbf{B}_p^d - c\frac{\text{dist}_p^2(f, \mathfrak{G})}{\|f\|^2_{L^p(\mathbb{R}^d)}} \right) \|f\|_{L^p(\mathbb{R}^d)}$$

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    - **number theory**: sphere packing in dimension 8 (Viazovska 2017) and in dimension 24 (Cohn–Kumar–Miller–Radchenko––Viazovska 2017)

Restriction theory

Given $1 \leq p \leq 2$, for which exponents $1 \leq q \leq \infty$ does

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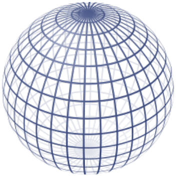

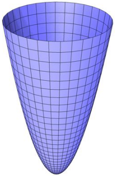

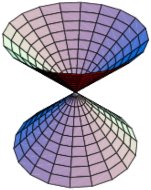

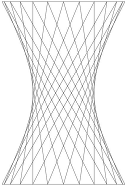

Bochner–Riesz

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What is the smallest area needed to rotate a unit line segment (a "needle") by 180 degrees in the plane?

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Also: local smoothing conjecture (solved in $\mathbb{R}^{2+1}$ only in 2020)

Why should *any* nontrivial restriction inequality hold?

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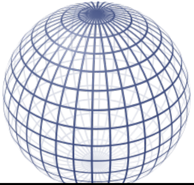
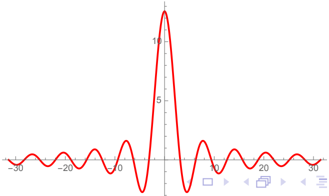
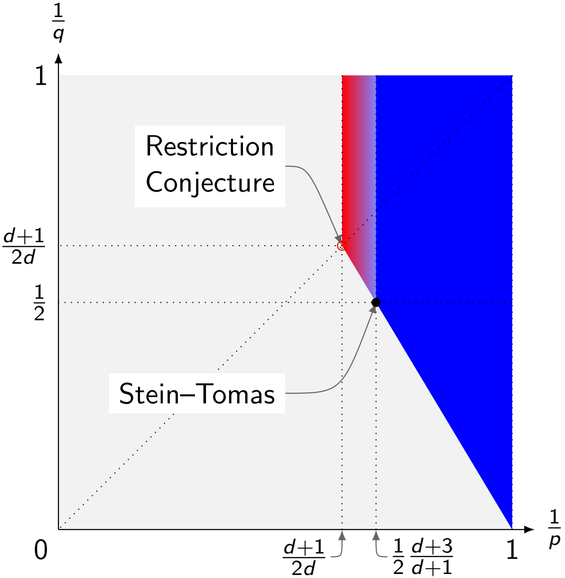




Riesz diagram for the restriction operator



Agmon–Hörmander (1975)

$$\frac{1}{\rho}\int_{B_\rho} |\widehat{f\sigma}(x)|^2 \,\frac{\mathrm{d}x}{(2\pi)^d} \leq \mathbf{A}_d(\rho) \int_{\mathbb{S}^{d-1}} |f(\omega)|^2 \,\mathrm{d}\sigma(\omega)$$

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Tao: $\mathcal{R}_{\mathbb{S}^{d-1}}(2 \to 2; \frac{1}{2})$; Ag–Hör: $\mathbf{A}_d(\rho) \to \frac{1}{\pi}$ as $\rho \to \infty$

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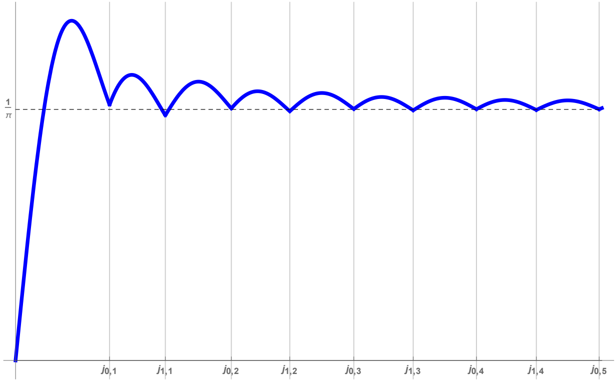

Negro–OS (2023): Best constant $\mathbf{A}_2(\rho)$

Sharpened Agmon–Hörmander: constructive stability

$$\delta_d(f;\rho) := \mathbf{A}_d(\rho)\|f\|^2_{L^2(\mathbb{S}^{d-1})} - \frac{1}{\rho}\int_{B_\rho} |\widehat{f\sigma}(x)|^2 \,\frac{\mathrm{d}x}{(2\pi)^d}$$

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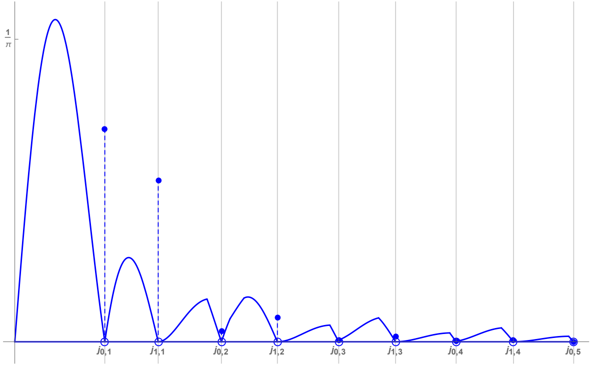

Negro–OS (2023): Stability constant $\mathbf{S}_2(\rho)$

Vega (1988)

Sharp mixed norm spherical restriction

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A B S T R A C T

Let $d \geq 2$ be an integer and let $2d/(d-1) < q \leq \infty$. In this paper we investigate the sharp form of the mixed norm Fourier extension inequality

$$\|\widehat{f\sigma}\|_{L^q_{\mathrm{rad}}L^2_{\mathrm{ang}}(\mathbb{R}^d)} \leq \mathbf{C}_{d,q}\,\|f\|_{L^2(\mathbb{S}^{d-1},\mathrm{d}\sigma)},$$

established by L. Vega in 1988. Letting $\mathcal{A}_d \subset (2d/(d-1),\infty]$ be the set of exponents for which the constant functions on $\mathbb{S}^{d-1}$ are the unique extremizers of this inequality, we show that: (i) $\mathcal{A}_d$ contains the even integers and ∞; (ii) $\mathcal{A}_d$ is an open set in the extended topology; (iii) $\mathcal{A}_d$ contains a neighborhood of infinity $(q_0(d),\infty]$ with $q_0(d) \leq \left(\frac{1}{2}+o(1)\right) d \log d$. In low dimensions we show that $q_0(2) \leq 6.76$; $q_0(3) \leq 5.45$; $q_0(4) \leq 5.53$; $q_0(5) \leq 6.07$. In particular, this breaks for the first time the even exponent barrier in sharp Fourier restriction theory. The crux of the matter in our approach is to establish a hierarchy between certain weighted norms of Bessel functions, a nontrivial question of independent interest within the theory of special functions.

Sharp Stein–Tomas

If $d \geq 2$ and $q \geq 2\frac{d+1}{d-1}$, then $\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq \mathbf{T}_{d,q}\|f\|_{L^2(\mathbb{S}^{d-1})}$.

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- Maximizers exist if $q > 2\frac{d+1}{d-1}$, or if $(d, q) \in \{(2, 6), (3, 4)\}$

  (Fanelli–Vega–Visciglia 2011, Christ–Shao 2012, Shao 2016, Frank–Lieb–Sabin 2016, Flock–Stovall 2022)

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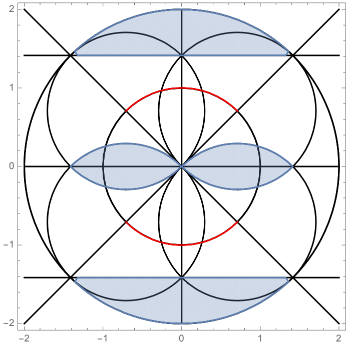

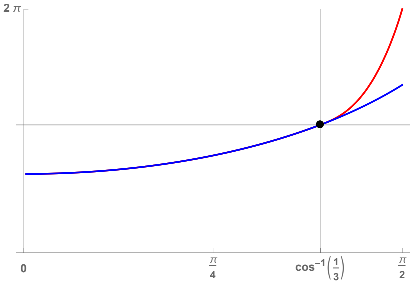

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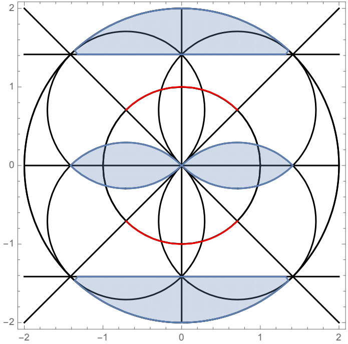

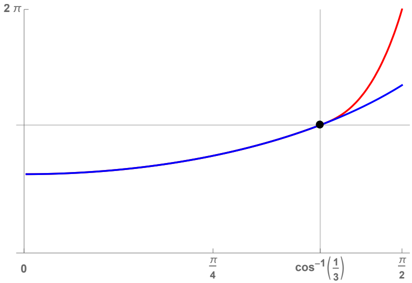

Left: The sumset $\mathcal{C}_\phi^\star + \mathcal{C}_\phi^\star$, where $\mathcal{C}_\phi^\star = -\mathcal{C}_\phi \cup \mathcal{C}_\phi$ and $\mathcal{C}_\phi \subset \mathbb{S}^2$ is the (red) cap centered at the north pole with half-angle $\phi = \frac{\pi}{4}$.

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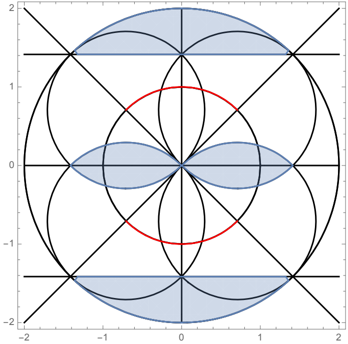

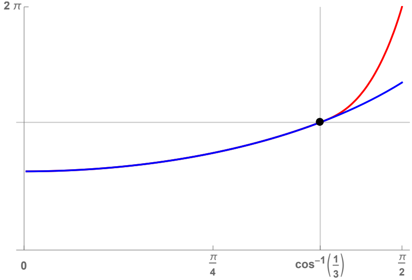

Right: The Stein–Tomas functional $\Phi(f) = \|f\|_2^{-2}\|f\sigma * f\sigma\|_2$ for $f = \mathbf{1}_{\mathcal{C}_\phi}$ and $0 < \phi < \frac{\pi}{2}$ (blue). It holds that $2\Phi(\mathbf{1}_{\mathcal{C}_\phi^\star}) = 3\Phi(\mathbf{1}_{\mathcal{C}_\phi})$ if and only if $\cos\phi \geq \frac{1}{3}$.

Sharp Stein–Tomas

If $d \geq 2$ and $q \geq 2\frac{d+1}{d-1}$, then $\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq \mathbf{T}_{d,q}\|f\|_{L^2(\mathbb{S}^{d-1})}$.

$$\mathbf{T}_{d,q} := \sup_{0 \neq f \in L^2} \Phi_{d,q}(f) := \sup_{0 \neq f \in L^2} \frac{\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{S}^{d-1})}}$$

- Maximizers exist if $q > 2\frac{d+1}{d-1}$, or if $(d, q) \in \{(2, 6), (3, 4)\}$

  (Fanelli–Vega–Visciglia 2011, Christ–Shao 2012, Shao 2016, Frank–Lieb–Sabin 2016, Flock–Stovall 2022)

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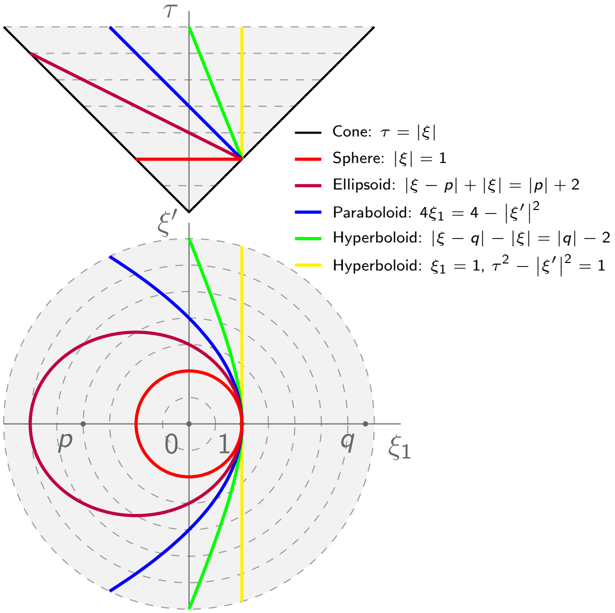
OS–Quilodrán (2021)

Let $d \in \{3,4,5,6,7\}$ *and* $q \geq 6$ *be an even integer. Then characters are the unique complex-valued maximizers of* $\Phi_{d,q}$.



τ
ξ'
p
0
1
q
ξ_1
Cone: $\tau = |\xi|$
Sphere: $|\xi| = 1$
Ellipsoid: $|\xi - p| + |\xi| = |p| + 2$
Paraboloid: $4\xi_1 = 4 - |\xi'|^2$
Hyperboloid: $|\xi - q| - |\xi| = |q| - 2$
Hyperboloid: $\xi_1 = 1,\ \tau^2 - |\xi'|^2 = 1$

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Gaussians are the unique maximizers

Conjecture (Lieb 1990 & Hundertmark–Zharnitsky 2006)

Gaussians maximize (1) *for all* $d \ge 1$.

Christ–Quilodrán (2014)

Let $1 < p < 2 + \frac{2}{d}$ *and* $q = \frac{d+2}{d}p'$. *Gaussians are critical points for the* $L^p \to L^q$ *extension inequality on the paraboloid,* $\|e^{it\Delta} f\|_{L^q(\mathbb{R}^{1+d})} \leq \mathbf{S}_{d,p}\|f\|_{L^p(\mathbb{R}^d)}$, *if and only if* $p = 2$.

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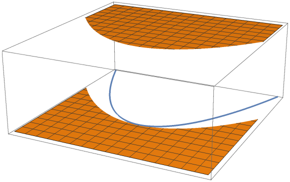
The inequality (non-sharp: Keel–Tao 1998)

$$\|(I + \Delta)^{\frac{1}{4}} e^{it(\Delta+\Delta^2)} f\|_{L^4(\mathbb{R}^{1+2})} \leq 2^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^2)}$$

is sharp but possesses no maximizers. Concentration occurs at 0.

Geometric comparison principle

If $\tau = |\xi|^2$, then the convolution $\mu * \mu$ is **constant** in its support:



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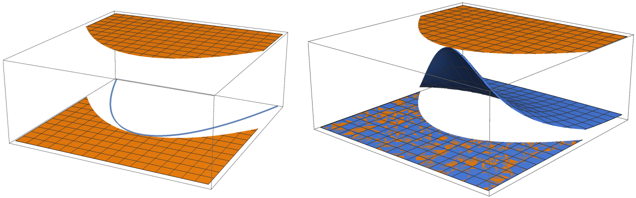

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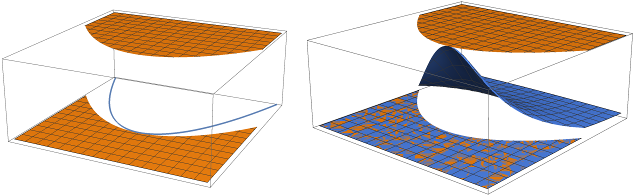

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Let $\phi : \mathbb{R}^d \to [0, \infty) \in C^1$ be a strictly convex function. Consider $|\cdot|^2$ and $|\cdot|^2 + \phi$ with projection measures ν_0 and ν. Then

$$\nu^{*(n)}(\xi, \tau) \le \nu_0^{*(n)}(\xi, \tau - n\phi(\xi/n)), \text{ for } \xi \in \mathbb{R}^d, \tau > n\psi(\xi/n),$$

with strict inequality in the interior of the support of $\nu^{*(n)}$.

Application to PDE (fourth order Schrödinger equations)

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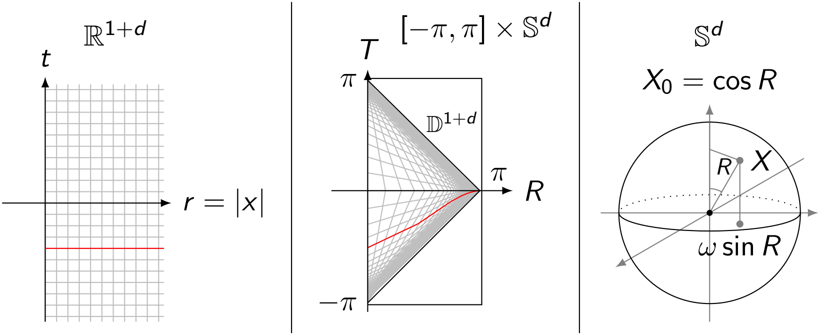
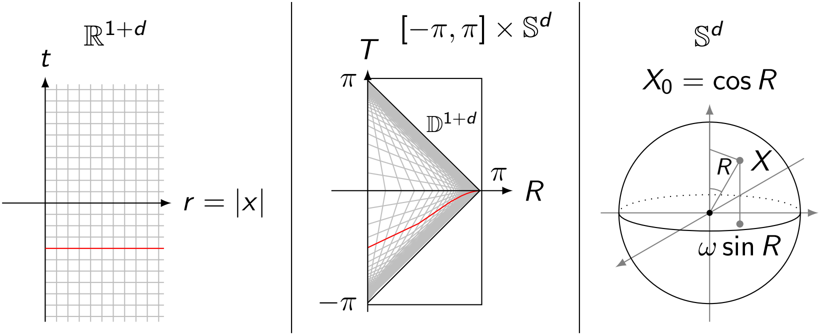
$\mathfrak{F}$**-functions**: $\widehat{f_\star}(\xi) = |\xi|^{-1} \exp(A|\xi| + b \cdot \xi + c)$, $|\mathrm{Re}(b)| < -\mathrm{Re}(A)$

Penrose transform



$$\mathcal{P} : \mathbb{R}^{1+d} \to \mathbb{D}^{1+d}$$

$$(T, R) = (\arctan(t + r) + \arctan(t - r), \arctan(t + r) - \arctan(t - r))$$



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Key : $(1 + |\cdot|^2)^{\frac{1-d}{2}}$ on $\mathbb{R}^d$ $\overset{\text{Penrose transform}}{\rightsquigarrow}$ $\mathbf{1}$ on $\mathbb{S}^d$

Spatial dimension d	**2-cone**	**1-cone**
2	NO	YES
3	YES	YES
4, 6, 8, ...	NO	Loc
5, 7, 9, ...	Loc	Loc

YES: Foschi 2007; **NO**: Negro 2018; **Loc**: Gonçalves–Negro 2022

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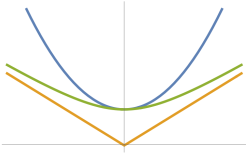
Negro–OS–Stovall–Tautges (2023)

Let $d \geq 2$ and $1 < p < \frac{2d}{d-1}$ and $q = \frac{d+1}{d-1}p'$. Then:

- *Maximizing sequences for (3) are precompact* mod symmetries*
- *$\mathfrak{F}$-functions are critical points if and only if $p = 2$*

What about hyperboloids?



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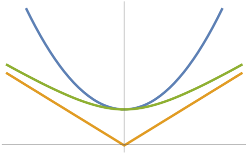

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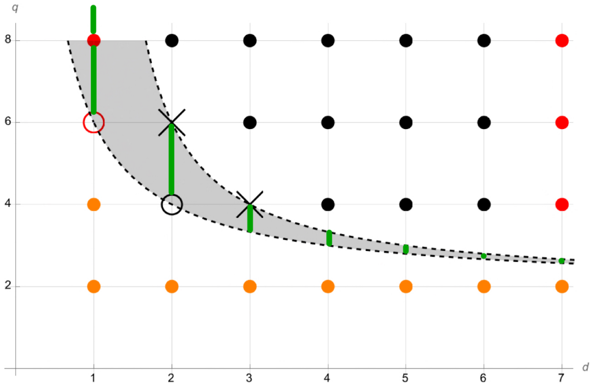

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Som recent sharp results

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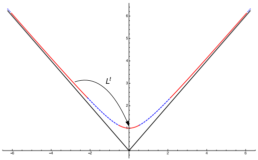
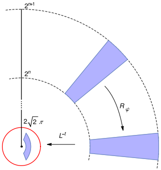
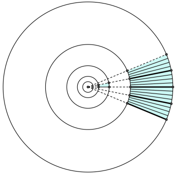
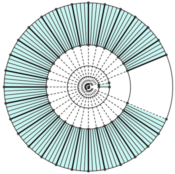
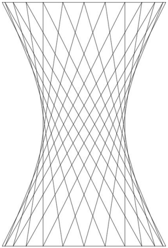


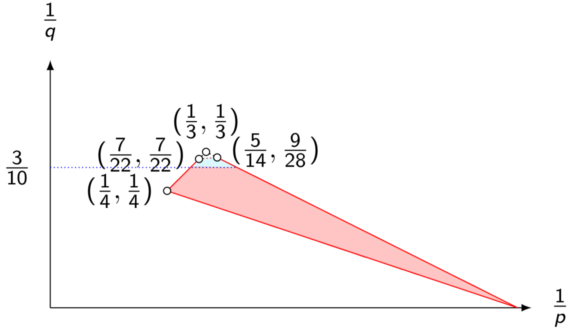






New L^p–L^q global restriction estimates to the one-sheeted hyperboloid $\{(\tau,\xi) \in \mathbb{R}^{1+2} : 1 + \tau^2 = |\xi|^2\}$





Bruce–OS–Stovall (2021)

Unconditional estimates in the bilinear range $q > \frac{10}{3}$.
Restriction conjecture for low frequency region implies global restriction estimates for exponent pairs within the red quadrilateral.

Thank you very much

Also to my collaborators:

• **G. Brocchi** (Chalmers) • **B. Bruce** (UBC) • **E. Carneiro** (ICTP) • **M. Christ** (UC Berkeley) • **D. Foschi** (Ferrara) • **F. Gonçalves** (IMPA) • **A. Guedes de Oliveira** (Porto) • **V. Kovač** (Zagreb) • **R. Mandel** (KIT) • **G. Negro** (IST) • **R. Quilodrán** • **J. P. Ramos** (ETH) • **J. Rupčić** (Zagreb) • **M. Sousa** (BCAM) • **S. Steinerberger** (UW Seattle) • **B. Stovall** (UW Madison) • **J. Tautges** (UW Madison) • **C. Thiele** (Bonn) • **P. Zorin-Kranich**