Some lattice subgroups that cannot act on the line (after Deroin and Hurtado)

Dave Witte Morris
University of Lethbridge, Alberta, Canada
https://deductivepress.ca/dmorris
dmorris@deductivepress.ca

Abstract: Deroin and Hurtado recently proved the 30-year-old conjecture that no lattice in $SL(3,\mathbb{R})$ can act faithfully (by homeomorphisms) on the real line. (The same is true for irreducible lattices in other semisimple Lie groups of real rank at least two.) We will discuss this theorem, and point out that the same methods apply to lattices in p-adic groups. In fact, the p-adic case is easier, because some of the technical issues do not arise.

https://deductivepress.ca/dmorris/talks/deroin-hurtado.pdf

Let $G = \mathrm{SL}(3,\mathbb{R}) = \{3 \times 3 \text{ mats} \mid \det = 1,\ \mathbb{R} \text{ entries}\}$
$=$ semisimple Lie group, with $\mathrm{rank}_{\mathbb{R}}\, G \geq 2$

Let Γ = irreducible **lattice** in G $(= \dot{\mathrm{S}}\mathrm{L}(3,\mathbb{Z}))$

- Γ is discrete (no accumulation points)
- G/Γ has finite volume

Zimmer program [1980s–now]

Show: if M is a compact mfld, and $\dim M$ is "small," then Γ cannot act faithfully on M ($\Gamma \not\curvearrowright M$) by diffeos.

Completed by Brown-Fisher-Hurtado [2020–2022$^+$].

But what about actions by **homeo**morphisms?
Assume $\dim M = 1$. (Higher dimensions wide open.)

Γ lattice in $\mathrm{SL}(3,\mathbb{R})$, $\dim M = 1$: $\Gamma \overset{?}{\curvearrowright} M$.

Thm [Witte, 1994]. $\dot{\mathrm{SL}}(3,\mathbb{Z}) \not\curvearrowright S^1$ *or* $\mathbb{R}$.

What about other latts in $\mathrm{SL}(3,\mathbb{R})$? or in other semi-simple Lie groups

Theorem (Ghys, Burger-Monod [1999])

If $\dot\Gamma \not\curvearrowright \mathbb{R}$, *then* $\Gamma \not\curvearrowright S^1$. (*unless* $\mathrm{SL}(2,\mathbb{R})$ *is a factor of* G)

Theorem (Deroin-Hurtado [2022$^+$])

$\Gamma \not\curvearrowright \mathbb{R}$. (*unless* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *is a factor of* G)

Γ is a lattice in $\mathrm{SL}(3,\mathbb{R})$, but same proof (easier):

$\Gamma \not\curvearrowright \mathbb{R}$ (or S^1) if Γ = lattice in $\mathrm{SL}(3,\mathbb{Q}_p)$. work in progress

Apparently(?): also lattices in $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{Q}_p)$.
($\Gamma = S$-arithmetic group, no p-adic factors of rank 1)

Almost-periodic space

Theorem (Deroin, Deroin et al. [2013, 2022$^+$])

If $\Gamma \curvearrowright \mathbb{R}$, then $\exists$ compact metrizable space Z:

- $\mathbb{R} \overset{free}{\curvearrowright} Z$ *and* $\Gamma \curvearrowright Z$ *with no global fixed point,*
- *each* $\mathbb{R}$*-orbit is* Γ*-invariant, and*
- *additional technical conditions are satisfied.*

Proof.

$\exists\, \Gamma \curvearrowright \mathbb{R}$, bi-Lipschitz, bdd displacement, etc.

$$Z \doteq \{\, \Gamma \overset{\varphi}{\curvearrowright} \mathbb{R} \mid \forall \text{gen } \gamma,\ |\varphi_\gamma(x) - x| < C,\ \cdots \,\}.$$

$\mathbb{R} \curvearrowright Z$: ${}^t\varphi_\gamma(x) = \varphi_\gamma(x - t) + t$. (conjugate by translation)

$\Gamma \curvearrowright Z$: ${}^\lambda\varphi = {}^{\varphi_\lambda(0)}\varphi$. □

$\mathbb{R} \curvearrowright Z,\ \Gamma \curvearrowright Z,$ *and each* $\mathbb{R}$*-orbit is* Γ*-invariant*

Induce to a G-action (classical)

Let $X = (G \times Z)/\Gamma$, where $(h, z) * \gamma = (h\gamma, \gamma^{-1}z)$.
So $G \curvearrowright X$ by $g[(h, z)] = [(gh, z)]$ and $X \simeq G/\Gamma \times Z$.

Let $K = \mathrm{SL}(3, \mathbb{Z}_p)$ = compact, open subgroup of G.

Since K is open, we know $K \backslash G$ is discrete.
Since G/Γ is compact, this implies $K \backslash G/\Gamma$ is finite.
For simplicity, assume $G = K\Gamma$.
So we can identify G/Γ with K: $X \simeq K \times Z$.

this is easier than the real case

Stationary measures

$\mathbb{R} \curvearrowright Z,\ \Gamma \curvearrowright Z,\ G \curvearrowright X,\ X \simeq G/\Gamma \times Z \simeq K \times Z$

Let μ_G = nice bi-K-invariant probability meas on G.

$G = K\Gamma \Rightarrow \mu_G = \mu_K * \mu_\Gamma$

μ_K = Haar on K,
μ_Γ = nice prob meas on Γ

Let μ_Z = an ergodic $\mathbb{R}$-inv't probability measure on Z.

Z can be constructed so mean displacement is 0:

$$\forall z \in Z,\ \sum_{\gamma \in \Gamma} (\gamma z - z)\, \mu_\Gamma(\gamma) = 0.$$

Then μ_Z is μ_Γ-stationary:

$$\sum_\gamma \mu_\Gamma(\gamma)\, \gamma_* \mu_Z = \mu_\Gamma * \mu_Z = \mu_Z.$$

So $\mu_X = \mu_K \times \mu_Z$ is μ_G-stationary.

harder to define
μ_X in real case

$$\mathbb{R} \looparrowright Z,\ \Gamma \looparrowright Z,\ G \looparrowright X,\ X \simeq G/\Gamma \times Z \simeq K \times Z$$

$\mu_X = \mu_K \times \mu_Z$ *is* μ_G*-stationary*

Let $P = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ and $A = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \subset P$.

For $a \in A$, $U_a^+ = \left\{ u \in G \;\middle|\; \begin{array}{c} a^n u a^{-n} \to 1 \\ \text{as } n \to -\infty \end{array} \right\}$.

Theorem (Furstenberg [1963] (real case))

$\exists!$ *P-inv't prob measure μ_P on X,* $\mu_X = \int_K k_* \mu_P \, dk$.

Key Proposition

If $U_a^+ \subseteq P$ *and* a *is "leafwise-contracting,"*
then μ_P *is* $C_G(a)$*-invariant.*

Before proving this, see how it gives a contradiction.

Key. $U_a^+ \subseteq P$ *(leafwise-contracting)* $\Rightarrow$ μ_P *is* $C_G(a)$*-inv't.*

Cor. μ_P *is* G*-invariant.* (*"propagating invariance"*)

Proof (ignore need to be leafwise-contracting).

$$a^n = \begin{bmatrix} \blacksquare & & \\ & \blacksquare & \\ & & \cdot \end{bmatrix} \Rightarrow U_a^+ = \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} * & * & \\ * & * & \\ & & 1 \end{bmatrix}\text{-inv't.}$$

$$a^n = \begin{bmatrix} \blacksquare & & \\ & \cdot & \\ & & \cdot \end{bmatrix} \Rightarrow U_a^+ = \begin{bmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & & \\ & * & * \\ & * & * \end{bmatrix}\text{-inv't.}$$

G is generated by these centralizers. □

This is where higher rank is used;: rank 1 $\Rightarrow$ $C_G(a) \doteq A \subset P$.
∴ Argument is more complicated if some simple factor has rank 1.

$\mu_K \times \mu_Z = \mu_X = \int_K k_* \, \mu_P \, dk = \int_K \mu_P \, dk = \mu_P$ is G-inv't.
So μ_Z is Γ-inv't, so $\Gamma \looparrowright$ $\mathbb{R}$-orbits by translations,
so $\Gamma \xrightarrow{\text{homo}} \mathbb{R}$. $\rightarrow\leftarrow$

Leafwise-Contracting (globally contracting)

Some half-plane of A is *leafwise-contracting*.

Action on each leaf is Lipschitz, so diff'ble a.e. Let

$$\chi(a) = \int_X \log D_{\text{leaf}}\, a(x)\, d\mu_P(x).$$

Then $\chi \colon A \to \mathbb{R}$ is a homomorphism.

Fact. χ is nontrivial: $\exists a,\ \chi(a) < 0$ and $U_a^+ \subset P$.

Idea of proof: $\mu_X(aX) = \mu_X(X)$, so $\int D_{\text{leaf}}\, a = 1$.

Jensen's Ineq: log is concave, so $\int \log D_{\text{leaf}} < \log 1$.

Since $\mu_X = \int_K k_* \mu_p\, dk$, can conclude also for μ_P.

Theorem

$\forall a \in \chi^{-1}(\mathbb{R}^-)$, *for a.e.* $x \in X$,

$\forall y \in \mathbb{R}x, \qquad d_{\text{leaf}}(a^n x, a^n y) \to 0.$

Key Proposition

If $U_a^+ \subseteq P$ *and* a *is leafwise-contracting,*
then μ_P *is* $C_G(a)$*-invariant.*

Proof. Let $c \in C_G(A)$. We wish to show $c_* \mu_P = \mu_P$.
Recall: μ_P is a P-inv't prob meas on $X \simeq G/\Gamma \times Z$.
Let x be a Birkhoff-generic point for a w.r.t. μ_P.
Then $a^k c x \approx x$ is Birkhoff-generic w.r.t. $c_* \mu_P$.

$$x_c = a^k c x \overset{G/\Gamma}{=} g^- u^+ x$$

technical issue

μ_P is U_a^+-inv't, so $x_0 = u^+ x$ is also generic. (a.e.)

- $d(a^n x_0, a^n g^- x_0) \prec \|g^-\| \approx 0$,
- $d(a^n x_c, a^n g^- x_0) = d_{\text{leaf}}(a^n x_c, a^n g^- x_0) \to 0$.

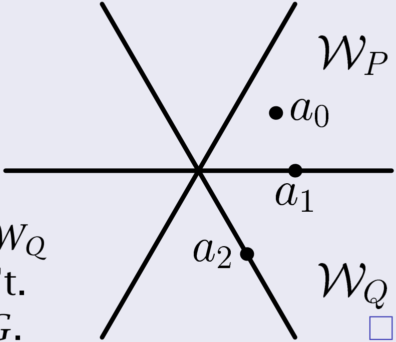
$\therefore$ x_0 and x_c have almost same Birkhoff averages.
So $\mu_P = c_* \mu_P$. □

Key. $U_a^+ \subseteq P$ *(leafwise-contracting)* $\Rightarrow$ *μ_P is $C_G(a)$-inv't.*

Cor. *μ_P is G-invariant.* *("propagating invariance")*

Proof.

Fix a_0 with $\chi(a_0) < 0$ and $U_{a_0}^+ \subset P$, so $a_0 \in \mathcal{W}_P$.
Contracting half-plane contains an adjacent $\mathcal{W}_Q$.
Choose a_1 on boundary:
μ_P is $C_G(a_1)$-inv't.
Weyl grp el't $w \in C_G(a_1)$
reflects across this side.
Then $\mu_P = w_* \mu_P = \mu_Q$.
Choose a_2 on other bdry of $\mathcal{W}_Q$
so $\mu_P = \mu_Q$ is $C_G(a_2)$-inv't.
These centralizers generate G.



□

Main reference:

Bertrand Deroin and Sebastian Hurtado:
Non left-orderability of lattices in higher rank semi-simple Lie groups.
https://arxiv.org/abs/2008.10687

Almost-periodic space:

Bertrand Deroin: Almost-periodic actions on the real line.
Enseign. Math. 59 (2013)183–194. MR 3113604

Bertrand Deroin, Victor Kleptsyn, Andrés Navas, Kamlesh Parwani:
Symmetric random walks on $\mathrm{Homeo}_+(\mathbb{R})$.
Ann. Probab. 41 (2013) 2066–2089. MR 3098067

Stationary measures:

Harry Furstenberg: Noncommuting random products.
Trans. Amer. Math. Soc. 108 (1963) 377–428. MR 0163345

Zimmer program:

Aaron Brown, David Fisher, Sebastian Hurtado:
Zimmer's conjecture: Subexponential growth, measure rigidity, and strong property (T). Ann. of Math. (2) 196 (2022) 891–940. MR 4502593

Aaron Brown, David Fisher, Sebastian Hurtado:
Zimmer's conjecture for non-uniform lattices and escape of mass.
`https://arxiv.org/abs/2105.14541`

David Fisher: Recent developments in the Zimmer program.
Notices Amer. Math. Soc. 67 (2020) 492–499. MR 4186267

Older papers on actions of lattices on 1-dimensional manifolds:

Étienne Ghys: Actions de réseaux sur le cercle.
Invent. Math. 137 (1999) 199–231. MR 1703323

Marc Burger and Nicolas Monod:
Bounded cohomology of lattices in higher rank Lie groups.
J. Eur. Math. Soc. (JEMS) 1 (1999) 199–235. MR 1694584

Marc Burger: An extension criterion for lattice actions on the circle,
in Geometry, rigidity, and group actions, ed. by B. Farb and D. Fisher.
Univ. Chicago Press, Chicago, IL, 2011. pp. 3–31. MR 2807827

Étienne Ghys: Groups acting on the circle.
Enseign. Math. (2) 47 (2001), no. 3-4, 329–407. MR 1876932

Dave Witte:
Arithmetic groups of higher $\mathbb{Q}$-rank cannot act on 1-manifolds.
Proc. Amer. Math. Soc. 122 (1994) 333–340. MR 1198459

Lucy Lifschitz and Dave Witte Morris:
Bounded generation and lattices that cannot act on the line,
Pure Appl. Math. Q. 4 (2008), no. 1, part 2, 99–126. MR 2405997