Cosmological perturbation theory

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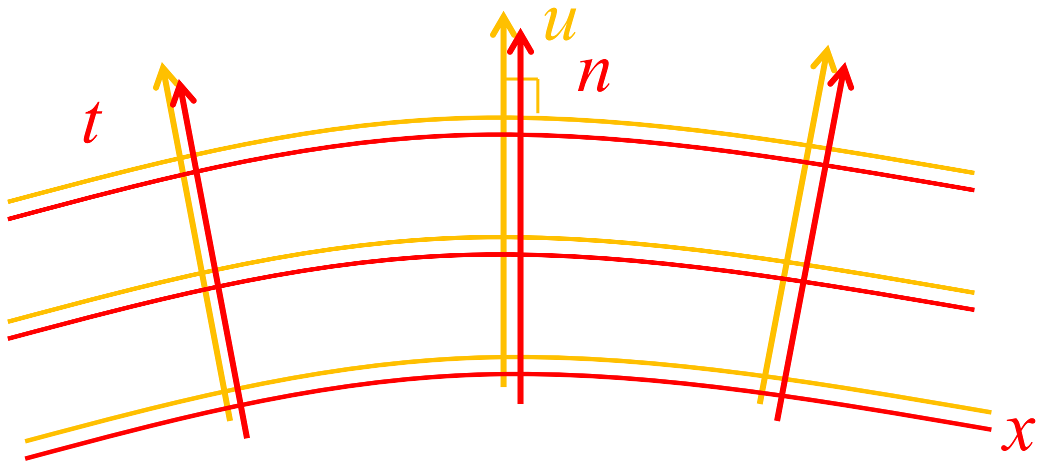
January 2022

Lecture 3

- Perturbation theory building-blocks
  - Perturbative expansion
  - Scalar/Vector/Tensor decomposition
  - Fourier transform
  - Statistical properties

- Metric perturbations
  - Gauge dependence
  - Gauge invariant quantities

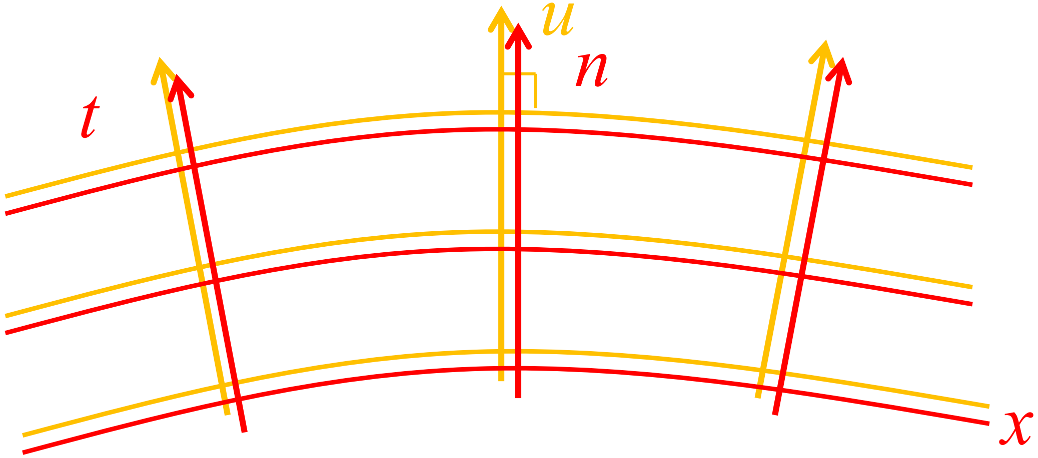
- Energy-momentum conservation
  - Conserved quantities on large scales
  - Adiabatic and entropy perturbations
- Einstein equations
  - Recovering Newtonian growth of structure in ΛCDM



FLRW cosmology

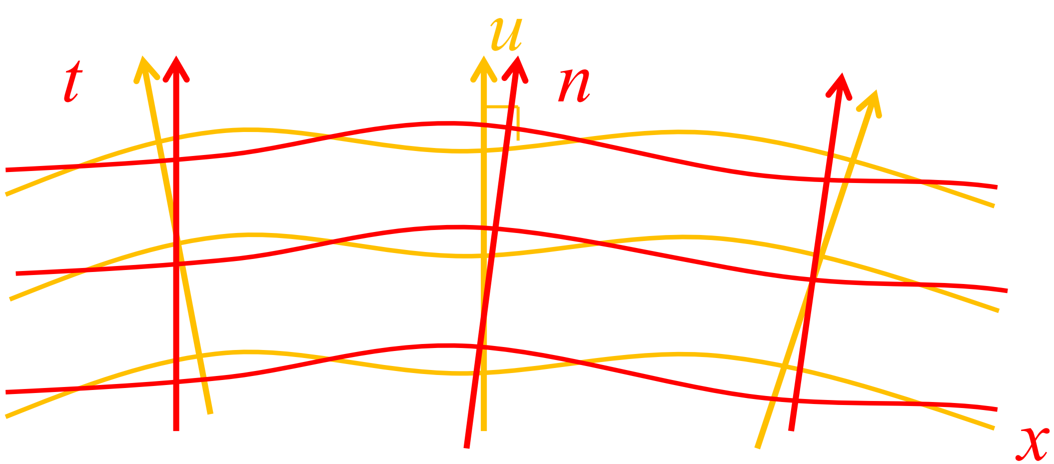
preferred coordinates for homogeneous and isotropic space

preferred space+time split in FRW cosmology breaks symmetry of Einstein's theory



FLRW cosmology

no unique choice of time (slicing) and space coordinates (threading) in an inhomogeneous universe



FLRW cosmology
+ perturbations

arbitrary gauge *(t,x)*

gauge problem: find different perturbations in different gauges

Relating gauge-invariant variables

- Gauge-invariant variables are *not unique*, and they are *not independent*
  - the curvature in the uniform-density gauge can be written in terms of the longitudinal gauge metric potential and density perturbation:

$$\zeta_\alpha \equiv \Phi + \frac{1}{3(1+w_\alpha)}\delta_{\alpha,\ell}$$

for example, for radiation and non-relativistic matter:

$$\zeta_\gamma \equiv \Phi + \frac{1}{4}\delta_\gamma \,, \qquad \zeta_m \equiv \Phi + \frac{1}{3}\delta_m$$

  - or could be written in terms of comoving curvature and density perturbation:

$$\zeta_\alpha \equiv \mathcal{R}_\alpha + \frac{1}{3(1+w_\alpha)}\delta_{\alpha,c}$$

Relating gauge-invariant variables

- Gauge-invariant variables are *not unique*, and they are *not independent*
  - the curvature perturbation in the uniform-density gauge is simply related to the density perturbation in the zero-curvature gauge:

$$\zeta_\alpha = -\frac{\mathcal{H}}{\rho'} \widetilde{\delta\rho}_{\alpha,flat}$$

  - they are both related to the perturbed expansion

$$\delta N \equiv \delta(\ln a) = \mathcal{H}\delta\tau$$

    from the zero-curvature to the uniform-α-density time-slices

$$\delta N_\alpha \equiv \zeta_\alpha = -\frac{\mathcal{H}}{\rho'} \widetilde{\delta\rho}_{\alpha,flat}$$

Energy-momentum conservation

Covariant conservation equation:

$$\nabla_\mu T^{\mu\nu} = 0$$

- FLRW background, $\rho(\tau)$, fluid continuity equation:

$$\rho' + 3\mathcal{H}(\rho + P) = 0$$

- Fluid continuity and Euler equations at first order:

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0\,,$$

$$(v + B)' + (1 - 3c_s^2)\mathcal{H}(v + B) + A + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0\,.$$

Conserved curvature perturbation, ζ

Fluid continuity equation in arbitrary gauge:

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0$$

simplifies in uniform-density gauge

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0$$

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simplifies in uniform-density gauge

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0$$

$$\Rightarrow \quad 3\mathcal{H}\delta P_{nad} + 3(\rho + P)\zeta' + (\rho + P)\nabla^2(v + E') = 0 \,,$$

Conserved curvature perturbation, ζ

Fluid continuity equation in arbitrary gauge:

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0$$

simplifies in uniform-density gauge

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0$$

$$\Rightarrow \quad 3\mathcal{H}\delta P_{\rm nad} + 3(\rho + P)\zeta' + (\rho + P)\nabla^2(v + E') = 0 \,,$$

$$\Rightarrow \quad \zeta' = -\frac{\mathcal{H}}{\rho + P}\delta P_{\rm nad} - \frac{1}{3}\nabla^2(v + E') \,,$$

$\zeta' = 0$ for adiabatic pertbns ($\delta P_{\rm nad} = 0$) on large scales ($\nabla^2 \to 0$)

Conserved curvature perturbation, ζ_α

- curvature perturbation in the uniform-α-density gauge

$$\zeta_\alpha = C + \frac{1}{3(1+w_\alpha)} \frac{\delta\rho_\alpha}{\rho_\alpha}$$

- continuity equation for each fluid (where $w_\alpha = P_\alpha/\rho_\alpha$) in background

$$\rho'_\alpha = -3(1+w_\alpha)\mathcal{H}\rho_\alpha$$

at first order in uniform-α-density gauge

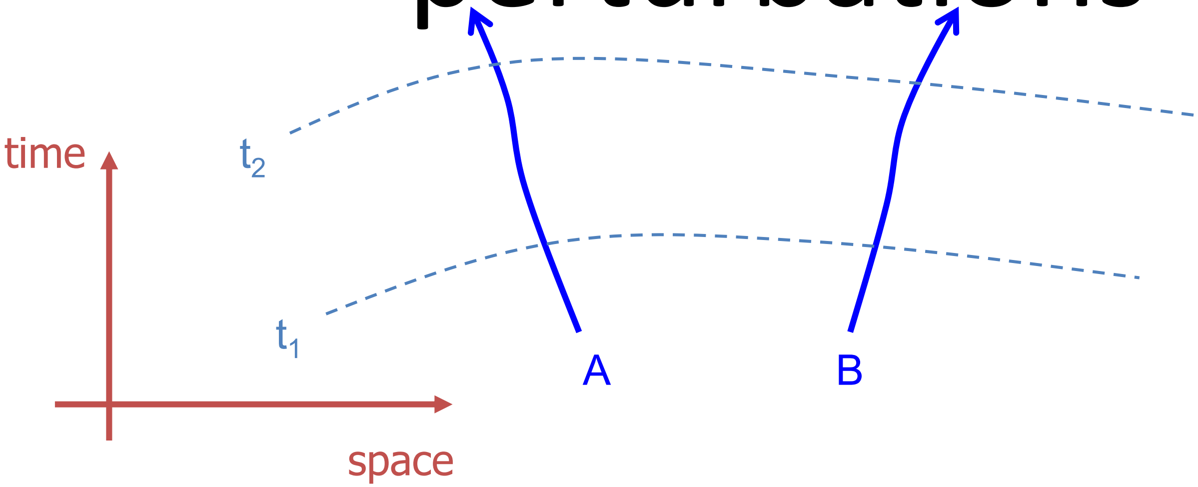
$$\delta\rho'_\alpha + 3\mathcal{H}(\delta\rho_\alpha + \delta P_\alpha) + 3(\rho+P)C' + (\rho+P)\nabla^2(v_\alpha + E') = 0$$

$$\Rightarrow \qquad \zeta'_\alpha = -\frac{1}{3}\nabla^2(v_\alpha + E')$$

$\zeta'_\alpha = 0$ *for barotropic fluids,* $P_\alpha(\rho_\alpha)$*, on large scales,* $\nabla^2 \to 0$.

Conserved cosmological perturbations

Lyth & Wands 2003



For every quantity, x, that obeys a **local conservation equation**

$$\frac{dx}{dN} = y(x) \quad , \qquad e.g. \quad \dot{\rho}_m = -3H\rho_m$$

where $dN = Hdt$ is the locally-defined expansion along comoving worldlines

there is a **conserved perturbation** $\zeta_x \equiv \delta N = \dfrac{\delta x}{y(x)}$

where perturbation $\delta x = x_A - x_B$ is a evaluated on hypersurfaces separated by uniform expansion $\Delta N = \Delta \ln a$

examples:

(i) total energy conservation:

$$\frac{d\rho}{dN} = H^{-1}\dot{\rho} = -3(\rho + P)$$

for perfect fluid / adiabatic perturbations, *P=P(ρ)*

$$\Rightarrow \quad \zeta_\rho = \frac{\delta\rho}{3(\rho + P)} \quad \text{conserved}$$

(ii) energy conservation for non-interacting perfect fluids:

$$H^{-1}\dot{\rho}_i = -3(\rho_i + P_i) \quad \text{where } P_i = P_i(\rho_i) \quad \Rightarrow \quad \zeta_i = \frac{\delta\rho_i}{3(\rho_i + P_i)}$$

(iii) conserved particle/quantum numbers (e.g., B, B-L,…)

$$H^{-1}\dot{n}_i = -3n_i \quad \Rightarrow \quad \zeta_i = \frac{\delta n_i}{3n_i}$$

Multi-fluid cosmology

- Primordial plasma (e.g. primordial nucleosynthesis, $T \approx 1$ MeV)
  - photons, baryons, neutrinos, cold dark matter
  - relative entropy/isocurvature perturbation:

$$S_\alpha = 3\left(\zeta_\alpha - \zeta_\gamma\right)$$

conserved for barotropic fluids on large scales

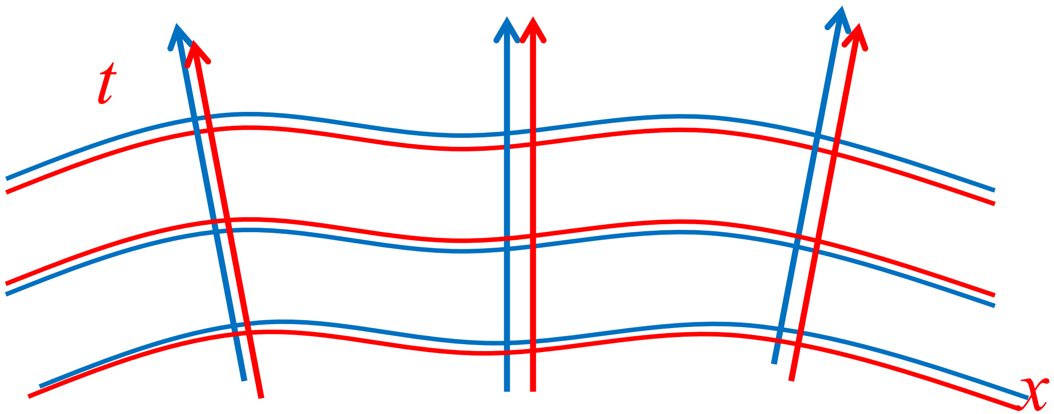
- for example, matter-isocurvature perturbation:

$$S_m = 3\left(\zeta_m - \zeta_\gamma\right) = \frac{\delta\rho_m}{\rho_m} - \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}$$

- total curvature/density perturbation:

$$\zeta = \sum_\alpha \frac{1+w_\alpha}{1+w}\zeta_\alpha$$

conserved on large scales for adiabatic pertbns, $S_\alpha = 0$

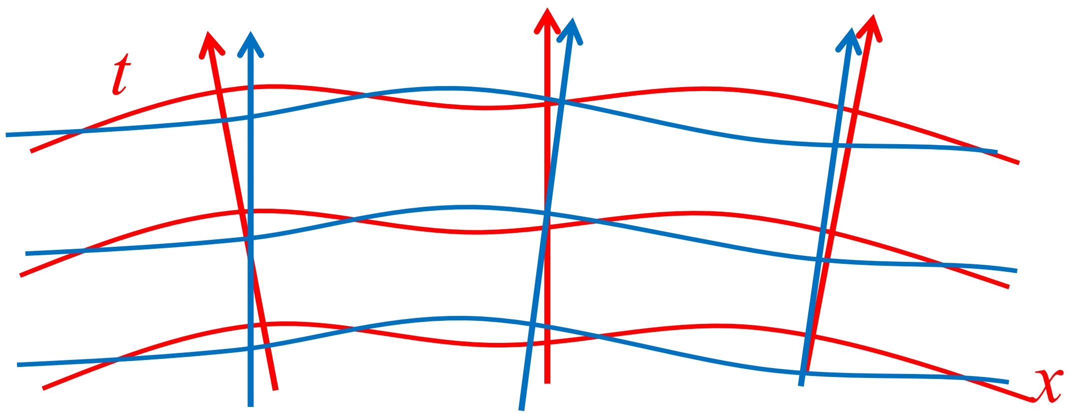


adiabatic perturbations

uniform-matter and uniform-radiation hypersurfaces coincide

$$\zeta_\gamma = \zeta_m$$

e.g., thermal equilibrium

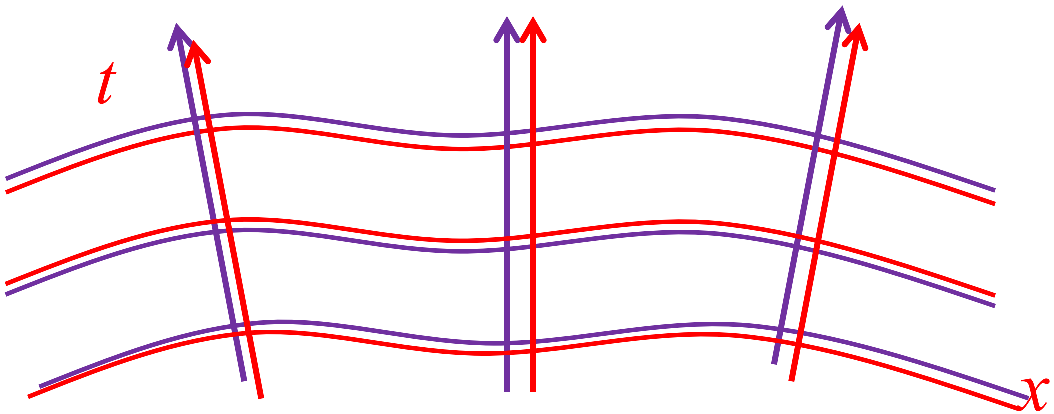


matter isocurvature perturbations

$$S_m = 3(\zeta_\gamma - \zeta_m)$$

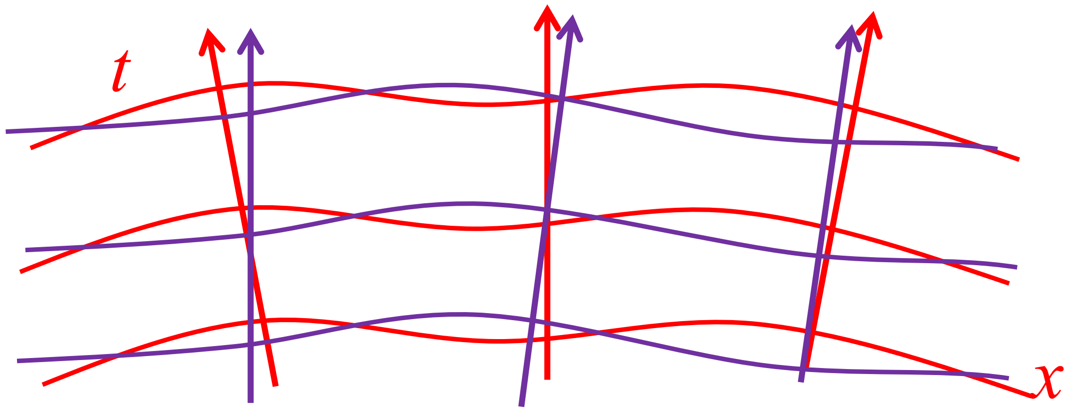
e.g., some multi-field reheating models

CMB temperature and polarization anisotropies place tight constraints on amplitude of isocurvature perturbations



adiabatic
perturbations

uniform-density and
uniform-pressure
hypersurfaces coincide



isocurvature
(non-adiabatic)
perturbations

multi-fluid, single clock cosmology

Consider inhomogeneous cosmology with only one degree of freedom:

- scalar field inflation with single attractor where $\varphi' = f(\varphi)$ (e.g., single-field slow roll)
- primordial plasma in full thermal equilibrium where $\rho_\alpha = f_\alpha(T)$ for all ρ_α

$$\zeta_\alpha = -\frac{\mathcal{H}}{\rho'}\widetilde{\delta\rho}_\alpha = -\frac{\mathcal{H}}{T'}\delta T$$

- relative entropy/isocurvature perturbation vanishes

$$S_\alpha = 3(\zeta_\alpha - \zeta_\gamma) = 0$$

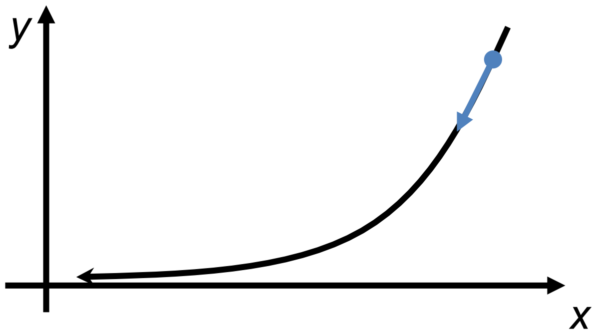
- total curvature perturbation *conserved on large scales*:

$$\zeta = \sum_\alpha \frac{1+w_\alpha}{1+w}\zeta_\alpha = -\frac{\mathcal{H}}{T'}\delta T$$

Cosmological perturbations on large scales

- **adiabatic perturbations** *e.g.,* $$\delta\left(\frac{n_\gamma}{n_B}\right) \propto \frac{\delta n_\gamma}{n_\gamma} - \frac{\delta n_B}{n_B} = 0$$
  - *perturb along the background trajectory*

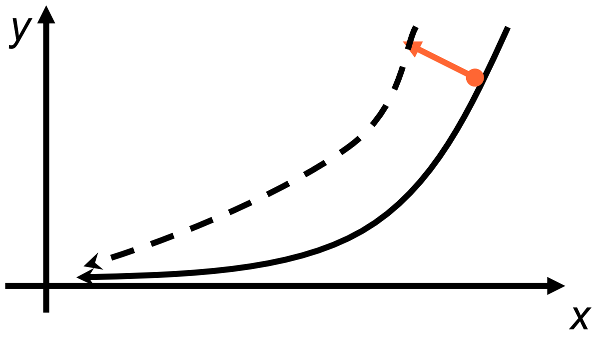
$$\frac{\delta x}{\dot{x}} = \frac{\delta y}{\dot{y}} = \delta T$$



  - *e.g, single-field perturbations along slow-roll attractor*
  - ***adiabatic perturbations stay adiabatic***

- **entropy perturbations**
  - *perturb off the background trajectory*

$$\frac{\delta x}{\dot{x}} \neq \frac{\delta y}{\dot{y}}$$



  - *e.g., baryon-photon* ***isocurvature*** *perturbation:*

Einstein equations

Covariant Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Einstein equations

Covariant Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- FLRW background, $a(\tau)$:
  - G^0_0: Energy constraint equation:

$$3\mathcal{H}^2 = 8\pi G a^2 \rho$$

  - G^i_0: Momentum constraint equation: trivial
  - G^j_i: Evolution equation:

$$\mathcal{H}' = -\frac{4\pi G}{3} a^2 (\rho + 3P)$$

Einstein equations at first order in an arbitrary gauge

Evolution equations

trace and trace-free spatial part of Einstein equations:

$$C'' + 2\mathcal{H}C' - \mathcal{H}A' - (2\mathcal{H}' + \mathcal{H}^2)A = -4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2\Pi\right) ,$$

$$\sigma' + 2\mathcal{H}\sigma - A - C = 8\pi G a^2 \Pi ,$$

Energy+momentum constraints

time-time and time-space components:

$$3\mathcal{H}(C' - \mathcal{H}A) - \nabla^2(C - \mathcal{H}\sigma) = 4\pi G a^2 \delta\rho ,$$

$$C' - \mathcal{H}A = 4\pi G a^2 (\rho + P)(v + B) .$$

Constraint equations in longitudinal gauge

$$A = \Psi \,, \quad B = 0 \quad , C = \Phi \,, \quad E = 0$$

Energy and momentum constraint equations

$$\begin{aligned} 3\mathcal{H}(\Phi' - \mathcal{H}\Psi) - \nabla^2\Phi &= 4\pi G a^2 \delta\rho \,, \\ \Phi' - \mathcal{H}\Psi &= -4\pi G a^2 (\rho + P) V \,. \end{aligned}$$

eliminate $\Phi' - \mathcal{H}\Psi$ gives:

$$\text{Poisson equation:} \quad \nabla^2\Phi = -4\pi G a^2 \delta\rho_c \,,$$

where gauge-invariant *comoving density perturbation*

$$\delta\rho_c \equiv \delta\rho + 3\mathcal{H}(\rho + P)(v + B)$$

for finite Φ, $\delta\rho_c$ becomes small on large scales ($\nabla^2 \to 0$)
$\Rightarrow$ comoving and uniform-density gauges coincide

$$\zeta = \mathcal{R} + \frac{\delta\rho_c}{3(1+w)\rho}$$

Another simplification in longitudinal gauge

$$A = \Psi\,, \quad B = 0 \quad , C = \Phi\,, \quad E = 0$$

trace-free spatial part of Einstein equations *in arbitrary gauge*:

$$\sigma' + 2\mathcal{H}\sigma - A - C \quad = \quad 8\pi G a^2 \Pi\,,$$

Another simplification in longitudinal gauge

$$A = \Psi \,, \quad B = 0 \quad , C = \Phi \,, \quad E = 0$$

trace-free spatial part of Einstein equations *in longitudinal gauge*:

$$-\Psi - \Phi \quad = \quad 8\pi G a^2 \Pi \,,$$

Another simplification in longitudinal gauge

$$A = \Psi \,, \quad B = 0 \quad , C = \Phi \,, \quad E = 0$$

trace-free spatial part of Einstein equations *in longitudinal gauge*:

$$-\Psi - \Phi \quad = \quad 8\pi G a^2 \Pi \,,$$

Hence for vanishing anisotropic stress, $\Pi = 0$ (e.g., perfect fluids):

$$\Psi \quad = \quad -\Phi \,,$$

Hence the only scalar metric potential in the longitudinal gauge is the "Newtonian" potential

$$\nabla^2 \Psi = 4\pi G a^2 \delta\rho_c$$

Recovering Newtonian equations

Euler equations *in arbitrary gauge*:

$$(v+B)' + (1-3c_s^2)\mathcal{H}(v+B) + A + \frac{1}{\rho+P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0$$

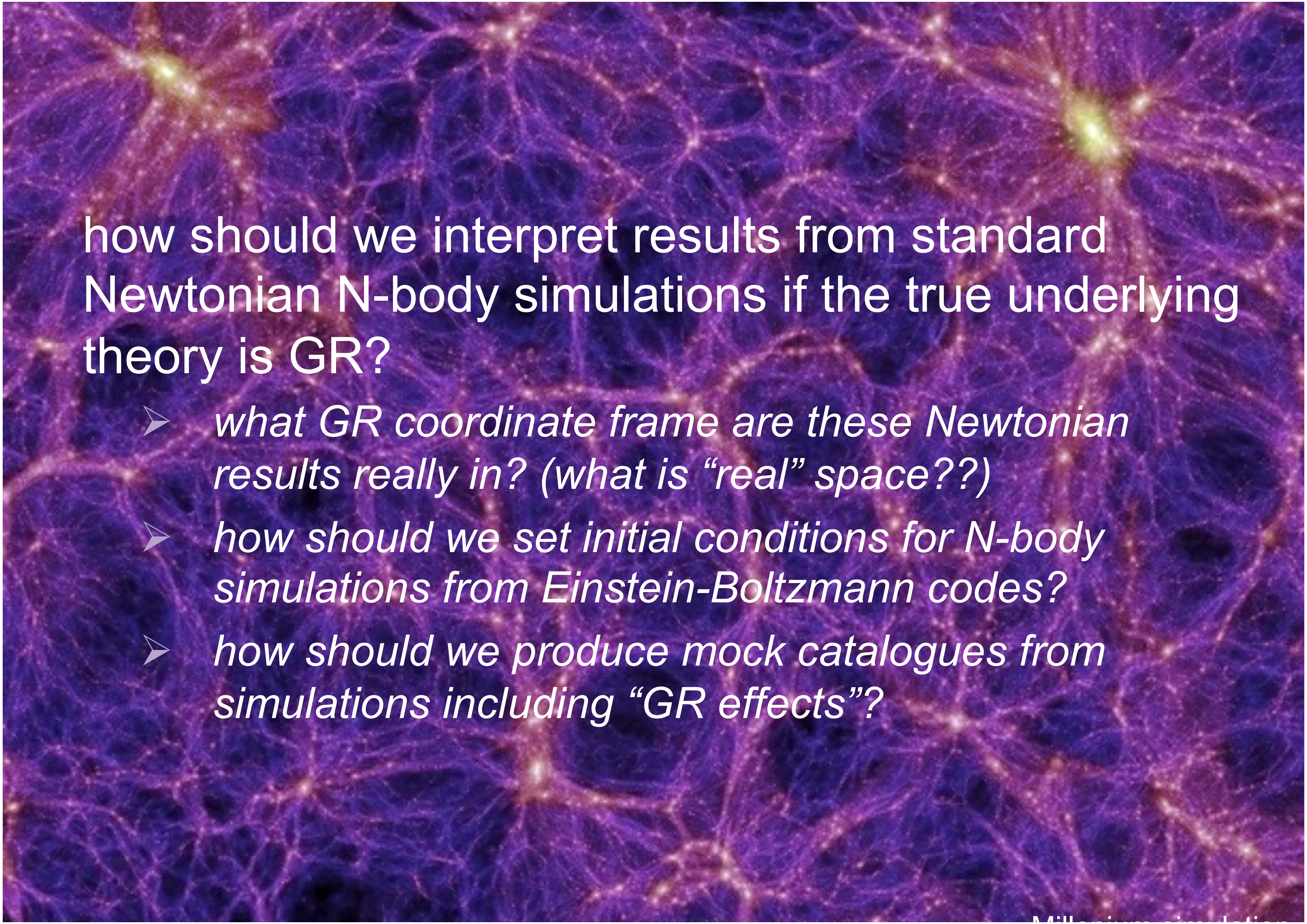
Euler equation for pressure-free perturbations (e.g., ΛCDM) in longitudinal gauge ($A=\Psi$, $B=0$)

$$\vec{v}' + \mathcal{H}\vec{v} \;=\; -\vec{\nabla}\Psi\ .$$

where $\vec{v} = \nabla v$, and

$$\nabla^2\Psi = 4\pi G a^2 \delta\rho_c$$

These are "Newtonian" equations for non-relativistic matter, but the source of the Newtonian potential is the comoving density perturbation, $\delta\rho_c$.


how should we interpret results from standard Newtonian N-body simulations if the true underlying theory is GR?
what GR coordinate frame are these Newtonian results really in? (what is “real” space??)
how should we set initial conditions for N-body simulations from Einstein-Boltzmann codes?
how should we produce mock catalogues from simulations including “GR effects”?

Recovering Newtonian equations

Fluid continuity and Euler equations

$$\begin{aligned}\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') &= 0\,, \\ (v + B)' + (1 - 3c_s^2)\mathcal{H}(v + B) + A + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) &= 0\,.\end{aligned}$$

Einstein energy and momentum constraints

$$\begin{aligned}3\mathcal{H}(C' - \mathcal{H}A) - \nabla^2(C - \mathcal{H}\sigma) &= 4\pi G a^2 \delta\rho\,, \\ C' - \mathcal{H}A &= 4\pi G a^2(\rho + P)(v + B)\,.\end{aligned}$$

Einstein evolution equations

$$\begin{aligned}C'' + 2\mathcal{H}C' - \mathcal{H}A' - (2\mathcal{H}' + \mathcal{H}^2)A &= -4\pi G a^2\left(\delta P + \frac{2}{3}\nabla^2\Pi\right)\,, \\ \sigma' + 2\mathcal{H}\sigma - A - C &= 8\pi G a^2\Pi\,,\end{aligned}$$

Recovering Newtonian eqns in ΛCDM ($\delta P = 0,\ \Pi = 0,\ c_s^2 = 0$)

Fluid continuity and Euler equations

$$
\begin{aligned}
\delta\rho' + 3\mathcal{H}\delta\rho + 3\rho_m C' + \rho_m \nabla^2 (v + E') &= 0 \,, \\
(v + B)' + \mathcal{H}(v + B) + A &= 0 \,.
\end{aligned}
$$

Einstein energy and momentum constraints

$$
\begin{aligned}
3\mathcal{H}(C' - \mathcal{H}A) - \nabla^2 (C - \mathcal{H}\sigma) &= 4\pi G a^2 \delta\rho \,, \\
C' - \mathcal{H}A &= 4\pi G a^2 \rho_m (v + B) \,.
\end{aligned}
$$

Einstein evolution equations

$$
\begin{aligned}
C'' + 2\mathcal{H}C' - \mathcal{H}A' - (2\mathcal{H}' + \mathcal{H}^2)A &= 0 \,, \\
\sigma' + 2\mathcal{H}\sigma - A - C &= 0 \,,
\end{aligned}
$$

Recovering Newtonian eqns in ΛCDM ($\delta P = 0,\ \Pi = 0,\ c_s^2 = 0$) in longitudinal gauge

Fluid continuity and Euler equations

$$\begin{aligned} \delta\rho' + 3\mathcal{H}\delta\rho + 3\rho_m C' + \rho_m \nabla^2 (v + E') &= 0\,, \\ \vec{\nabla} V' + \mathcal{H}\vec{\nabla} V &= -\vec{\nabla}\Psi\,. \end{aligned}$$

Einstein energy and momentum constraints

$$\begin{aligned} -\nabla^2\Phi &= 4\pi G a^2 \left(\delta\rho + 3\mathcal{H}\rho_m V\right)\,, \\ C' - \mathcal{H}A &= 4\pi G a^2 \rho_m (v + B)\,. \end{aligned}$$

Einstein evolution equations

$$\begin{aligned} C'' + 2\mathcal{H}C' - \mathcal{H}A' - (2\mathcal{H}' + \mathcal{H}^2)A &= 0\,, \\ \Psi &= -\Phi\,, \end{aligned}$$

Recovering Newtonian eqns in ΛCDM $(\delta P = 0,\ \Pi = 0,\ c_s^2 = 0)$ in longitudinal+comoving gauges

Fluid continuity and Euler equations

$$\begin{aligned} \delta\rho_c' + 3\mathcal{H}\delta\rho_c + \rho_m \nabla^2 E_c' &= 0\,, \\ \vec{\nabla} V' + \mathcal{H}\vec{\nabla} V &= -\vec{\nabla}\Psi\,. \end{aligned}$$

Einstein energy and momentum constraints

$$\begin{aligned} -\nabla^2\Phi &= 4\pi G a^2 \left(\delta\rho + 3\mathcal{H}\rho_m V\right)\,, \\ \mathcal{R}' &= 0\,. \end{aligned}$$

Einstein evolution equations

$$\begin{aligned} \mathcal{R}'' + 2\mathcal{H}\mathcal{R}' &= 0\,, \\ \Psi &= -\Phi\,, \end{aligned}$$

Recovering Newtonian eqns in ΛCDM ($\delta P = 0$, $\Pi = 0$, $c_s^2 = 0$) in longitudinal+comoving gauges

Fluid continuity and Euler equations

$$\begin{aligned} \delta\rho_c' + 3\mathcal{H}\delta\rho_c + \rho_m \vec{\nabla}\cdot\vec{\nabla}V &= 0\,, \\ \vec{\nabla}V' + \mathcal{H}\vec{\nabla}V &= -\vec{\nabla}\Psi\,. \end{aligned}$$

Einstein energy and momentum constraints

$$\begin{aligned} -\nabla^2\Phi &= 4\pi G a^2 \delta\rho_c\,, \\ \mathcal{R}' &= 0\,. \end{aligned}$$

Einstein evolution equations

$$\begin{aligned} \mathcal{R}'' + 2\mathcal{H}\mathcal{R}' &= 0\,, \\ \Psi &= -\Phi\,, \end{aligned}$$

Recovering Newtonian eqns in ΛCDM ($\delta P = 0,\ \Pi = 0,\ c_s^2 = 0$) in longitudinal+comoving gauges

Fluid continuity and Euler equations

$$\begin{aligned} \delta\rho_c' + 3\mathcal{H}\delta\rho_c + \rho_m \vec{\nabla}\cdot\vec{\nabla}V &= 0\,, \\ \vec{\nabla}V' + \mathcal{H}\vec{\nabla}V &= -\vec{\nabla}\Psi\,. \end{aligned}$$

Poisson constraint

$$\nabla^2\Psi = 4\pi G a^2 \delta\rho_c\,.$$

Growth of structure in ΛCDM in GR is same as in Newtonian gravity *in the absence of radiation* and *at first order*

- can be extended to second and higher order for evolution equations but GR constraints for initial data are non-linear
- can add relativistic effects of radiation on spacetime metric at first order by constructing *Newtonian motion gauge* (Fidler *et al* 2016)

Evolution equation for δ

Continuity equation for comoving density contrast:

$$\delta_c' + \nabla^2 V = 0$$

Taking time derivative

$$\delta_c'' + \nabla^2 V' = 0$$

plus Euler equation

$$V' + \mathcal{H} V = -\Psi$$

eliminating V' and V gives

$$\delta_c'' + \mathcal{H}\delta_c' - \nabla^2 \Psi = 0$$

Using $\Psi = -\Phi$ and Poisson equation

$$\nabla^2 \Phi = -\frac{3}{2}\Omega_m \mathcal{H}^2 \delta_c$$

gives linear second-order differential equation

$$\delta_c'' + \mathcal{H}\delta_c' - \frac{3}{2}\Omega_m \mathcal{H}^2 \delta_c = 0$$

Growth of structure

In absence of any pressure perturbations, e.g., ΛCDM

$$\delta_c'' + \mathcal{H}\delta_c' - \frac{3}{2}\Omega_m \mathcal{H}^2 \delta_c = 0$$

Growth of structure is independent of scale in ΛCDM cosmology.

$$\delta_k(\tau) = D_+(\tau)\delta_{k,0} + D_-(\tau)\tilde{\delta}_{k,0}$$

- Growing mode $D_+(\tau)$ normalised so that $D_+(\tau_0) = 1$ today
- Decaying mode $D_-(\tau)$ assumed negligible at late times
- In matter-dominated cosmology ($\Omega_m = 1$) then $D_+ \propto a$

$$f \equiv \frac{d \ln D_+}{d \ln a} = 1$$

- In ΛCDM

$$f \simeq \Omega_m^{6/11}$$

First-order density in Fourier space

$$\delta_{\vec{k}}(z) = \int d\vec{x}\, e^{i\vec{k}.\vec{x}} C_1(\vec{x}) D_+(z)$$

linear transfer function through radiation era from Einstein-Boltzmann code

$$\delta_{\vec{k}}(z) = T_{\vec{k}}(z)\Phi_{\vec{k},\mathrm{ini}} = T_{\vec{k}}(z)\frac{3}{5}\zeta_{\vec{k}}$$

- Power spectrum