Day 2 Geometry of the global nilpotent cone

2.1 Donagi-Pantev's programme

<u>Geometric Langlands</u> X R.S. $g \geq 2$

$\mathbb{C}^\times$ Rk ~~one~~ n local system on X

$GL(n, \mathbb{C})$ (L, ∇) ~~extends~~ yields a D-module uniquely

to $\mathrm{Jac}^0(X)$

$N_X(n,0) = \check{N}_X(GL(n,\mathbb{C}))$

$$\begin{tikzcd}
\pi_1(X) \arrow[rr, "\rho"] \arrow[dr] & & \mathbb{C}^\times \\
& \pi_1(X)_{ab} = \pi_1(\mathrm{Jac}^0(X)) \arrow[ur, "\rho_{Jac}"'] &
\end{tikzcd}$$

$\pi_1(X)_{ab} = H_1(X, \mathbb{Z})$

So ρ_{Jac} corresponds to $(\mathbb{L}, \nabla) \to \mathrm{Jac}$

Recall from <u>Day 1</u> that $\exists$

$$r : M_X(n,0) \dashrightarrow N_X(n,0)$$

Through the Hitchin system we may dominate $N_X(n,d)$ by $\mathrm{Jac}^0(X_b)$ (see § 2.2)

$$r_b : \mathrm{Jac}^0(X_b) \dashrightarrow N_X(n,d)$$

$$(\mathcal{L}, \nabla) \longrightarrow X \quad \text{of rk } n$$

$$\Big\updownarrow \text{NAHC}$$

$$(E, \varphi) \in h^{-1}(b) \rightsquigarrow \mathcal{L} \in \mathrm{Jac}^0(X_b)$$

$$\Big\downarrow GLC_{\mathbb{C}^\times} \text{ (+ mirror symmetry)}$$

$$\check{\mathcal{L}} \longrightarrow \mathrm{Jac}^0(X_b)$$

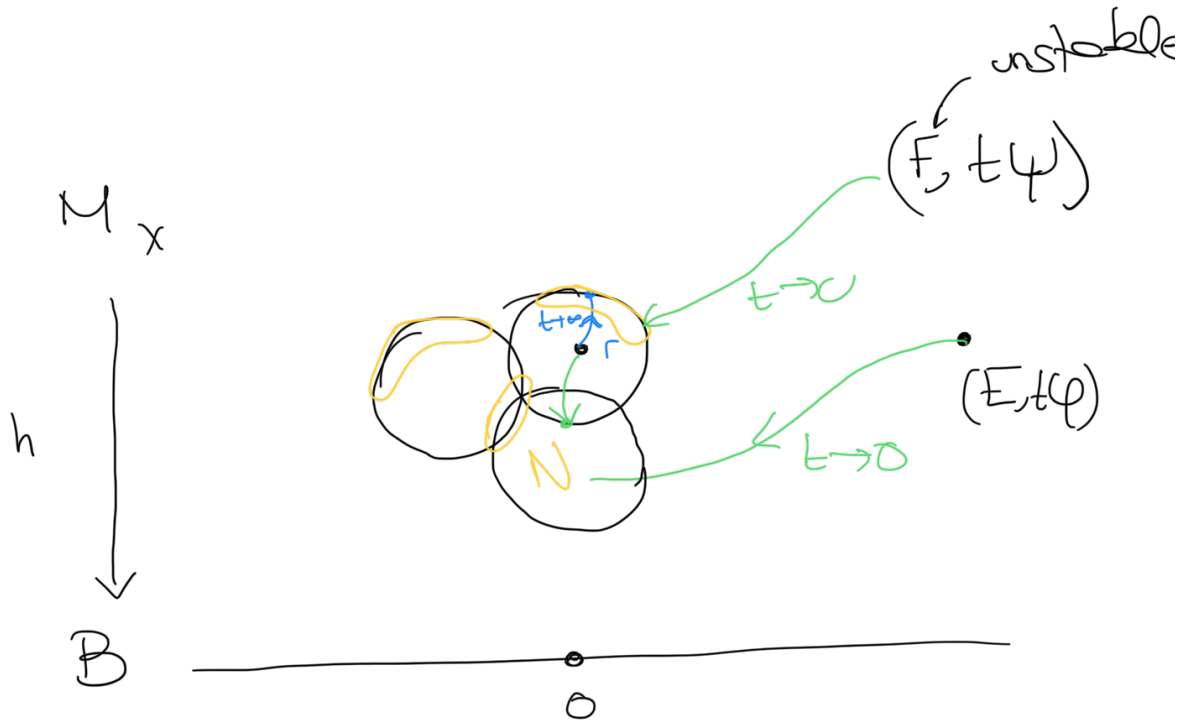
- Identify $\mathrm{Jac}^0(X_b)$ w/ spectral variety for $N_X(n,d)$
- Resolve $r_b \rightsquigarrow r_b{}_* \check{\mathcal{L}}$ Higgs bundle w/ extra (parabolic) structure
- Mochizuki D-module on $N_X(n,d)$

<u>Step 0</u> : understand r & r_b & in particular, their locus of indefinition

2.2. The nilpotent cone in all this

<u>Definition</u> : the nilpotent cone is $h^{-1}(0)$.

<u>Rk</u> : (E, φ) is nilpotent $\Leftrightarrow$ $h(E, \varphi) = 0$



$\to$ h is $\mathbb{C}^\times$ equivariant for the action $t \cdot (E, \varphi) = (E, t\varphi)$

$$t \cdot (b_1, \dots, b_n) = (tb_1, \dots, t^n b_n)$$

$\to h$ is proper

$$\lim_{t \to 0} h(E, t\varphi) \in h(0)$$

$h^{-1}(0)$ not smooth

Lagrangian
of pure dimension $\frac{1}{2}$]

<u>Rk</u> $N_X(n,d) \subset h^{-1}(0)^{\mathbb{C}^\times}$

$$E \longmapsto (E, 0)$$

$h^{-1}(0)$ complete

<u>Def</u> (Drinfeld) a vector bundle E is very stable if $\nexists\ \varphi \in H^0(\mathrm{End}\, E \otimes K)$ nilpotent $\neq 0$.

<u>Laumon</u>: very stable bundles are stable and generic inside N_X^s

"<u>Proof</u>"

φ nilpotent $\rightsquigarrow$ iterated kernel filtration

$$0 = E_{-1} \subsetneq E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell = E$$

$$F_i = E_i / E_{i-1} \qquad d_i = \deg F_i$$
$$n_i = \deg F_i$$

<u>Def</u> the type of (E, φ) is the 2n uple $(\bar{n}, \bar{d})$

<u>Thus</u>: $\mathcal{C}_{\bar{n},\bar{d}} = \{(E,\varphi) \in h^{-1}(0) \mid \text{type}(E,\varphi) = (\bar{d},\bar{n})\}$

is of pure dimension $\frac{1}{2} \dim M_X$, locally closed

$$\mathcal{C}_{n,0} = N_X^{v.s}$$

LL

<u>More generally</u> $\overline{\mathcal{C}_{\bar{n},\bar{d}}} = \bigcup$ irreducible components

(follows from flatness of h Ginzburg)

Work by Ginzburg, Laumon, Bradlow Heinloth, Garcia-Prada, Gothen, Bozec

$\rightsquigarrow$ $\overline{\mathcal{C}_{\bar{n},\bar{d}}}$ are the components

v.s.

<u>Def</u>: $W = N_X \setminus N_X^*$

Drinfeld announced that it is of pure codimension one.

<u>Conjecture</u> (Donagi-Pantev) resolve r/r_b by a finite number of blowups along $U_n/U_{n_b} = \left\{ (E,\varphi) \in M/h^{-1}(b) \;\middle|\; E \text{ is not ss} \right\}$

Then the image of the exceptional divisor is the wobbly locus.

<u>Rk</u>: a priori no need for the nilpotent cone to define wobbly bundles but they are intimately related. Indeed

$$W^s = \bigcap_{(\bar{n},\bar{d}) \neq (n,d)} \overline{C_{\bar{n},\bar{d}}} \cap \overline{C_{n,d}}$$

<u>Counter example</u>: the bundle

$$E = \mathcal{O} \oplus \mathcal{O} \in N_X(2,0)$$

is ss $\Rightarrow$ NOT very stable. But

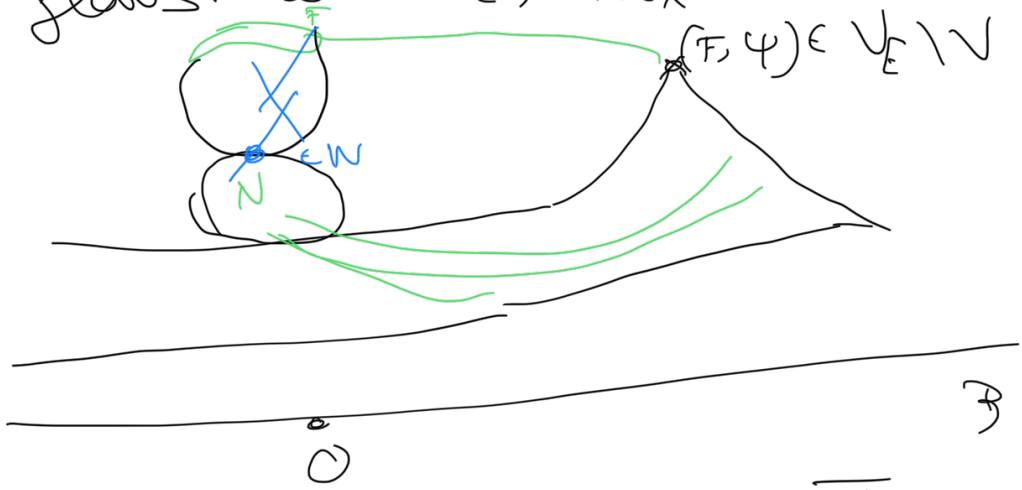
$\forall\ \varphi \in H^0(\mathrm{End}\,E \otimes \mathcal{L}) \cap h^{-1}(0)$

$$(E,\varphi) \sim_s (\mathcal{O}\oplus\mathcal{O}, 0)$$

Thm (Pauly–P.N., Zelaci) $E \in N_X^S$ is very stable $\Leftrightarrow h_E : H^0(\mathrm{End}\, E \otimes K) \longrightarrow B$ is proper.

$\Leftarrow$) Known. Proper between affines of the same dimension $\Rightarrow$ finite. But E wobbly & stable $\Rightarrow (E, \mathbb{C}\varphi) \subset h_E^{-1}(0)$

$\Rightarrow$) One proves that if $H^0(\mathrm{End}\, E \otimes K)$ not proper then $\overline{V_E} \setminus V_E \subset U_n \Rightarrow$ it flows down to $h^{-1}(0) \setminus N_X$.



In fact, one proves that $V_E \subsetneq \overline{V_E}$ $\Leftrightarrow V_E \cap h^{-1}(0) \neq \overline{V_E} \cap h^{-1}(0)$.

Corollary (BNR) $h^{-1}(b) \dashrightarrow N_X$ is dominant.

Proof

$\forall E \in N_X^{v.s.}$ $\quad V_E \cap h^{-1}(b)\ \forall b$

(h_E finite $\Rightarrow$ surjective)

$\llcorner$

<u>Rk</u> BNR prove it otherwise for generic fibers

$$T^*N \xrightarrow{\lambda} N_X \times B$$
$$(E,\varphi) \qquad (E, h(E,\varphi))$$

$$\lambda^{-1}(E,0) = (E,0) \qquad \forall E \in N_X^{v.s.}$$

$\Rightarrow$ $h^{-1}(E,\varphi)$ grically finite (dimension thm)
+ diff is an iso
(rk is maximal for v.s.)

2.3 Proof of Donagi-Pantev's conjecture

Let $(n,d) = 1$

$$U_n \subset M_X(n,d) \overset{r}{\dashrightarrow} N_X(n,d)$$

$$\begin{tikzcd} M_X(n,d) \arrow[r, dashed, "r"] & N_X(n,d) \\ \text{[illegible]} \arrow[u] \arrow[ur, "\tilde{r}"'] & \end{tikzcd}$$

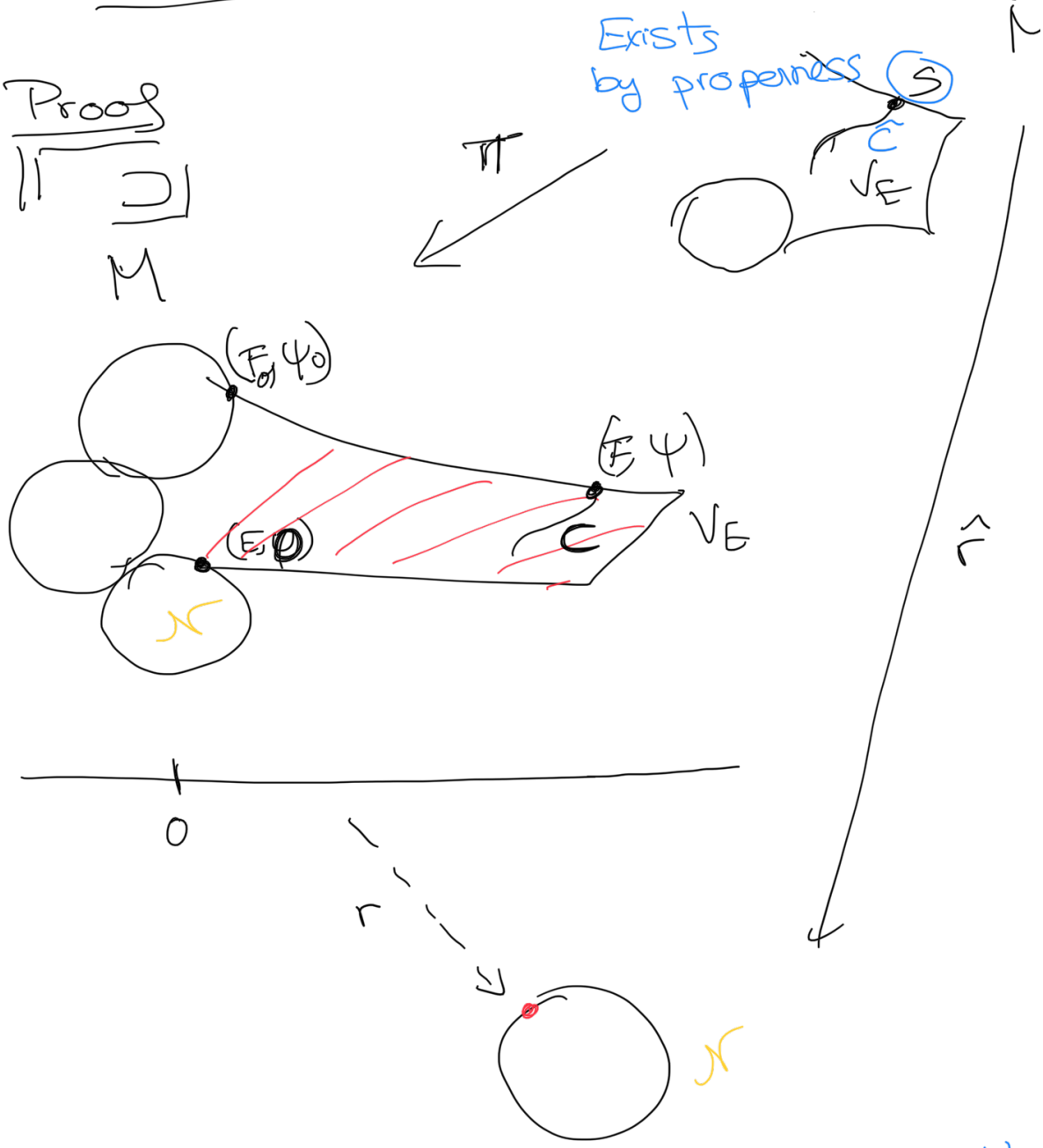
$Bl_{U_n}(M) = M_X(n,d)$

≥ 1

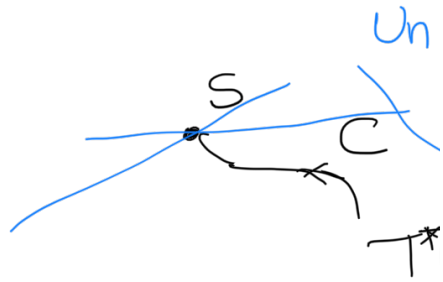
Theorem (PN'20) $\hat{r}(\hat{U}_n) = W$.

Proof

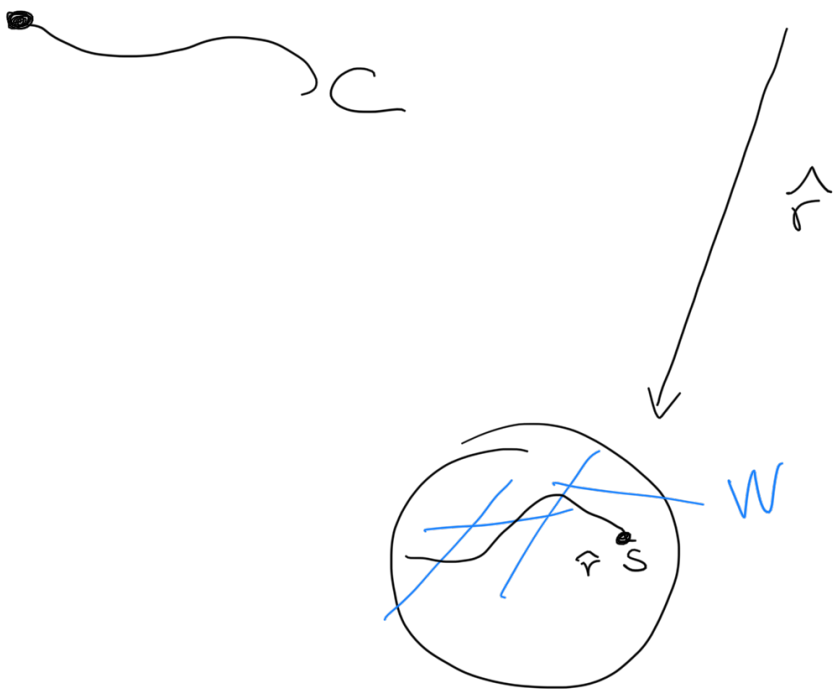
$\boxed{\supseteq}$



$\boxed{\subseteq}$



(F, ψ) π $\hat{M}$



$\hat{r}(S)$ v.s. $\Rightarrow$ $\hat{r}(C)^{v.s.} \subset C$ dense

Flow $\pi(C)$ to the nilpotent cone to find that $s \in N \cap C_{\bar{n},\bar{d}}$ $(\bar{n},\bar{d}) \neq (n,c$

$\boxtimes$

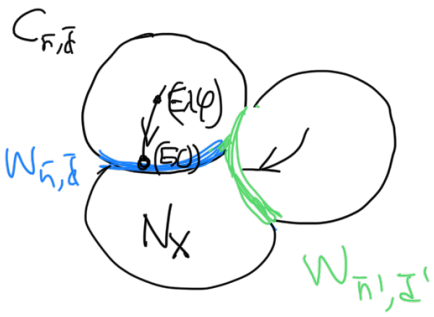
Day 3

Dringeld's "conjecture"

$$\boxed{\text{The wobbly locus is of pure codim 1}}$$

2.0 Preliminary remarks

$(E,\varphi) \in C_{\bar{n},\bar{d}}$ Est. smooth



$$T_{(E,\varphi)} M_X \longleftrightarrow H^0(\mathrm{End}\, E \otimes K)$$

$$\varphi \perp H^1(\mathrm{End}(E)) = T_E N$$

When E appears in a unique $C_{\bar{n},\bar{d}}$ & has a unique nilp H.f. (up to constants) $\Rightarrow$ we have $T_E(C_{n,d} \cap N_X) = T_E W = \langle \varphi \rangle^{\perp}$

2.1 Higgs bundle extensions

Let $(E_1,\varphi_1), (E_2,\varphi_2)$ be H.b. Consider

$$(E_1,\varphi_1) \hookrightarrow (E,\varphi) \twoheadrightarrow (E_2,\varphi_2)$$

Such extensions are parametrised by

$\mathbb{H}^1(C_\bullet)$ $\qquad C_\bullet : E_2^* E_1 \longrightarrow E_2^* E_1 K$

$$s \longmapsto s \circ \varphi_2 - \varphi_1 \circ s$$

[Thaddeus, Defs of par]

<u>In particular</u> (E,φ) nilp $\Rightarrow (E,\varphi)$ is an ex

$$(E_0,\varphi_0) \to (E,\varphi) \to (E/E_0, \overline{\varphi}_0)$$

2.2 The rank 2 case (Pal-Pauly)

We want to parametrise non trivial ext.

$$(F_0,0) \hookrightarrow (E,\varphi) \twoheadrightarrow (F_1,0)$$

<u>RK</u> φ can be defined in terms of

$\psi \in H^0(F_1^* F_0 K)$

$$\begin{tikzcd}
E_0 \arrow[r, "i"] & E \arrow[r, "\pi"] & F_1 \arrow[dll, "\psi"'] \\
E_0K \arrow[r] & EK \arrow[r] & F_1K
\end{tikzcd}$$

$\varphi = i \circ \psi \circ \pi$

Riemann Roch

So we are looking for

$$\begin{tikzcd}
\mathcal{Z}_\delta^{(1,1)} \arrow[r] \arrow[d] & \left\{\begin{array}{ll} \mathrm{Jac}^\delta \times \mathrm{Jac}^{d} & \delta \geq \\ \mathrm{Sym}^\delta \times \mathrm{Jac}^{d} & \delta \leq \end{array}\right. \arrow[d] \\
\mathrm{Jac}^{d_1}(X) \times \mathrm{Jac}^{d_0}(X) \arrow[r] & \mathrm{Jac}^\delta(X) \times \mathrm{Jac}^{d}(
\end{tikzcd}$$

$$(F_1, F_0) \longmapsto (F_1^* F_0 K, F_0)$$

$$\delta = d_0 - d_1 + 2g - 2$$

Necessary

<u>Bounds on δ</u>: (E,φ) semistable $\Rightarrow$

$$d_0 < d_1 \Rightarrow \delta \leq 2g-2$$

$$H^0(F_1^* F_0 K) \neq 0 \Rightarrow 0 < \delta$$

<u>Rk</u> $\delta = 2g-2 \Rightarrow (E,\varphi) = (F_0, 0) \oplus (F_1, 0)$

$$\Rightarrow (E,\varphi) \in N_X \setminus N_X^s$$

So for higher components $\delta < 2g-2$

We will see that

(INEQ) $0 < \delta < 2g-2$ $\quad \delta \equiv d \mod$

are sufficient for the $\exists$ $C_\delta^{(1)} \subset h^{-1}(0)$ $\neq \emptyset$

Now, starting from $(F_1, F_0) \in Z_\delta$ we need

- An extension type $\xi \in H^1(F_1^* F_0)$
- A Higgs field $\varphi \in H^0(F_1^* F_0 K)$

In this simple case we may just proceed directly (original approach), but let us analyse the H.b. extension approach:

$$C_\delta : F_0 F_1^* \xrightarrow{0} F_0 F_1^* K$$

$$\Downarrow$$

$$\mathbb{H}^i = H^{i-1}(F_0 F_1^* K) \oplus H^i(F_0 F_1^*)$$

$$\mathbb{H}^0 \longrightarrow H^0(F_0 F_1^*) \xrightarrow{0} H^0(F_0 K F_1^*) \longrightarrow \mathbb{H}^1(\mathcal{G}) \longrightarrow$$

$$H^1(F_0 F_1^*) \xrightarrow{0} H^1(F_0 F_1^*) \longrightarrow 0$$

<u>So</u> $\mathbb{H}^1$ is indeed given by a pair $(\xi, \varphi) \in H^1(F_1^* F_0) \oplus H^0(F_1^* F_0 K)$.

<u>Lemma</u>: $\exists\ \mathcal{H}^1 \dashrightarrow Z_\delta^{1,1}$ rational v.b. w/ fiber $\mathbb{H}^1(AK^* \to A$

$$\mathcal{H}^1_{(A,B)} = \mathbb{H}^1(AK^* \xrightarrow{0} A)$$

<u>Thm</u> (Pal-Pauly) $\exists$ comm. rational diagram

$$\begin{tikzcd}
\mathbb{P}(\mathcal{H}^1_\delta) \arrow[r, dashed, "\tilde{f}"] \arrow[d, dashed, "\tilde{r}"'] & M_X(2,d) \arrow[d, dashed, "r"] \\
\mathbb{P}(\mathcal{E}xt_\delta) \arrow[r, dashed, "f"] & N_X(2,d)
\end{tikzcd}$$

s.t.

(1) $\overline{\tilde{f}(\mathbb{P}(\mathcal{H}^1))} = C_\delta^{(1,1)}$ $\quad 0 < \delta < 2g$

(2) $\overline{\tilde{f}(\mathbb{P}(\mathcal{E}xt_\delta))} - W^{(1,1)}$ is of

codimension 1 $\quad \delta \leq g$

(3) $\overline{\tilde{f}(\mathbb{P}(\mathcal{E}xt_\delta))} \subset W^{(1,1)}_{\delta_{max}}$ $\quad \delta_{max} = \max\{0 < \delta \leq g\} = g$ or $g-1$

Here $\mathcal{E}xt_\delta$ is the bundle on Z_δ constructed from the universal ext. bundle as follows:

$$\begin{array}{c} \mathcal{E}xt^u_\delta \\ \downarrow \\ X \times \mathcal{E}xt^1(p_{02}^*\mathcal{P}_1, p_{01}^*\mathcal{P}_0 p_\delta^* k \end{array}$$

where

$$\begin{tikzcd} X \times \mathrm{Jac}^{d_0} \times \mathrm{Jac}^{d_1} \arrow[rr, "p_{ij}"] \arrow[d, "p_0"'] \arrow[d, shift left=6, "p_1"] \arrow[d, shift right=6, "p_2"'] & & X \times \mathrm{Jac}^{d_{i+}} \\ X \quad \mathrm{Jac}^{d_0} \quad \mathrm{Jac}^{d_1} & & \end{tikzcd}$$

Now

$$\begin{tikzcd} X \times \mathcal{E}xt^1(\quad) \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\ X & \mathcal{E}xt^1(\quad) \end{tikzcd}$$

$$Rq_{2*}(\mathcal{E}xt^u_\delta) \longrightarrow Rq_{2*}(\mathcal{E}xt^u_\delta)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ Z_\delta & \longrightarrow & \mathrm{Jac}^{d_0} \times \mathrm{Jac}^{d_1} \end{array}$$

← locally trivial relative to the Jac.

"Proof"

Dimensional count:

Z_δ	$\mathbb{H}^1(C_0)$ (*) 0	$H^0(F_1^* F_0 K)$	
$2g$	$2g-2+h^1(F_1^* F_0 K)$	$\delta-g+1$	$\delta \geq g$
$\delta+g$	$2g-2-\delta+g-1+1$	1	$\delta < g$

generic!

Generically

(*) generically $H^1(F_1^* F_0 K) = H^0(F_0^* F_1) \underset{\text{gric}}{=} 0$

$\deg \leq g-2$

Altogether $\rightsquigarrow$ $4g-2 = \dim \mathcal{H}^1$

$\exists$ rational map

$$\mathbb{P}(\mathcal{H}^1) \underset{\text{grically inj.}}{\dashrightarrow} M_X \cap h^{-1}(0)$$

moduli space $\Rightarrow$ correpr. moduli problem

By dimension $\rightsquigarrow$ component

PUT STABILIT CONDITION

Now, also

$$\begin{tikzcd} \mathbb{P}(\mathcal{H}^1_\delta) \arrow[r, dashed, "\tilde{\rho}_\delta"] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & M_X \arrow[d, dashed] \\ \mathbb{P}(\mathcal{H}^1_\delta) \arrow[r, dashed, "\rho_\delta"] & N_X \end{tikzcd}$$

By definition $W_\delta = \mathrm{Im}(\rho_\delta)$

$\mathcal{H}^1 \dashrightarrow \mathcal{H}^1_\delta$ grcc bundle of rk δ

$\Longrightarrow$ Divisor only if $\delta \leq g$

To check pure codim

$$W^{\nu}_\delta \subset W^{\nu}_{\delta_{max}}$$

$\blacksquare$