How close are shell models to the 3D Navier–Stokes equations?

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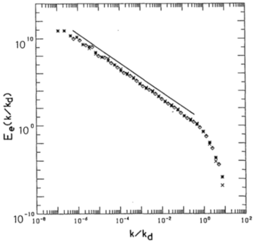
Shell models of turbulence

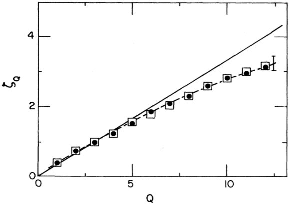
Sabra shell model (L'vov, Podivilov, Pomyalov, Procaccia, *Phys. Rev.* E, 1998)

$$\dot{u}_j = \mathrm{i}(ak_{j+1}u^*_{j+1}u_{j+2} + bk_j u_{j+1}u^*_{j-1} - ck_{j-1}u_{j-1}u_{j-2}) - \nu k_j^2 u_j + f_j$$

$$k_j = k_0\lambda^j,\ \ \lambda > 1,\ \ j = 1, 2, \ldots$$



Yamada, Ohkitani, *J. Phys. Soc. Japan* (1987)



$$\langle |u_j|^Q \rangle \sim k_j^{-\zeta_Q}$$

Jensen, Paladin, Vulpiani, *Phys. Rev.* A (1991)

Frisch, *Turbulence: The legacy of AN Kolmogorov* (1995)

Bohr, Jensen, Paladin, Vulpiani, *Dynamical systems approach to turbulence* (1998)

Biferale, *Annu. Rev. Fluid Mech.* (2003)

Verma, *Energy transfers in fluid flows* (2019)

Mathematical analysis of shell models

Constantin, Levant, and Titi *Physica* D (2006); *J. Stat. Phys,* (2007)

- Global regularity for weak and strong solutions
- Exponential decay of $|u_j|$ with k_j
- Finite-dimensional global attractor and inertial manifold

Stochastically forced
Barbato, Barsanti, Bessaih, Flandoli *J. Stat. Phys,* (2006)

Inviscid
Constantin, Levant, and Titi *Phys. Rev.* E (2007)
Barbato, Flandoli, Morandin *Proc. Am. Math. Soc.* (2011), *Trans. Am. Math. Soc.* (2011), *Ann. Appl. Probab.* (2011)

A comparison between shell models and the 3D NSEs

How the powers of the velocity derivatives depend on *Re*?

3D Navier–Stokes equations

Gibbon *J. Nonlin. Sci.* (2019)

$$\left\langle \|\nabla^n \mathbf{u}\|_{2m}^{\alpha_{n,m}} \right\rangle_T \leqslant c_{n,m}\, L^{-1} \nu^{\alpha_{n,m}} Re^3$$

$$V = [0, L]^3,\ \alpha_{n,m} = 2m/[2m(n+1) - 3]$$

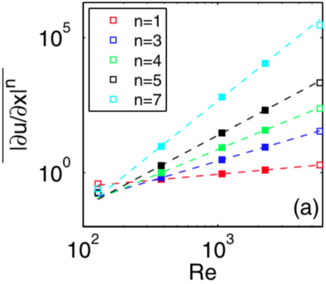
$$1 \leqslant n, 1 \leqslant m \leqslant \infty \quad \text{or} \quad n = 0, 3 \leqslant m \leqslant \infty$$

Shell models

?

Velocity gradients in turbulence

$$\langle |\nabla \mathbf{u}|^p \rangle \sim Re^{\chi_p}$$



Schumacher et al. *NJP* (2007)

Multifractal model:

Nelkin *Phys. Rev.* A (1990)
Benzi, Biferale, Paladin, Vulpiani, Vergassola *Phys. Rev. Lett.* (1991)

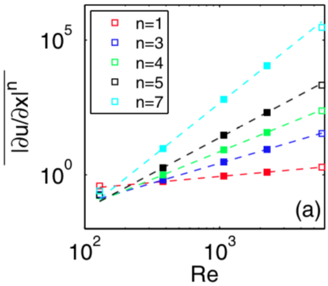
DNS: Schumacher, Sreenivasan, Yakhot *New J. Phys.* (2007); *PNAS* (2014)

Burgers equation: Chakraborty, Frisch, Pauls, Ray *Phys. Rev.* E (2012)

Velocity gradients in turbulence

$$\langle |\nabla \mathbf{u}|^p \rangle \sim Re^{\chi_p}$$

$$\left\langle \left\langle |\nabla^n \mathbf{u}|^p \right\rangle_V^{\alpha_{n,p}} \right\rangle_T$$



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Energy $E(t) = \sum_{j=1}^{\infty} |u_j(t)|^2$

Energy conservation: $a + b + c = 0$

Initial energy: $E(0) < \infty$

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Forcing $f_j = F\phi_j$ with $\phi_j = 0$ if $j < j_f$ and $j > j_{\max}$

Characteristic and maximum wave numbers: $k_f = k_0\lambda^{j_f}$ and $k_{\max} = k_0\lambda^{j_{\max}}$

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r.m.s. velocity $U = \sqrt{\langle E \rangle_T}$ with $\langle \cdot \rangle_T = \frac{1}{T} \int_0^T \cdot \, \mathrm{d}t$

$$Re = \frac{U}{\nu k_f}$$

$$Gr = \frac{|F|}{\nu^2 k_f^3}$$

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$$Re = \frac{U}{\nu k_f} \qquad Gr = \frac{|F|}{\nu^2 k_f^3}$$

Numerics $a = 1$, $b = c = -1/2$, $k_0 = 2^{-4}$, $\lambda = 2$

$f_j = F\delta_{j,1}$, $F = 5\times 10^{-3}(1+\mathrm{i})$, $10^{-7} \leqslant \nu \leqslant 6\times 10^2$, $8 \leqslant N \leqslant 27$

slaved Adams–Bashforth [Pisarenko, Biferale, Courvoisier, Frisch, Vergassola *Phys. Fluids* (1993)]

Strategy

- Adapted from the study of the 3D Navier–Stokes equations

  Doering & Gibbon *Applied Analysis of the Navier–Stokes equations*, 1995

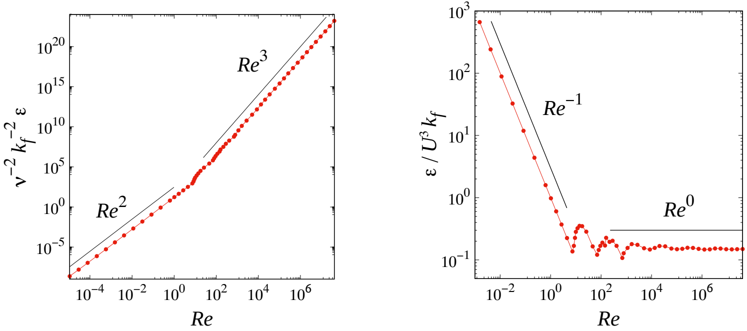
- Use the estimates for the lower-order derivatives to obtain estimates for the higher-order ones

Energy dissipation rate

$$\epsilon = \nu \left\langle \sum_{j=1}^{\infty} k_j^2 |u_j|^2 \right\rangle_T$$

$$\epsilon \leqslant \nu^3 k_f^4 \left(c_1 Re^2 + c_2 Re^3 \right)$$

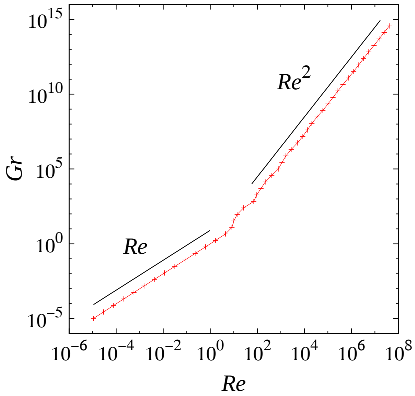
The constants c_1, c_2 depend on a, b, c, λ and the 'shape' of the forcing, but are uniform in ν, k_0, k_f, $k_{\max}$, $|F|$.



3D Navier–Stokes equations → Doering and Foias *J. Fluid Mech.* (2005)

Grashof vs Reynolds relation

$$Gr \leqslant c_1 Re + c_2 Re^2$$



3D Navier–Stokes equations → Doering and Foias *J. Fluid Mech.* (2005)

Higher-order derivatives

$$H_n = \sum_{j=0}^{\infty} k_j^{2n} |u_j|^2, \quad n \geqslant 1$$

$$H_0 = E$$
$$\langle H_1 \rangle_T = \epsilon/\nu$$

Ladder inequalities

$$\frac{1}{2}\frac{\mathrm{d}H_n}{\mathrm{d}t} \leqslant -\nu H_{n+1} + c_n H_n \sup_{j\geq 1}(k_j|u_j|) + H_n^{1/2}\Phi_n^{1/2} \qquad (n \geqslant 0)$$

where $\Phi_n = \sum_{j=0}^{\infty} k_j^{2n}|f_j|^2$

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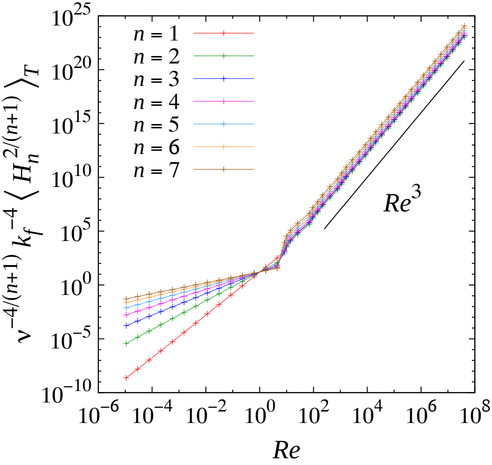
3D Navier–Stokes equations → Bartuccelli, Doering, Gibbon *Nonlinearity* (1991)

$$H_n = \|\nabla^n \mathbf{u}\|_2^2, \qquad \tfrac{1}{2}\dot{H}_n \leqslant -\nu H_{n+1} + c_n H_n \|\nabla \mathbf{u}\|_\infty + H_n^{1/2} \Phi_n^{1/2}, \qquad n \geqslant 1$$

Higher-order velocity derivatives



$$H_n = \sum_{j=1}^{\infty} k_j^{2n} |u_j|^2$$

For $n \geqslant 1$ and $Re \gg 1$

$$\left\langle H_n^{\frac{2}{n+1}} \right\rangle_T \leqslant c_n \, \nu^{\frac{4}{n+1}} k_f^4 \, Re^3$$

Higher-order velocity derivatives

Shell model

$$H_n = \sum_{j=1}^{\infty} k_j^{2n} |u_j|^2$$

$$\left\langle H_n^{\frac{2}{n+1}} \right\rangle_T \leqslant c_n \, \nu^{\frac{4}{n+1}} \, k_f^4 \, Re^3$$

3D Navier–Stokes

Foias, Guillopé, Temam
Comm. Part. Differ. Equat. (1981)

$$H_n = \|\nabla^n \mathbf{u}\|_2^2 = \int_V |\nabla^n \mathbf{u}|^2 \mathrm{d}\mathbf{x}$$

$$\left\langle H_n^{\frac{2}{2n-1}} \right\rangle_T \leqslant c_n \, L^{-1} \nu^{\frac{2}{2n-1}} \, Re^3$$

In the shell model, the higher-order derivatives of the velocity display a weaker dependence on *Re* compared to the 3D Navier–Stokes equations

Absorbing ball for H_n

Alternative ladder inequality

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}H_n}{\mathrm{d}t} &\leqslant -\frac{\nu}{2}H_{n+1} + \frac{d_n}{\nu}H_n \sup_j |u_j|^2 + H_n^{\frac{1}{2}}\Phi_n^{\frac{1}{2}} \\ &\leqslant -2H_n\left[\frac{\nu}{2}\frac{H_n^{\frac{1}{n}}}{H_0^{\frac{1}{n}}} - \frac{d_n}{\nu}H_0 - \frac{\Phi_n^{\frac{1}{2}}}{H_n^{\frac{1}{2}}}\right]\end{aligned}$$

By using $\Phi_n = C_n \nu^4 k_f^{2n+6} Gr^2$ and

$$\overline{\lim_{t\to\infty}}\, H_0 \leqslant \nu^2 \left(\frac{k_f}{k_1}\right)^4 k_f^2\, Gr^2$$ (Levant, Constantin, Titi, *Physica* D, 2006)

$$\overline{\lim_{t\to\infty}}\, H_n \leqslant \nu^2 k_f^{2(n+1)} \left[c_n\, Gr^{2(n+1)} + d_n\, Gr^2\right]$$

Higher-order moments

Shell model

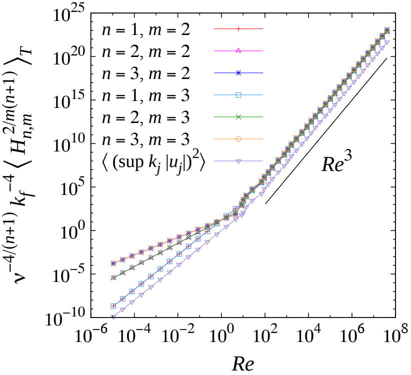
$$H_{n,m} = \sum_{j=1}^{\infty} (k_j^n |u_j)^{2m}, \quad H_{n,1} = H_n$$

$$H_{n,m} \leqslant H_n^m$$

For $n, m \geqslant 1$ and $Re \gg 1$

$$\left\langle H_{n,m}^{\beta_{n,m}^{\mathrm{sh}}} \right\rangle_T \leqslant c_n \, \nu^{\frac{4}{n+1}} \, k_f^4 \, Re^3$$

$$\beta_{n,m}^{\mathrm{sh}} = \frac{2}{m(n+1)}$$



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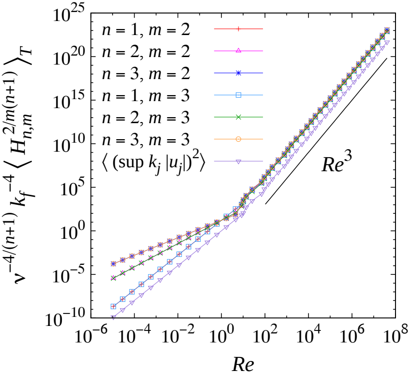

$$\left\langle \left(\sup_j k_j^n |u_j| \right)^{\frac{4}{n+1}} \right\rangle \leqslant \hat{c}_n \, \nu^{\frac{4}{(n+1)}} k_f^4 \, Re^3$$

Higher-order moments

Shell model

$$H_{n,m} = \sum_{j=1}^{\infty} k_j^{2nm} |u_j|^{2m}$$

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3D Navier–Stokes

$$H_{n,m} = \|\nabla^n \mathbf{u}\|_{2m}^{2m} = \int_V |\nabla^n \mathbf{u}|^{2m} \mathrm{d}\mathbf{x}$$

Gibbon *J. Nonlin. Sci.* (2019)

For $1 \leqslant n$, $1 \leqslant m \leqslant \infty$ and $Re \gg 1$

$$\left\langle H_{n,m}^{\beta_{n,m}^{\text{ns}}} \right\rangle_T \leqslant c_n \, L^{-1} \, \nu^{2m\beta_{n,m}^{\text{ns}}} Re^3$$

$$\beta_{n,m}^{\text{ns}} = \frac{1}{2m(n+1) - 3}$$

$\beta_{n,m}^{\text{sh}} > \beta_{n,m}^{\text{ns}}$ for all $n > 1$

Velocity gradient averaged NSEs

Shell model

$$\tfrac{1}{2}\dot{H}_n \leqslant -\nu H_{n+1} + c_n H_n \sup_j (k_j|u_j|) + H_n^{1/2}\Phi_n^{1/2}$$

3D Navier–Stokes

$$\tfrac{1}{2}\dot{H}_n \leqslant -\nu H_{n+1} + c_n H_n \|\nabla \mathbf{u}\|_\infty + H_n^{1/2}\Phi_n^{1/2}$$

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$$\sup_{j\geqslant 1}(k_j|u_j|) \leqslant \left[\sum_{j=1}^{\infty}(k_j|u_j|)^2\right]^{\frac{1}{2}} = H_1^{\frac{1}{2}}$$

3D Navier–Stokes

$$\frac{1}{2}\dot{H}_n \leqslant -\nu H_{n+1} + c_n H_n \|\nabla\mathbf{u}\|_\infty + H_n^{1/2}\Phi_n^{1/2}$$

$$\|\nabla\mathbf{u}\|_\infty \leqslant c_n H_{n+1}^{\frac{3}{4n}} H_1^{\frac{2n-3}{4n}}$$

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$$\|\nabla \mathbf{u}\|_\infty \leqslant c_n H_{n+1}^{\frac{3}{4n}} H_1^{\frac{2n-3}{4n}}$$

Velocity gradient averaged Navier–Stokes equations

Assumption: $\|\nabla \mathbf{u}\|_\infty \approx c\, L^{-3/2}\|\nabla \mathbf{u}\|_2 = c\,\langle|\nabla \mathbf{u}|^2\rangle_V^{1/2}$

$$\tfrac{1}{2}\dot{H}_n \leqslant -\nu H_{n+1} + c_n L^{-3/2} H_n H_1^{1/2} + H_n^{1/2}\Phi_n^{1/2}$$

$\beta_{n,m}^{\mathrm{sh}} = \beta_{n,m}^{\mathrm{vga}}$ for all $n, m \geqslant 1$

Moments of the energy

3D NSEs (Gibbon, *JNLS*, 2019)

$$H_{0,m} = \|\mathbf{u}\|_{2m}^{2m} = \int_V |\mathbf{u}|^{2m} d\mathbf{x} \qquad \left\langle H_{0,m}^{\frac{1}{2m-3}} \right\rangle_T \leqslant c_m L^{-1} \nu^{\frac{2m}{2m-3}} Re^3 \qquad (3 < m \leqslant \infty)$$

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VGA NSEs

$$\|\mathbf{u}\|_{2m} \leqslant c\,L\|\nabla\mathbf{u}\|_{2m} \leqslant c\,L^{\frac{2m+3}{2m}}\|\nabla\mathbf{u}\|_\infty$$

$$\|\nabla\mathbf{u}\|_\infty \approx c\,L^{-3/2}\|\nabla\mathbf{u}\|_2 \implies \|\mathbf{u}\|_{2m} \leqslant c\,L^{-\frac{m-3}{2m}}\|\nabla\mathbf{u}\|_2$$

$$\left\langle H_{0,m}^{\frac{1}{m}} \right\rangle_T \leqslant c_m L^{-\frac{2m-3}{m}} \nu^2 Re^3 \qquad (1 \leqslant m \leqslant \infty)$$

Moments of the energy

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VGA NSEs

$$\|\mathbf{u}\|_{2m} \leqslant c\, L \|\nabla \mathbf{u}\|_{2m} \leqslant c\, L^{\frac{2m+3}{2m}} \|\nabla \mathbf{u}\|_\infty$$

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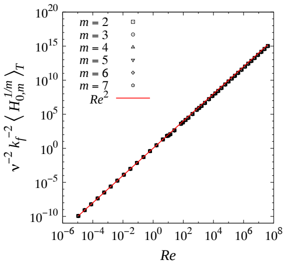
$$\left\langle H_{0,m}^{\frac{1}{m}} \right\rangle_T \leqslant c_m L^{-\frac{2m-3}{m}} \nu^2 Re^3 \qquad (1 \leqslant m \leqslant \infty)$$

Shell model

$$H_{0,m} = \sum_{j=1}^{\infty} |u_j|^{2m} \leqslant H_0^m$$

$$\left\langle H_{0,m}^{\frac{1}{m}} \right\rangle_T \leqslant \nu^2 k_f^2 Re^2$$

$$(1 \leqslant m \leqslant \infty)$$



Equivalent to assuming $\|\mathbf{u}\|_{2m} \approx c_m\, L^{\frac{3(m-1)}{2m}} \|\mathbf{u}\|_2$

Shell models as NSEs "on a point"

Navier–Stokes equations in d dimensions (Gibbon *EPL*, 2020)

$$\left\langle H_{n,m}^{\beta^{\text{ns}}_{n,m,d}} \right\rangle_T \leqslant c_n\, L^{-1}\, \nu^{2m\beta^{\text{ns}}_{n,m,d}} Re^3$$

$$\beta^{\text{ns}}_{n,m,d} = \frac{(4-d)}{2m(n+1)-d}$$

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$$\beta^{\rm ns}_{n,m,d} = \frac{(4-d)}{2m(n+1)-d}$$

Shell model

$$\beta^{\rm sh}_{n,m} = \frac{2}{m(n+1)} = \lim_{d\to 0} \beta^{\rm ns}_{n,m,d}$$

Conclusions

- The estimates of the energy dissipation rate in terms of *Re* are the same in the shell model and in the 3D NSEs
- However, the estimates of the velocity and its higher-order derivatives display a weaker dependence on *Re* than in the 3D NSEs
- The estimates of the velocity derivatives for the shell model are equivalent to those corresponding to a velocity gradient averaged version of the 3D NSEs

D. Vincenzi & J.D. Gibbon, `arXiv:2009.01583`