Physics of the Early Universe, ICTS, Bengaluru, January 2022

Cosmological perturbation theory

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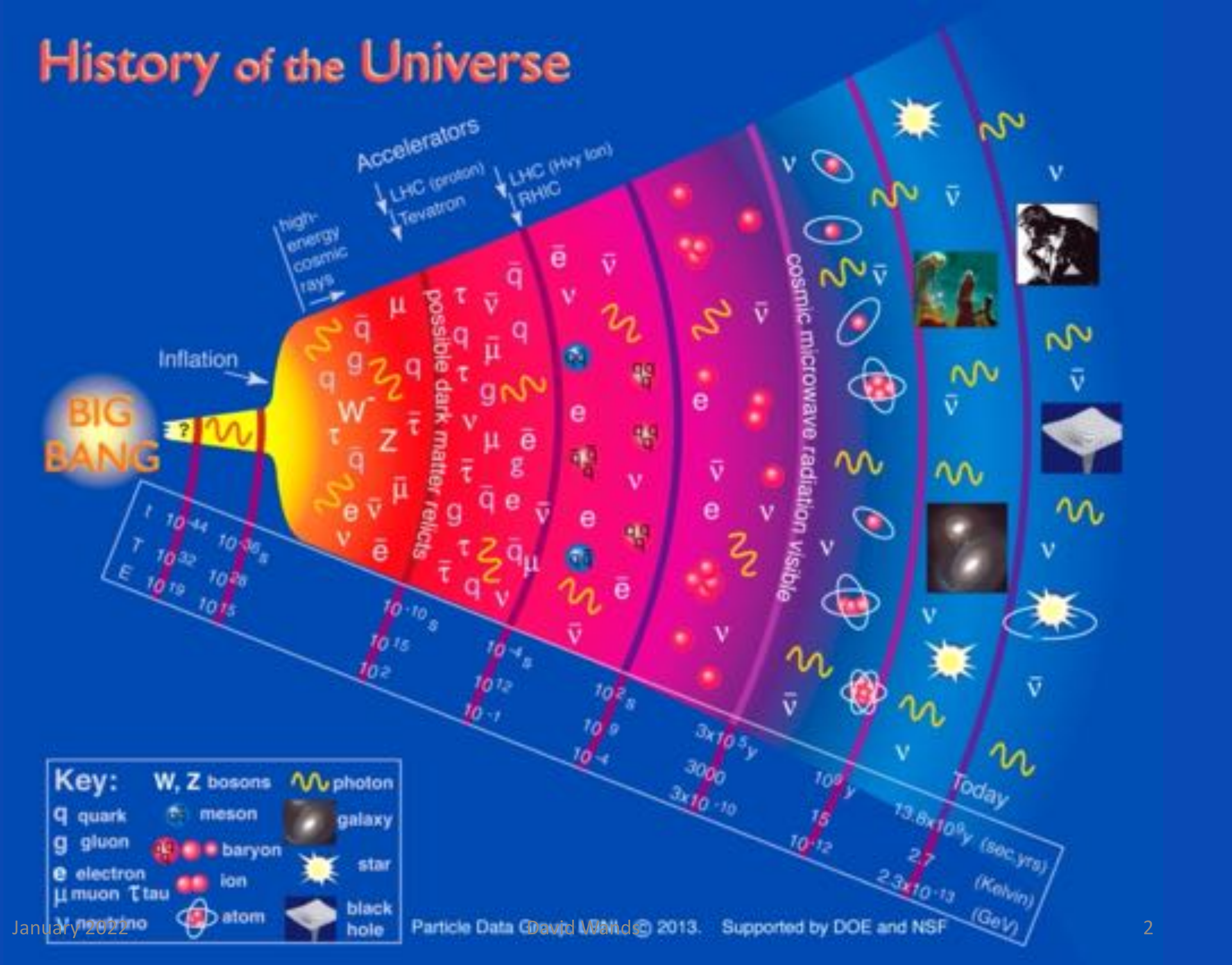
Outline

Motivation

Homogeneous cosmology

Perturbation theory

Metric perturbations


History of the Universe
Accelerators
LHC (proton)
Tevatron
LHC (Hvy Ion)
RHIC
high-energy cosmic rays
Inflation
BIG BANG
possible dark matter relicts
cosmic microwave radiation visible
Today
Key:
q quark
g gluon
e electron
μ muon τ tau
ν neutrino
W, Z bosons
meson
baryon
ion
atom
photon
galaxy
star
black hole
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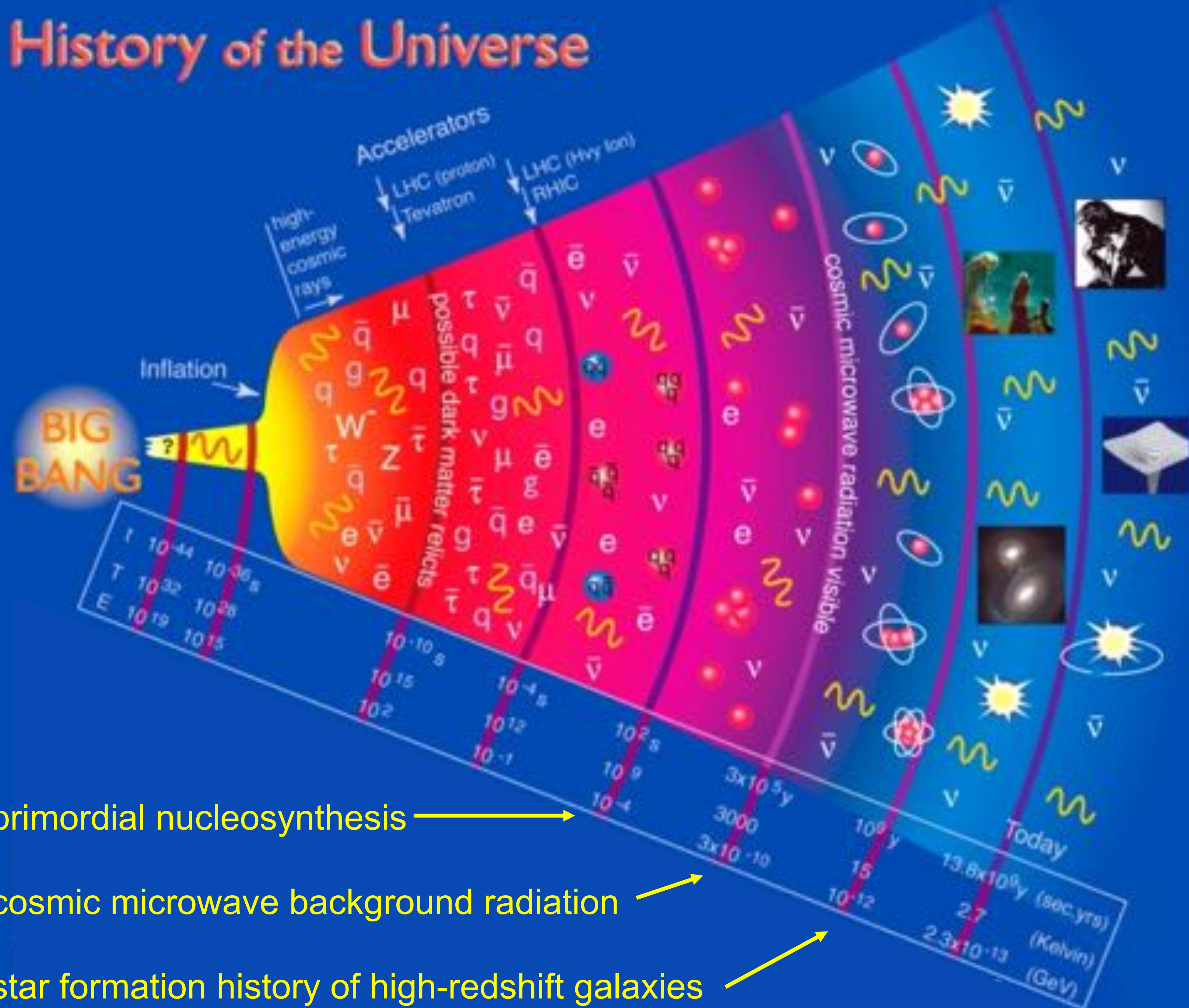
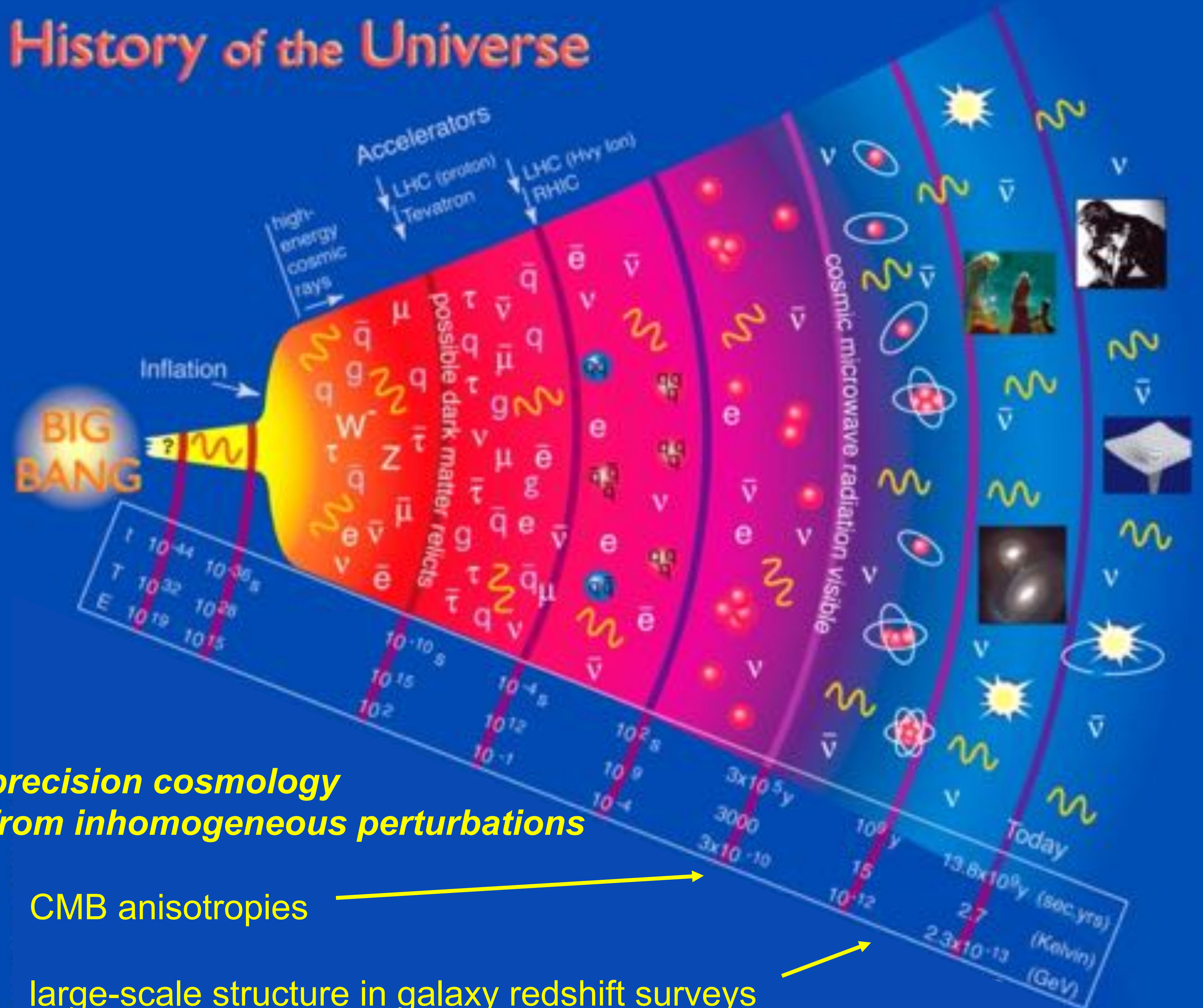
(sec,yrs)
(Kelvin)
(GeV)



primordial nucleosynthesis

cosmic microwave background radiation

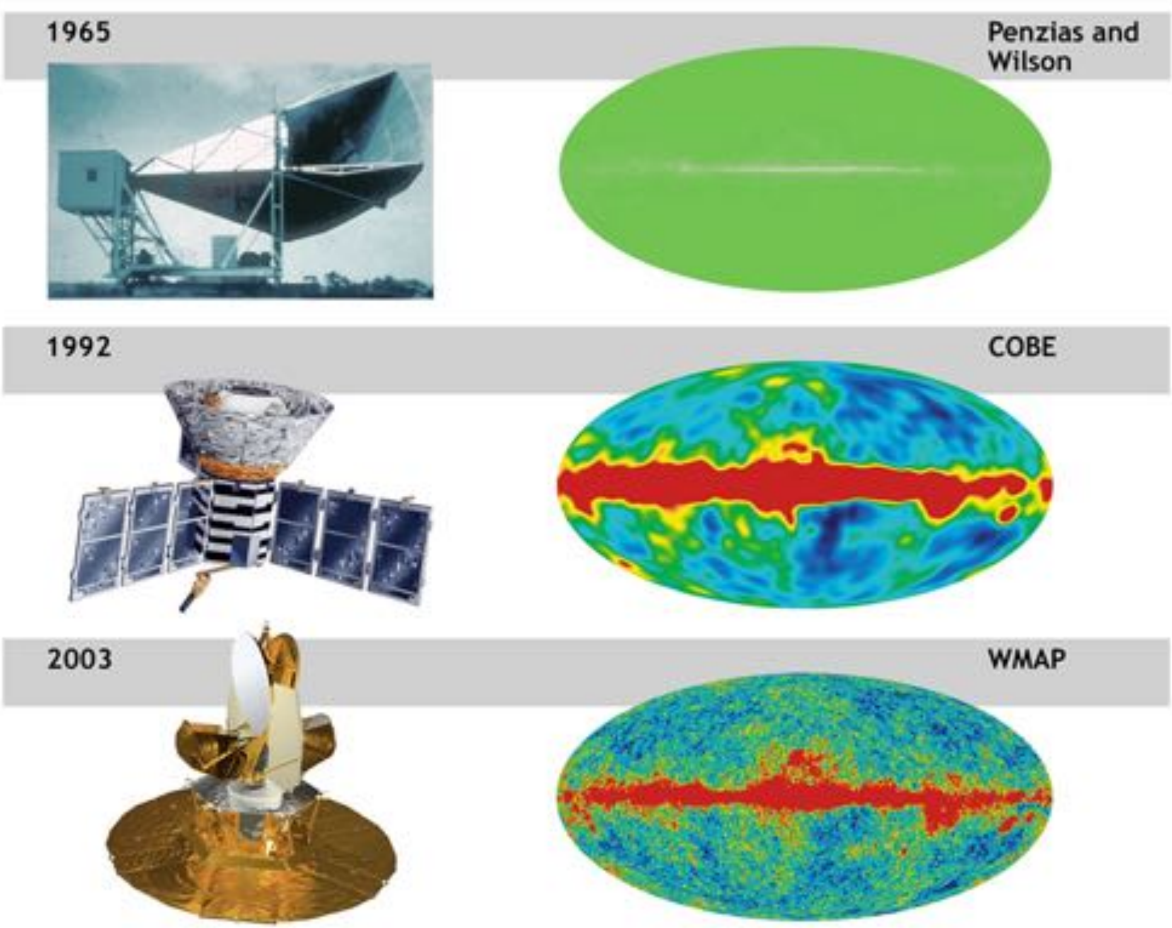
star formation history of high-redshift galaxies



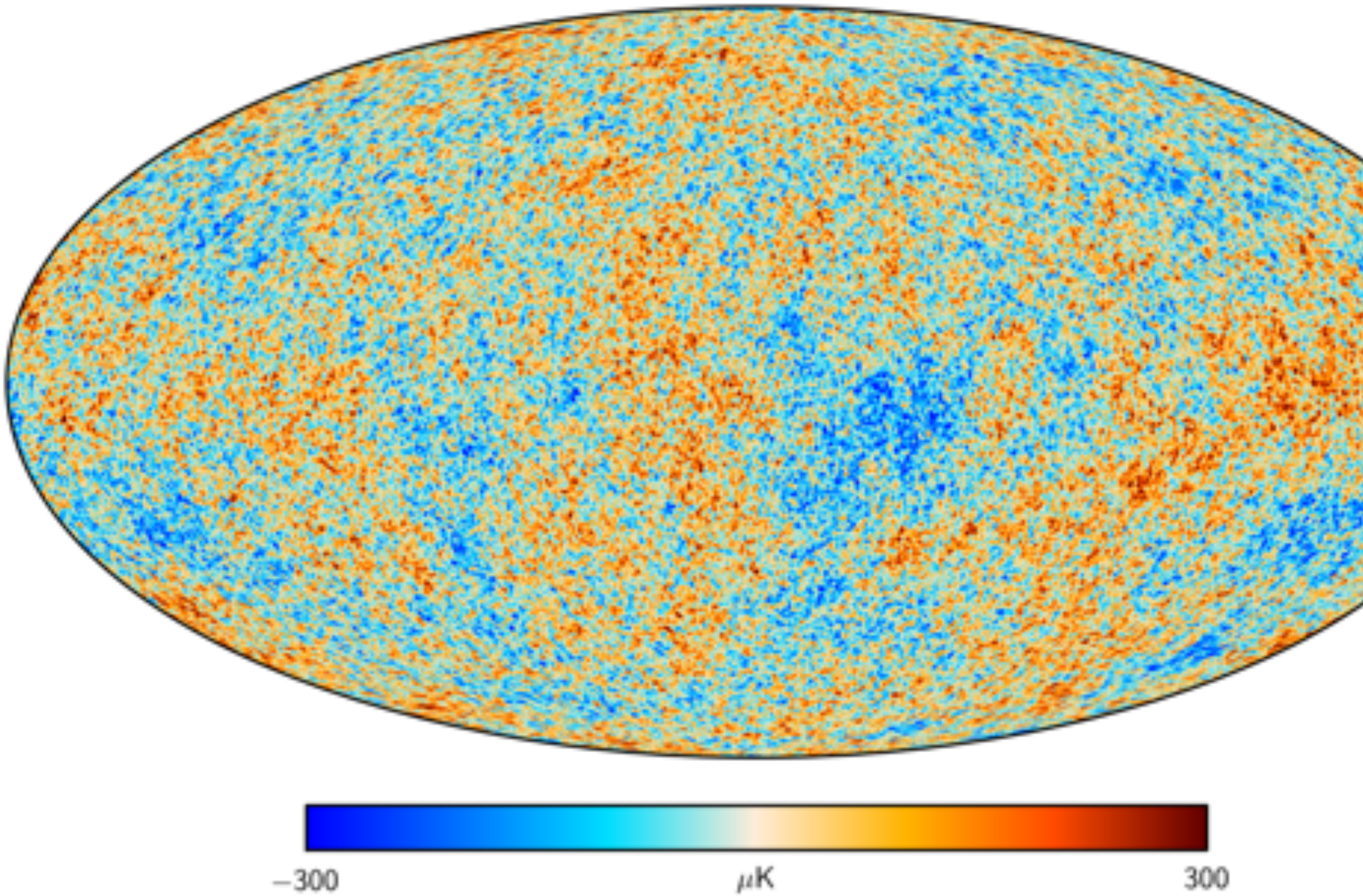
precision cosmology
from inhomogeneous perturbations

- CMB anisotropies
- large-scale structure in galaxy redshift surveys

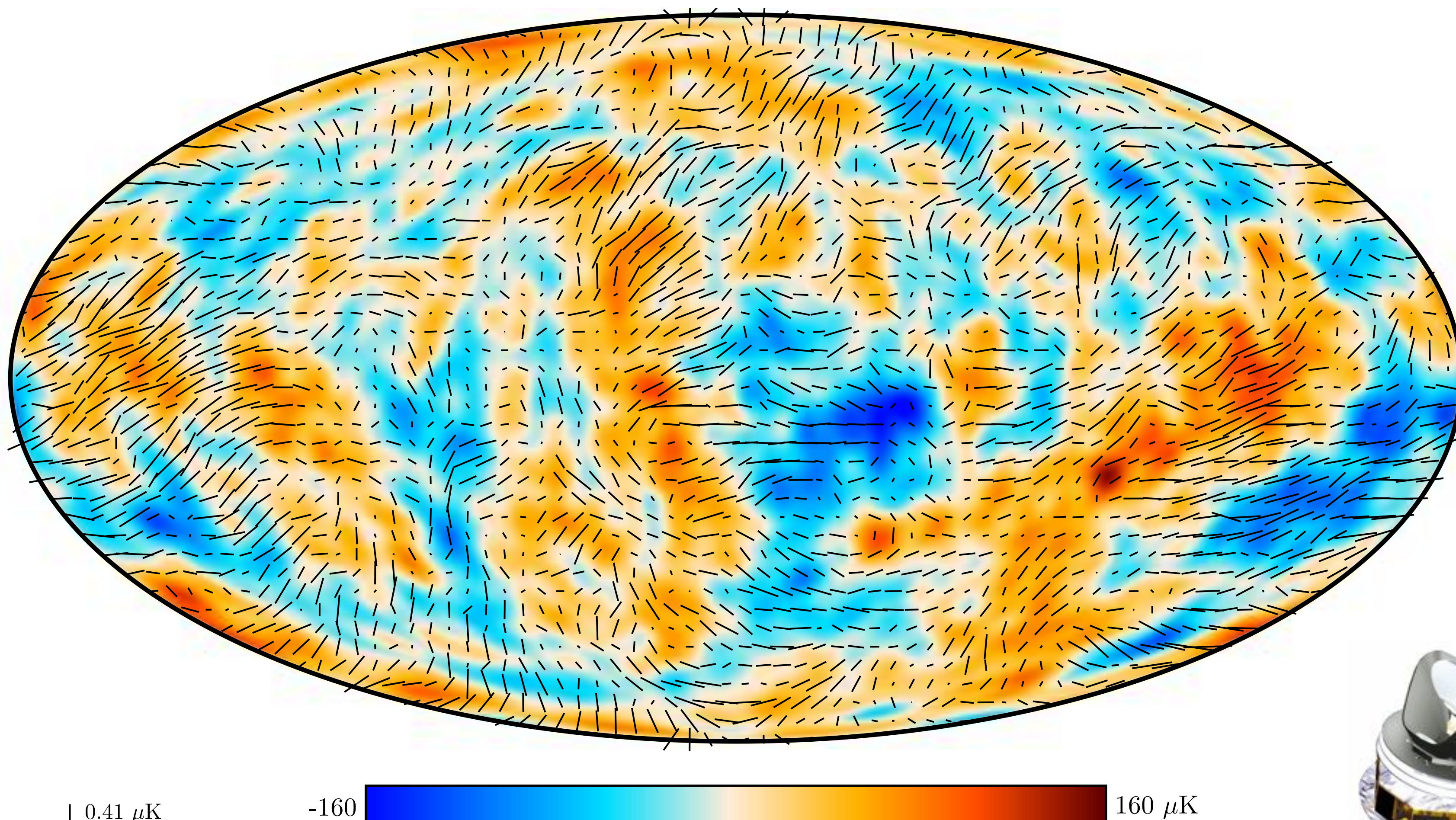
temperature anisotropies in CMB



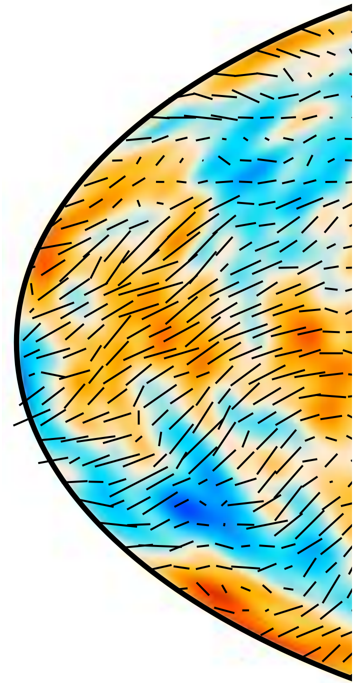
ESA Planck CMB temperature map

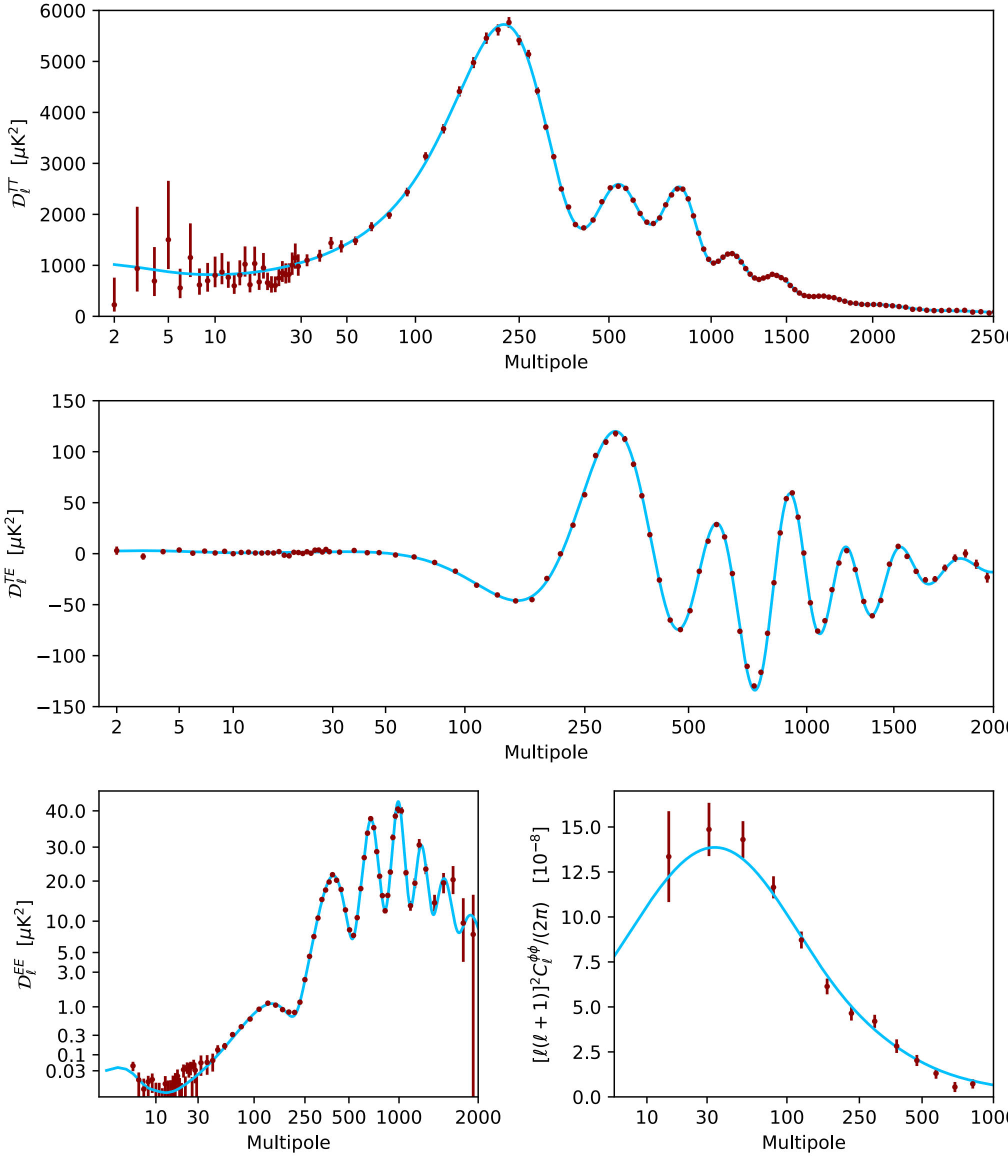


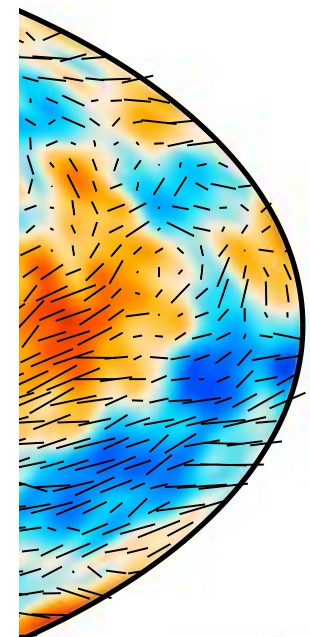
ESA Planck CMB polarization map



Planck CMB angular power spectra

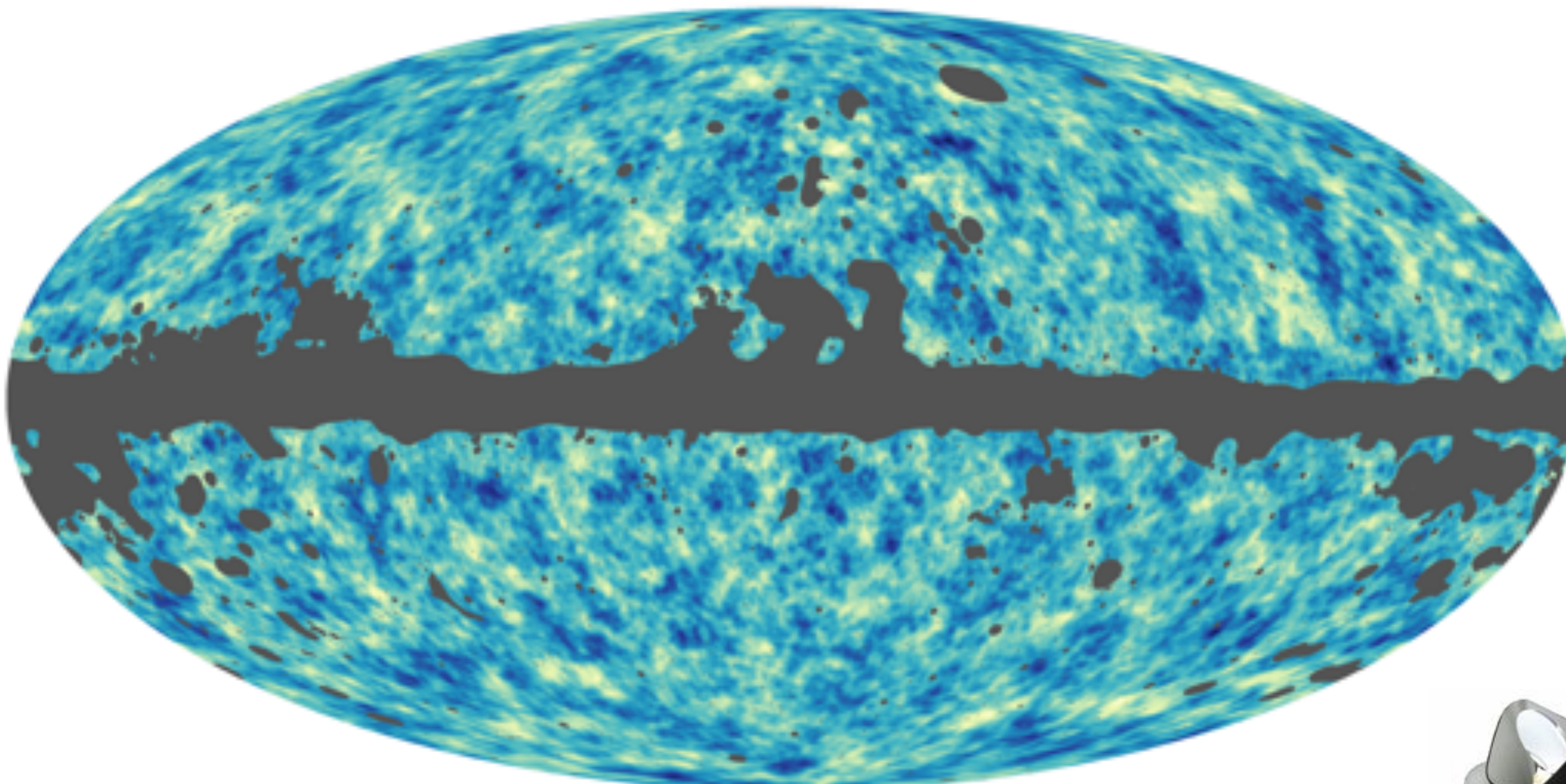




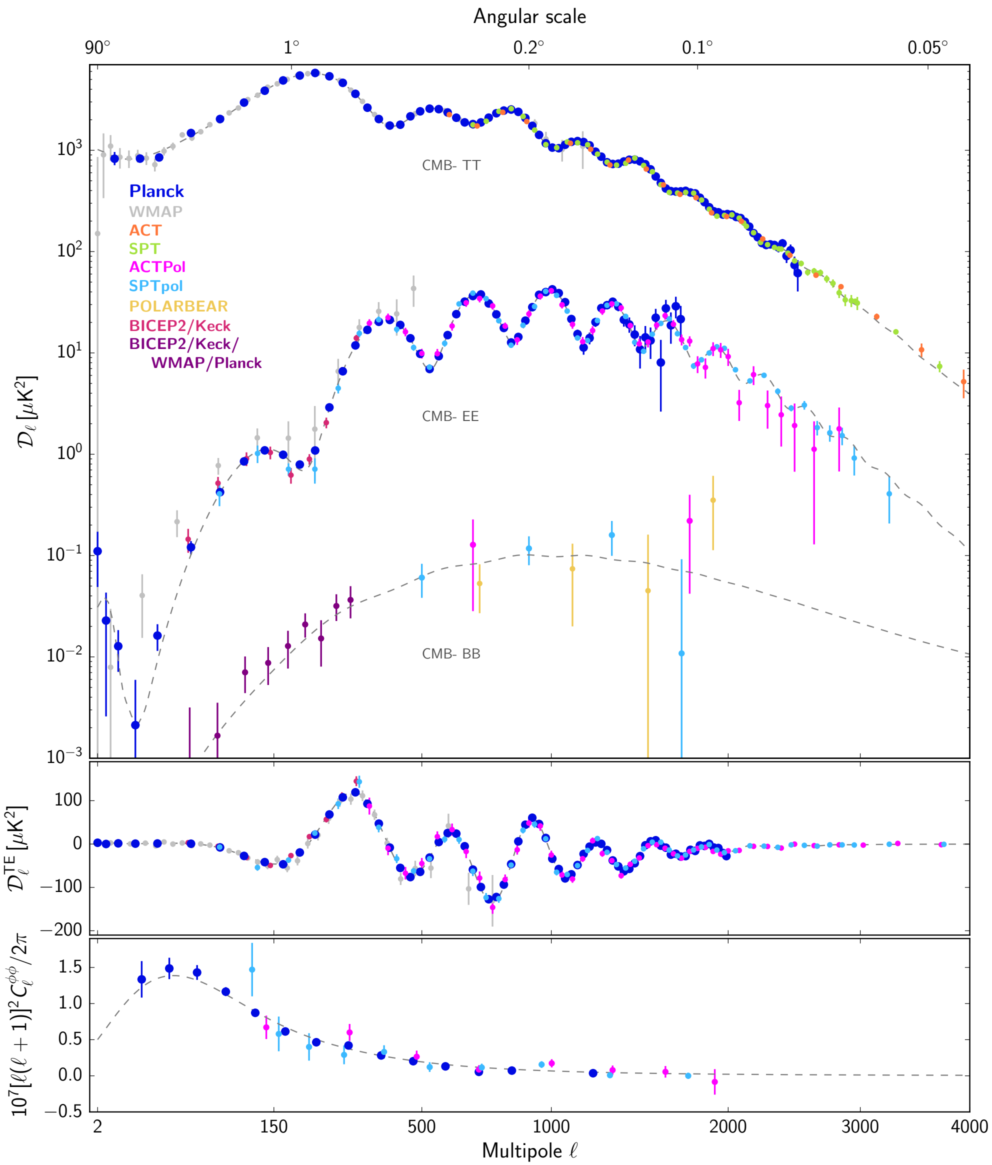


| 0.41 μK

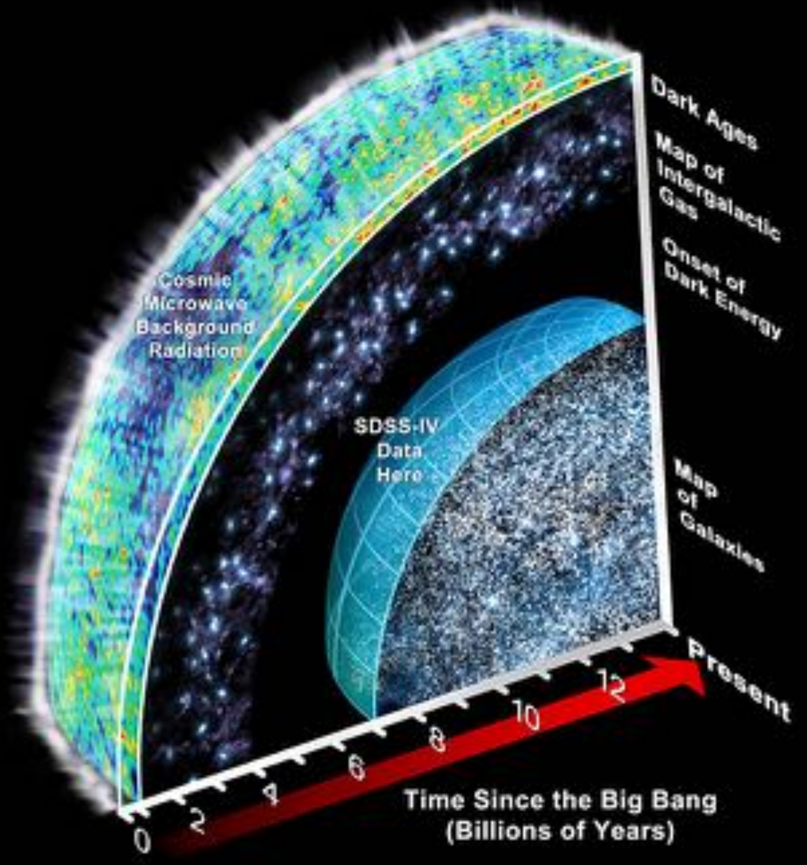
ESA Planck CMB lensing map



combined CMB power spectra

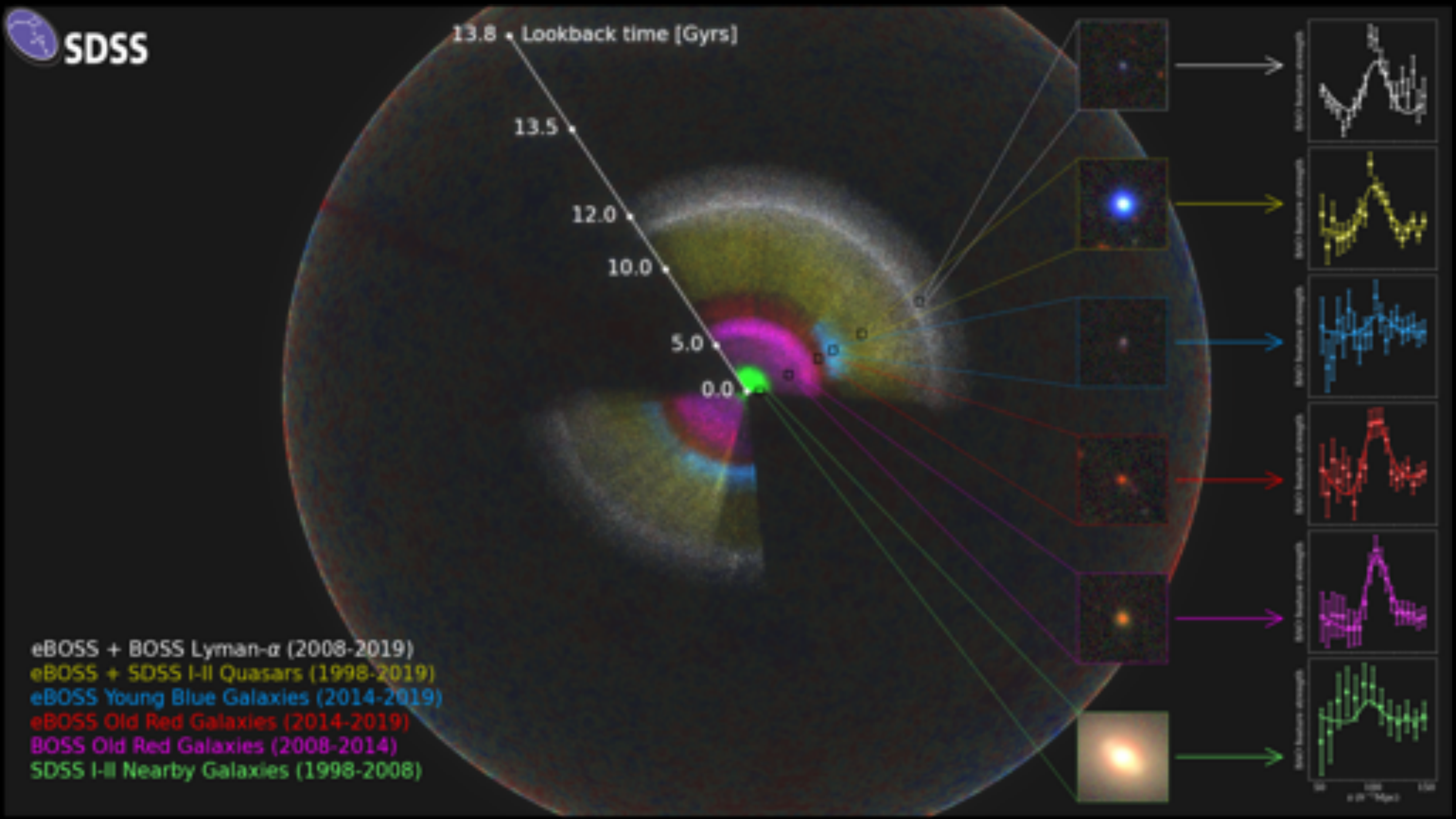


galaxy redshift surveys

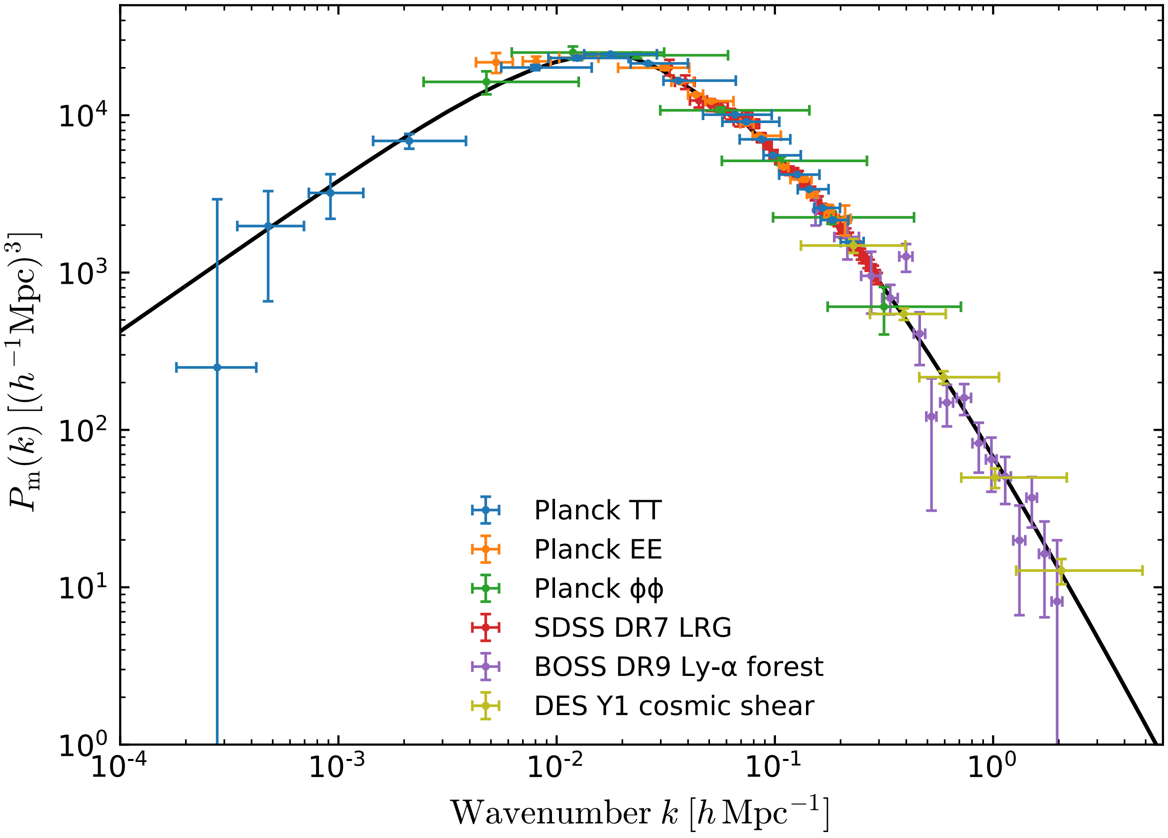


SDSS

eBOSS galaxy+quasar survey



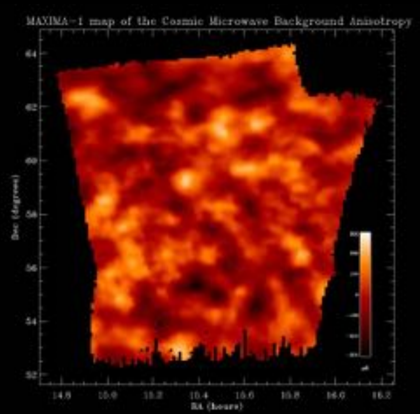
matter power spectrum, $P(k)$



standard model of structure formation

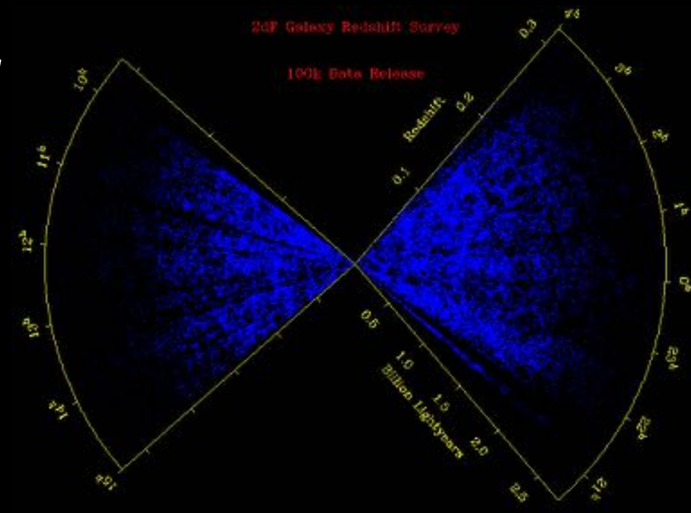
primordial perturbations

in cosmic microwave background



gravitational

instability



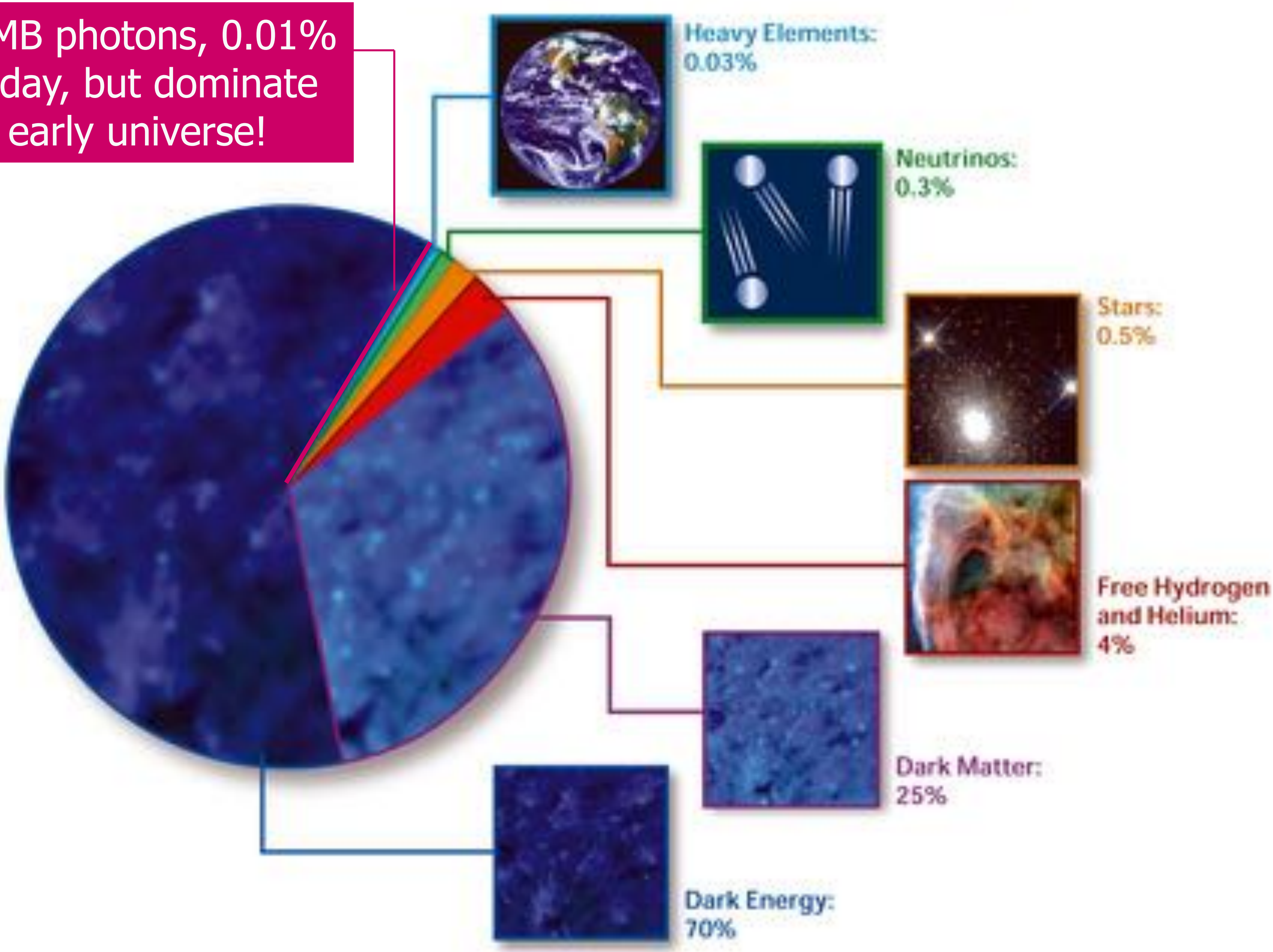
large-scale structure of our Universe

observational data offers precision tests of

- composition of the cosmos
- the nature of the primordial perturbations

COMPOSITION OF THE COSMOS TODAY

CMB photons, 0.01% today, but dominate in early universe!



key questions

- how do we characterise the inhomogeneous (time and space dependent) structures that we see all around us?
  - *homogeneous background cosmology*
  - ***cosmological perturbations*** *in general relativity*
- how can we probe the history of our Universe through its large-scale structure?
- how does cosmic structure originate in the very early Universe?

outline:

0. Homogeneous and isotropic cosmology

1. Inhomogeneous scalar, vector and tensor perturbations

2. Metric perturbations and gauge transformations

3. Einstein equations

4. Adiabatic and isocurvature, Gaussian and non-Gaussian perturbations

References:

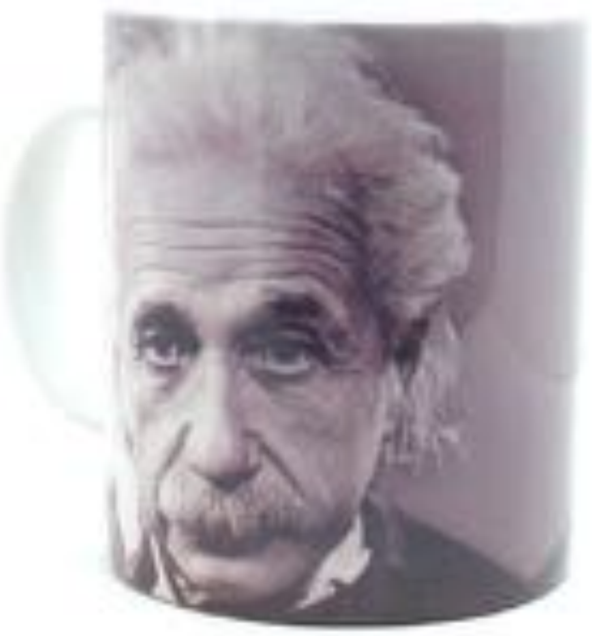
- Bardeen, Phys Rev D22, 1882 (1980)
- Kodama and Sasaki, Prog Theor Phys Supp 78, 1 (1984)
- Mukhanov, Feldman and Brandenberger, Phys Rep 215, 203 (1992)
- Malik and Wands, Phys Rep 475, 1 (2009), arXiv:0809:4944

Einstein's theory of gravity: General Relativity

spacetime tells matter
how to move

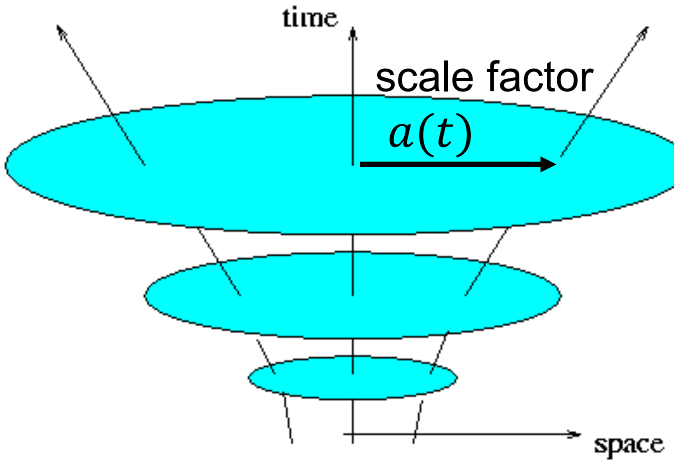
&

matter tells spacetime
how to curve



Friedmann's dynamic cosmology

- slice up 4D spacetime into expanding 3D space uniform matter density and spatial curvature



FLRW metric

- 4D spacetime split into 1+3
- Friedmann-Lemaitre-Robertson-Walker (FLRW) line element:

$$ds^2 = -c^2 dt^2 + a^2(t) dX^2$$

  - time + homogeneous and isotropic space
  - dynamical scale factor, $a(t)$, where $a_0 = 1$ today
  - maximally-symmetric 3-space, curvature K

$$dX^2 = \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

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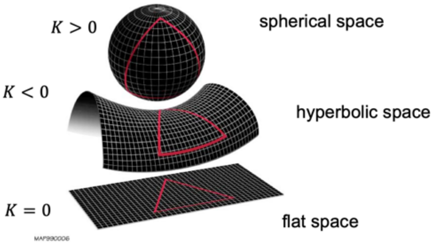

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$$dX^2 = \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

  - assume $K = 0$, flat space, unless otherwise stated

conformal time, τ

- alternative/conformal time coordinate, $d\tau = c\, dt/a$
- Friedmann-Lemaitre-Robertson-Walker (FLRW) line element:

$$\begin{aligned} ds^2 &= -c^2 dt^2 + a^2(t) dX^2 \\ &= a^2(\tau)\left[-d\tau^2 + dX^2\right] \end{aligned} \tag{1}$$

  - conformal rescaling of Minkowski spacetime
  - conformal time = comoving distance travelled by light

$$X = \int d\tau = \int \frac{c\, dt}{a(t)}$$

natural units:

for photons (and other relativistic quanta)

frequency = (wavelength)$^{-1}$ = wavenumber

= momentum = mass = energy = temperature

by convention they have different units (seconds, metres, etc) which we have to relate using fundamental constants

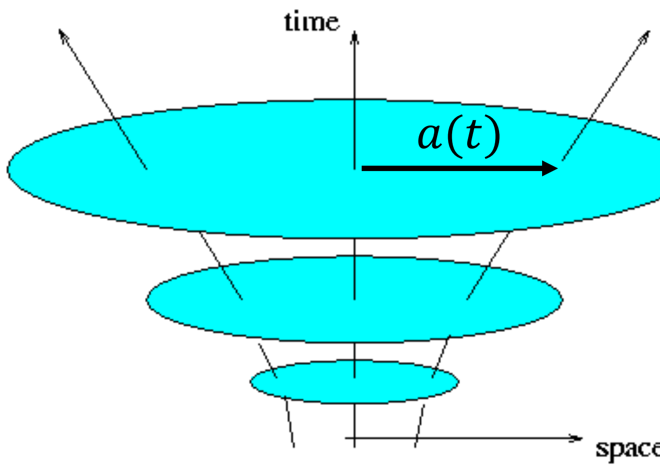
much easier to use natural units, such that fundamental constants

$$\boxed{\hbar = k_B = c = 1}$$

leaves only one dimensional constant = Newton's constant, *G*

Friedmann's dynamic cosmology

- slice up 4D spacetime into expanding 3D space
  uniform matter density and spatial curvature



scale factor $a(t)$

Hubble rate $H \equiv \dot{a} / a$

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$$

$$h \approx 0.7$$

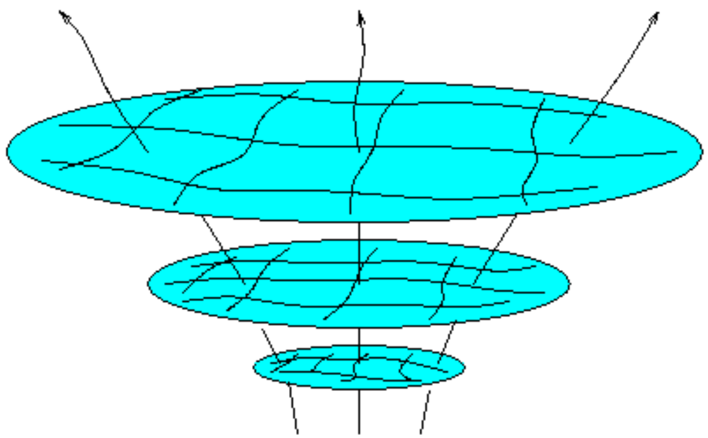
- Friedmann equation
  - from Einstein's energy constraint:

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2}$$

$$\Rightarrow 1 = \Omega_m + \Omega_\Lambda + \Omega_\kappa$$

breaking spatial symmetry

inhomogeneous pressure waves in the primordial plasma
or clustering of matter in the late-time universe
break the symmetry of the spatially homogeneous cosmology



full nonlinear numerical solutions have limited dynamic range

perturbative approach valid while inhomogeneities are small
(typically this applies at early times and/or large scales)

Scalar perturbations

- Scalar quantity, e.g., density at fixed point P is invariant under change of coordinates
- Split into background (homogeneous) part and a perturbation (inhomogeneous):

$$\rho(t,\vec{x}) = \bar{\rho}(t) + \delta\rho(t,\vec{x})$$

  - expand perturbation order-by-order in a small parameter, ε:

$$\delta\rho(t,\vec{x}) = \varepsilon\delta_1\rho(t,\vec{x}) + \frac{1}{2}\varepsilon^2\delta_2\rho(t,\vec{x}) + \dots$$

  - keep only terms at first order in $\varepsilon \Rightarrow$ *linear peturbations*

$$\delta\rho(t,\vec{x}) = \varepsilon\delta_1\rho(t,\vec{x})$$

Expanding equations order-by-order

- e.g., non-relativistic continuity equation for density $\rho(t,\vec{x})$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \qquad (2)$$

expand density and velocity order-by-order

$$\begin{aligned}
\rho(t,\vec{x}) &= \bar{\rho}(t) + \varepsilon\delta_1\rho(t,\vec{x}) + \frac{1}{2}\varepsilon^2\delta_2\rho(t,\vec{x}) + \ldots \\
\vec{v}(t,\vec{x}) &= \varepsilon\delta_1\vec{v}(t,\vec{x}) + \frac{1}{2}\varepsilon^2\delta_2\vec{v}(t,\vec{x}) + \ldots
\end{aligned}$$

substitute into Eq. (2)

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\bar{\rho} + \varepsilon\delta_1\rho + \frac{1}{2}\varepsilon^2\delta_2\rho + \ldots\right) \\
&+\vec{\nabla}\cdot\left[\left(\bar{\rho} + \varepsilon\delta_1\rho + \frac{1}{2}\varepsilon^2\delta_2\rho + \ldots\right)\right. \\
&\left.\times\left(\varepsilon\delta_1\vec{v} + \frac{1}{2}\varepsilon^2\delta_2\vec{v} + \ldots\right)\right] = 0
\end{aligned}$$

Perturbation equations order-by-order

- collect terms order-by-order in ε

$$\begin{aligned}&\frac{\partial}{\partial t}\bar{\rho}\\&+\varepsilon\left\{\frac{\partial}{\partial t}\delta_1\rho+\vec{\nabla}\cdot(\bar{\rho}\delta_1\vec{v})\right\}\\&+\frac{1}{2}\varepsilon^2\left\{\frac{\partial}{\partial t}\delta_2\rho+\vec{\nabla}\cdot(\bar{\rho}\delta_2\vec{v}+2\delta_1\rho\delta_1\vec{v})\right\}+\ldots=0\end{aligned}$$

- solve order-by-order in ε

$$\begin{aligned}\frac{\partial}{\partial t}\bar{\rho}&=0\qquad\Rightarrow\qquad\bar{\rho}=C\\\frac{\partial}{\partial t}\delta_1\rho+C\vec{\nabla}\cdot\delta_1\vec{v}&=0\\\frac{\partial}{\partial t}\delta_2\rho+C\vec{\nabla}\cdot\delta_2\vec{v}&=-2C\vec{\nabla}\cdot(\delta_1\rho\,\delta_1\vec{v})\end{aligned}$$

Fourier transform

- Field in real space is an integral over Fourier modes:

$$\delta\rho(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \delta\rho_{\vec{k}}(t)\, e^{i\vec{k}.\vec{x}}$$

- Fourier modes are eigenfunctions of the spatial Laplacian:

$$\nabla^2 \left(e^{i\vec{k}.\vec{x}} \right) = -k^2 e^{i\vec{k}.\vec{x}}$$

  which provide a complete orthonormal basis:

$$\int d^3x\, e^{i\vec{k}_1.\vec{x}}\, e^{i\vec{k}_2.\vec{x}} = (2\pi)^3 \delta^{(3)} \left(\vec{k}_1 - \vec{k}_2 \right)$$

- Coefficient in Fourier space is integral over real space:

$$\delta\rho_{\vec{k}}(t) = \int d^3x\, \delta\rho(t,\vec{x})\, e^{-i\vec{k}.\vec{x}}$$

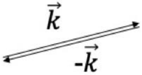
Statistical distribution

- *theory*
  describes properties of distribution = ensemble, which we will assume to be isotropic

  $\langle \ldots \rangle$ = ensemble average

- *observations*
  describe one realisation from the distribution

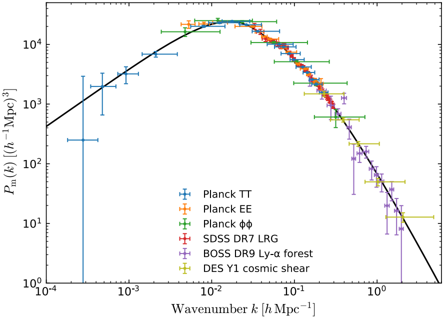
Power spectrum



- defined by the correlation of two modes in Fourier space:

$$\langle \delta\rho_{\vec{k}_1} \delta\rho_{\vec{k}_2} \rangle = (2\pi)^3 \; P_\rho(k_1)\, \delta^3 \left(\vec{k}_1 + \vec{k}_2 \right)$$

  - $P_\rho(k)$ only a function of wavenumber, k, not wavevector, $\vec{k}$, for an isotropic distribution



Power spectrum

- defined by the correlation of two modes in Fourier space:

$$\langle \delta\rho_{\vec{k}_1} \delta\rho_{\vec{k}_2} \rangle = (2\pi)^3 \, P_\rho(k_1) \delta^3 \left(\vec{k}_1 + \vec{k}_2 \right)$$

- Variance in real space:

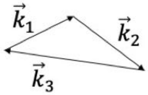
$$\begin{aligned}
\langle \delta\rho^2(\vec{x}) \rangle &= \langle \int \frac{d^3\vec{k}_1 d^3\vec{k}_2}{(2\pi)^6} \delta\rho_{\vec{k}_1} \delta\rho_{\vec{k}_2} e^{i\left(\vec{k}_1+\vec{k}_2\right)\cdot\vec{x}} \rangle \\
&= \int \frac{d^3\vec{k}_1 d^3\vec{k}_2}{(2\pi)^6} \langle \delta\rho_{\vec{k}_1} \delta\rho_{\vec{k}_2} \rangle e^{i\left(\vec{k}_1+\vec{k}_2\right)\cdot\vec{x}} \\
&= \int \frac{d^3\vec{k}_1}{(2\pi)^3} P_\rho(k_1) = \int d\ln k_1 \, \mathcal{P}_\rho(k_1)
\end{aligned}$$

- dimensionless power spectrum per log k:

$$\mathcal{P}_\rho(k) = \frac{4\pi k^3}{(2\pi)^3} P_\rho(k)$$

Higher-order statistics

- Bispectrum



$$\langle \delta\rho_{\vec{k}_1} \delta\rho_{\vec{k}_2} \delta\rho_{\vec{k}_3} \rangle = (2\pi)^3 \, B_\rho(k_1, k_2, k_3) \, \delta^3 \left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 \right)$$

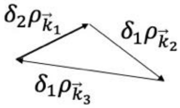
  Bispectrum is zero for *Gaussian* perturbations

- We usually take *first-order* perturbations to be *Gaussian*:

$$\langle \delta_1\rho_{\vec{k}_1} \delta_1\rho_{\vec{k}_2} \delta_1\rho_{\vec{k}_3} \rangle = 0$$

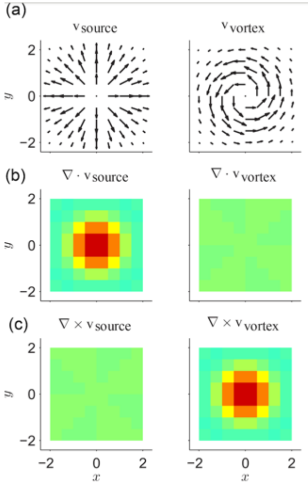
  Second- and higher-order perturbations are *non-Gaussian*:

$$\langle \delta_2\rho_{\vec{k}_1} \delta_1\rho_{\vec{k}_2} \delta_1\rho_{\vec{k}_3} \rangle \neq 0$$



Vector perturbations

- decompose any 3-vector: $\vec{V} = \vec{\nabla} V^{(s)} + \vec{V}^{(v)}$
  - *scalar* (longitudinal/potential) flow: $\vec{\nabla} \times \vec{\nabla} V^{(s)} = 0$
  - *vector* (transverse/divergence-free) flow: $\vec{\nabla} \cdot \vec{V}^{(v)} = 0$



Vector perturbations

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  - *scalar* (longitudinal/potential) flow: $\vec{\nabla} \times \vec{\nabla} V^{(s)} = 0$
  - *vector* (transverse/divergence-free) flow: $\vec{\nabla} \cdot \vec{V}^{(v)} = 0$

- *scalar* (longitudinal/potential) flow *coupled* to scalar density perturbations *at first order*, e.g.,

$$\frac{\partial}{\partial t}\delta_1\rho + C\vec{\nabla} \cdot \delta_1\vec{v} \quad = \quad 0$$

- *vector* (transverse/divergence-free) flow *independent* of scalar density perturbations *at first order*

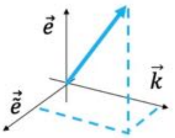
  but coupled at second and higher order, e.g.,

$$\frac{\partial}{\partial t}\delta_2\rho + C\vec{\nabla} \cdot \delta_2\vec{v} \quad = \quad -2C\vec{\nabla} \cdot (\delta_1\rho\,\delta_1\vec{v})$$

Vector perturbations

- decompose any 3-vector: $\vec{V} = \vec{\nabla} V^{(s)} + \vec{V}^{(v)}$
  - *scalar* = potential flow: $\vec{\nabla} \times \vec{\nabla} V^{(s)} = 0$
  - *vector* = divergence-free flow: $\vec{\nabla} \cdot \vec{V}^{(v)} = 0$

- Fourier transform each part



  - *scalar* (longitudinal)

$$\vec{\nabla} V^{(s)}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, i\vec{k}\, V^{(s)}_{\vec{k}}(t)\, e^{i\vec{k}.\vec{x}}$$

  - *vector* (transverse)

$$\vec{V}^{(v)}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ \vec{e}_{\vec{k}} V^{(v)}_{\vec{k}}(t) + \tilde{\vec{e}}_{\vec{k}} \tilde{V}^{(v)}_{\vec{k}}(t) \right\} e^{i\vec{k}.\vec{x}}$$

Vector perturbations

- putting parts back together

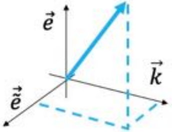
$$\vec{V}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ i\vec{k}\, V^{(s)}_{\vec{k}}(t) + \vec{e}_{\vec{k}} V^{(v)}_{\vec{k}}(t) + \vec{\tilde{e}}_{\vec{k}} \tilde{V}^{(v)}_{\vec{k}}(t) \right\} e^{i\vec{k}.\vec{x}}$$

where $\vec{e}_{\vec{k}}$ and $\vec{\tilde{e}}_{\vec{k}}$ are orthonormal *polarisation* vectors:

$$\vec{e}_{\vec{k}} \cdot \vec{e}_{\vec{k}} = \vec{\tilde{e}}_{\vec{k}} \cdot \vec{\tilde{e}}_{\vec{k}} = 1\,, \quad \vec{e}_{\vec{k}} \cdot \vec{\tilde{e}}_{\vec{k}} = 0$$

transverse to wavevector $\vec{k}$:

$$\vec{k} \cdot \vec{e}_{\vec{k}} = \vec{k} \cdot \vec{\tilde{e}}_{\vec{k}} = 0$$



- for *linear perturbations* each component (longitudinal plus two transverse components) evolve independently for each wavevector $\vec{k}$

Cosmological perturbation theory

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Motivation

Homogeneous cosmology

Perturbation theory

Scalar perturbations

Fourier transforms

Power spectrum

Bispectrum

Vector perturbations

Tensor perturbations

Metric perturbations

Tensors

Vectors

Scalars

Geometrical interpretation

Tensor perturbations

- decompose any 3-tensor (e.g., anisotropic stress):

$$T_{ij} = \delta_{ij} C + \nabla_i \nabla_j S + (1/2)\left(\nabla_i V_j + \nabla_j V_i\right) + h_{ij}$$

  - *scalars* C and S are longitudinal/potential
  - *vector* V_i is transverse: $\nabla^i V_i = 0$
  - *tensor* h_{ij} is transverse and trace-free:

$$\nabla^i h_{ij} = \nabla^j h_{ij} = 0 \,, \qquad h^i_i = 0$$

Tensor perturbations

- Fourier transform:

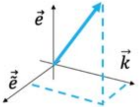
$$h_{ij}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ h^{(+)}_{\vec{k}}(t) q^{(+)}_{\vec{k}\,ij} + h^{(\times)}_{\vec{k}}(t) q^{(\times)}_{\vec{k}\,ij} \right\} e^{i\vec{k}.\vec{x}}$$

where $e_{\vec{k}\,i}$ and $\tilde{e}_{\vec{k}\,i}$ are orthonormal, transverse vectors, such that *polarisation* tensors

$$\begin{aligned} q^{(+)}_{\vec{k}\,ij} &= \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} e_{\vec{k}\,j} - \tilde{e}_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} \right) \\ q^{(\times)}_{\vec{k}\,ij} &= \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} + \tilde{e}_{\vec{k}\,i} e_{\vec{k}\,j} \right) \end{aligned}$$

are tracefree $q^{(+)i}_{\vec{k}\;\;i} = q^{(\times)i}_{\vec{k}\;\;i} = 0$ and transverse to $\vec{k}$:

$$k^i q^{(+)}_{\vec{k}\,ij} = k^i q^{(\times)}_{\vec{k}\,ij} = 0$$



Metric perturbations

- Split metric into background and perturbation:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,.$$

- Spatially-flat FLRW background:

$$\bar{g}_{00} = a^2 \,, \quad \bar{g}_{0i} = 0 \,, \quad \bar{g}_{ij} = a^2 \delta_{ij}$$

- split perturbations into scalars, vectors and tensors:

$$\begin{aligned} \delta g_{00} &= 2a^2 A \\ \delta g_{0i} &= a^2 \left(\nabla_i B - S_i \right) \\ \delta g_{ij} &= a^2 \left(2C\delta_{ij} + 2\nabla_i \nabla_j E + \nabla_i F_j + \nabla_j F_i + h_{ij} \right) \end{aligned}$$

- S/V/T evolve independently for linear perturbations :
  - 4 scalars: A, B, C, E
  - 2 vectors: S_i, F_i
  - 1 tensor: h_{ij}