ON THE BRUMER–STARK CONJECTURE AND REFINEMENTS

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SETUP FOR BRUMER-STARK CONJECTURE

$F =$ totally real field.

$H =$ finite Galois extension, CM field.

$G = \mathrm{Gal}(H/F)$, abelian.

$S =$ {infinite places, ramified places}.

$T =$ finite set of places, disjoint from S.

$$L_{S,T}(\chi, s) = \prod_{\mathfrak{p} \notin S} \frac{1}{1 - \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s}} \prod_{\mathfrak{p} \in T} (1 - \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{1-s})\,.$$

THE BRUMER-STARK CONJECTURE

Fix a prime $\mathfrak{p}$ of F and a prime $\mathfrak{P}$ of H above $\mathfrak{p}$.

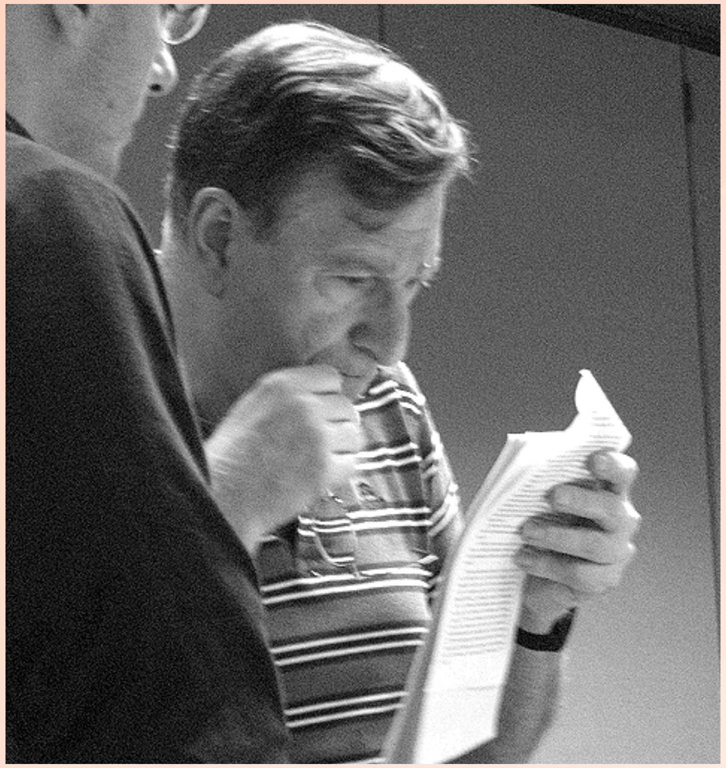
Conjecture (Tate-Brumer-Stark).

There exists $u \in \mathcal{O}_H[1/\mathfrak{p}]^*$ such that $|u| = 1$ under each embedding $H \hookrightarrow \mathbf{C}$,

$$L_{S,T}(\chi,0) = \sum_{\sigma \in G} \chi^{-1}(\sigma)\,\mathrm{ord}_{\mathfrak{P}}(\sigma(u))$$

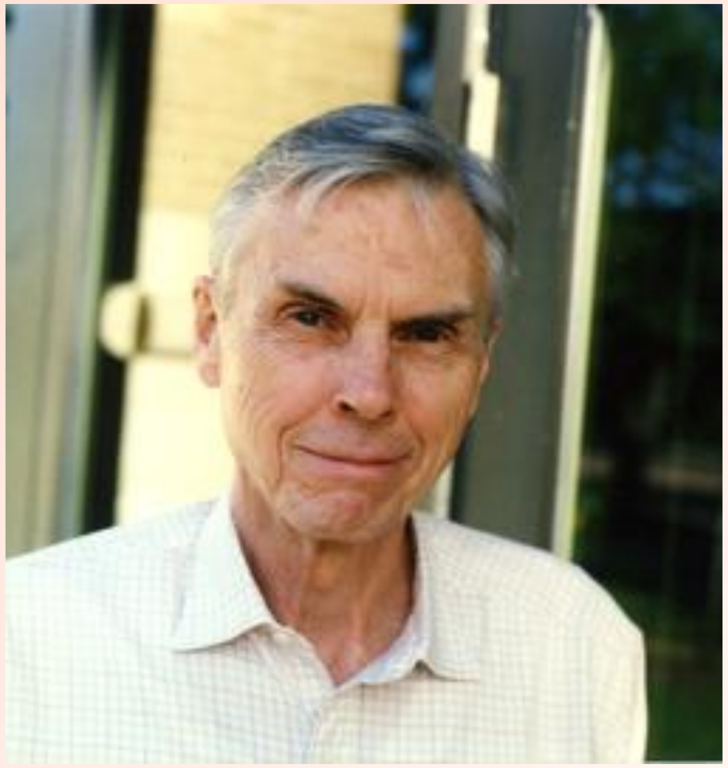
for all characters χ of G, and $u \equiv 1 \pmod{\mathfrak{q}\mathcal{O}_H}$ for all $\mathfrak{q} \in T$.

Ludwig Stickelberger



Armand Brumer

Harold Stark



John Tate

RESULTS

Theorem (D-Kakde). There exists

$$u \in \mathcal{O}_H[1/\mathfrak{p}]^* \otimes \mathbf{Z}[1/2]$$

satisfying the conditions of the Brumer-Stark conjecture.

There is a “higher rank” version of the Brumer-Stark conjecture due to Karl Rubin. We obtain this result as well, after tensoring with $\mathbf{Z}[1/2]$.

GROUP RINGS AND STICKELBERGER ELEMENTS

Theorem. (Deligne-Ribet, Cassou-Noguès)

There is a unique $\Theta \in \mathbf{Z}[G]$ such that

$$\chi(\Theta) = L_{S,T}(\chi^{-1}, 0)$$

for all characters χ of G.

CLASS GROUP

Define

$$\mathrm{Cl}^T(H) = I(H)/\langle (u) : u \equiv 1 \pmod{T} \rangle.$$

This is a G-module.

Brumer-Stark states: Θ annihilates $\mathrm{Cl}^T(H)$.

For this, it suffices to prove

$$\Theta \in \mathrm{Ann}_{\mathbf{Z}_p[G]}(\mathrm{Cl}^T(H) \otimes \mathbf{Z}_p)$$

for all primes p.

DEFINITION OF FITTING IDEAL

Let R be a commutative ring and M a finitely presented R-module.

$$R^n \xrightarrow{f} R^m \longrightarrow M \longrightarrow 0$$

Definition. $\mathrm{Fitt}_R(M)$ is the ideal of R generated by all $m \times m$ minors of the matrix representing the map f.

Example. If $n = m$ then we say M is quadratically presented over R, and $\mathrm{Fitt}_R(M) = (\det(f))$.

PROPERTIES OF FITTING IDEAL

Exercises.

- $\mathrm{Fitt}_R(R/I) = I$ for any ideal $I \subset R$.
- $\mathrm{Fitt}_R(M) \subset \mathrm{Ann}_R(M)$.
- If $R = \mathbf{Z}$, and M is a finitely generated abelian group, then $\mathrm{Fitt}_{\mathbf{Z}}(M) = 0$ if M is infinite, and $\mathrm{Fitt}_{\mathbf{Z}}(M) = (\#M)$ if M is finite.
- If $M \twoheadrightarrow M'$, then $\mathrm{Fitt}_R(M') \supset \mathrm{Fitt}_R(M)$.
- Base Change: If S is an R-algebra, then

$$\mathrm{Fitt}_S(M \otimes_R S) = \mathrm{Fitt}_R(M) \cdot S$$

PROJECT TO MINUS SIDE

Let c = complex conjugation in G.

Define

$$\mathbf{Z}_p[G]_- = \mathbf{Z}_p[G]/(1+c), \qquad M_- = M/(1+c)M$$

for any $\mathbf{Z}[G]$-module M.

REFINEMENTS: CONJECTURES OF KURIHARA, BURNS, AND SANO

Theorem. For odd primes p, we have

$$\Theta \in \mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(\mathrm{Cl}^T(H)_-^\vee).$$

Theorem. For odd primes p, we have

$$\boxed{\mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_S^T(H)_-) = (\Theta)}$$

RITTER-WEISS MODULE

$\nabla_S^T(H) = G$-module made from Class Field Theory.

There is a canonical map

$$\nabla_S^T(H)_- \twoheadrightarrow \mathrm{Cl}^T(H)_-^\vee$$

Therefore the Kurihara-Burns-Sano conjecture implies the

Brumer-Stark conjecture.

$$\begin{aligned}\mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_S^T(H)_-) = (\Theta) &\Longrightarrow \Theta \in \mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(\mathrm{Cl}^T(H)_-^\vee) \\ &\Longrightarrow \Theta \in \mathrm{Ann}_{\mathbf{Z}_p[G]_-}(\mathrm{Cl}^T(H)_-^\vee) \\ &\Longrightarrow \Theta \in \mathrm{Ann}_{\mathbf{Z}_p[G]_-}(\mathrm{Cl}^T(H)_-)\,.\end{aligned}$$

COHOMOLOGICAL INTERPRETATION

Theorem. For a $\mathbf{Z}[1/2][G]_{-}$-module N, a surjection

$$\nabla_S^T(H)_{-} \twoheadrightarrow N$$

is equivalent to a surjective cohomology class $\kappa \in H^1(G_F, N)$ satisfying certain local conditions.

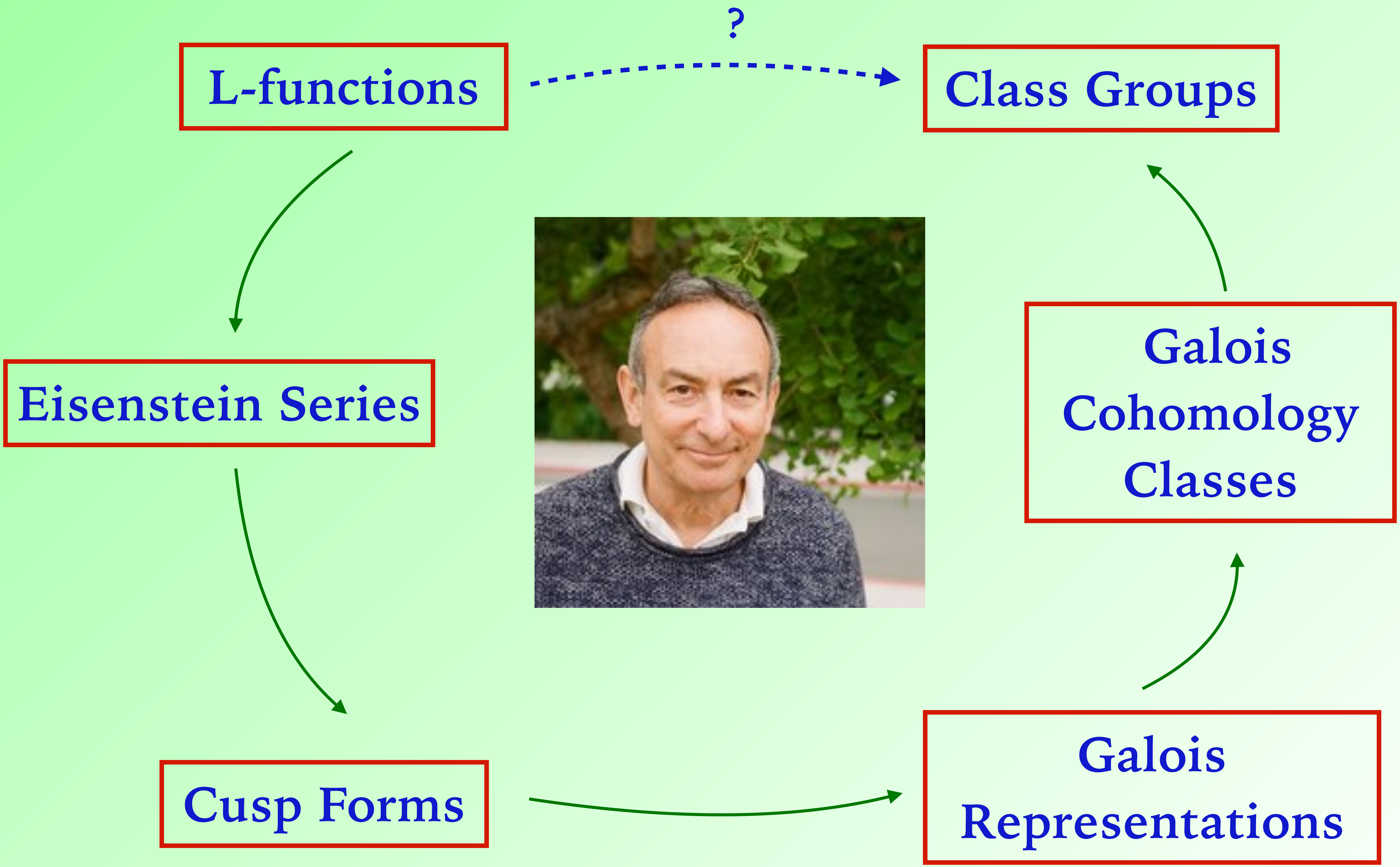
INCLUSION IMPLIES EQUALITY

Corollary. To prove $\mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_S^T(H)_-) \subset (\Theta)$, it suffices to construct a $\mathbf{Z}_p[G]_-$-module N, and a surjective cohomology class $\kappa \in H^1(G_F, N)$ satisfying the local conditions such that $\mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(N) \subset (\Theta)$.

Theorem. The inclusion above implies an equality:

$$\mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(\nabla_S^T(H)_-) = (\Theta).$$

RIBET'S METHOD



HILBERT MODULAR FORMS

$F =$ totally real field of degree n. If $\#\mathrm{Cl}^+(F) = 1$, a Hilbert modular form for F is a holomorphic function

$$f\colon \mathscr{H}^n \longrightarrow \mathbf{C}$$

such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathscr{O}_F)$,

we have $f|_\gamma = f$, where

$$f|_\gamma(z_1, \ldots, z_n) = \prod_{i=1}^{n} (c_i z_i + d_i)^{-k} f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right)$$

FORMS WITH LEVEL

More generally, we can let $\mathfrak{n} \subset \mathcal{O}_F$ be an ideal and define

$$\Gamma_0(\mathfrak{n}) = \{\gamma \in \mathrm{GL}_2^+(\mathcal{O}_F) \colon c \in \mathfrak{n}\}.$$

For

$$\chi \colon (\mathcal{O}_F/\mathfrak{n})^* \longrightarrow \mathbf{C}^*,$$

we define the forms $M_k(\mathfrak{n}, \chi)$ of nebentypus χ to be the holomorphic $f \colon \mathcal{H}^n \longrightarrow \mathbf{C}$ such that

$$f|_\gamma = \chi(d)f$$

for $\gamma \in \Gamma_0(\mathfrak{n})$.

FOURIER EXPANSION

In the general case $h = \#\mathrm{Cl}^+(F) \geq 1$, a Hilbert modular form will be an h-tuple of holomorphic functions $\mathscr{H}^n \to \mathbf{C}$, each with a modularity property with respect to a certain congruence subgroup. A more natural definition is as a function on a certain adelic space.

A Hilbert modular form is f described by its Fourier coefficients,

$$c(\mathfrak{m}, f),\ \mathfrak{m} \subset \mathcal{O}_F \text{ nonzero}, \qquad c_\lambda(0,f),\ \lambda \in \mathrm{Cl}^+(F)\,.$$

HILBERT MODULAR FORMS OVER A RING

$$M_k(\mathbf{Z}) = \{f \in M_k \colon c(\mathfrak{m}, f), c_\lambda(0, f) \in \mathbf{Z} \text{ for all } \mathfrak{m}, \lambda\}.$$

$$M_k(R) = M_k(\mathbf{Z}) \otimes R.$$

This makes sense, as for $R \subset \mathbf{C}$, we have

$$M_k(R) = \{f \in M_k \colon c(\mathfrak{m}, f), c_\lambda(0, f) \in R \text{ for all } \mathfrak{m}, \lambda\}.$$

GROUP RING VALUED MODULAR FORMS

$$M_k(G) = \{f \in M_k(\mathbf{Z}_p[G]\colon \chi(f) \text{ has nebentypus } \chi \text{ for all characters } \chi \text{ of } G\}.$$

Example: Eisenstein Series.

$E_1(G) \in M_1(G)$ defined by

$$c(\mathfrak{m}, E_1(G)) = \sum_{\mathfrak{a} \supset \mathfrak{m}, (\mathfrak{a}, S)=1} \sigma_{\mathfrak{a}}, \quad c_\lambda(0, E_1(G)) = \frac{1}{2^n}\Theta$$

This must be modified in level 1.

GROUP RING CUSP FORM

$$f = E_1(G)V_k - \frac{\Theta}{2^n}H_{k+1}(G)$$

is cuspidal at infinity, where V_k and $H_{k+1}(G)$ have constant term 1.

Choose $V_k \equiv 1 \pmod{p^N}$, where $\Theta \mid p^N$ away from trivial zeroes.

$$\boxed{f \equiv E_1(G) \pmod{\Theta}.}$$

The existence of V_k and $H_{k+1}(G)$ are non-trivial theorems of Jesse Silliman, generalizing results of Hida and Chai.

This can be modified to yield a cusp form f satisfying $\boxed{f \equiv E.}$

HOMOMORPHISM FROM HECKE ALGEBRA

The congruence $f \equiv E_1(G) \pmod{\Theta}$ yields a homomorphism

$$\varphi\colon \mathbf{T} \longrightarrow \mathbf{Z}_p[G]_-/(\Theta), \quad T_\ell \mapsto 1 + \chi(\ell)$$

where $\chi\colon G_F \to G$ is the canonical character.

GALOIS REPRESENTATION: DELIGNE, CARAYOL, HIDA, WILES

Theorem. There exists a continuous irreducible representation

$$\rho\colon G_F \longrightarrow \mathrm{GL}_2(K), \quad K = \mathrm{Frac}(\mathbf{T}),$$

such that:

- ρ is unramified outside $p\mathfrak{n}$, where $\mathfrak{n} = \mathrm{cond}(\mathrm{H/F})$ is the level.
- For prime ideals $\ell \nmid p$, $\mathrm{char}(\rho(\mathrm{Frob}_\ell)) = x^2 - T_\ell x + \mathrm{N}\ell^{k-1}$.
- $\rho|_{G_p} \cong \begin{pmatrix} \eta^{-1}\epsilon^{k-1} & * \\ 0 & \eta \end{pmatrix}$.

USING THE EISENSTEIN CONGRUENCE

Recall

$$\varphi\colon \mathbf{T} \longrightarrow \mathbf{Z}_p[G]_-/(\Theta), \quad T_\ell \mapsto 1 + \chi(\ell).$$

If $I = \ker(\varphi)$, we have

$$\mathrm{char}(\rho(\mathrm{Frob}_\ell)) \equiv (x-1)(x-\chi(\ell)) \pmod{I}.$$

RIBET'S LEMMA

Theorem. $\mathbf{T}$ = reduced complete local Noetherian ring, $I \subset \mathbf{T}$ ideal. Suppose given a compact group G and a continuous representation

$$\rho\colon G \longrightarrow \mathrm{GL}_2(K), \quad K = \mathrm{Frac}(\mathbf{T}),$$

such that

$$\mathrm{char}(\rho(g)) \equiv (x - \chi_1(g))(x - \chi_2(g)) \pmod{I}.$$

for characters $\chi_1, \chi_2\colon G \longrightarrow \mathbf{T}^*$ with $\chi_1 \not\equiv \chi_2 \pmod{\mathfrak{m}}$. Suppose that for each projection $K \to k$ onto a field, the projection of ρ is irreducible. Then there exists a fractional ideal $B \subset K$ and a surjective cohomology class $\kappa \in H^1(G, B/IB(\chi_2\chi_1^{-1}))$.

COHOMOLOGY CLASS κ

Ribet's Lemma therefore yields a fractional ideal $B \subset K = \mathrm{Frac}(\mathbf{T})$ and a surjective cohomology class

$$\kappa \in H^1(G_F, B/IB(\chi^{-1}))\,.$$

We have $\mathrm{Fitt}_{\mathbf{T}/I}(B/IB) = 0$, whence $\mathrm{Fitt}_{\mathbf{Z}_p[G]_-/\Theta}(B/IB) = 0$, so

$$\mathrm{Fitt}_{\mathbf{Z}_p[G]_-}(B/IB) \subset (\Theta).$$