Arithmetic groups of higher real rank are not left-orderable (after Deroin and Hurtado)

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Abstract: B. Deroin and S. Hurtado recently proved the 30-year-old conjecture that if G is an almost-simple algebraic $\mathbb{Q}$-group, and the real rank of G is at least two, then no arithmetic subgroup of G is left-orderable. We will discuss this theorem, and explain some of the main ideas of the proof, by illustrating them in the simpler case where the real field $\mathbb{R}$ is replaced with a p-adic field. Harmonic functions and continuous group actions are key tools.

https://deductivepress.ca/dmorris/talks/deroin-hurtado.pdf

Γ = arithmetic group (or countable group)

Question

¿ $\exists$ *left-invariant total order* $\prec$ *on* Γ ?

- *total:* $x \prec y$ *or* $x \succ y$ *or* $x = y$
- *left-invariant:* $x \prec y \implies ax \prec ay, \ \forall x, y, a$

"Is Γ ***left-orderable****?"*

Example

$\mathbb{Z}$ is left-orderable (namely, $<$).

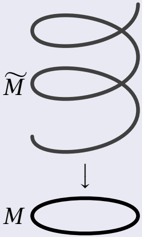
left-orderable: $\exists \prec, \quad x \prec y \implies ax \prec ay$

Motivation

Γ is left-orderable $\implies$

- Γ has an action on $\mathbb{R}$ $\begin{pmatrix}\text{faithful, continuous,}\\ \text{orientation-preserving}\end{pmatrix}$ (iff)
- Group ring $\mathbb{Z}[\Gamma]$ has no zero divisors
  (conjectured for all torsion-free groups)
- $\mathbb{Z}[\Gamma] \cong \mathbb{Z}[\Lambda] \implies \Gamma \cong \Lambda$ [Lagrange-Rhemtulla]
- If $\Gamma = \pi_1(M)$:

$$\begin{array}{ccc} \widetilde{M} & \hookrightarrow & M \times \mathbb{R} \\ \downarrow & \swarrow & \\ M & & \end{array}$$



(iff)

Example

$\Gamma = \text{free group} \doteq \mathrm{SL}_2(\mathbb{Z}) \doteq \mathrm{SO}(1,2)_{\mathbb{Z}}$

$\implies$ every (f.g.) subgroup of Γ maps onto $\mathbb{Z}$.

I.e., Γ is **locally indicable**.

Proposition (Burns-Hale 1972)

Γ *locally indicable* $\implies$ Γ *is left-orderable.*

Rough idea of proof.

For $x, y \in \Gamma$, choose $\varphi_{x,y} \colon \langle x, y \rangle \twoheadrightarrow \mathbb{Z}$.

Define $x \prec y$ if $\varphi_{x,y}(x) < \varphi_{x,y}(y)$.

Transitive: For $x, y, z \in \Gamma$, assume (Zorn's Lemma)

$\varphi_{x,y} =$ restriction of $\varphi_{x,y,z}$ to $\langle x, y \rangle$

or $x, y \in \ker \varphi_{x,y,z}$ □

Example

$\dot{\mathrm{SL}}_2(\mathbb{Z})$ is left-orderable.

Theorem (Witte 1994)

*$\dot{\mathrm{SL}}_3(\mathbb{Z})$ is **not** left-orderable.*

Conjecture (1990's)

Arith subgroup of $\mathbf{G}$ *left-orderable* $\implies \mathrm{rank}_{\mathbb{R}}\,\mathbf{G} = 1$.
I.e., $\mathbf{G}(\mathbb{R}) \doteq \mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$ *or* $\mathrm{Sp}(1,n)$ *or* $F_{4,1}$.
(if $\mathbf{G}$ *is almost* $\mathbb{Q}$*-simple)*

Note: arithmetic subgroups of $\mathrm{Sp}(1,n)$ and $F_{4,1}$ have **Kazhdan's property (T)**.

Open question: ¿ ∃ left-orderable Kazhdan group ?

Conjecture (1990's)

Arith subgrp Γ *of* $\mathbf{G}$ *left-orderable* $\implies$ $\operatorname{rank}_{\mathbb{R}} \mathbf{G} = 1$.

The conjecture is true [Deroin-Hurtado 2024+].

Proof uses group actions and real analysis,
including **harmonic functions**.

Definition (Furstenberg 1963)

Assume:

- μ is a Borel measure on G with $\mu(G) = 1$.
- $\varphi \colon G \to \mathbb{R}$.

φ is μ-**harmonic** if $\forall x \in G$:

$$\varphi(x) = \int_G \varphi(gx)\, d\mu(g) \quad \text{for all } x \in G.$$

Γ = arithmetic subgroup of a simple $\mathbb{Q}$-group $\mathbf{G}$.
Let $G = \mathbf{G}(\mathbb{R}) = \mathrm{SL}_3(\mathbb{R})$ and assume G/Γ is compact.
I.e. $\mathbf{G}$ is $\mathbb{Q}$-anisotropic.

harmonic: $\varphi(x) = \int_G \varphi(gx)\, d\mu(g)$

Theorem (e.g., Ledrappier 1985, Ballmann-Ledrappier 1996)

Every bounded μ_Γ-harmonic function on Γ extends to a unique μ_G-harmonic function on G.
($\exists$ μ_G on G and μ_Γ on Γ)

Proof. To simplify, replace $\mathbb{R}$ with $\mathbb{Q}_p$:
$G = \mathbf{G}(\mathbb{Q}_p) = \mathrm{SL}_3(\mathbb{Q}_p)$ (and G/Γ is compact)
(Γ is S-arithmetic, with $S = \{\infty, p\}$)

$G = \mathrm{SL}_3(\mathbb{Q}_p)$ and G/Γ is compact

Let $K = \mathrm{SL}_3(\mathbb{Z}_p)$ = compact, open subgroup of G.

K is open, so $K\backslash G$ is discrete, so $K\backslash G/\Gamma$ is finite.

For simplicity, assume $G = K\Gamma$: $G \simeq K \times \Gamma$.

Choose μ_G to be bi-K-invariant:

$\mu_G(A) = \mu_G(kA) = \mu_G(Ak)$ for all $k \in K$.

Exer. φ μ_G-harmonic $\left(\varphi(x) = \int_G \varphi(gx)\, d\mu_G(g)\right)$

$\Rightarrow$ $\varphi(kx) = \varphi(x)$: φ is left K-invariant.

Corollary (uniqueness)

φ, ψ μ_G-harmonic, $\varphi|_\Gamma = \psi|_\Gamma \Rightarrow \varphi = \psi$.

$$\varphi(g) = \varphi(k\gamma) = \varphi(\gamma) = \psi(\gamma) = \psi(k\gamma) = \psi(g)$$

$G \simeq K \times \Gamma$ and μ_G is bi-K-invariant

Fubini's Theorem: $\mu_G = \mu_K \times \mu_\Gamma$.

Exercise (existence)

For $\varphi \colon \Gamma \to \mathbb{R}$, define $\hat{\varphi} \colon G \to \mathbb{R}$ by $\hat{\varphi}(k\gamma) = \varphi(\gamma)$.
Show: φ is μ_Γ-harmonic $\Rightarrow$ $\hat{\varphi}$ is μ_G-harmonic.

Hint. For $x \in \Gamma$:

$$\begin{aligned} \int_G \hat{\varphi}(gx)\, dg &= \int_K \int_\Gamma \hat{\varphi}(k\gamma x)\, d\gamma\, dk \\ &= \int_K \int_\Gamma \hat{\varphi}(\gamma x)\, d\gamma\, dk \\ &= \int_\Gamma \hat{\varphi}(\gamma x)\, d\gamma \\ &= \varphi(x) \\ &= \hat{\varphi}(x). \end{aligned}$$

This is easier than the real case!

Vague idea of Deroin-Hurtado proof

Theorem (Deroin-Hurtado 2024^+)

Arith subgrp Γ *of* $\mathbf{G}$ *left-orderable* $\implies \operatorname{rank}_{\mathbb{R}} \mathbf{G} = 1$.

Assume Γ is left-orderable. So Γ acts on $\mathbb{R}$.

$G = \mathbf{G}(\mathbb{R})$ has no (nontrivial, continuous) action on $\mathbb{R}$:

$\operatorname{Stab}_G(x)$ is a subgroup of codimension 1. $\rightarrow\leftarrow$

To get a contradiction, might try to show that the action of Γ extends to an action of G.

Unfortunately, it is more complicated than that.

- Assume Γ is left-orderable. So Γ acts on $\mathbb{R}$.
- Compactify this Γ-action:
  (Deroin et al. [2013, 2013, 2024$^+$])
  construct a compact set C, with an action of Γ,
  s.t. C is a union of Γ-inv't copies of $\mathbb{R}$, ("foliation")
  and the Γ-action has no fixed points.
- Every continuous action on a compact space has a "stationary" measure μ: (Func Analysis)
  $\gamma \mapsto \mu(\gamma A)$ is harmonic.
- **Induce** Γ-action on C to action of G: (classical)
  embed C in larger $\hat{C}$, such that G acts on $\hat{C}$
  (and same action of Γ on the subset C).
- Harmonic functions on Γ extend to G:
  $\exists$ stationary measure $\hat{\mu}$ for the G-action.

$\exists$ stationary measure $\hat{\mu}$ for the G-action on $\hat{C}$.
$g \mapsto \hat{\mu}(gA)$ is harmonic

- **Hard part:** $\hat{\mu}$ is G-invariant: $\hat{\mu}(gA) = \hat{\mu}(A)$.
  (Assumes $\operatorname{rank}_{\mathbb{R}} \mathbf{G} > 1$)
- By a restriction process, $\exists$ Γ-inv't measure on C, and, hence, a Γ-invariant measure on (a.e.) $\mathbb{R}$.
- This implies Γ acts by translations:
  $$\forall \gamma, \ \exists c = c(\gamma) \in \mathbb{R}, \ \gamma(x) = x + c.$$
- Then $c \colon \Gamma \to \mathbb{R}$ is a homomorphism.
  So $c(\gamma) = 0$. (Assumes $\operatorname{rank}_{\mathbb{R}} \mathbf{G} > 1$)
- *Contradiction:* Γ-action has no fixed points.

Proof of the hard part: $\hat{\mu}$ is G-invariant

$$\text{Let} \quad P = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \subset P.$$

$$\text{For } a \in A, \quad U_a^+ = \left\{ u \in G \;\middle|\; \begin{matrix} a^n u a^{-n} \to 1 \\ \text{as } n \to -\infty \end{matrix} \right\}.$$

Theorem (Furstenburg 1963)

$\exists$ P-invariant measure μ_P, $\quad \hat{\mu} = \int_K k_* \, \mu_P \, dk.$

Key. $U_a^+ \subseteq P$ *and a is leafwise-contracting* $\Rightarrow$ *μ_P is $C_G(a)$-inv't.*

$$a^n = \begin{bmatrix} \blacksquare & & \\ & \blacksquare & \\ & & \cdot \end{bmatrix} \Rightarrow U_a^+ = \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} * & * & \\ * & * & \\ & & 1 \end{bmatrix}\text{-inv't.}$$

$$a^n = \begin{bmatrix} \blacksquare & & \\ & \cdot & \\ & & \cdot \end{bmatrix} \Rightarrow U_a^+ = \begin{bmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & & \\ & * & * \\ & * & * \end{bmatrix}\text{-inv't.}$$

These centralizers generate G.

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