COSMOLOGICAL BACKGROUNDS of GRAVITATIONAL WAVES

DANIEL G. FIGUEROA

IFIC, Valencia, Spain

PHYSICS OF THE EARLY UNIVERSE, ICTS Bangalore, India (Jan. 3rd - 12th 2022)

PHYSICS OF THE EARLY UNIVERSE (HYBRID)

SCHEDULE

PHYSICS OF THE EARLY UNIVERSE (HYBRID)
3 - 12 January 2022

Note that all dates and times are in Indian Standard Time (IST). IST is behind Japan Standard Time by 3:30 hours, ahead of Central European Winter Time by 4:30 hours, Greenwich Mean time by 5:30 hours and Eastern Standard Time by 10:30 hours.

Introductory remarks by the Organisers on 3rd January at 11:15 hrs

Welcome remarks by Centre Director, Prof. Rajesh Gopakumar on 4th January at 11:30 hrs

	09:30-11:00	11:30-13:00	14:30-16:00	16:30-18:00	18:30-20:00
Monday, January 3		**Introductory remarks (11:15 hrs)** CPT I		Inflation I	CPT T
Tuesday, January 4		**Welcome remarks** CPT II	Bounces I	Inflation II	Inflation T
Wednesday, January 5		CPT III	Bounces II	GW I	Inflation III
Thursday, January 6	PPEU I	CPT IV	Bounces III	GW II	Inflation IV
Friday, January 7	PPEU II	CWD I	PPEU T	GW III	GW T
Monday, January 10	PPEU III	CWD II	DEMG I	DM I	Bounces T
Tuesday, January 11	PPEU IV	CWD III	DEMG II	DM II	DEMG T
Wednesday, January 12	CWD T	CWD IV	DEMG III	DM III	DM T **Closing remarks (20:00 hrs)**

GW

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GW

CPT	- Cosmological perturbation theory - by **David Wands**
Inflation	- The inflationary paradigm - by **William Kinney**
Bounces	- The bouncing scenario - by **Patrick Peter**
PPEU	- Particle physics in the early universe - by **Masahide Yamaguchi**
CWD	- Comparison of models of the early universe with the cosmological data - by **Shiv Sethi**
GWs	- Generation and imprints of primordial gravitational waves - by **Daniel Figueroa**
DM	- Dark matter - by **Katelin Schutz**
DEMG	- Dark energy and modified gravity - by **Alessandra Silvestri**

GWs - Generation and imprints of primordial gravitational waves - by **Daniel Figueroa**

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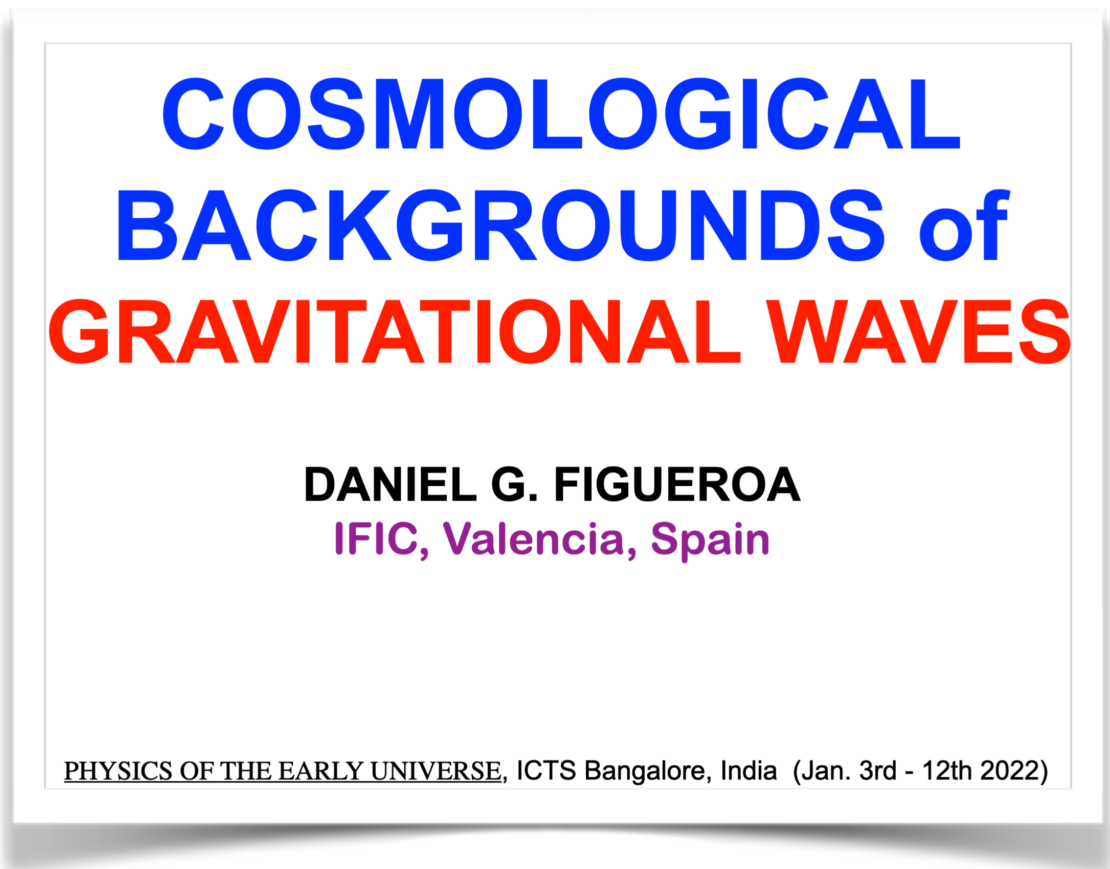
GWs - Generation and imprints of primordial gravitational waves - by **Daniel Figueroa**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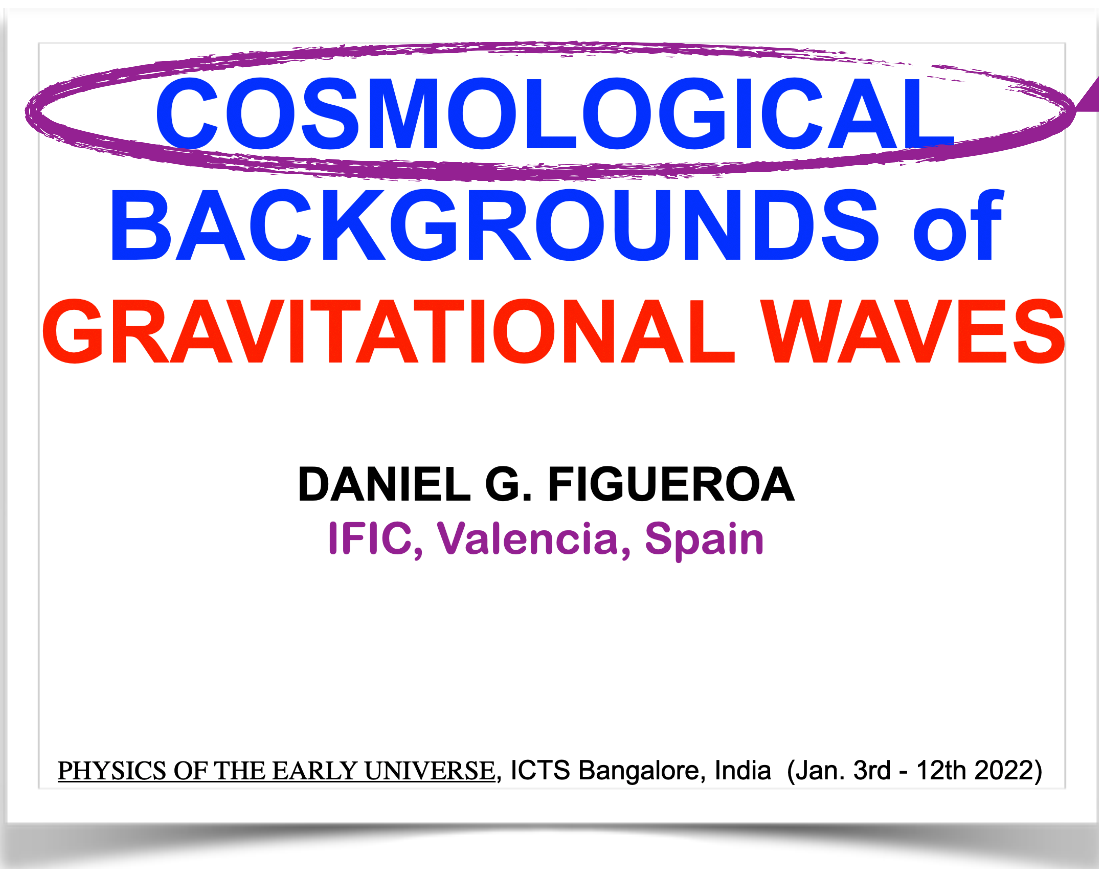

GWs → - Generation and imprints of primordial gravitational waves - by **Daniel Figueroa** ←

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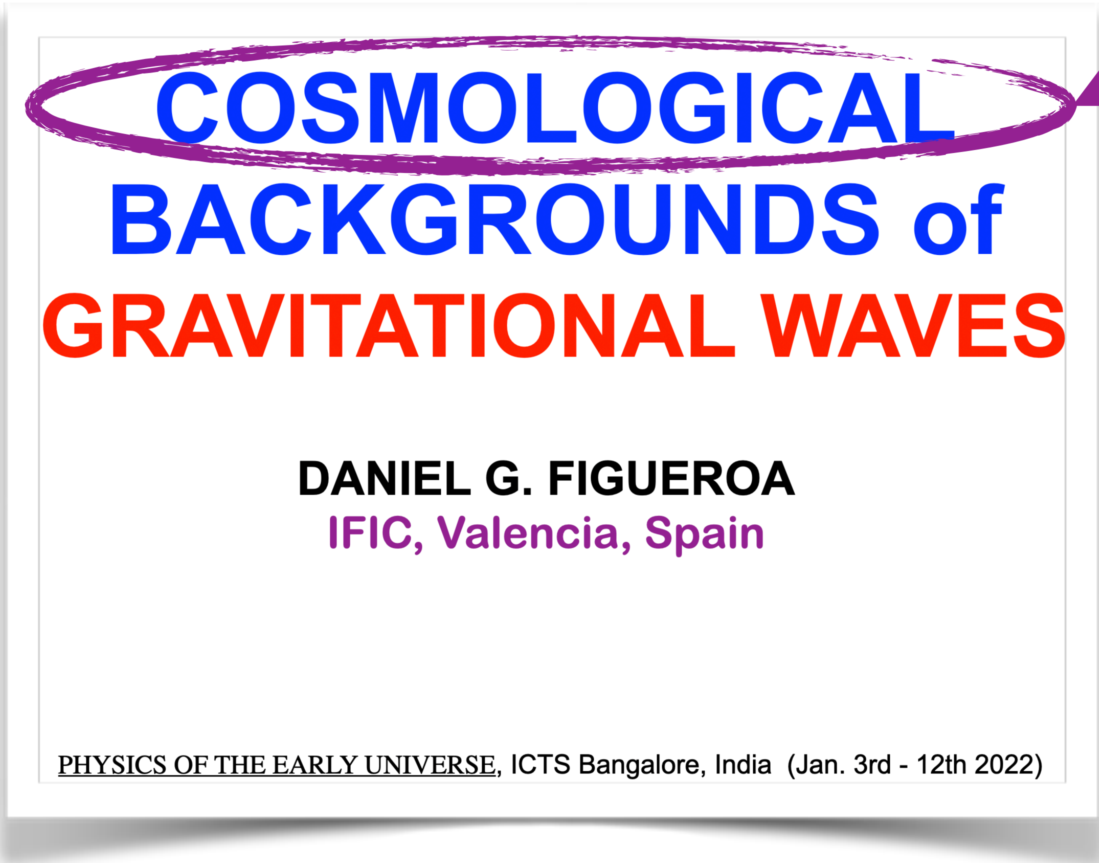
COSMOLOGICAL BACKGROUNDS

$\equiv$

'PRIMORDIAL' BACKGROUNDS

GWs → - Generation and imprints of primordial gravitational waves - by **Daniel Figueroa** ←



?

COSMOLOGICAL BACKGROUNDS

$$\equiv$$

'PRIMORDIAL' BACKGROUNDS

GW = Gravitational Waves

GWs → - Generation and imprints of primordial gravitational waves - by **Daniel Figueroa** ←

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COSMOLOGICAL BACKGROUNDS

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Early Universe

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GWs - Generation and imprints of primordial gravitational waves - by **Daniel Figueroa**

?

COSMOLOGICAL BACKGROUNDS of GRAVITATIONAL WAVES

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COSMOLOGICAL BACKGROUNDS

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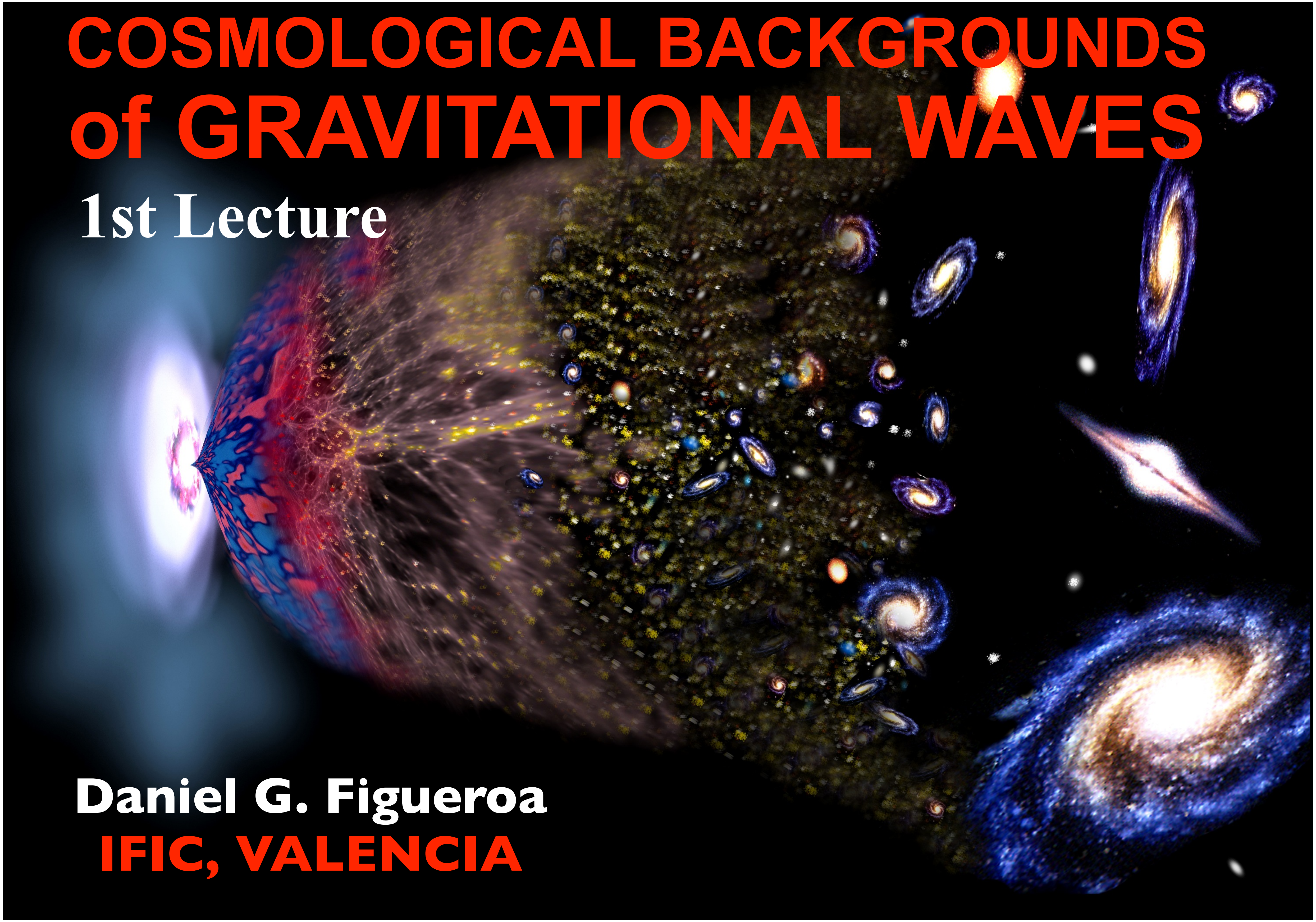
STOCHASTIC

COSMOLOGICAL BACKGROUNDS of GRAVITATIONAL WAVES

1st Lecture



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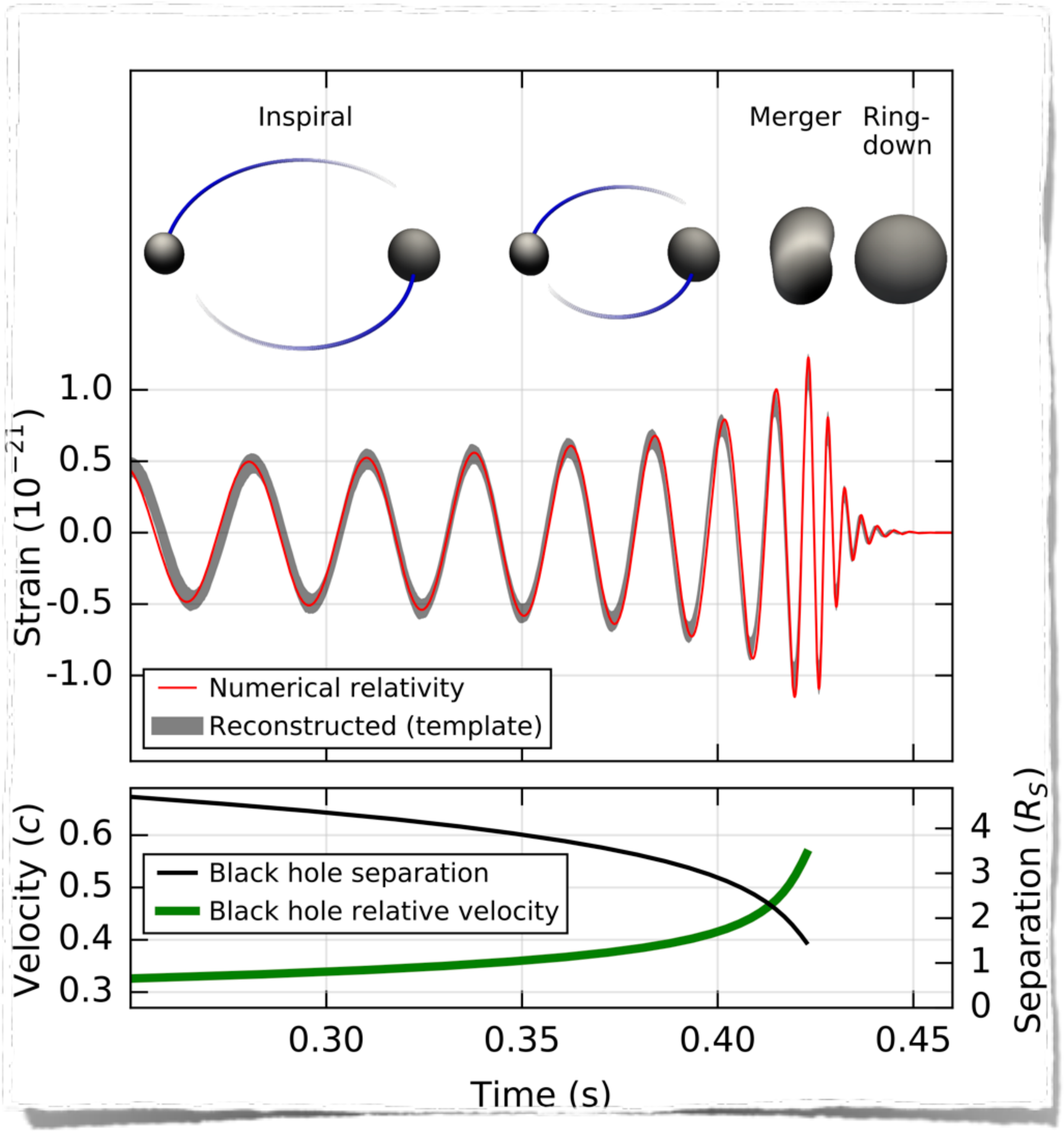
MOTIVATION

(cosmologist biased)

Let us celebrate …

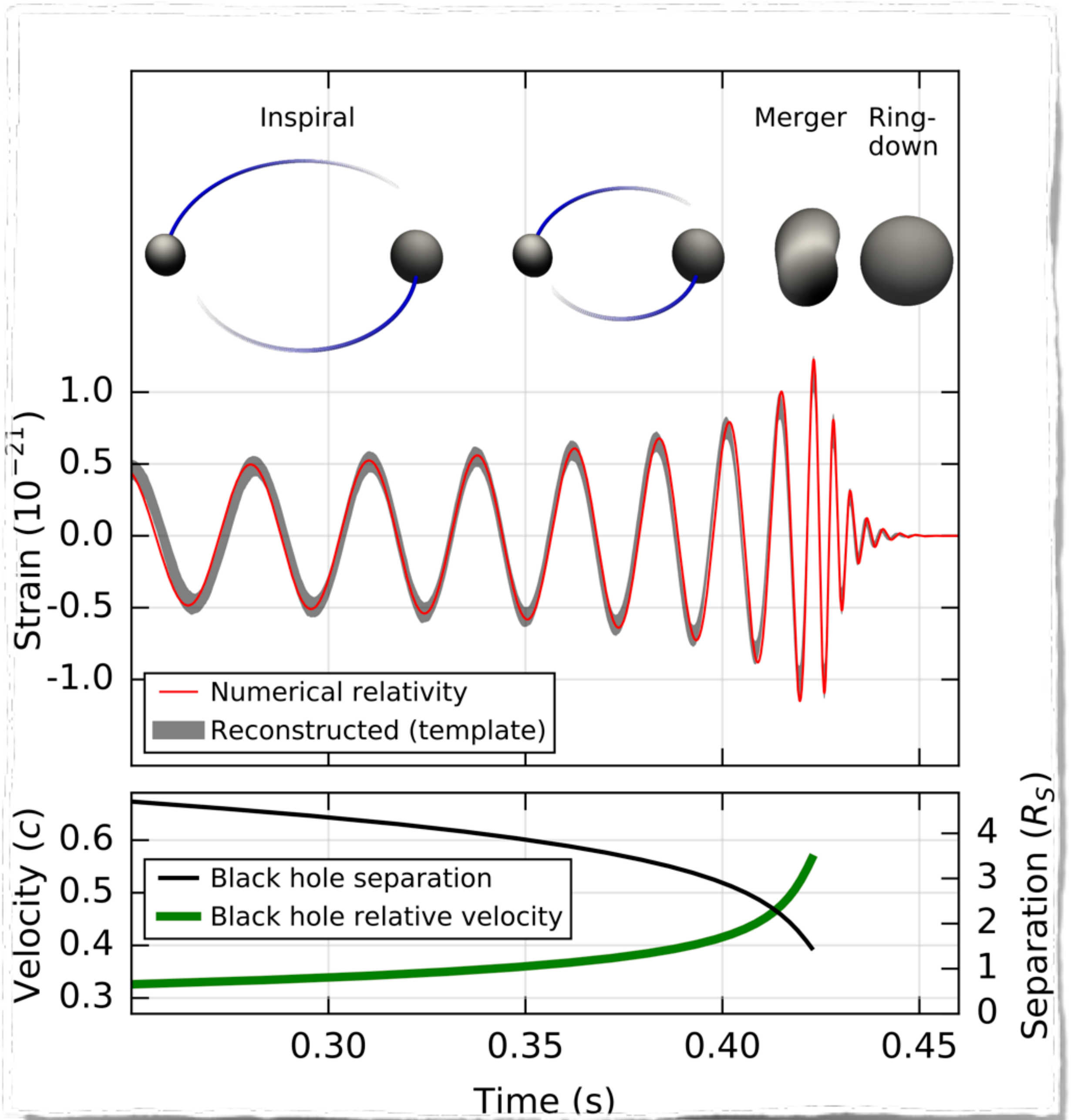


[LIGO & Virgo Scientific Collaborations (arXiv:1602.03841)]

Gravitational Waves (GWs) detected !

[by LIGO/VIRGO]

Let us celebrate …



[LIGO & Virgo Scientific Collaborations (arXiv:1602.03841)]

Einstein 1916 … LIGO/VIRGO 2015-2021

Let us celebrate …

*** O(10) Solar mass Black Holes (BH) exist**

*** We can test the BH's paradigm and Neutron Star physics**

*** We can further test General Relativity (GR)**
[so far no deviation]

*** We can observe the Universe through GWs**

*** …**

Gravit… …al …)

[b… …O/VIRGO]

milestone in physics

Let us celebrate …

* O(10) Solar mass Black Holes (BH) exist

* We can test the BH's paradigm and Neutron Star physics

* We can further test General Relativity (GR)
[so far no deviation]

(binaries) → **Extremely interesting !**

BUT …

… **We will focus on something else !**

* We can observe the Universe through GWs

* …

Let us celebrate …

* O(10) Solar m s
Black Ho st

N (binaries)

* W test
Gen lativity (GR)
[so far no deviation]

* We can observe the
Universe through GWs

* …

Stay tuned !

more fun
guaranteed
to come …

Extremely
interesting !

BUT …

… We will focus
on something else !

Let us celebrate …

* O(10) Solar mass Black Holes (BH) exist

* We can test the BH's paradigm and Neutron Star physics

* We can further test General Relativity (GR) [so far no deviation]

* We can observe the Universe through GWs

* …

(binaries)

Extremely interesting !

BUT …

… We will focus on something else !

* We can observe the Universe through GWs

* We can observe the Universe through GWs

* We can observe the Universe through GWs

* Cosmology with GWs

* We can observe the Universe through GWs

* Cosmology with GWs

* Late Universe:

* Early Universe:

*** We can observe the Universe through GWs**

* Cosmology with GWs

*** Late Universe:** Standard sirens: distances in cosmology; Measuring H0 and EoS dark energy; cosmological parameters; modify gravity, lensing, …

*** We can observe the Universe through GWs**

* Cosmology with GWs

*** Late Universe:**

*** Early Universe:** High Energy Particle Physics

* **We can observe the Universe through GWs**

* Cosmology with GWs

* **Late Universe:**

* **Early Universe:** High Energy Particle Physics

* We can observe the Universe through GWs

* **Cosmology with GWs**

* **Late Universe:**

* **Early Universe:** High Energy Particle Physics

Can we really probe High Energy Physics using Gravitational Waves (GWs) ? How ?

GWs: probe of the early Universe

Why ?

One and ONLY One reason …

GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

   ADVANTAGE: GW DECOUPLE upon Production
   DISADVANTAGE: DIFFICULT DETECTION

GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

   ADVANTAGE: GW DECOUPLE upon Production
   DISADVANTAGE: DIFFICULT DETECTION

2. **ADVANTAGE**: GW $\longrightarrow$ Probe for Early Universe

$$\longrightarrow \begin{cases} \textbf{Decouple} \longrightarrow \text{Spectral Form Retained} \\ \textbf{Specific HEP} \;\; \Leftrightarrow \;\; \text{Specific GW} \end{cases}$$

GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

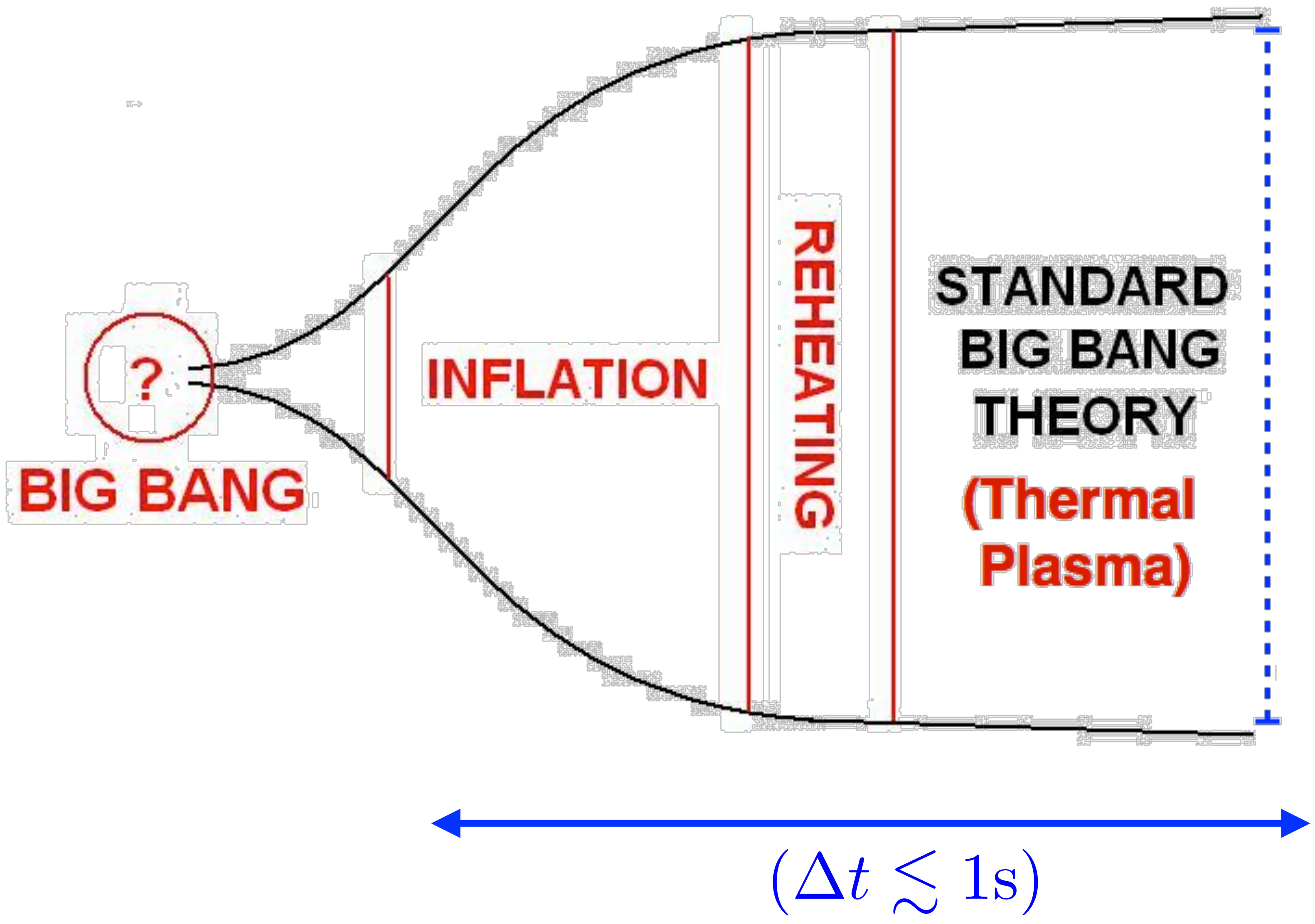
   ADVANTAGE: GW DECOUPLE upon Production
   DISADVANTAGE: DIFFICULT DETECTION

2. **ADVANTAGE**: GW $\longrightarrow$ Probe for $\boxed{\text{Early Universe}}$

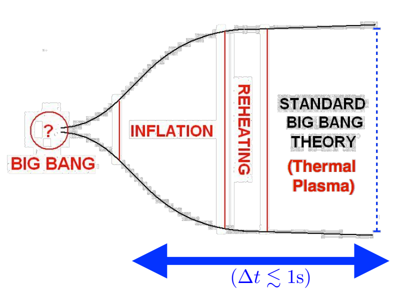
$$\longrightarrow \left\{ \begin{array}{l} \textbf{Decouple} \longrightarrow \text{Spectral Form Retained} \\ \textbf{Specific HEP} \;\; \Leftrightarrow \;\; \text{Specific GW} \end{array} \right.$$

What processes of the early Universe ?

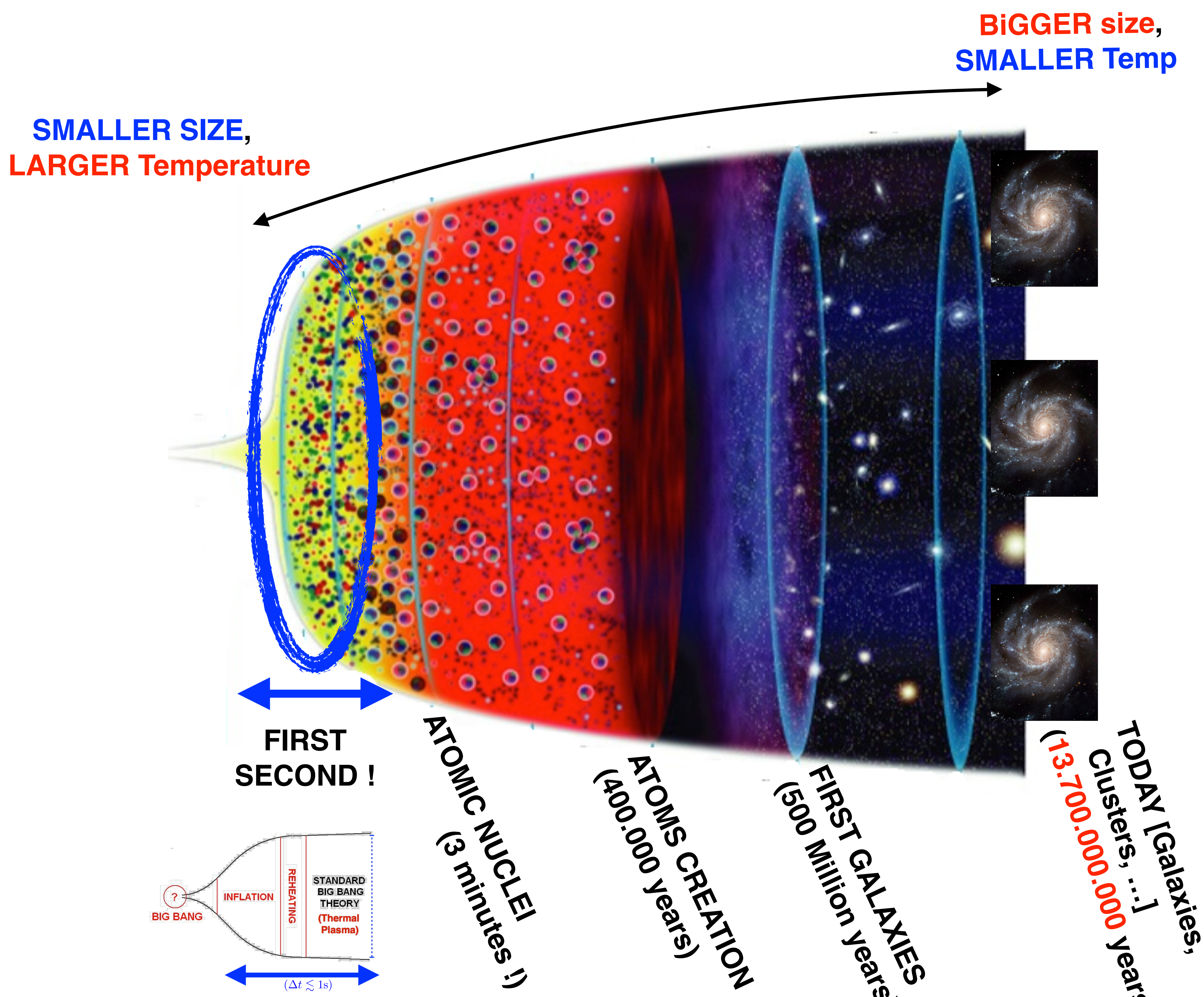
The Early Universe



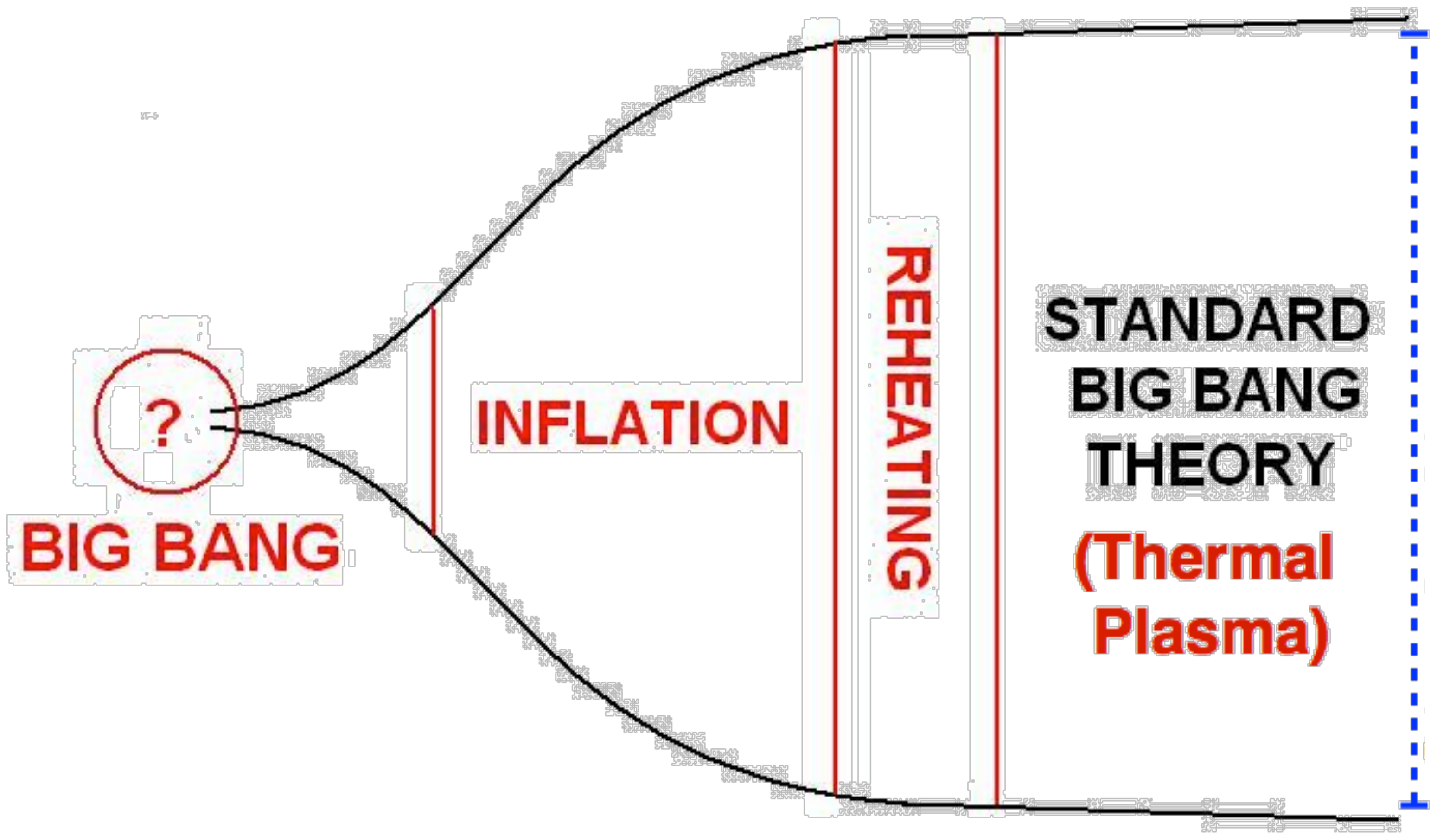
The Early Universe



The Early Universe

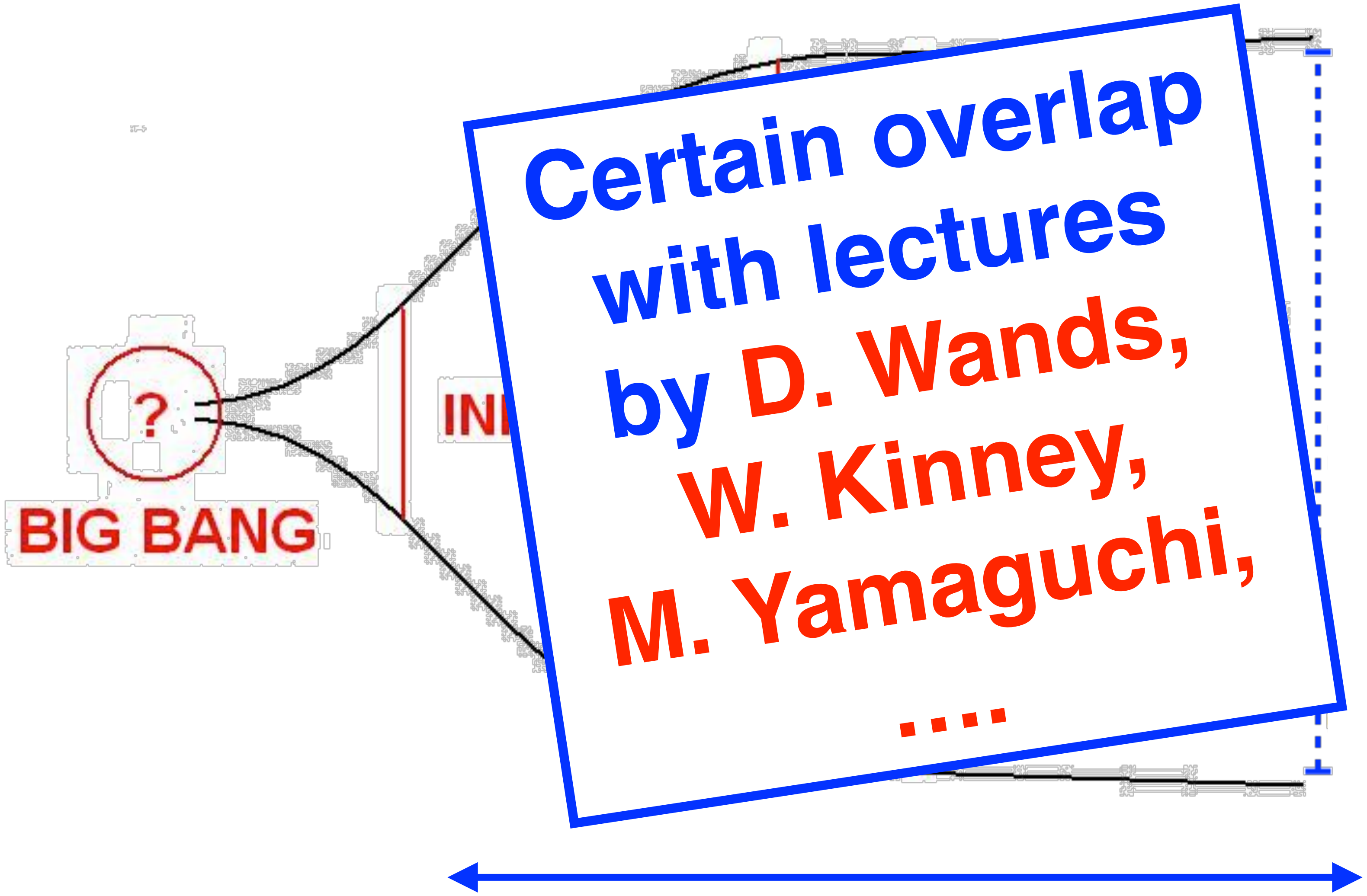


The Early Universe



$(\Delta t \lesssim 1\mathrm{s})$

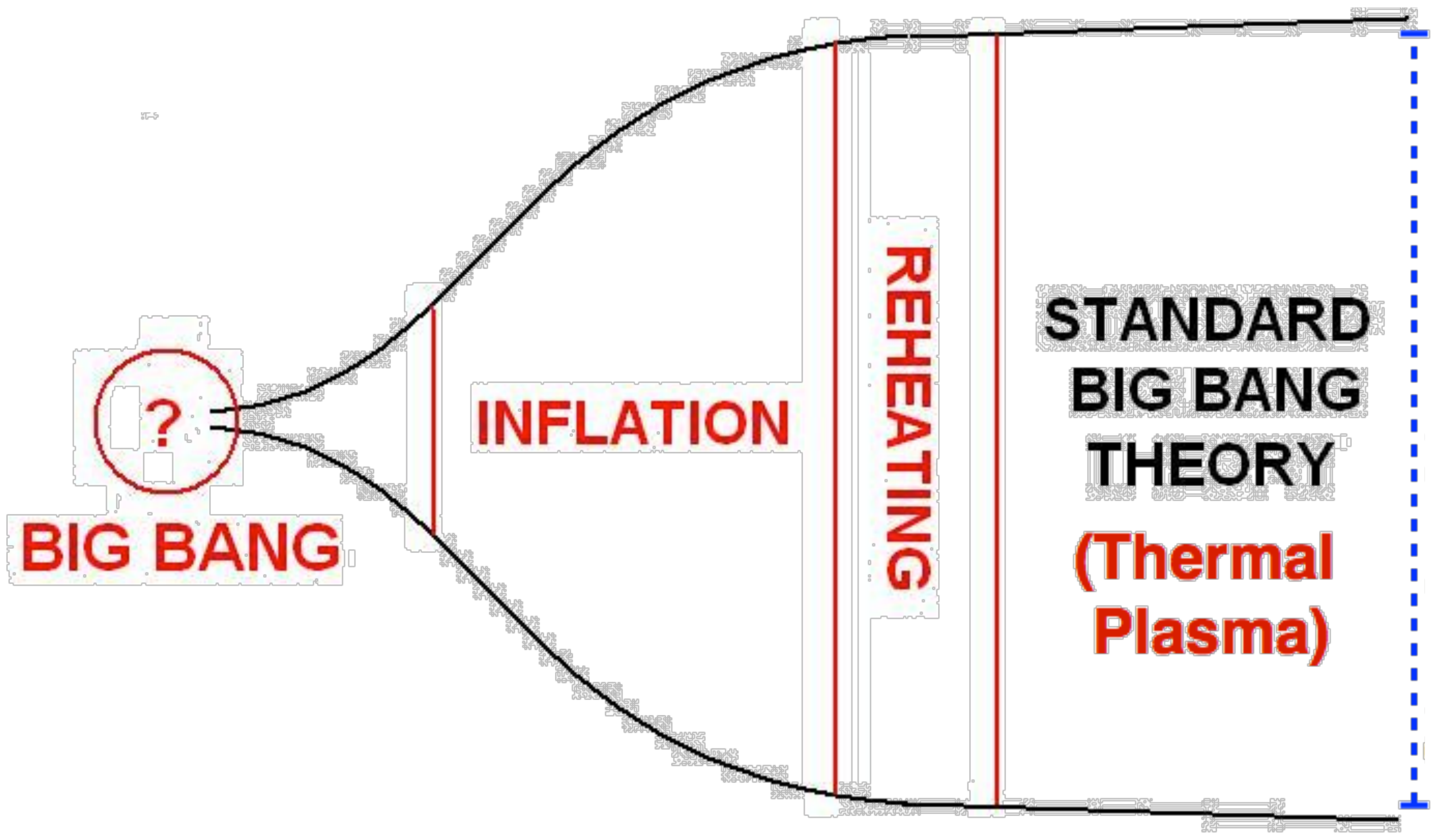
The Early Universe



Certain overlap with lectures by D. Wands, W. Kinney, M. Yamaguchi, ….

$(\Delta t \lesssim 1\mathrm{s})$

The Early Universe



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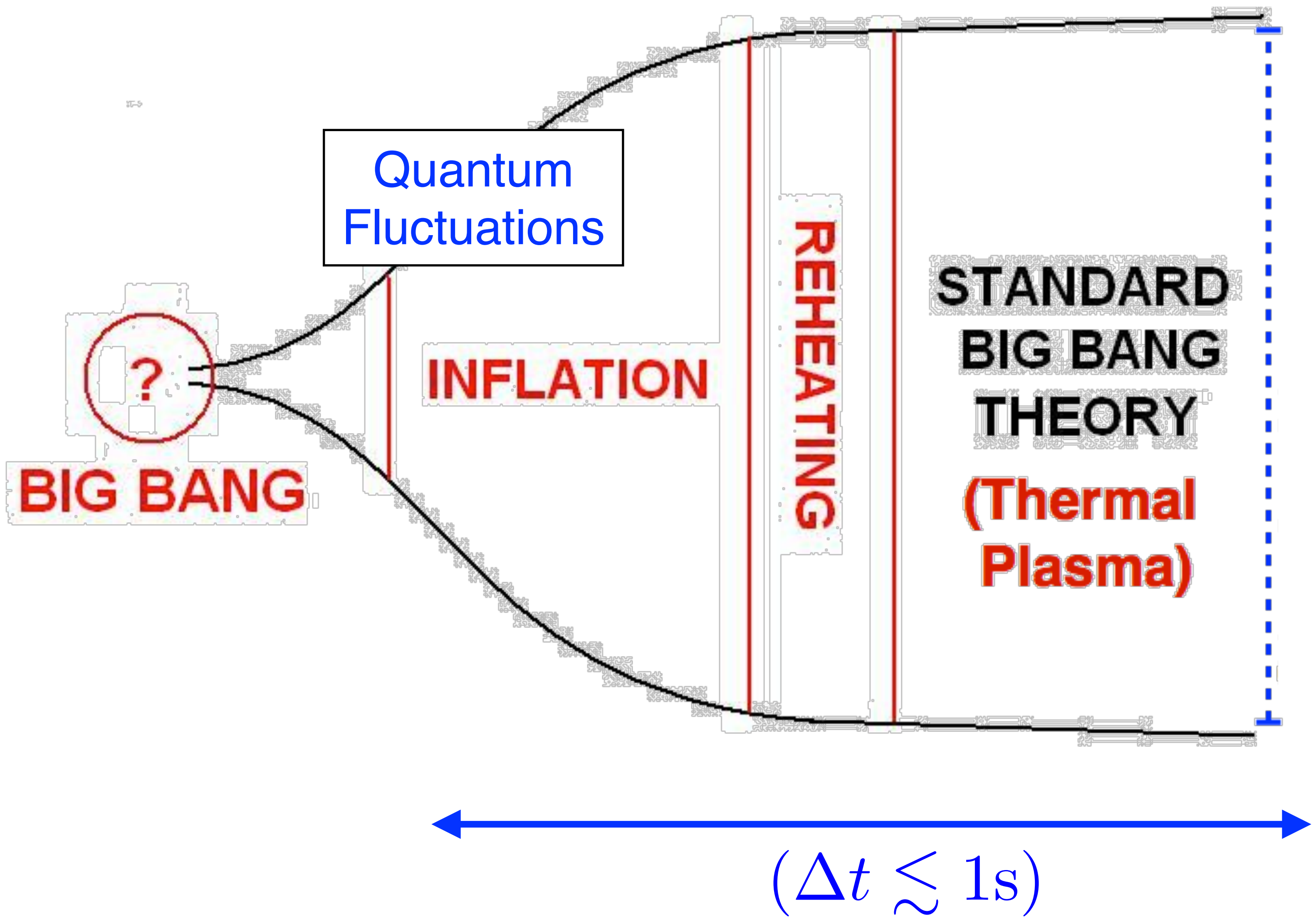
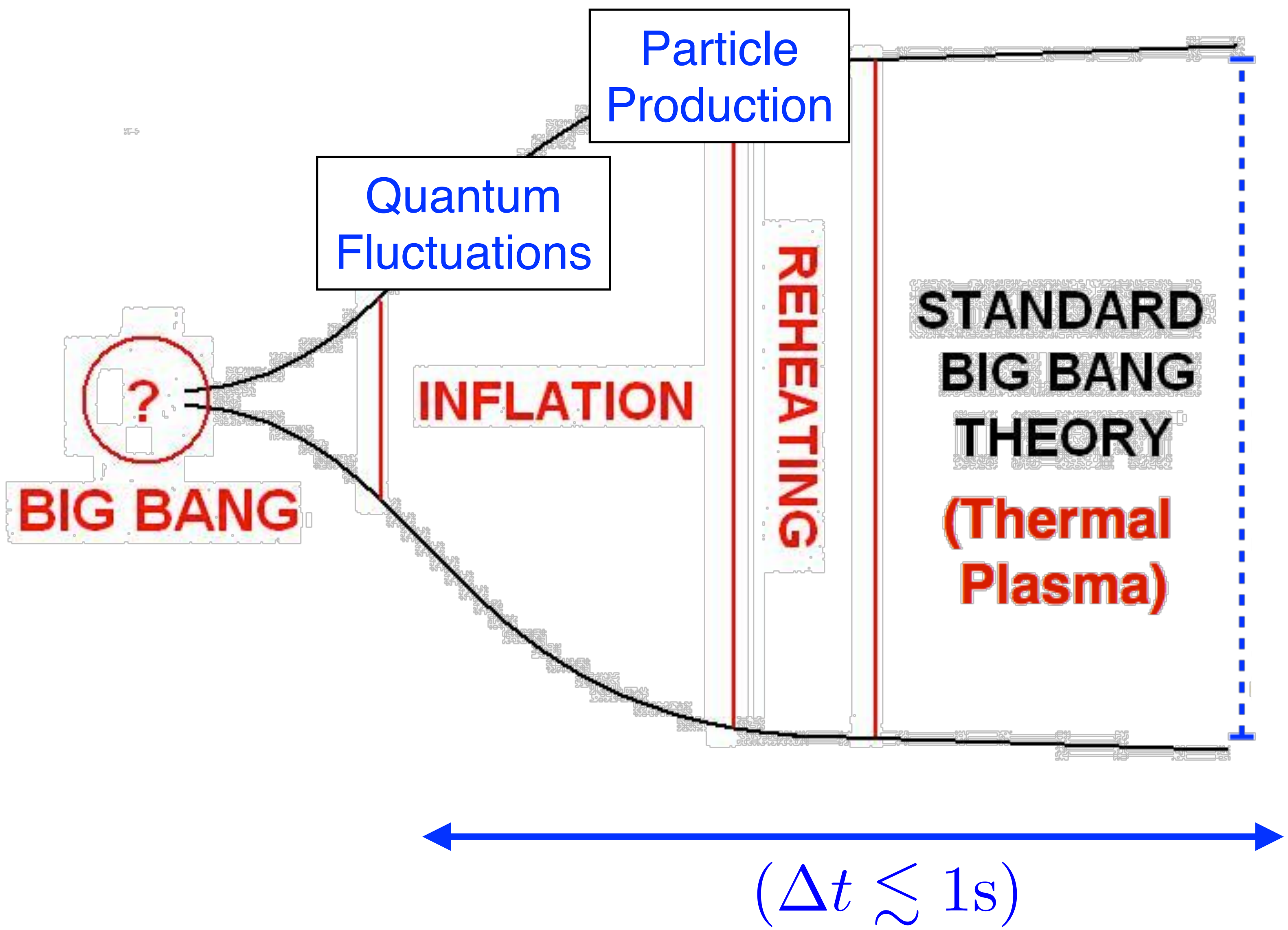


The Early Universe



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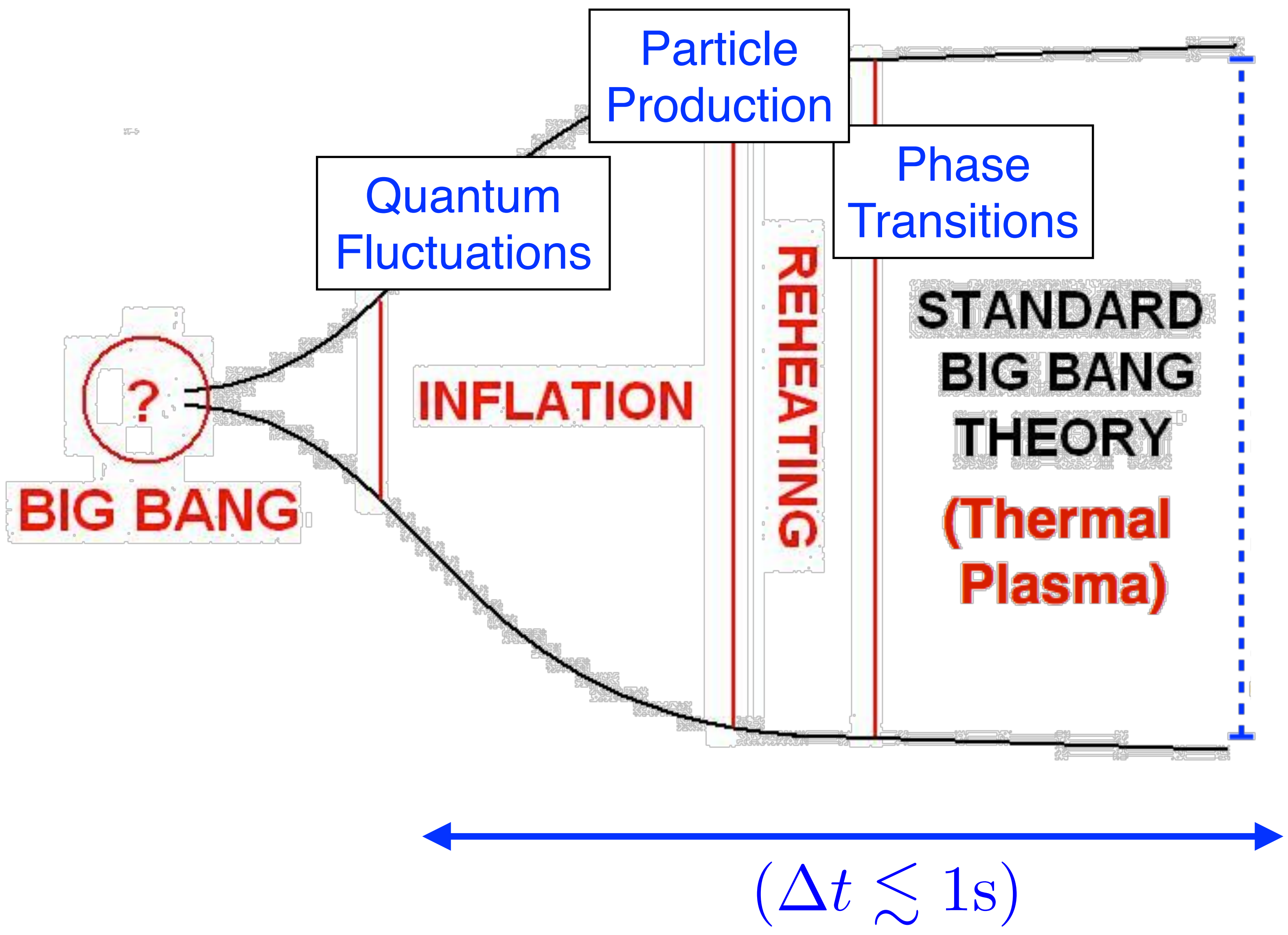
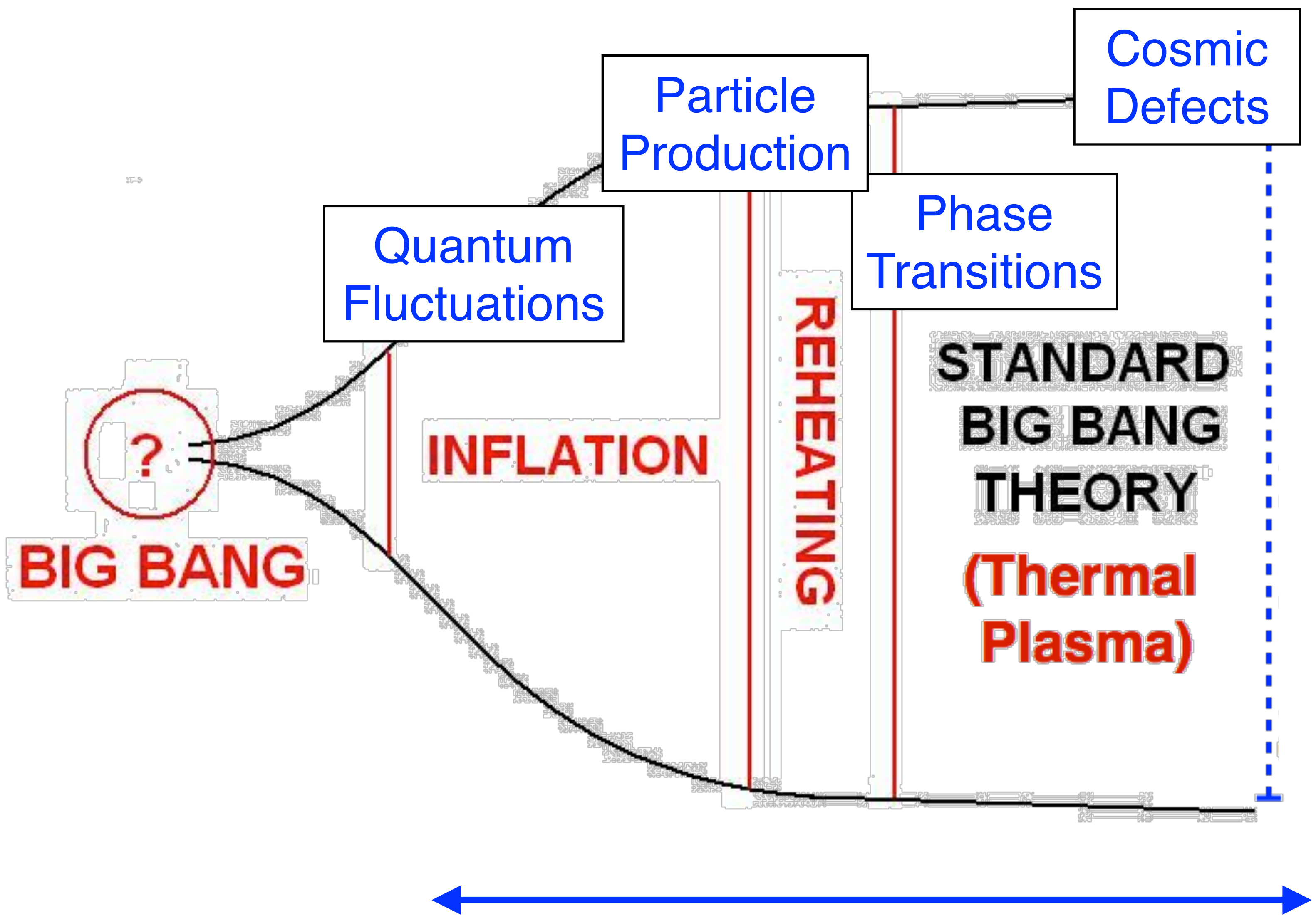


The Early Universe



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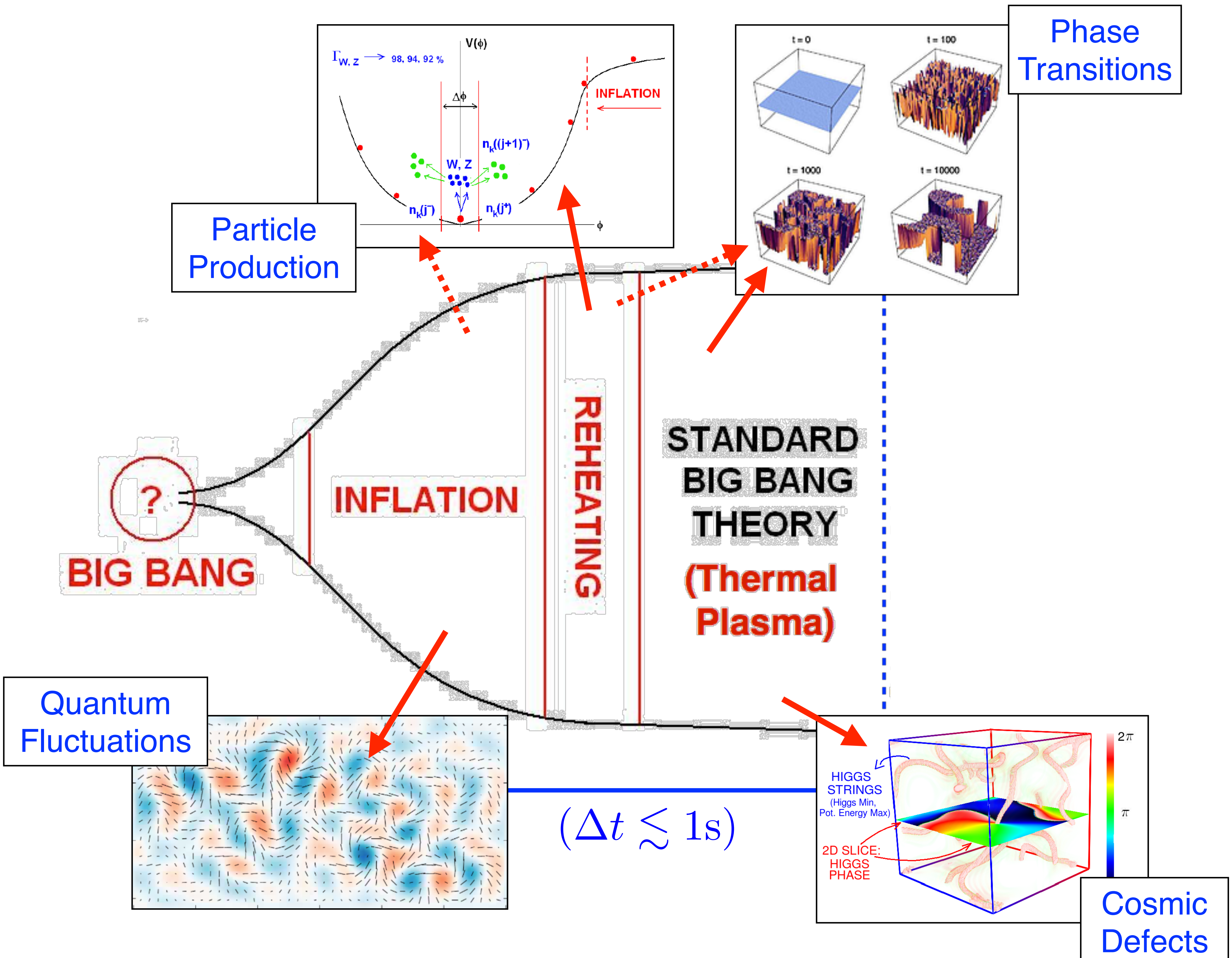
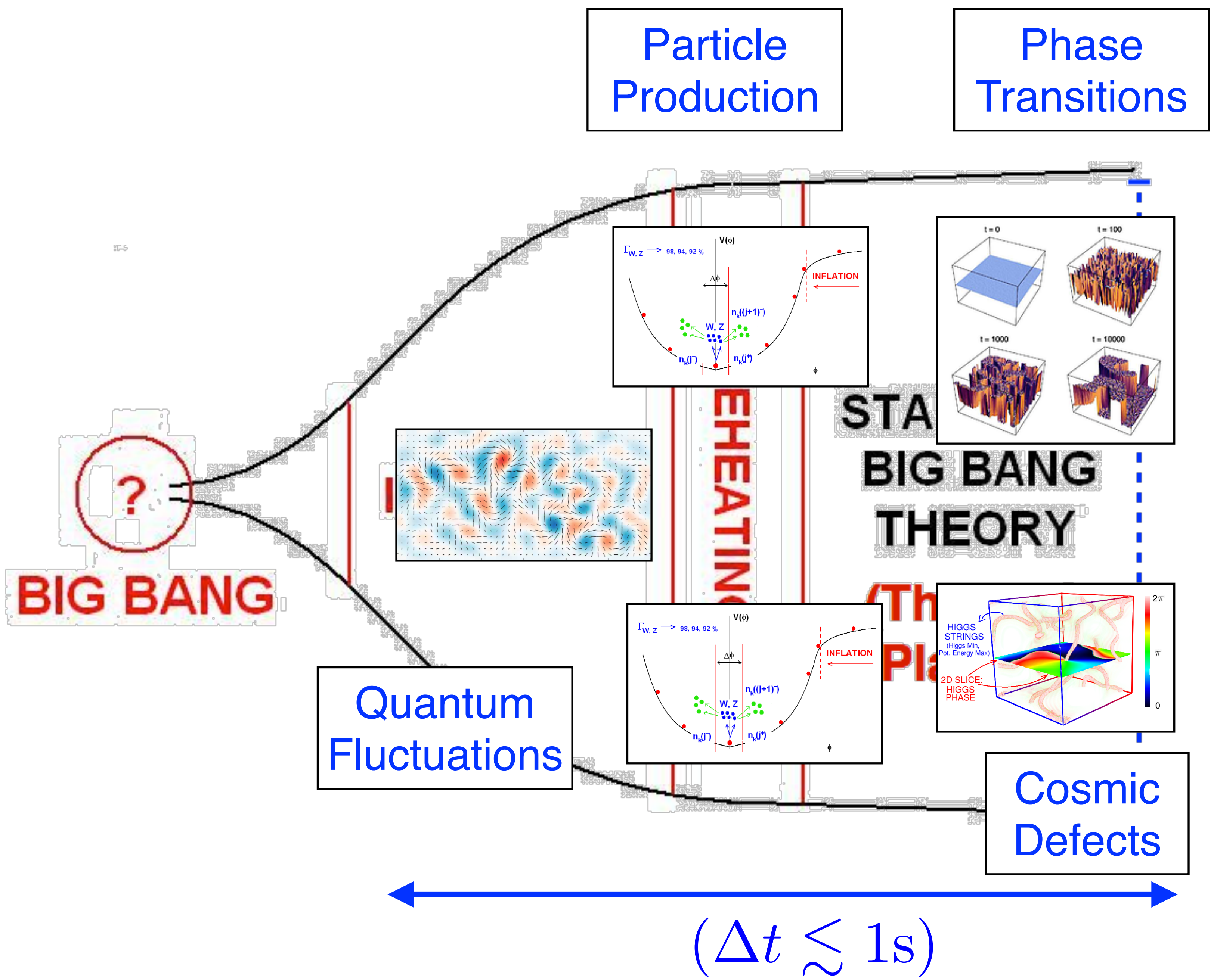
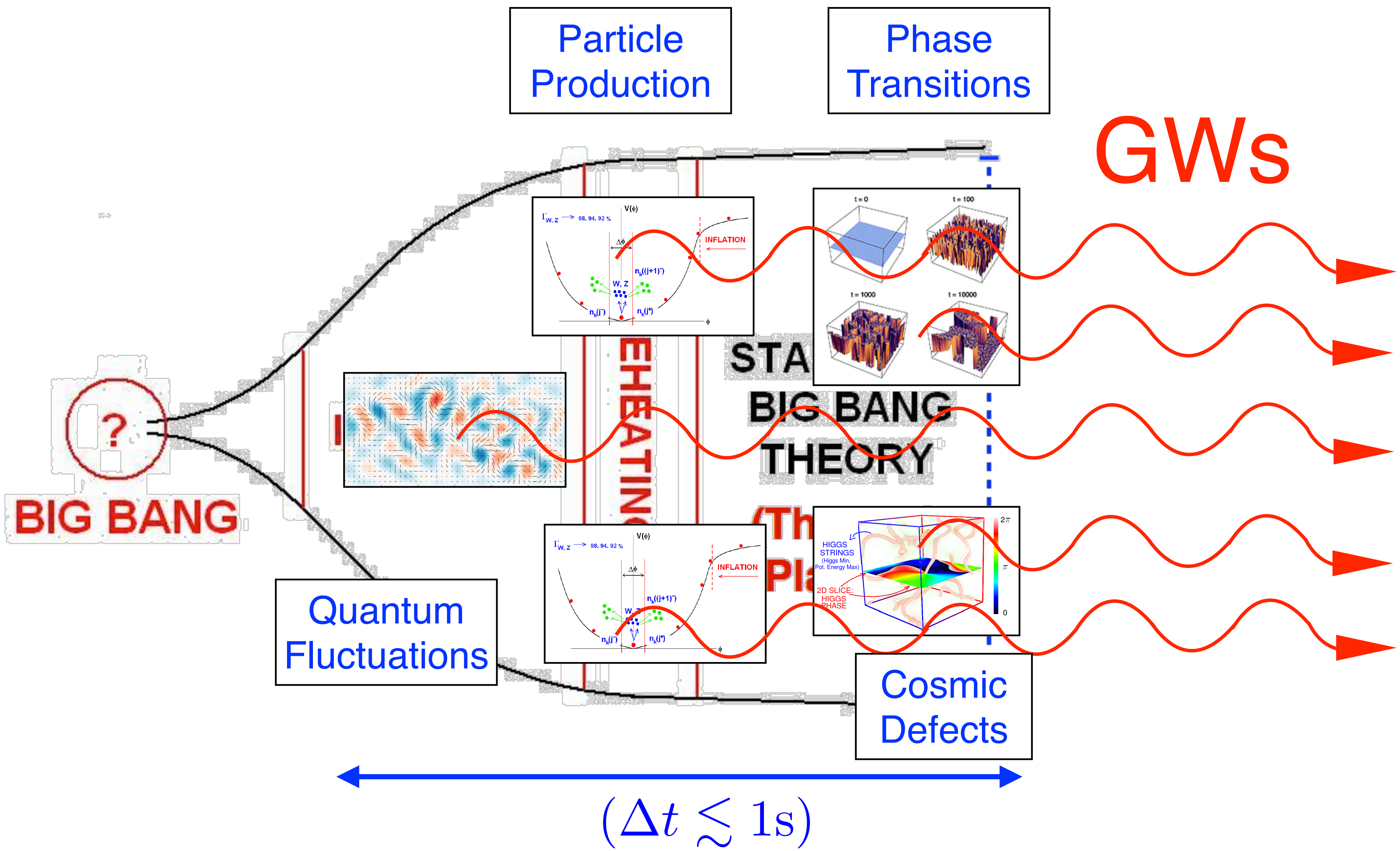


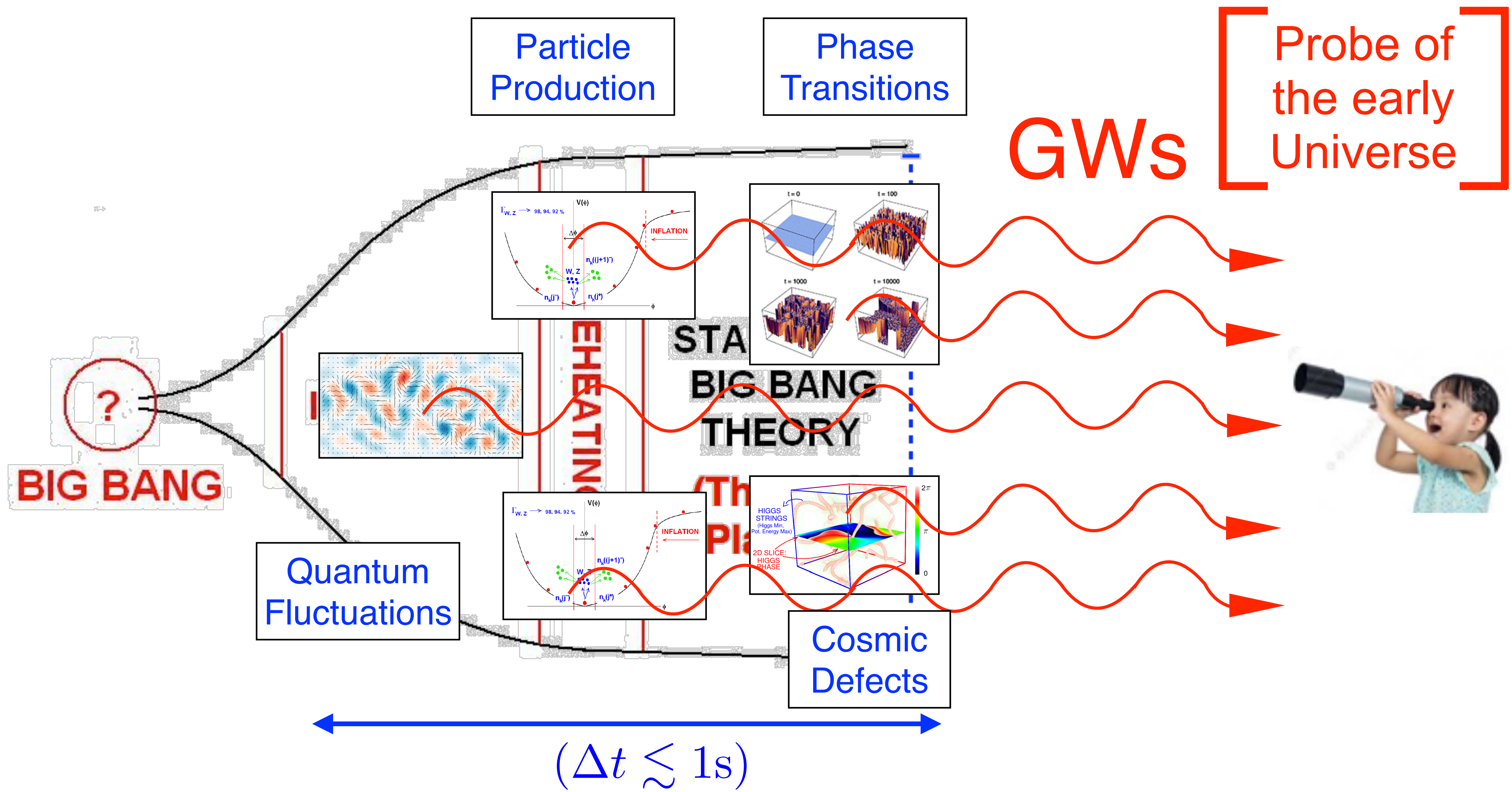
The Early Universe



The Early Universe

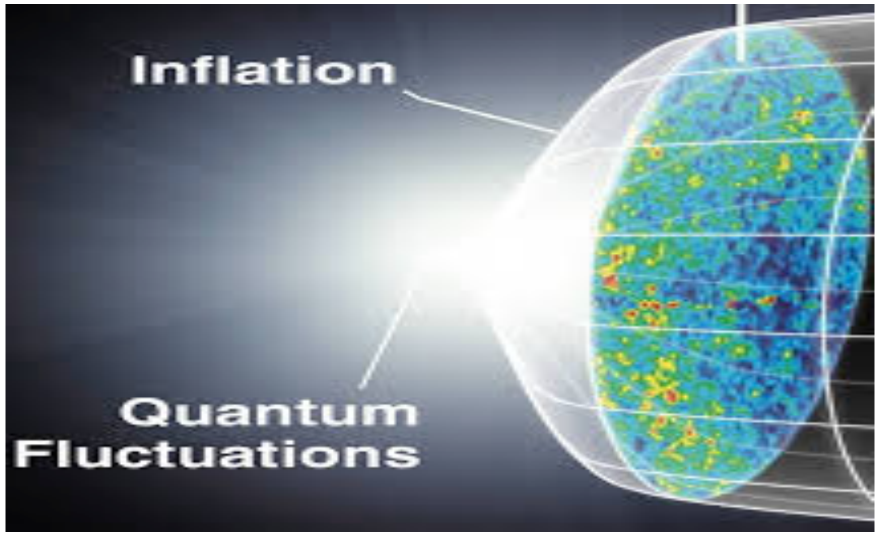


The Early Universe



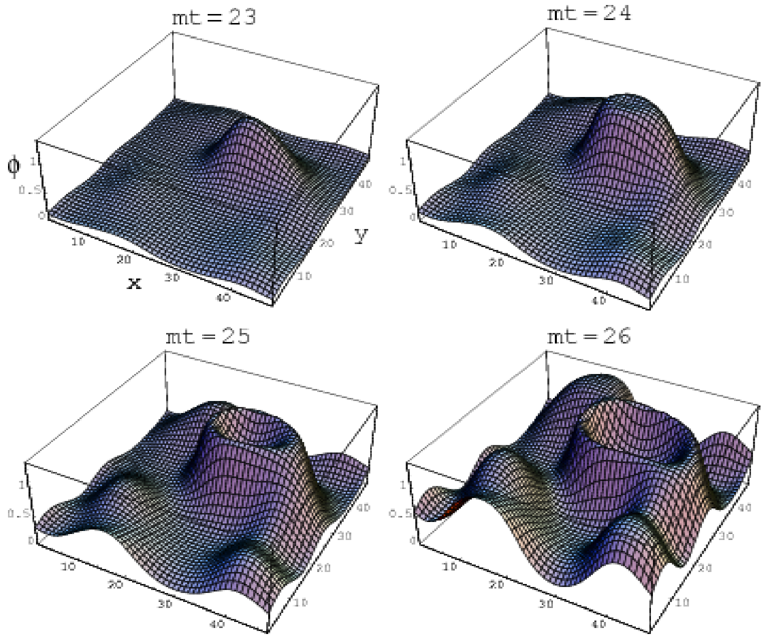
COSMOLOGICAL GRAVITATIONAL WAVES

Inflationary Period



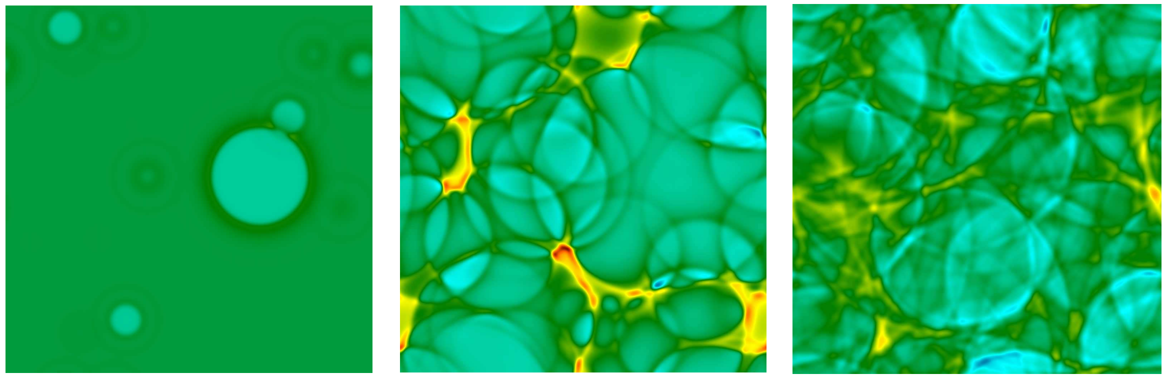
(Image: Google Search)

(p)Reheating



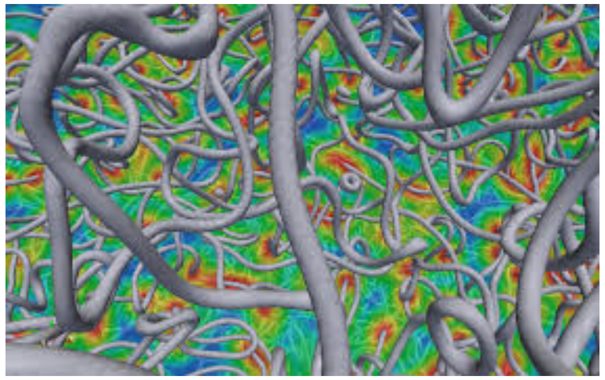
(Fig. credit: Phys.Rev. D67 103501)

Phase Transitions



(Image: PRL 112 (2014) 041301)

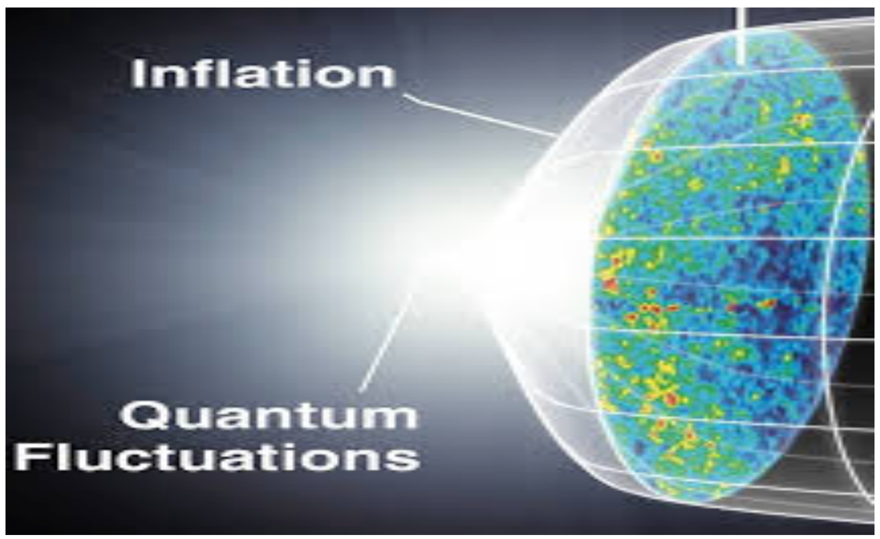
Cosmic Defects



(Image: Daverio et al, 2013)

COSMOLOGICAL GRAVITATIONAL WAVES

Inflationary Period



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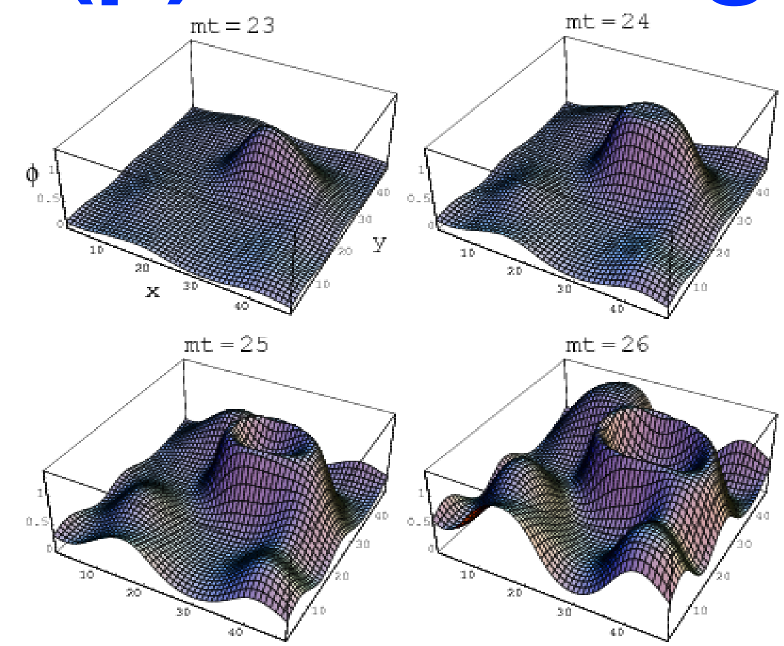
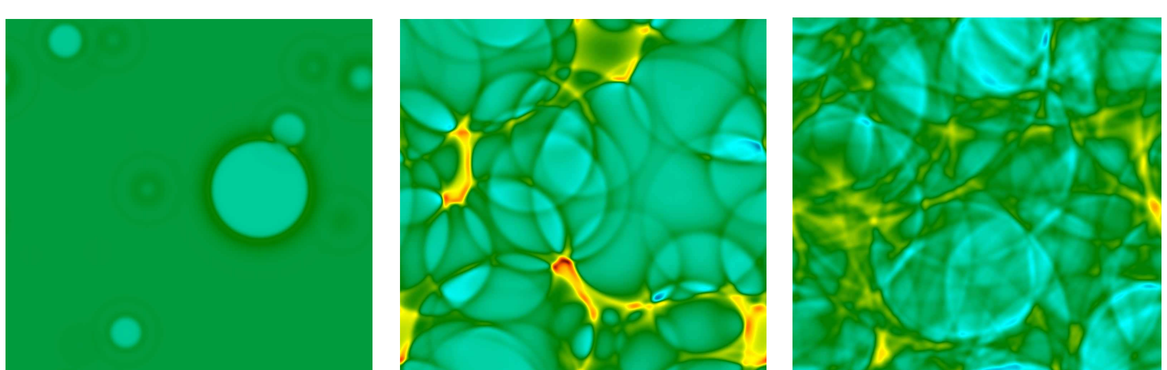
(p)Reheating



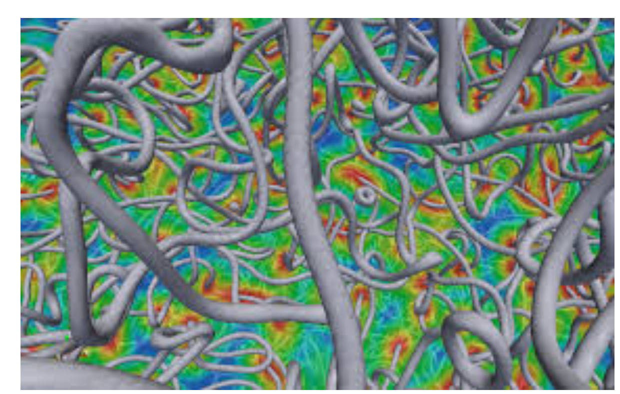
(Fig. credit: Phys.Rev. D67 103501)

Phase Transitions



(Image: PRL 112 (2014) 041301)

Cosmic Defects



(Image: Daverio et al, 2013)

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE

Early Universe

1) GWs from Inflation

2) GWs from Preheating

3) GWs from Phase Transitions

4) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

Early Universe

1) GWs from Inflation $\longrightarrow$ ~ 1 h

2) GWs from Preheating $\longrightarrow$ ~ 1 h

3) GWs from Phase Transitions $\longrightarrow$ ~ 1 h

4) GWs from Cosmic Defects $\longrightarrow$ ~ 1 h

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

0) Gravitational Waves definition

Early Universe

1) GWs from Inflation

2) GWs from Preheating

3) GWs from Phase Transitions

4) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

Early Universe

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

1st lecture (~ 1.5 h)

Early Universe

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

1st lecture (~ 1.5 h)

Early Universe

2) GWs from Inflation

3) GWs from Preheating

2nd lecture (~ 1.5 h)

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

COSMOLOGICAL GRAVITATIONAL WAVES

OUTLINE (~ 4.5 h)

1) Gravitational Waves definition

1st lecture (~ 1.5 h)

Early Universe

2) GWs from Inflation

3) GWs from Preheating

2nd lecture (~ 1.5 h)

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

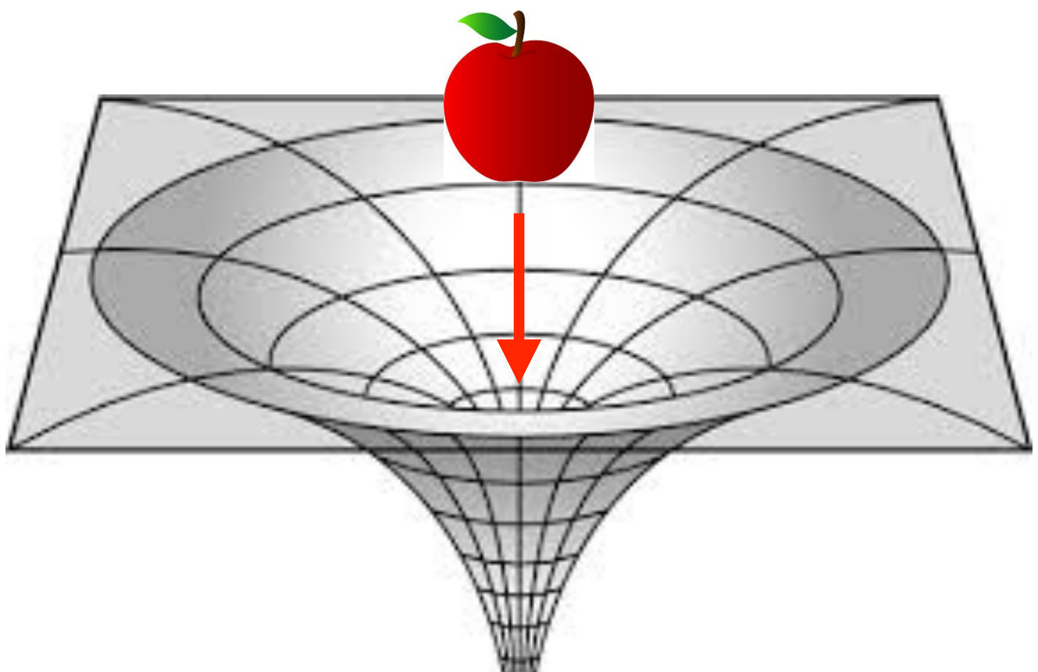
3rd lecture (~ 1.5 h)

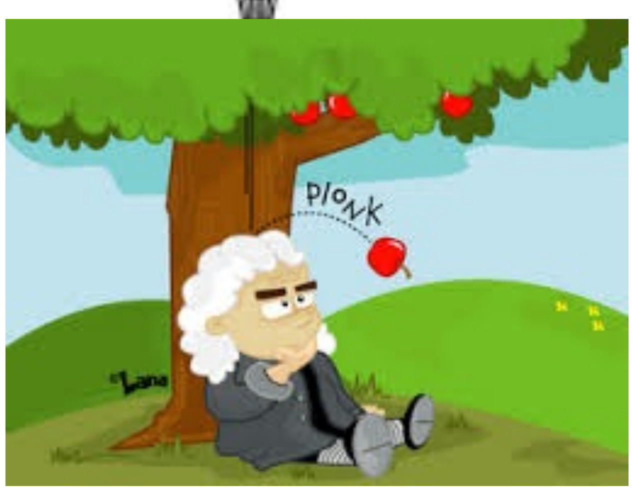
Let’s Start !

A PRIMER ON GRAVITATIONAL WAVES

Gravitational Framework

General Relativity (GR)





Gravitational Framework

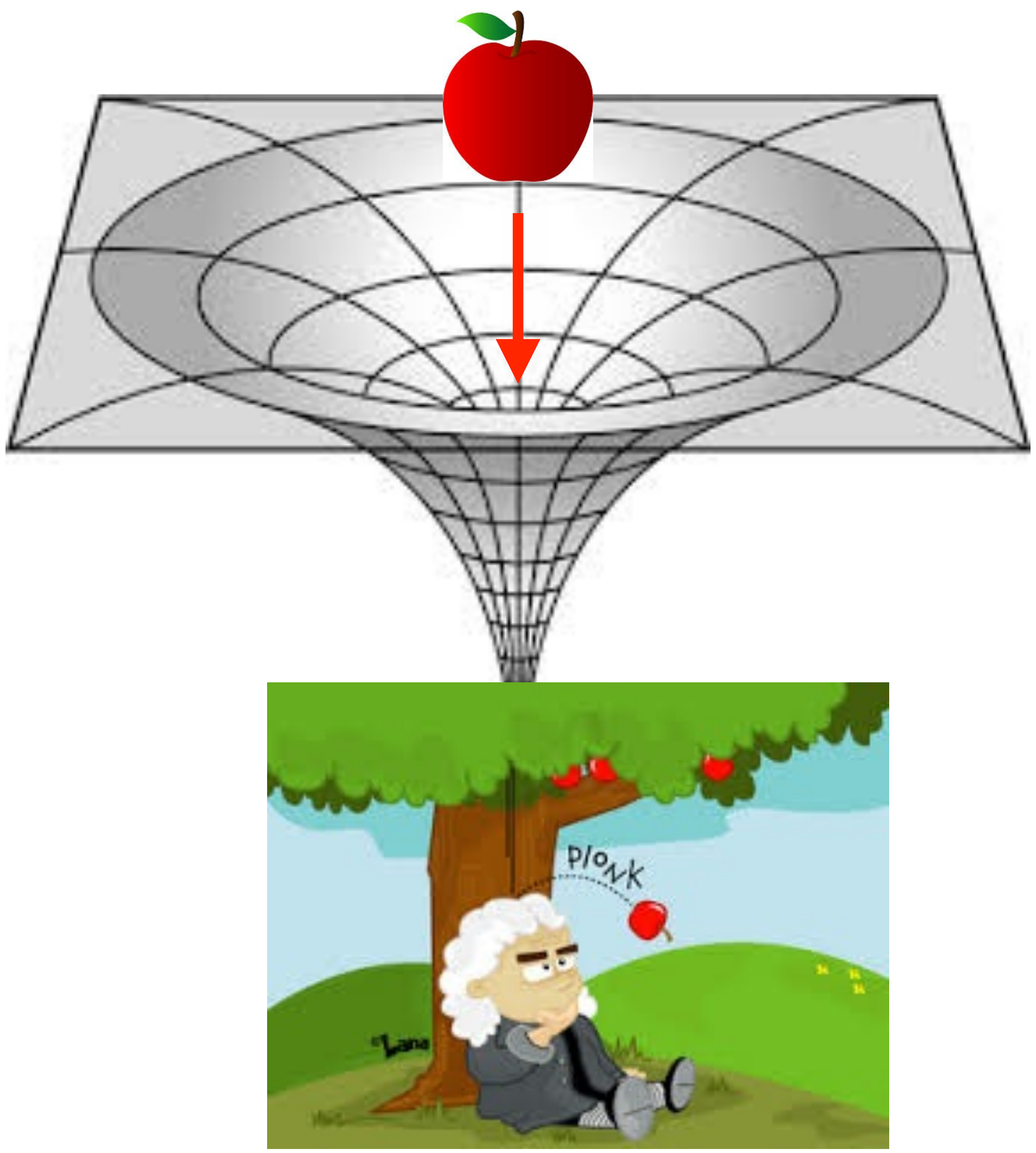
General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass



Gravitational Framework

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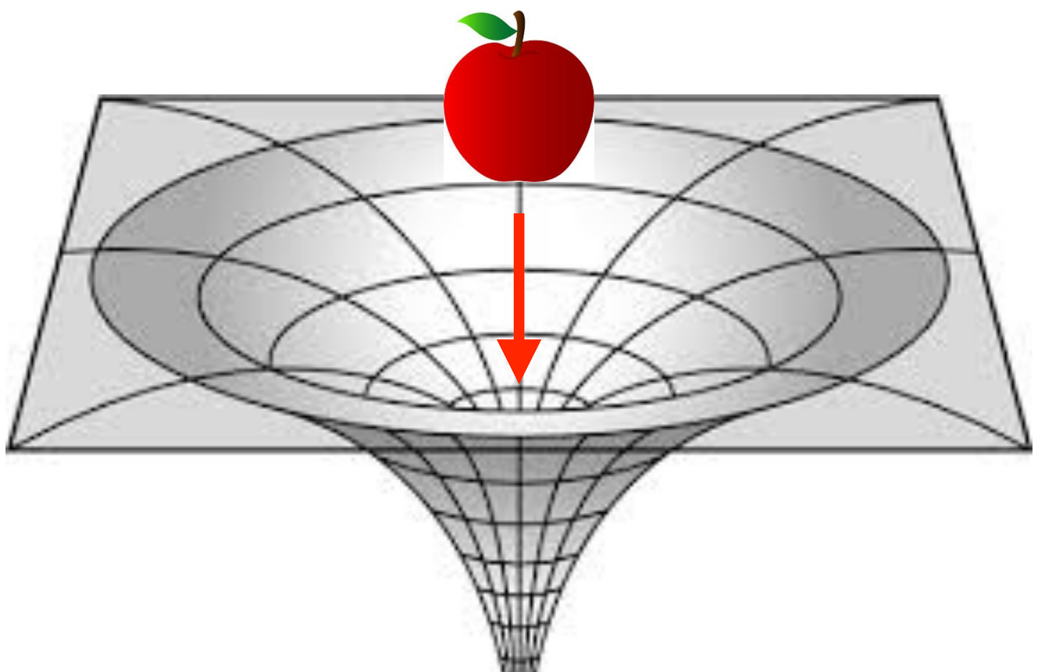
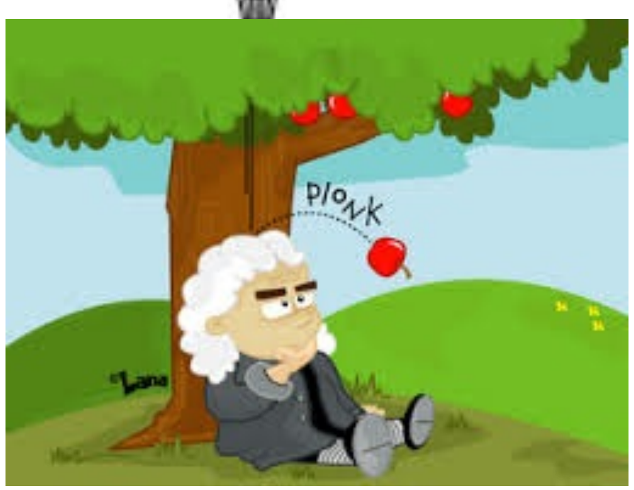
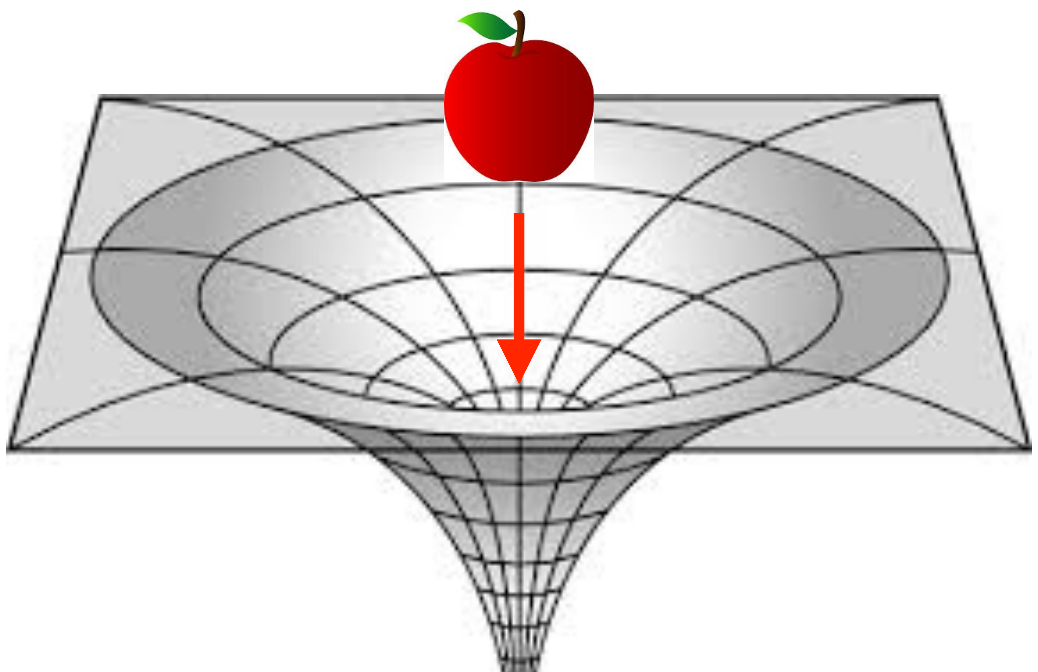


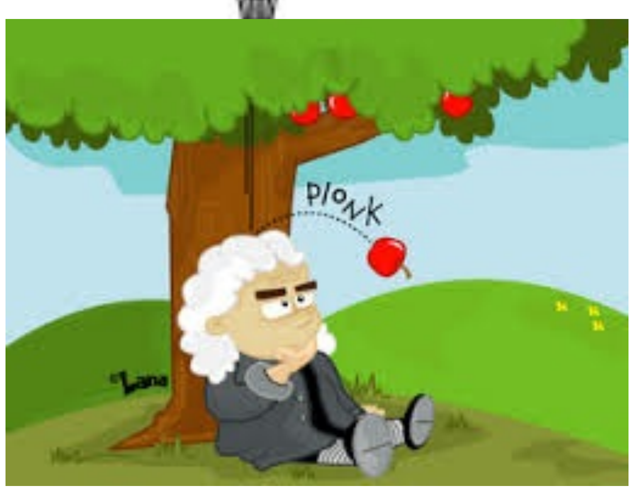


$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Gravitational Framework

General Relativity (GR)





$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$\mathrm{DIFF}: \quad x^\mu \rightarrow x'^\mu(x)$$

symmetry

skip

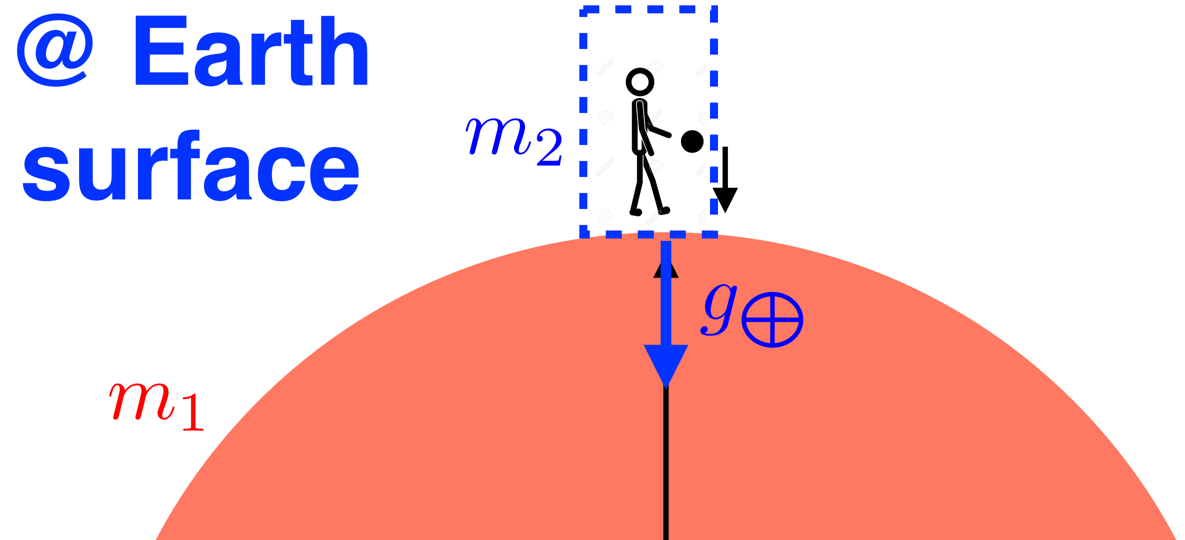
(

skip

A PRIMER ON GENERAL RELATIVITY

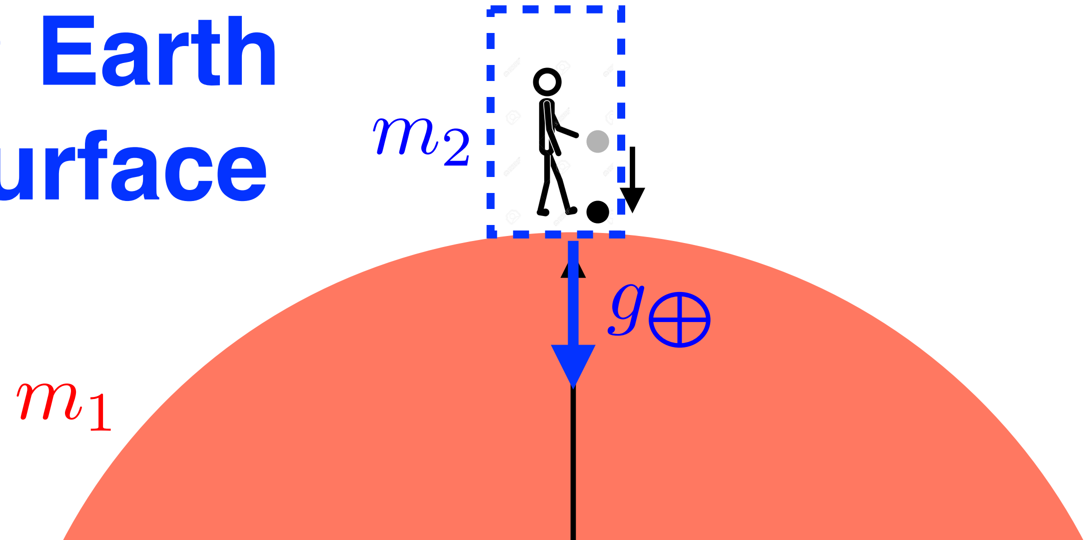
skip

The Equivalence Principle

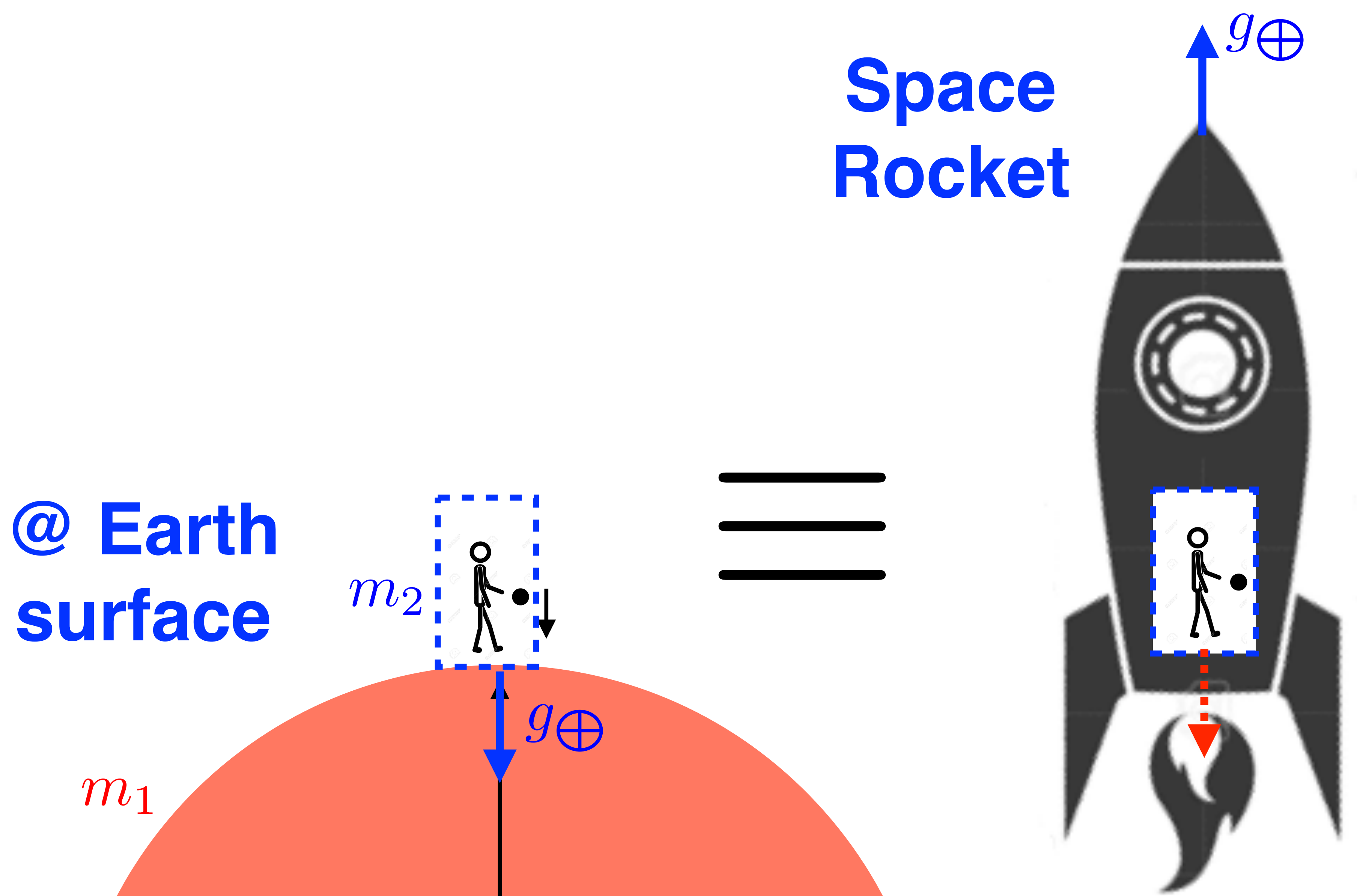


The Equivalence Principle

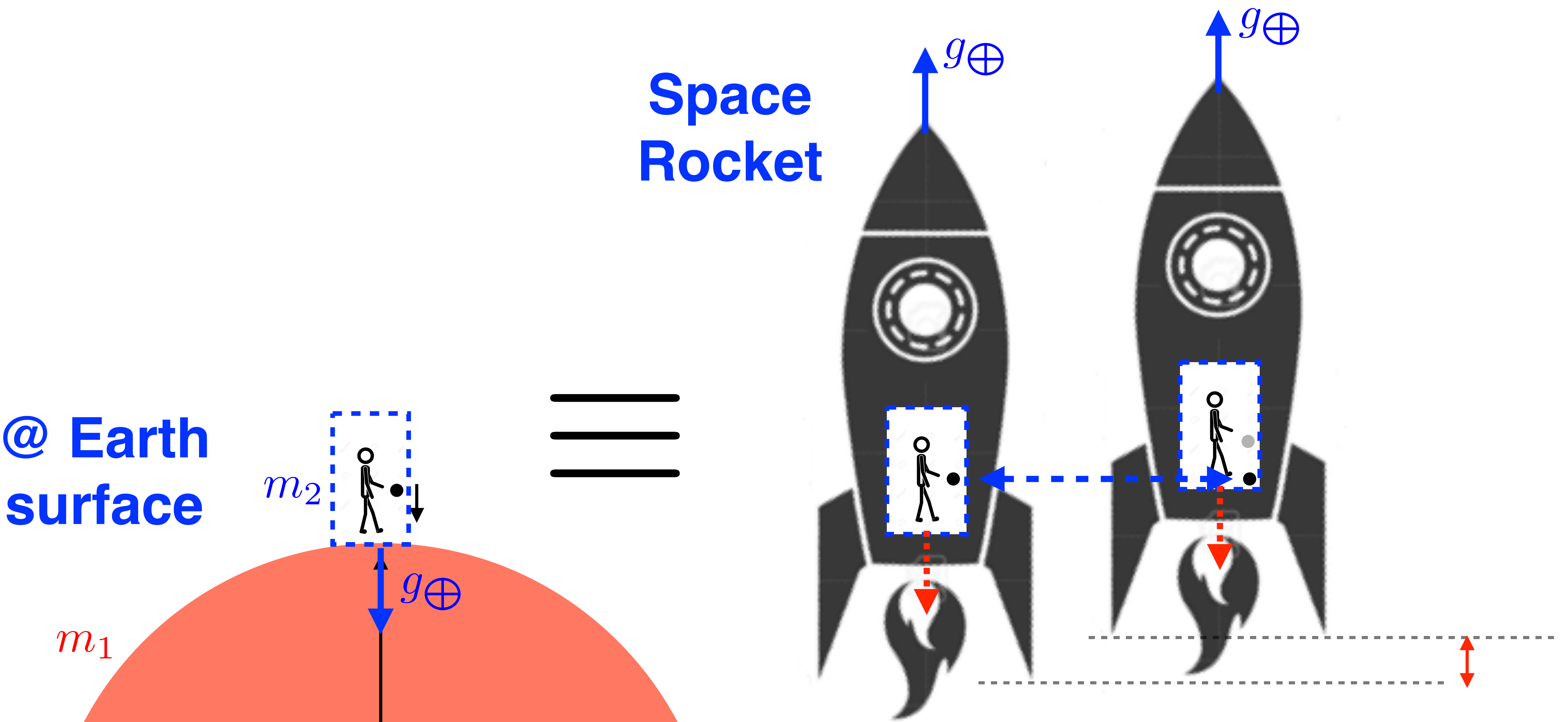
@ Earth surface



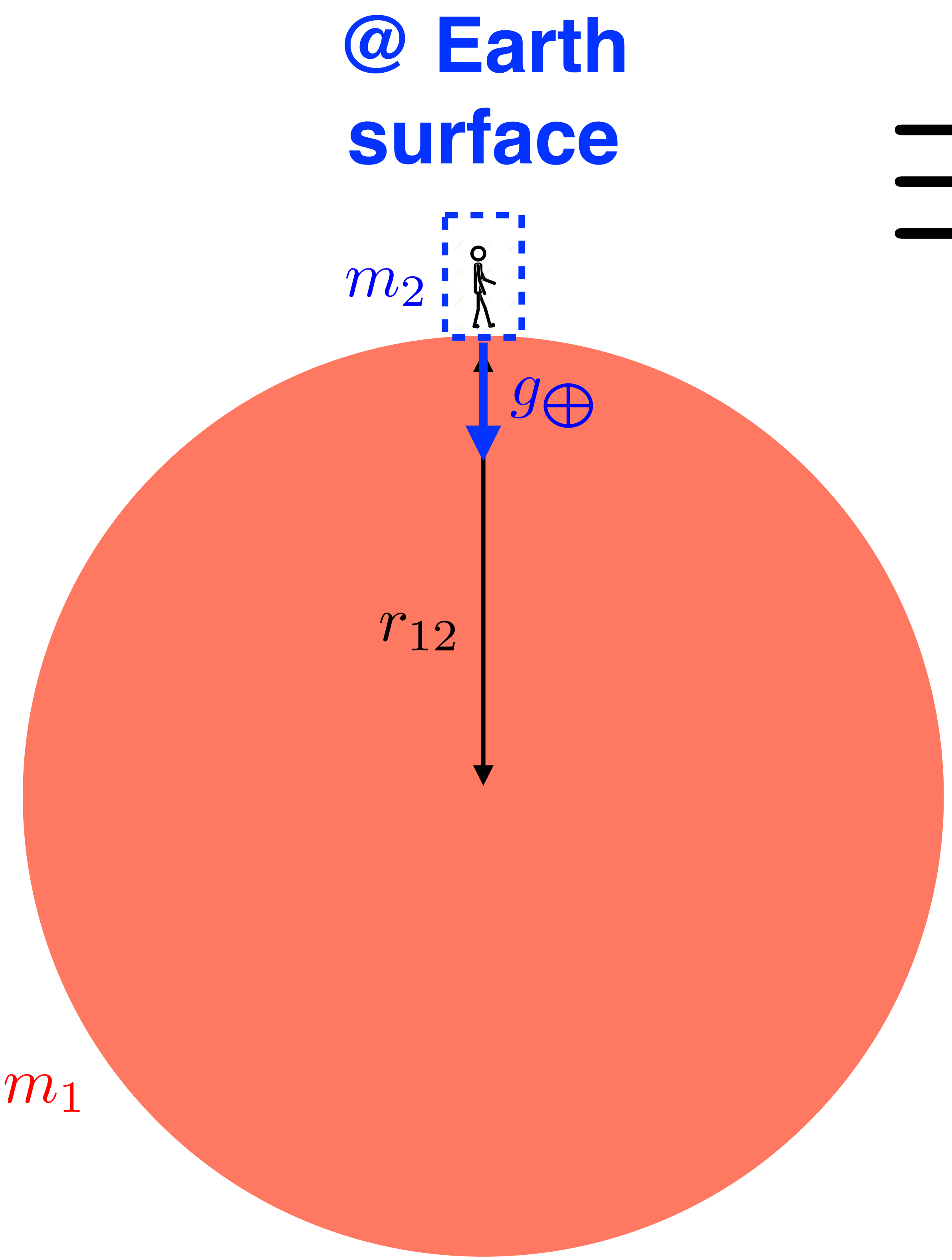
The Equivalence Principle



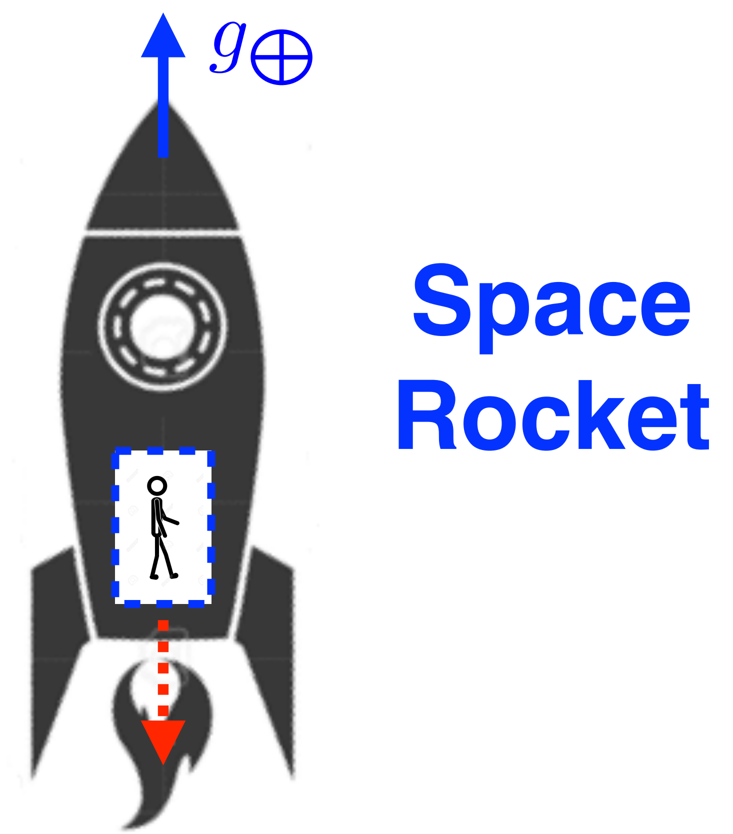
The Equivalence Principle



The Equivalence Principle



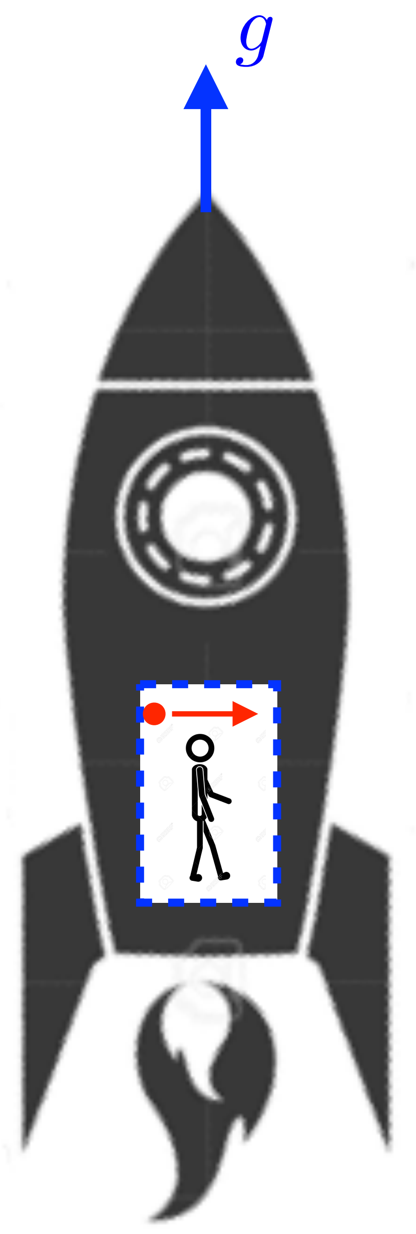
$\equiv$



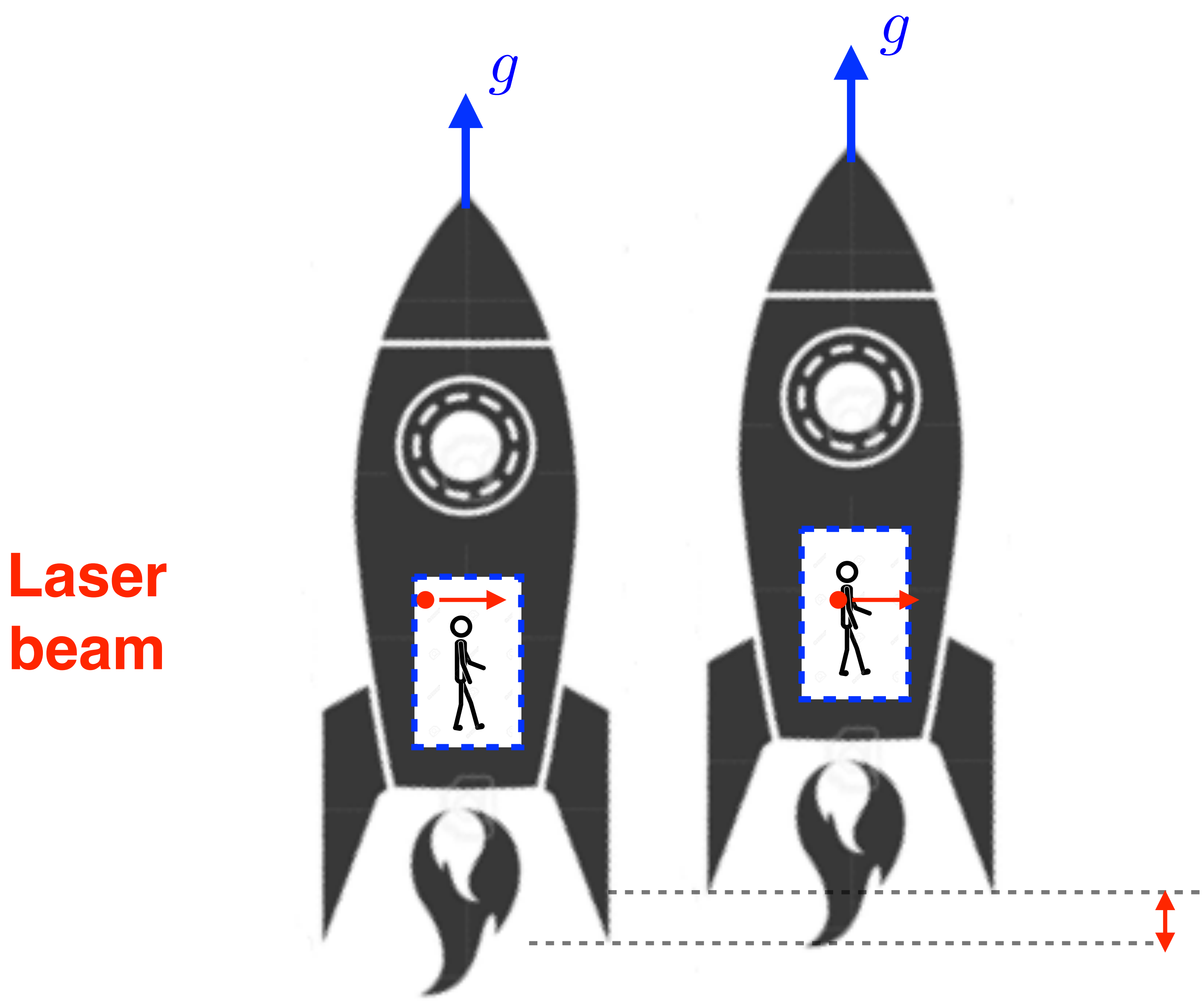
Situations appear to be identical !

The Equivalence Principle

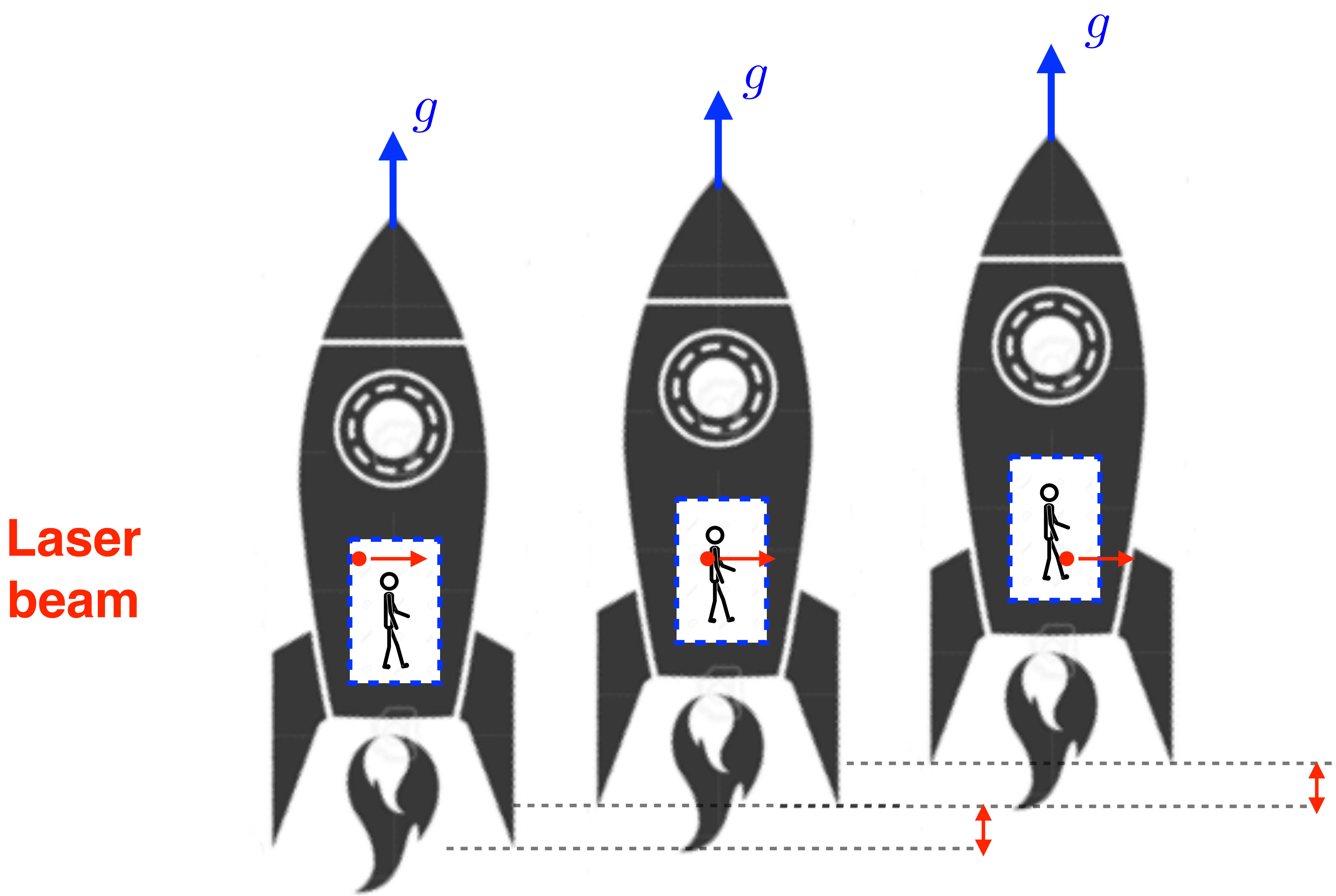
Laser
beam



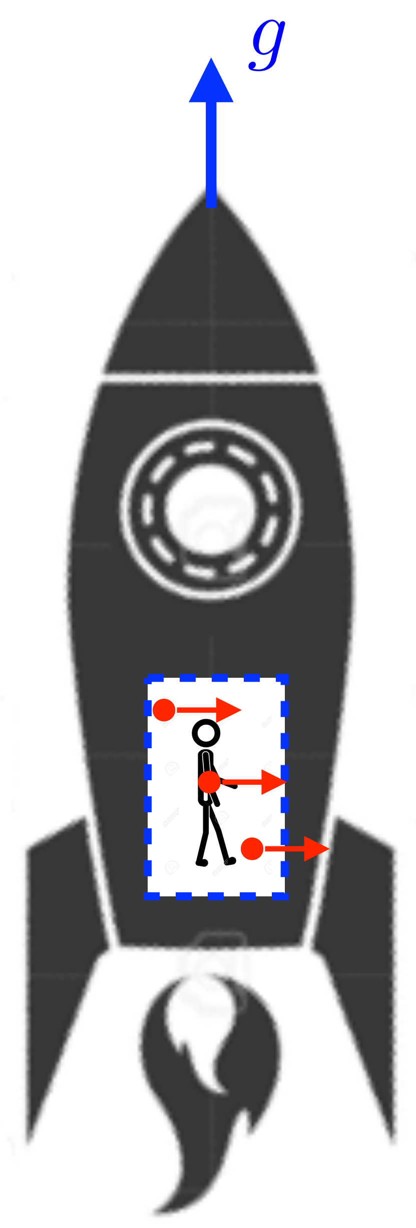
The Equivalence Principle



The Equivalence Principle

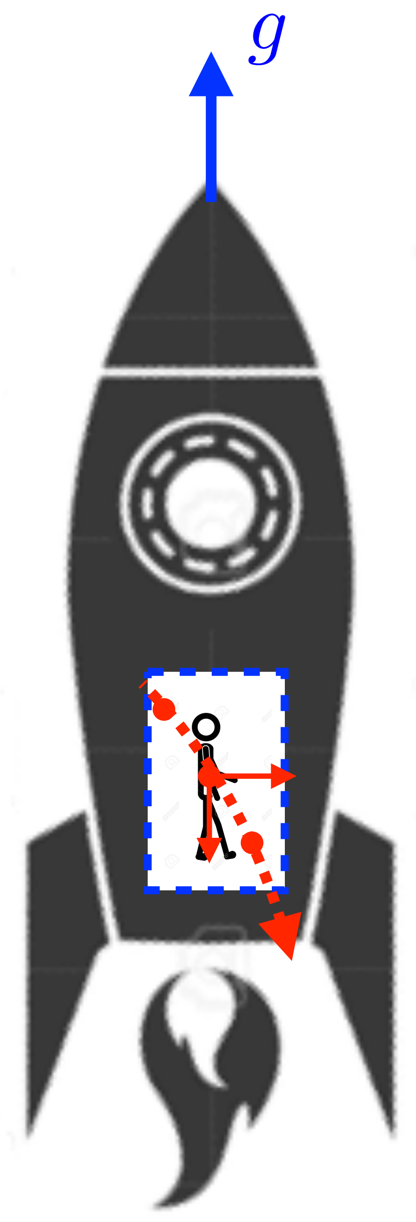


The Equivalence Principle



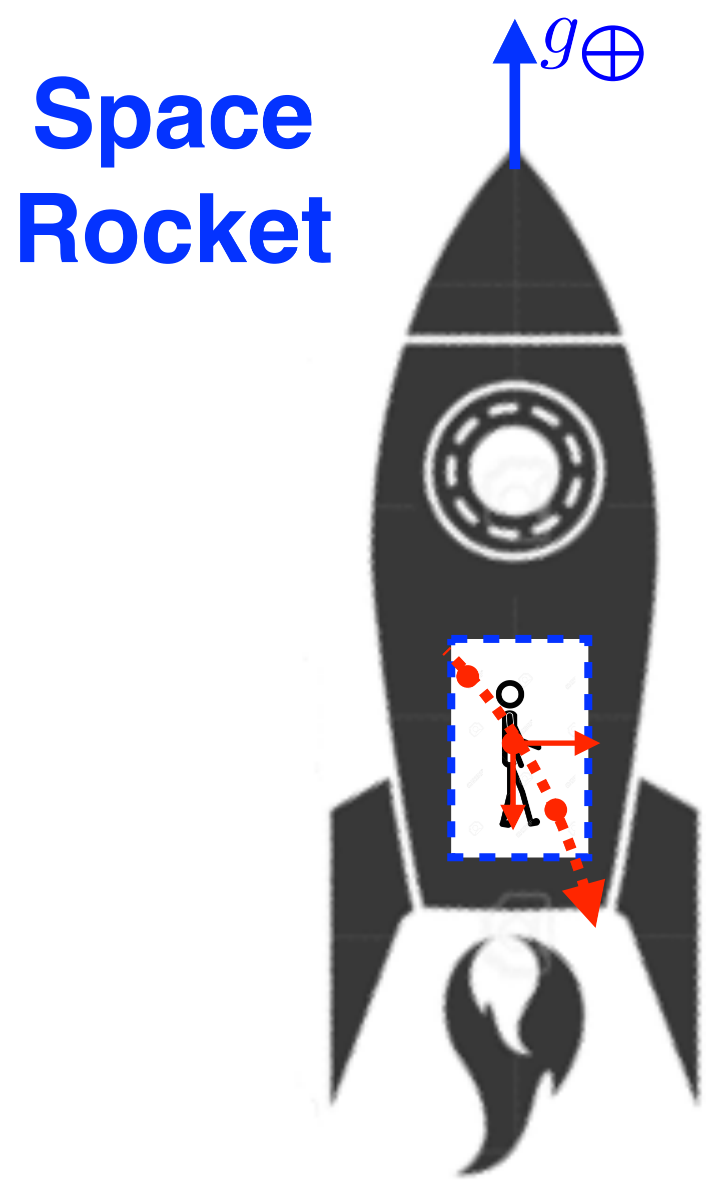
Laser beam

The Equivalence Principle

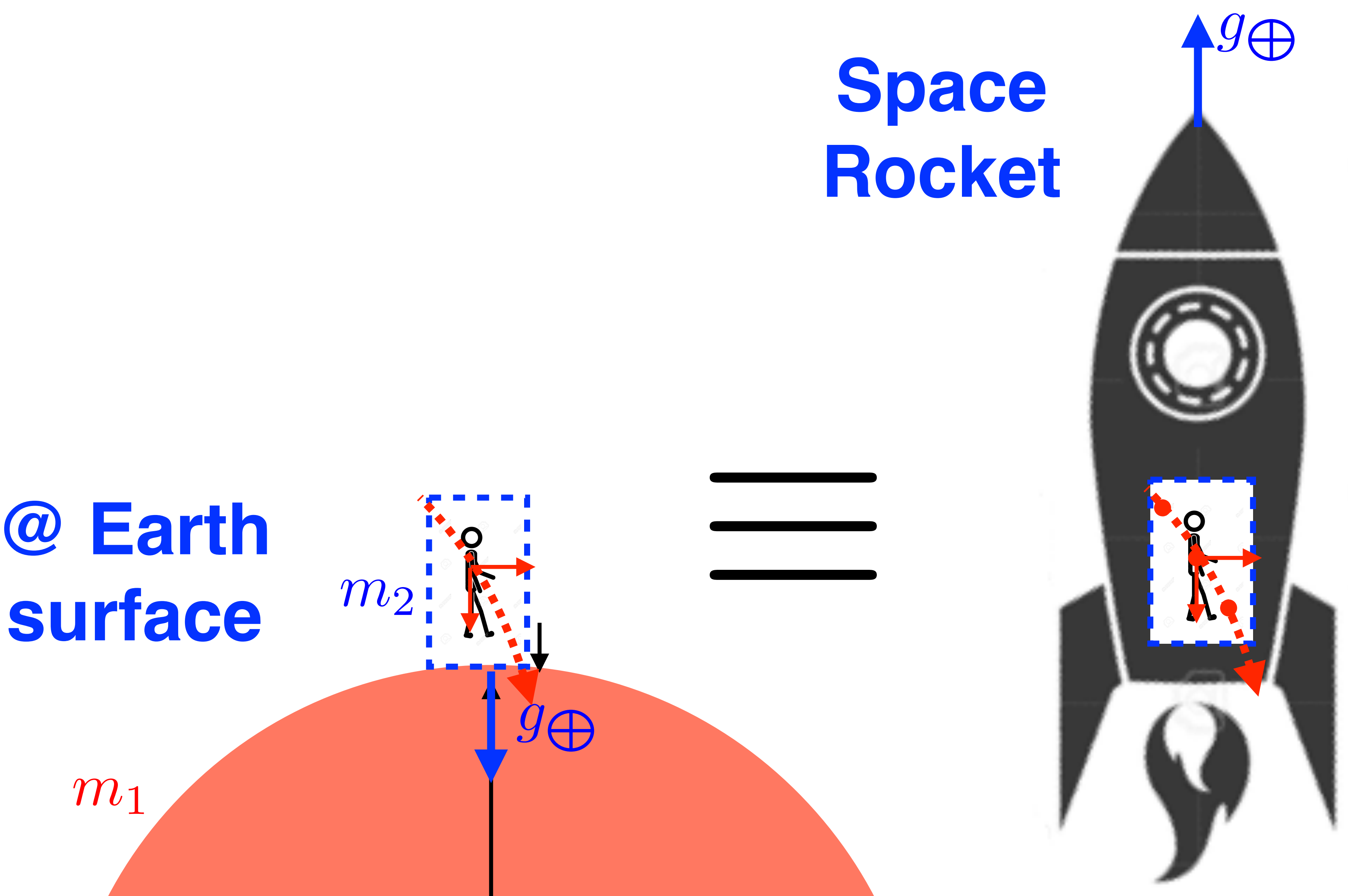


Laser beam

The Equivalence Principle



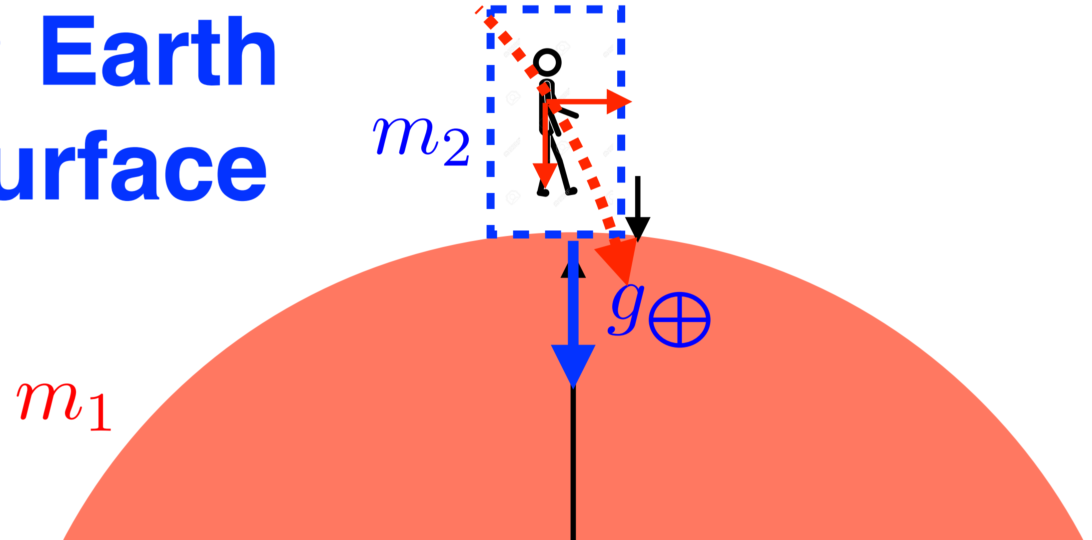
The Equivalence Principle



The Equivalence Principle

Gravitational field
must bend light !

@ Earth surface

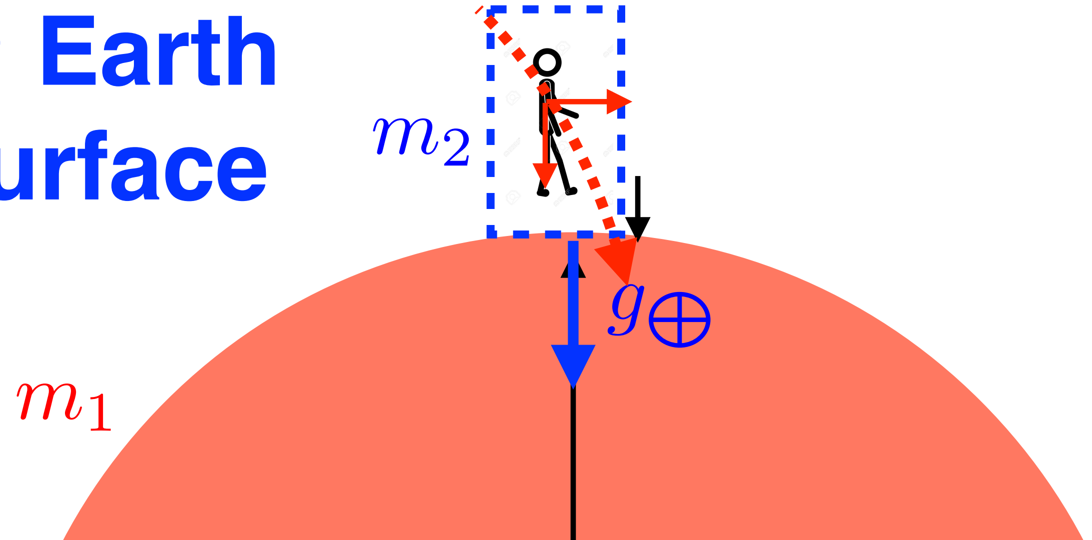


The Equivalence Principle

(Earth gravity too weak to observe the effect, but …)

Gravitational field must bend light !

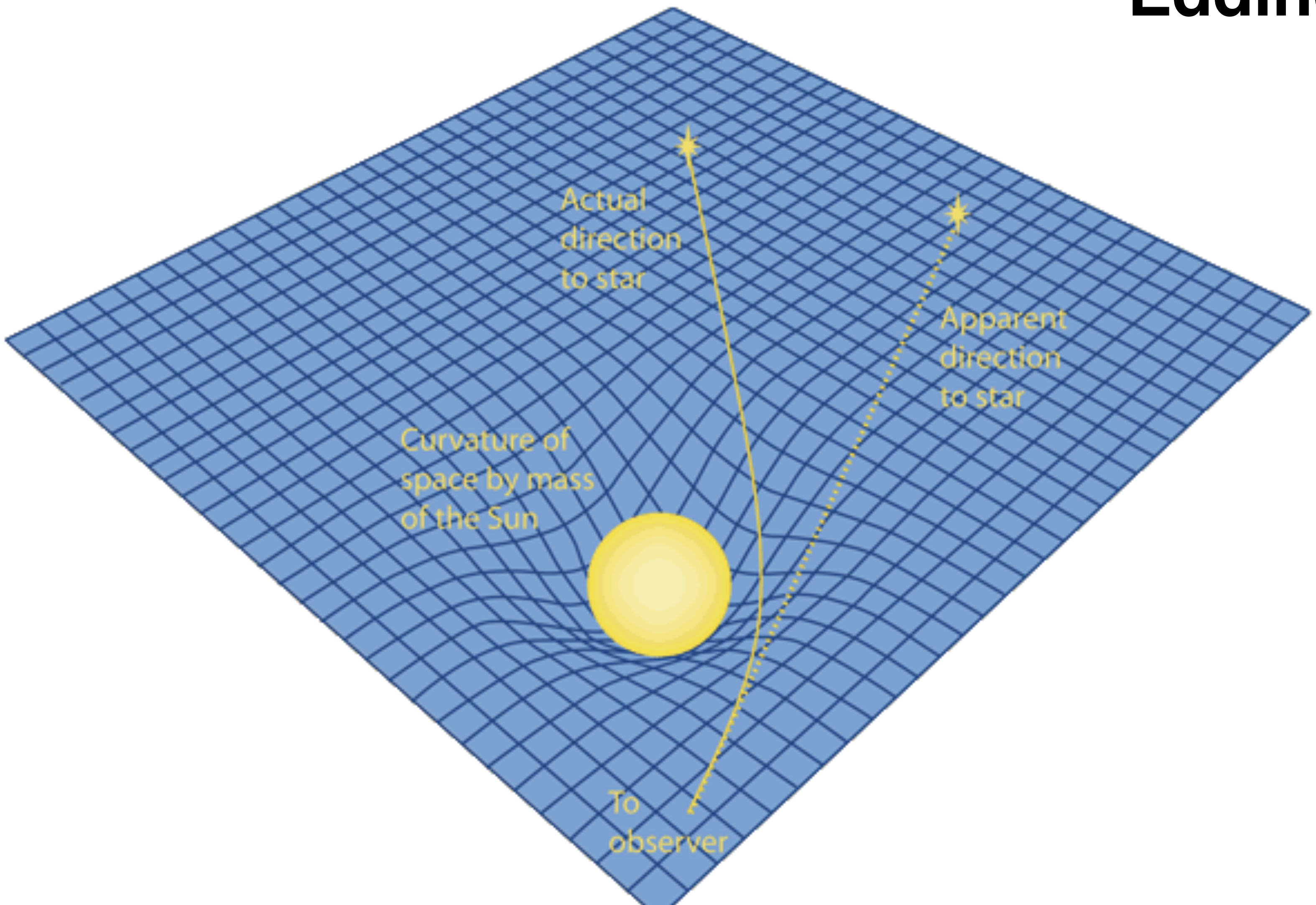
@ Earth surface



The Equivalence Principle

(Earth gravity too weak to observe the effect, but …) → **e.g. the sun does a better job !**

Eddington 1919



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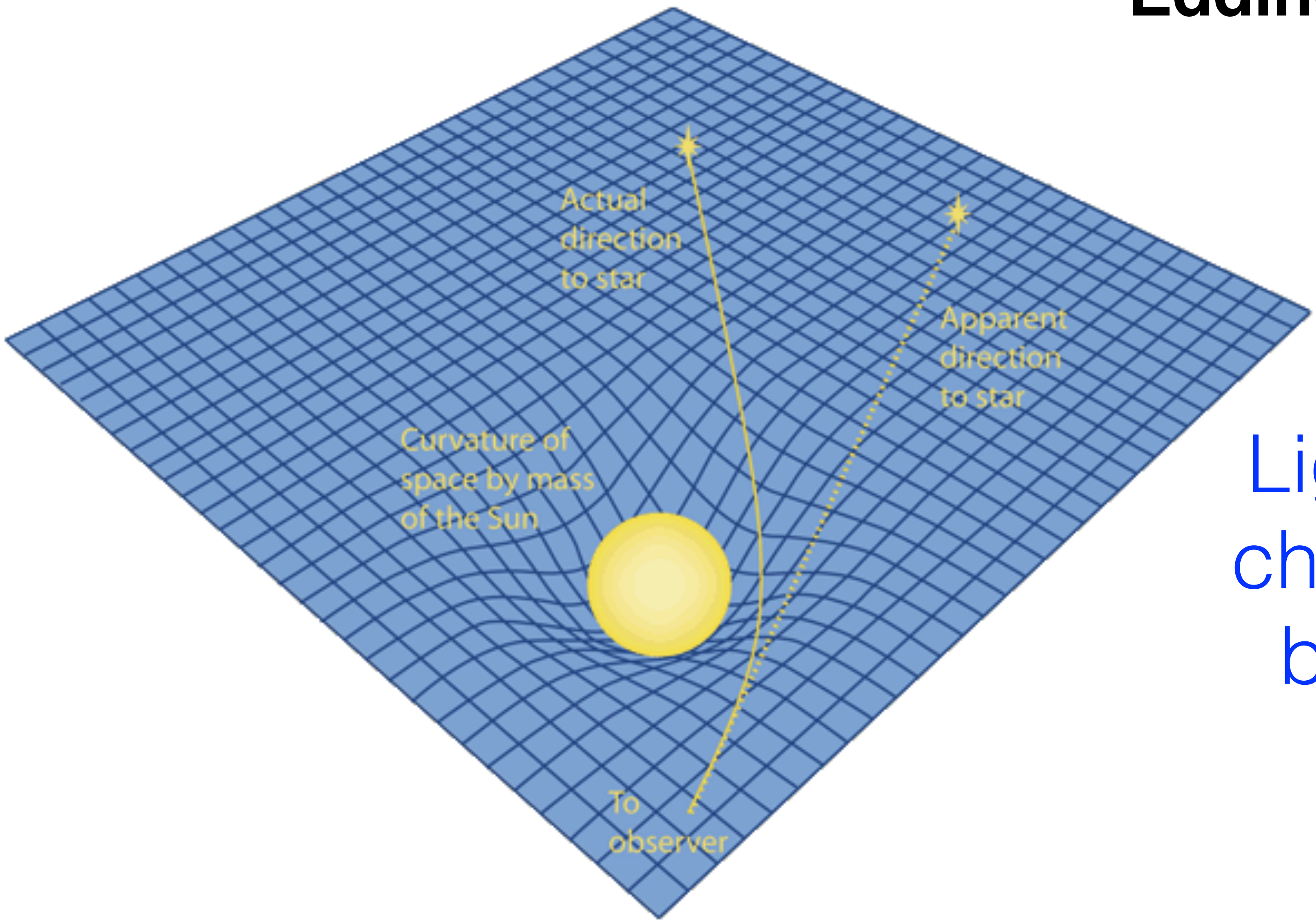


Light does not change speed, but direction

The Equivalence Principle

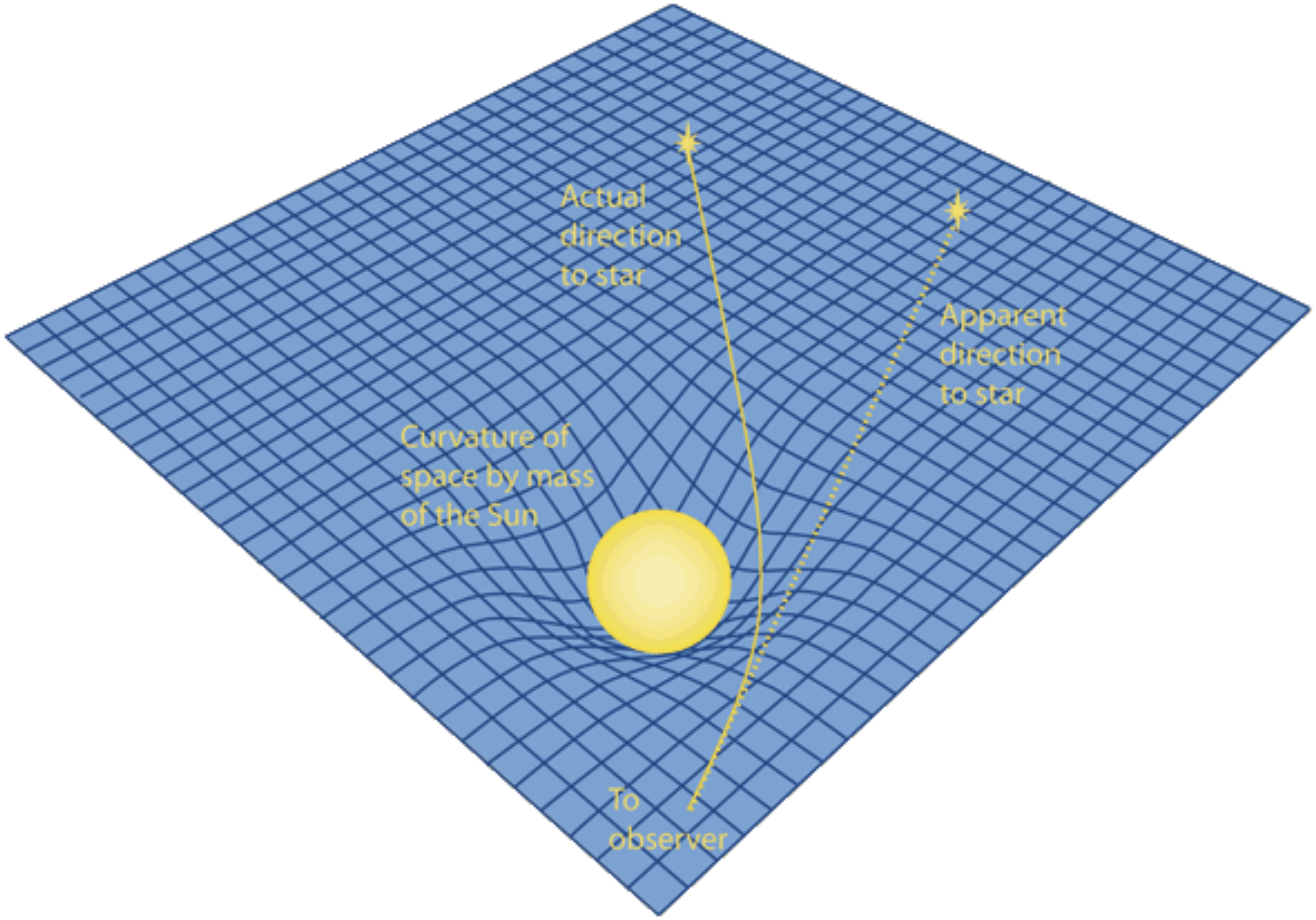
Einstein understood like this…

light bending,

light red/blue-shifting,

gravitational time dilation,

…

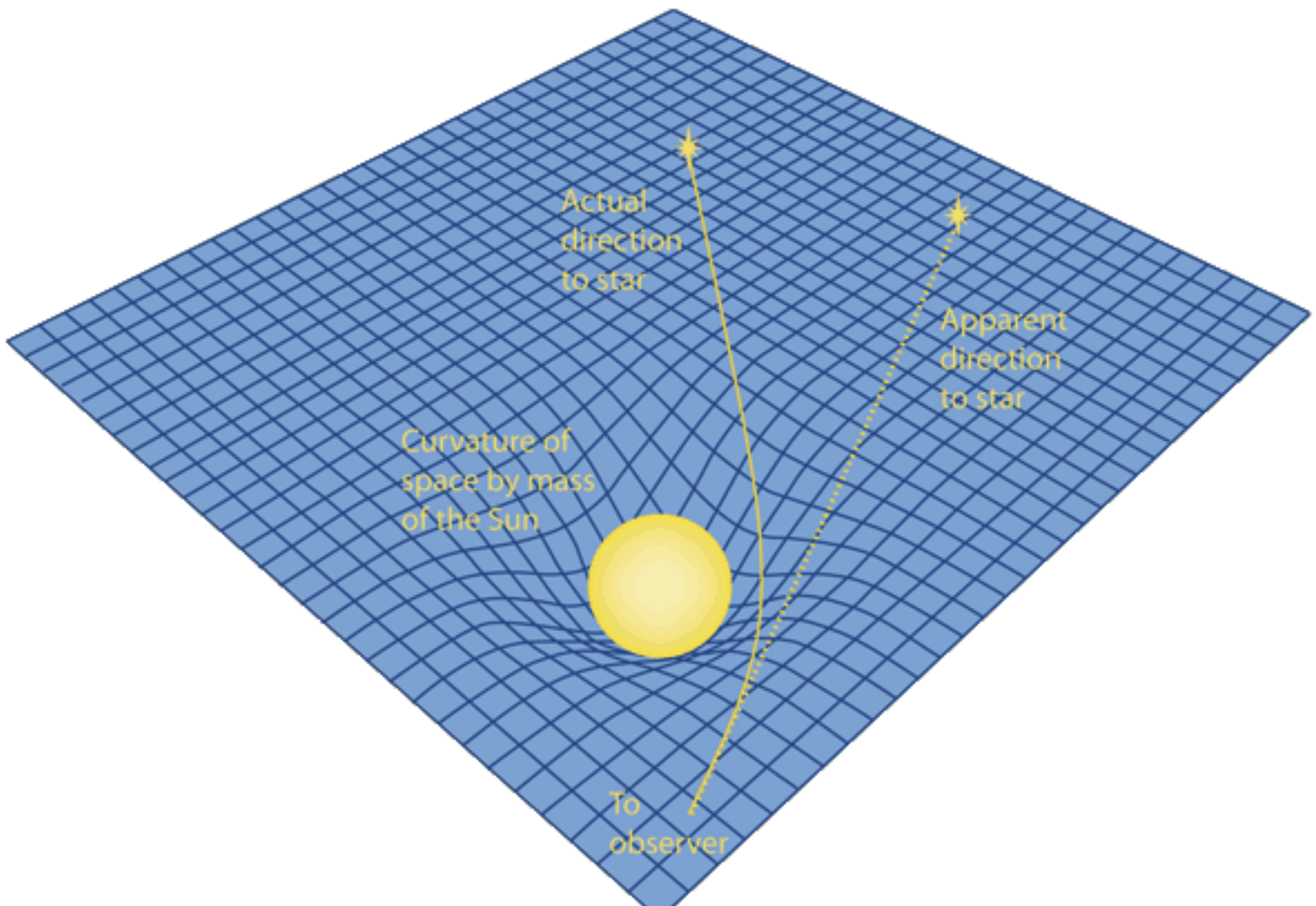


The Equivalence Principle

Einstein understood like this…

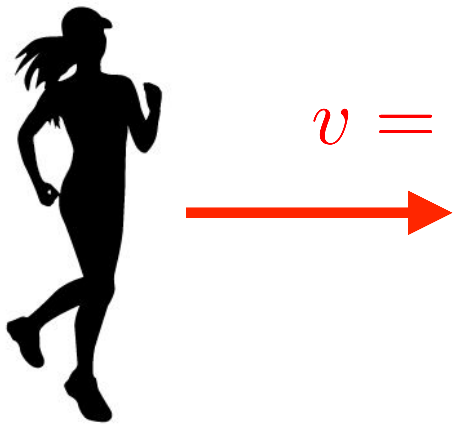
a mathematical formulation was needed !

…

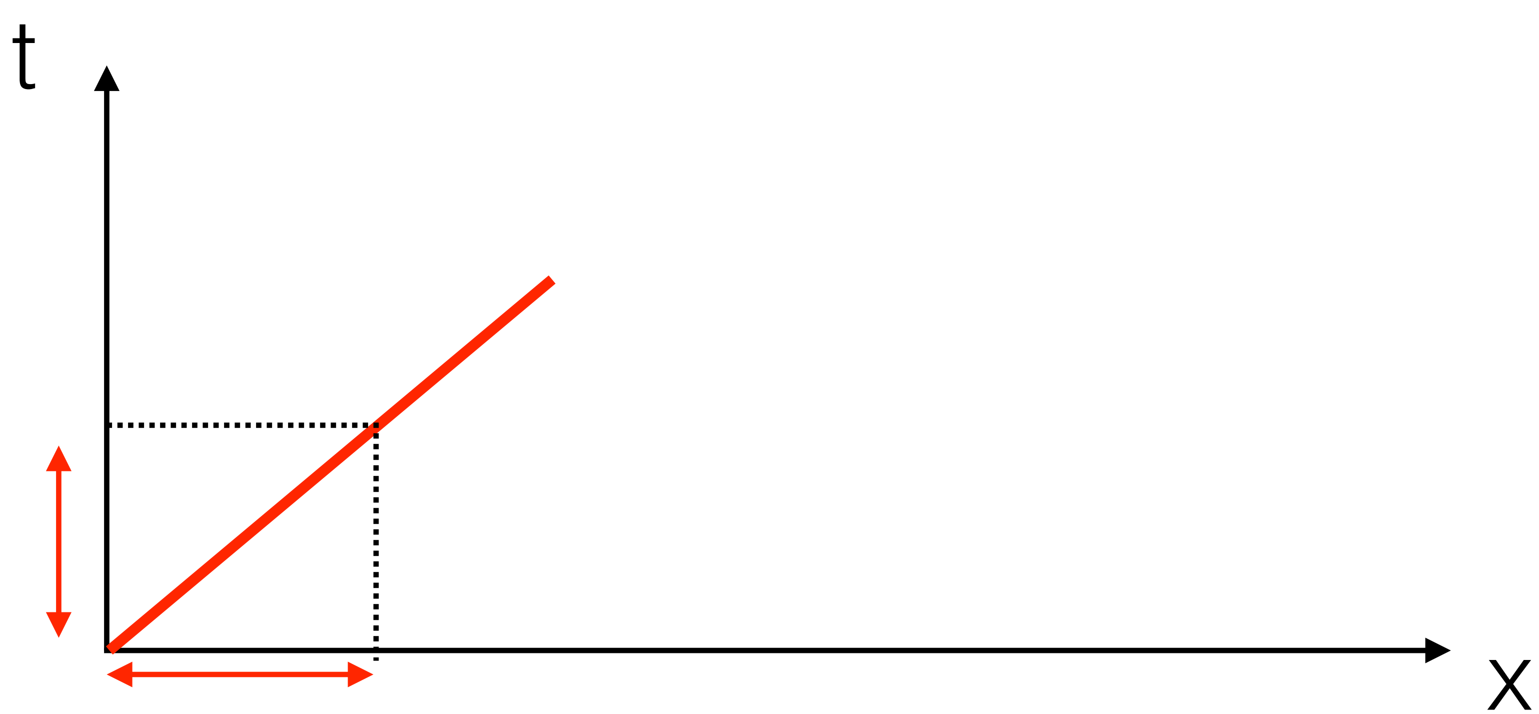


Mathematical formulation of General Relativity (GR)

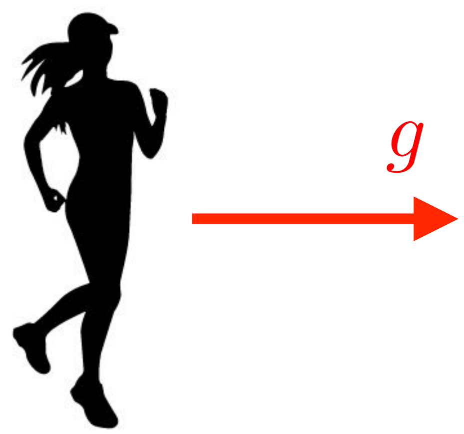
General Relativity Equations

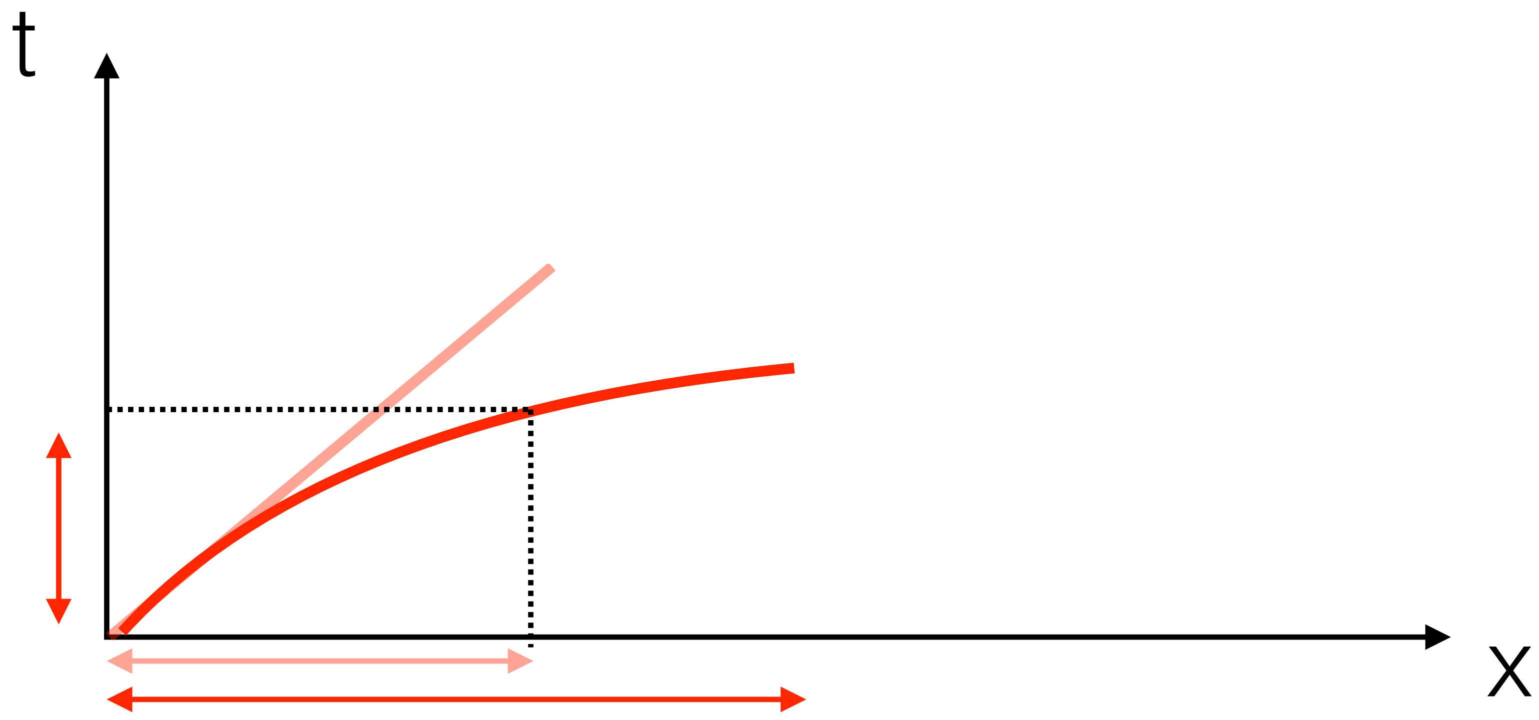


$v = const.$

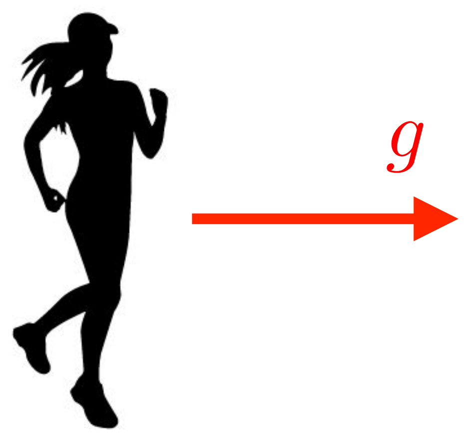


General Relativity Equations

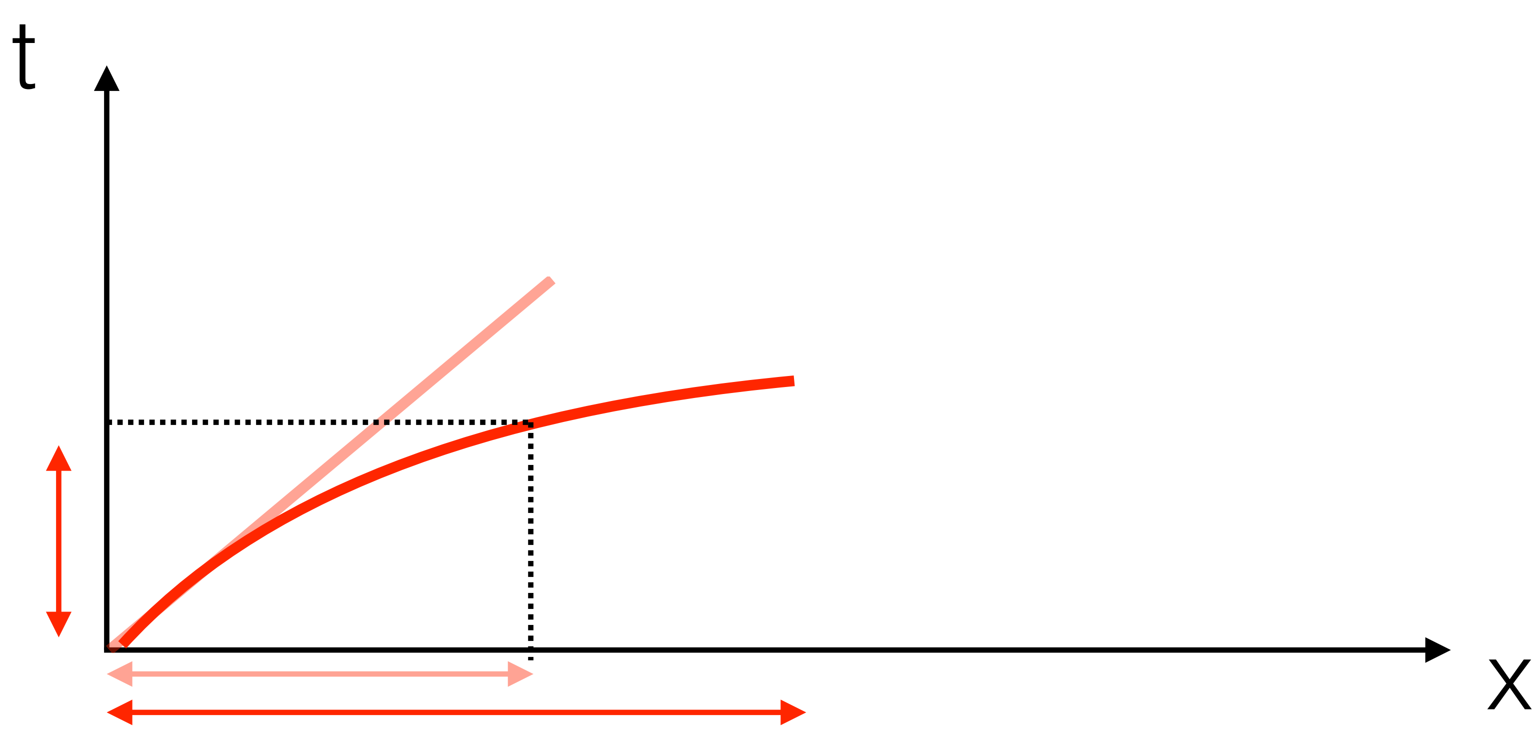




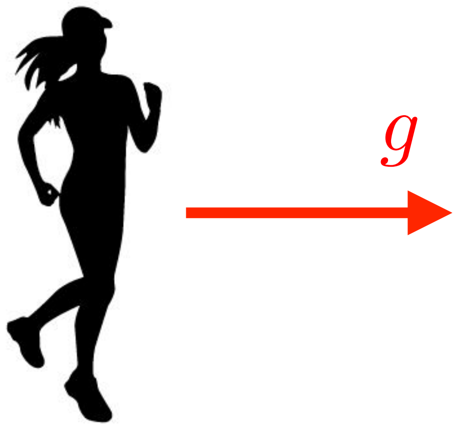
General Relativity Equations



Acceleration $\equiv$ Gravity



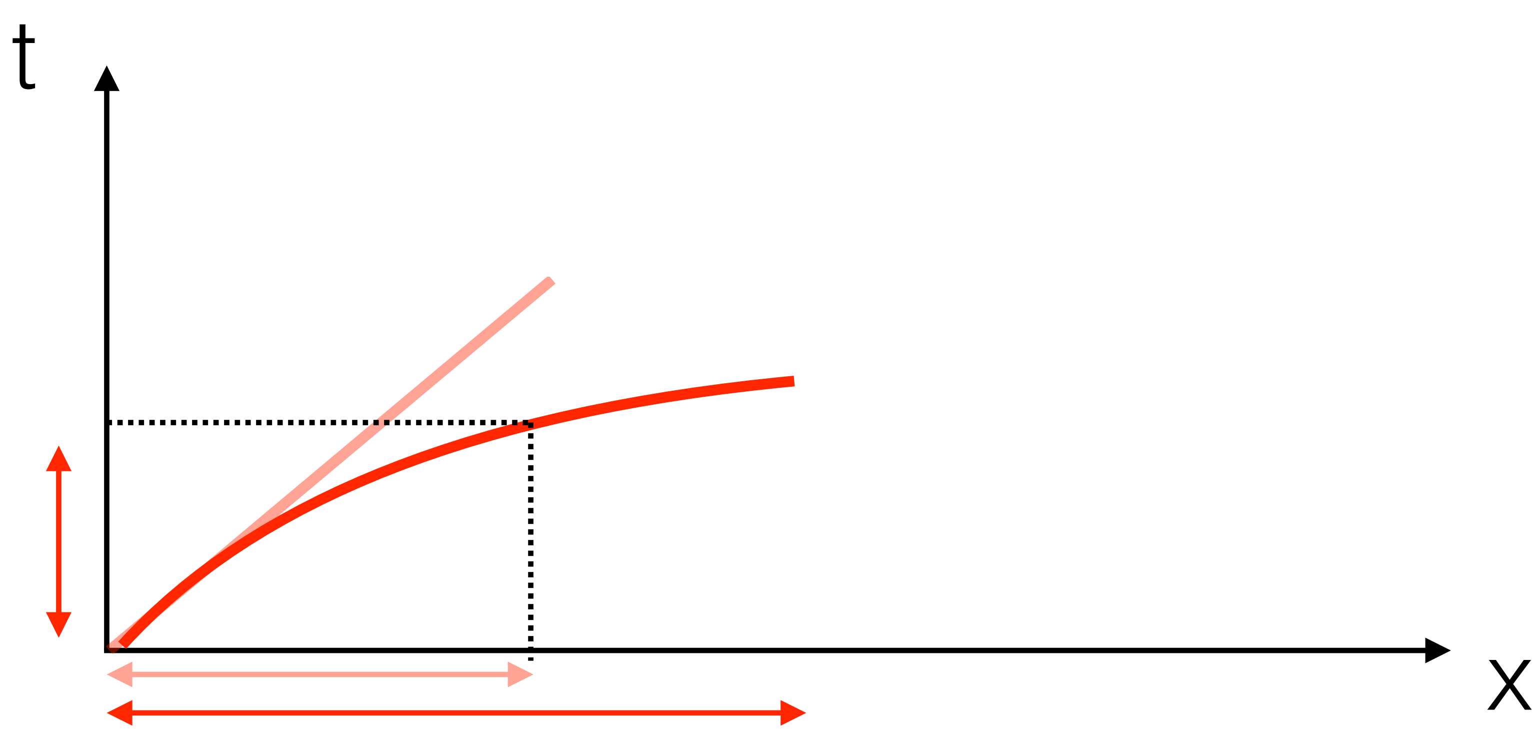
General Relativity Equations



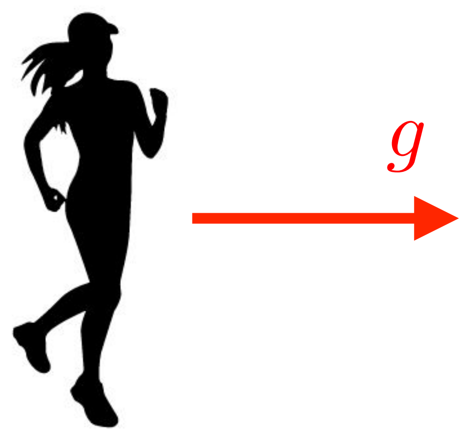
Acceleration $\equiv$ Gravity

$\equiv$

curved space-time trajectory



General Relativity Equations



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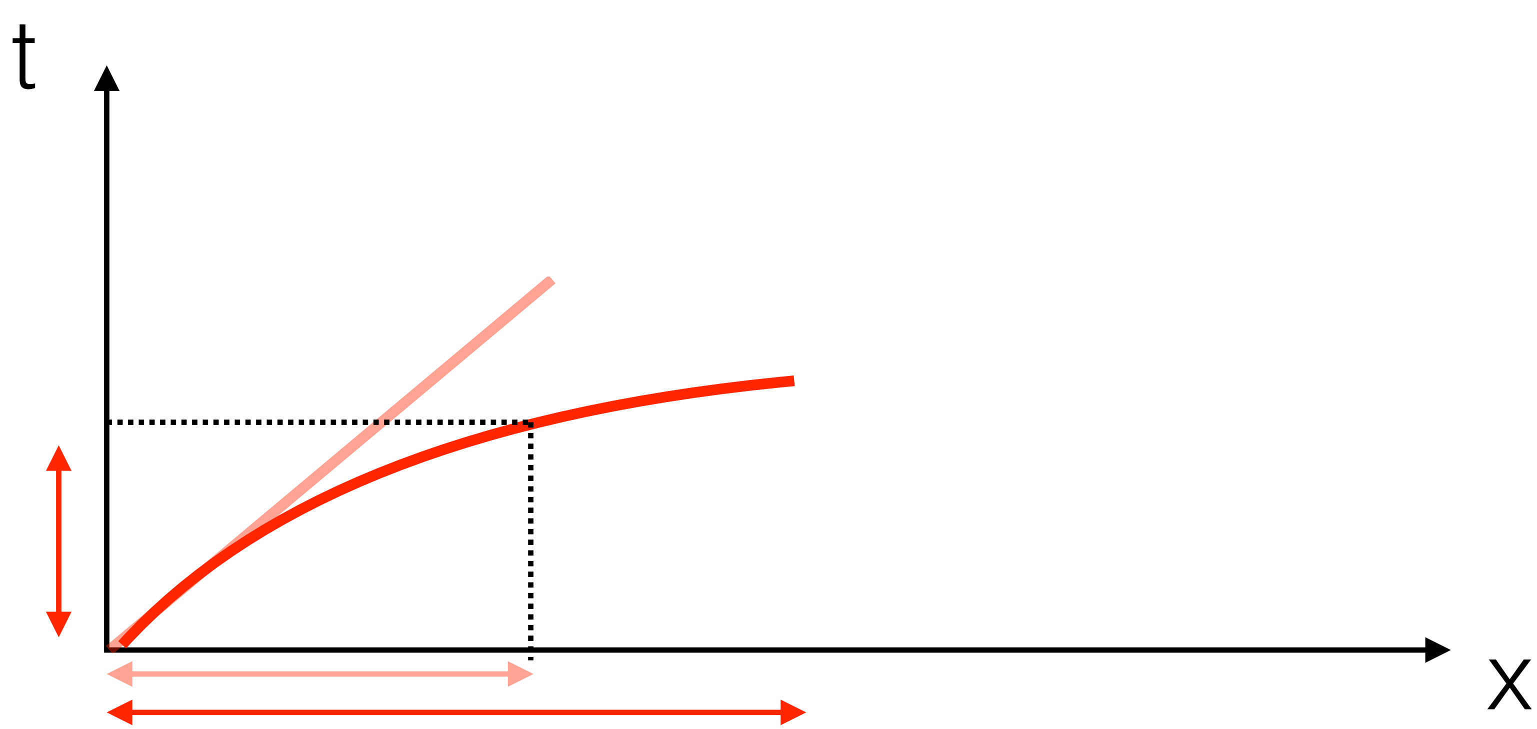
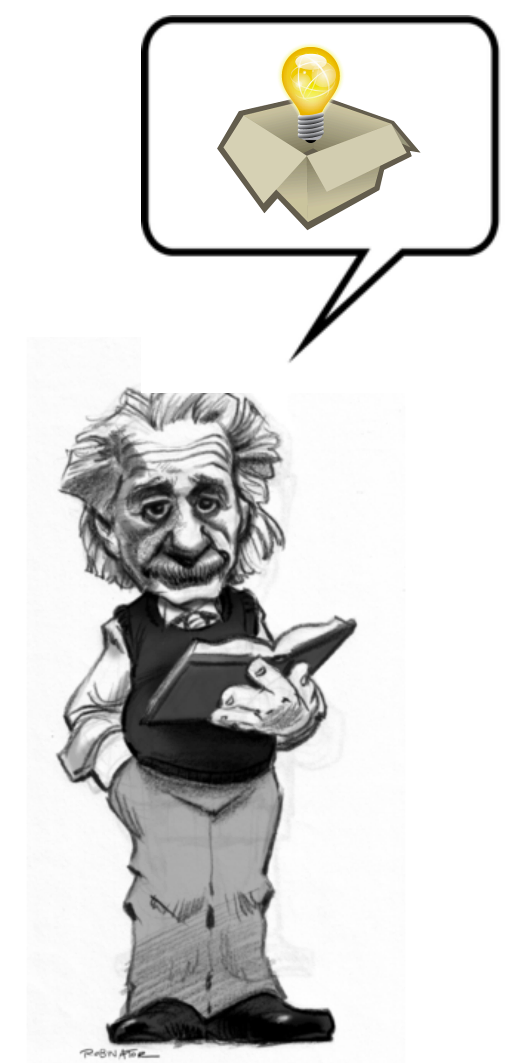
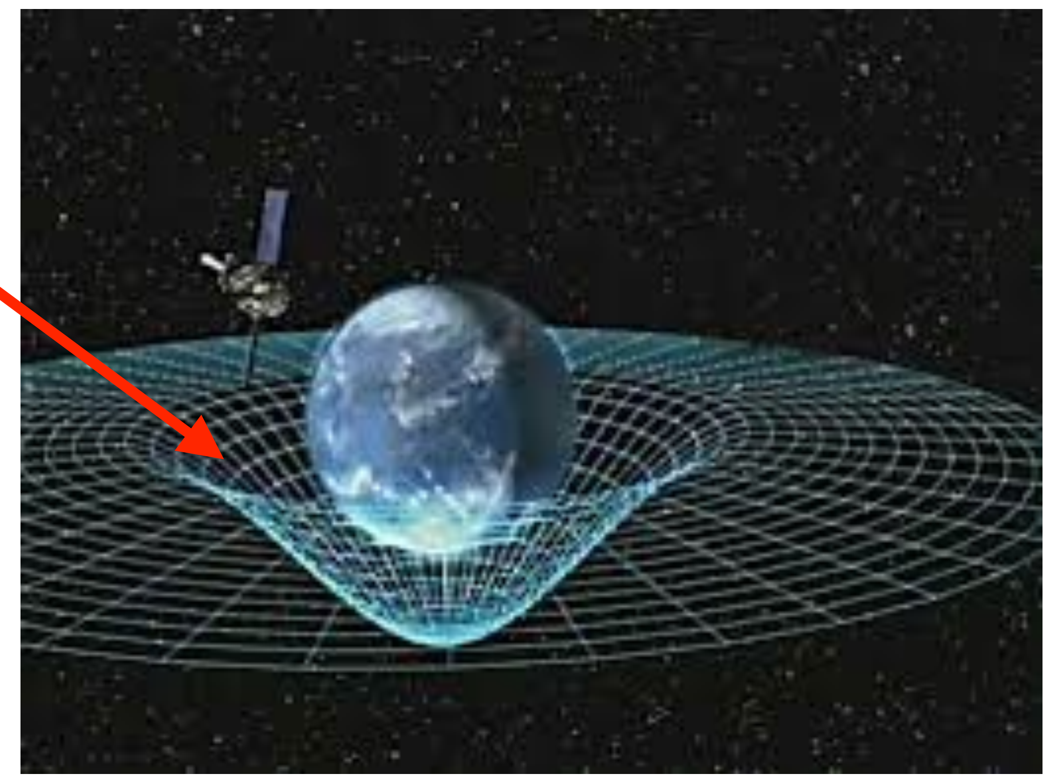
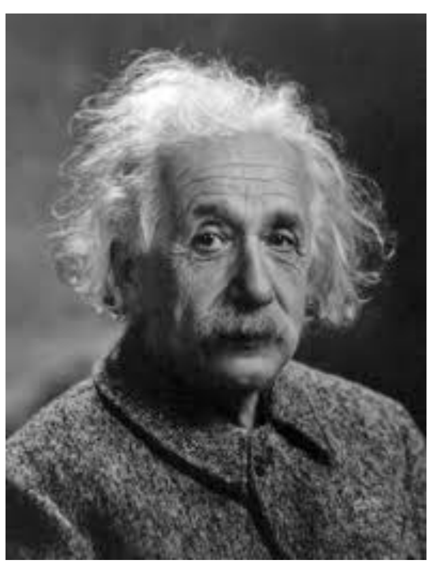




General Relativity Equations

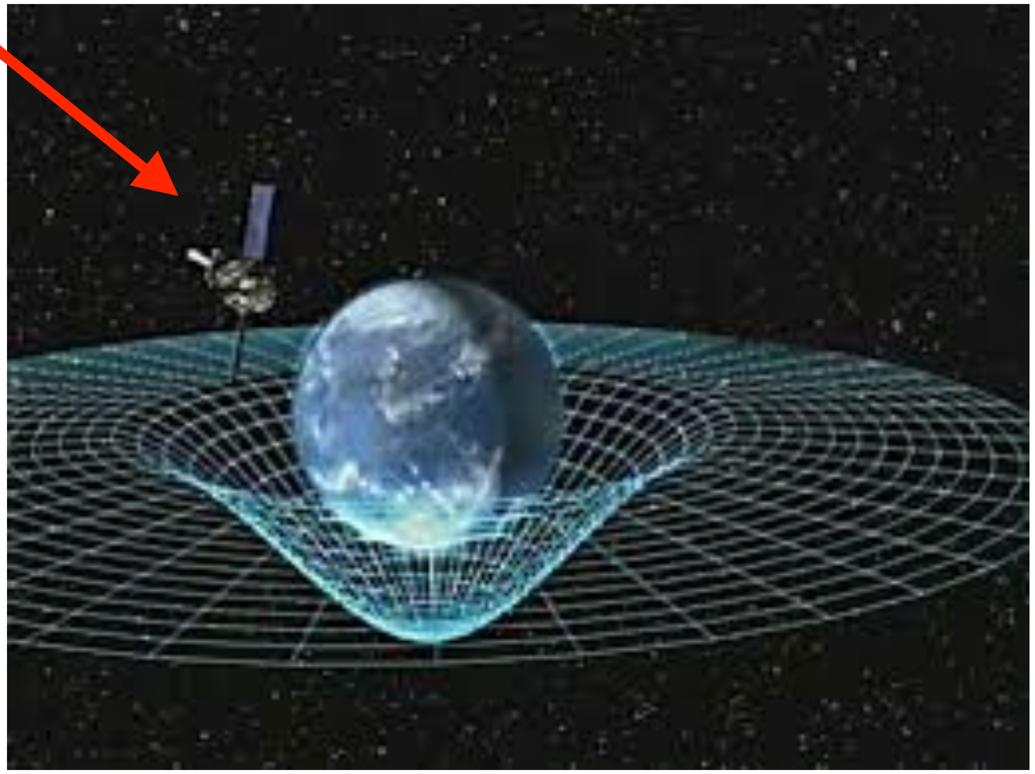
Presence of Matter (Energy/p)
dictates 'Space-Time' Geometry

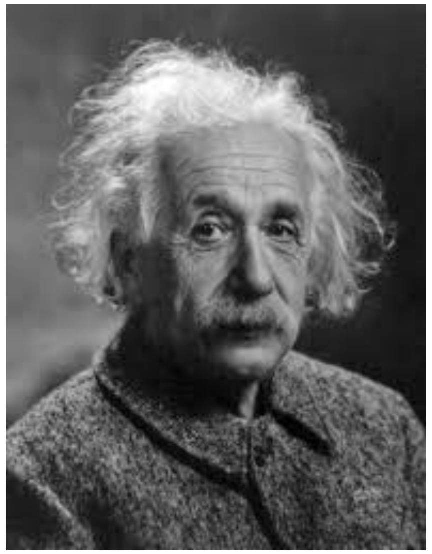




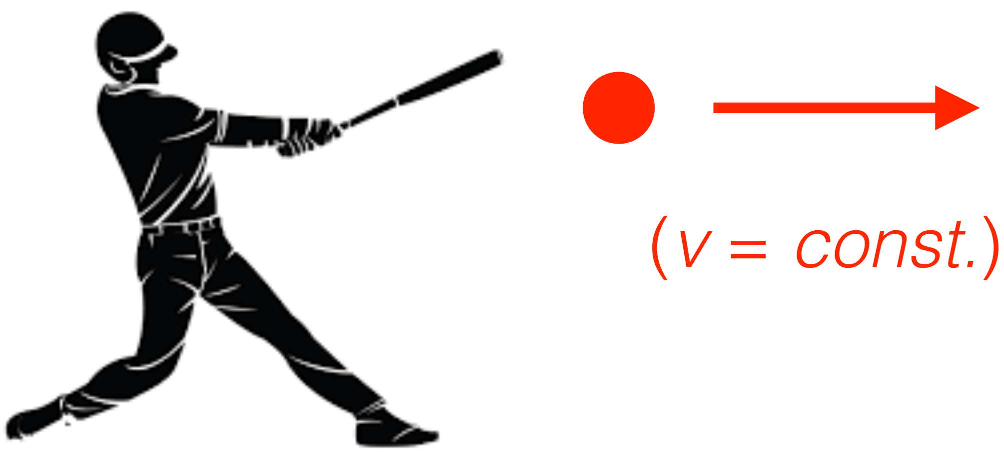
General Relativity Equations

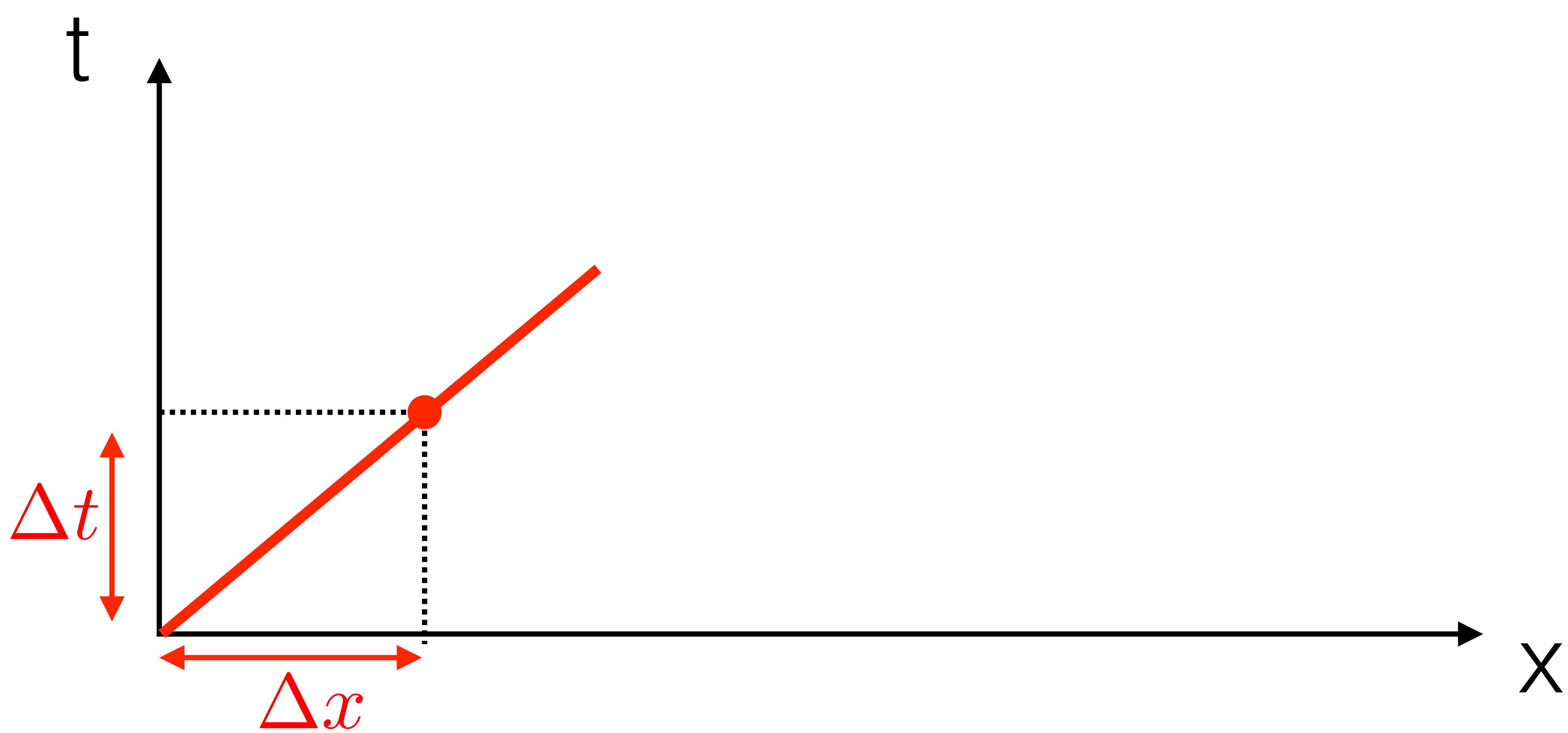
'Space-Time' Geometry
dictates Movement of Matter



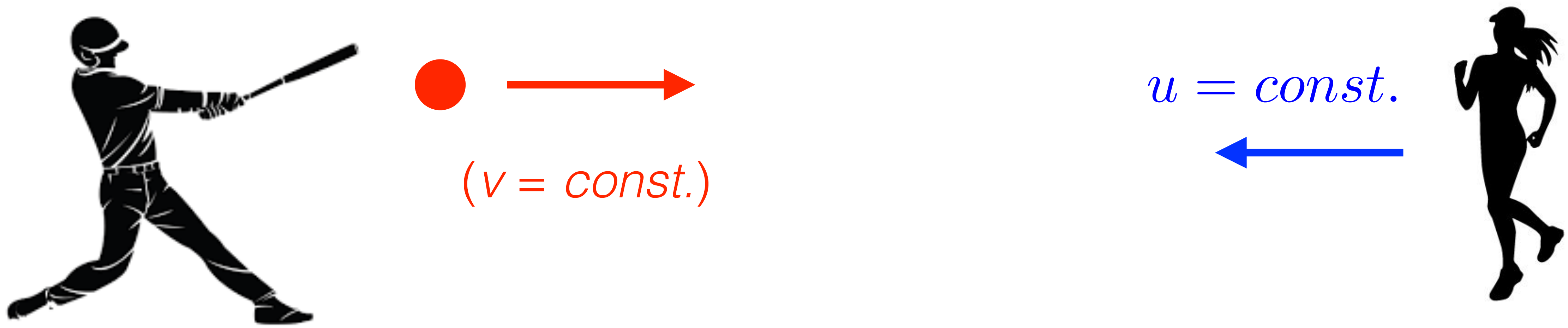


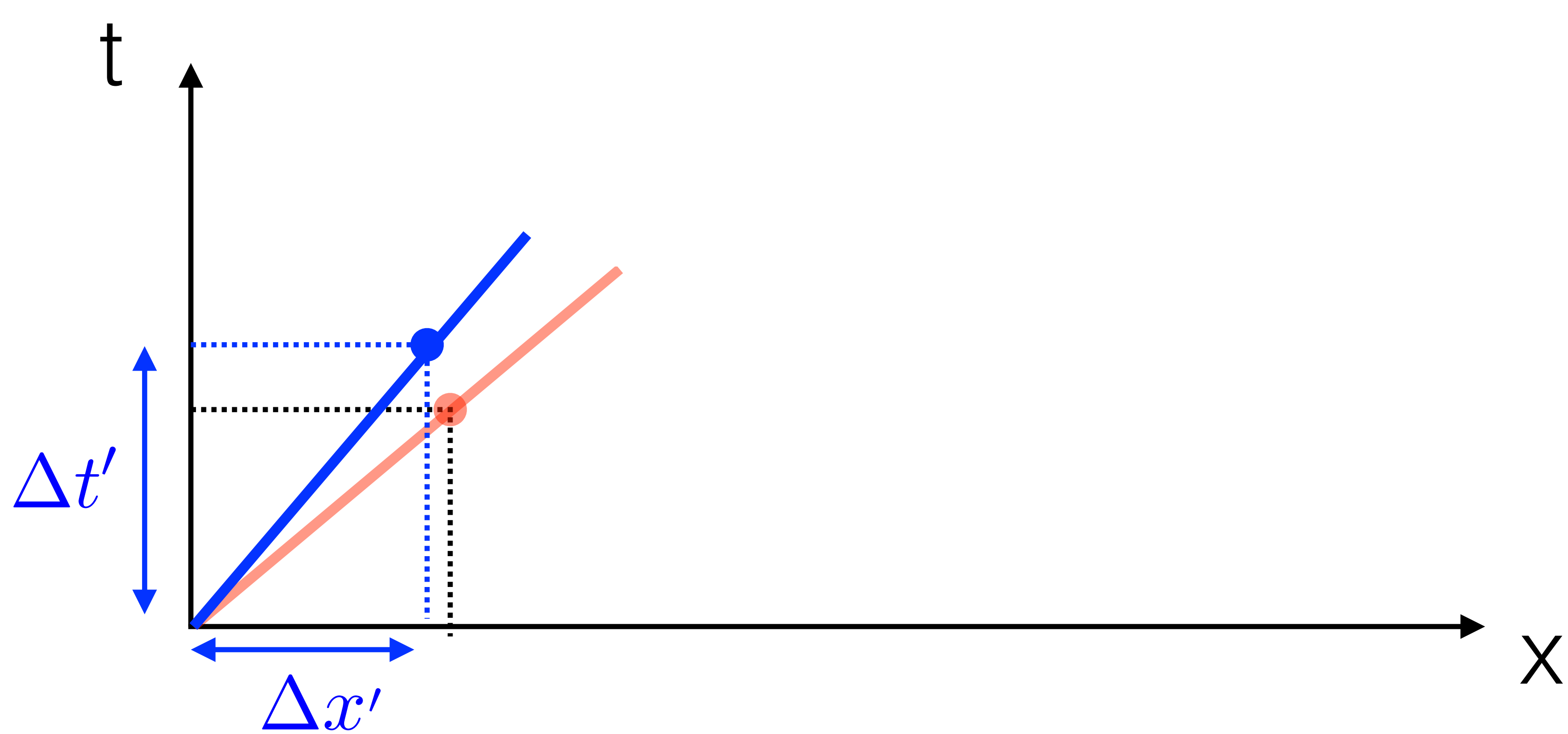
General Relativity Equations





General Relativity Equations



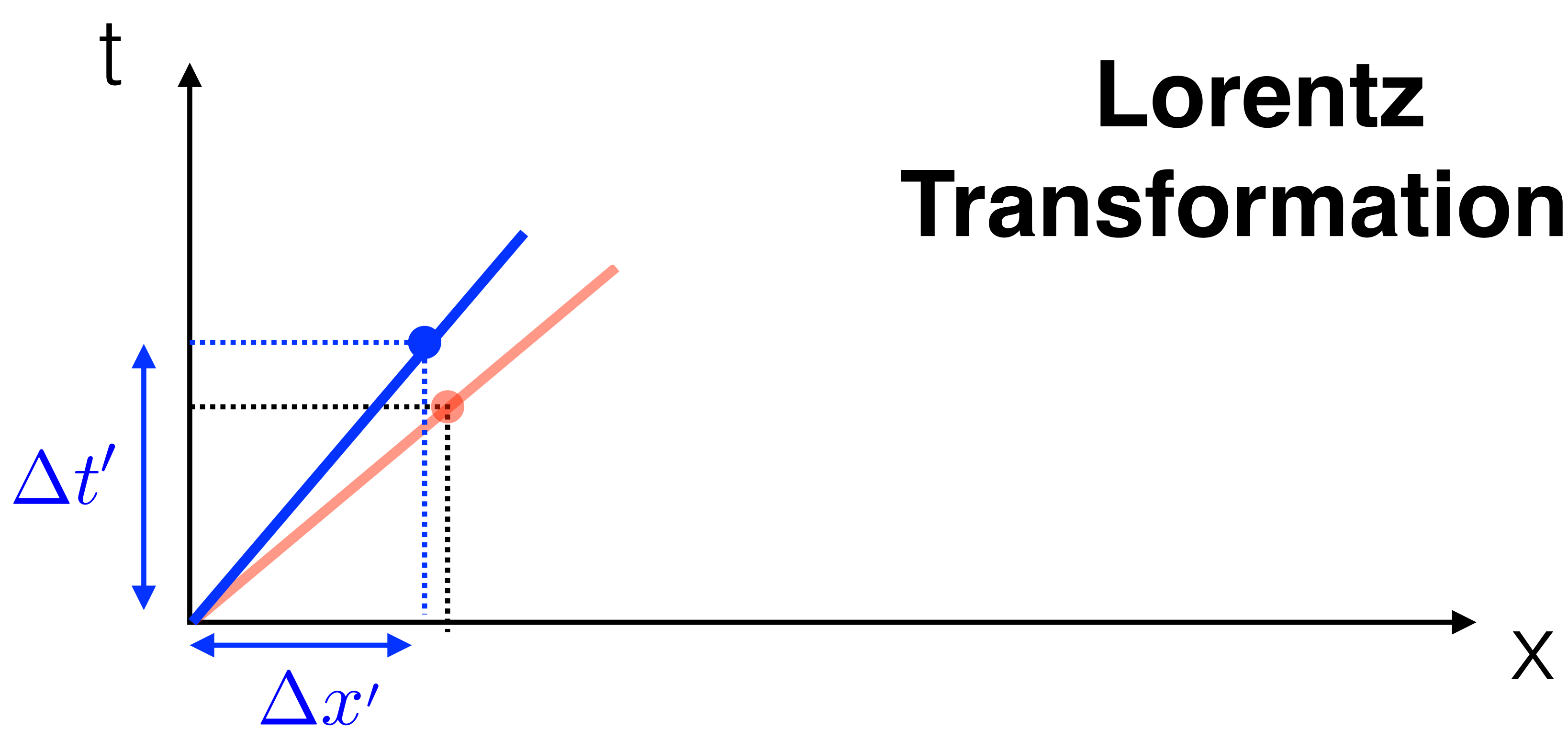


General Relativity Equations

$(v = const.)$

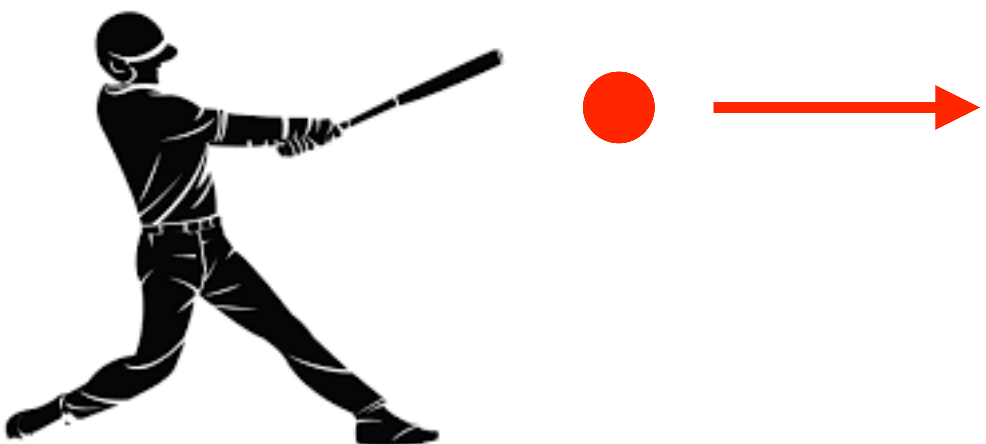
$u = const.$

$$(\Delta x, \Delta t) \neq (\Delta x', \Delta t')$$



Lorentz Transformations

General Relativity Equations



$u = const.$

$$(\Delta x, \Delta t) \neq (\Delta x', \Delta t')$$

$$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 = s'^2 = (c\Delta t')^2 - (\Delta x')^2$$

General Relativity Equations

$$u = const.$$

$$(\Delta x, \Delta t) \neq (\Delta x', \Delta t')$$

$$s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 = s'^2 = (c\Delta t')^2 - (\Delta x')^2$$

$$\boxed{ds^2 = c^2 dt^2 - \sum_j dx_j dx^j}$$

Special Relativity

Space-time interval invariant

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j$$

Space-time invariant
interval (**Special Relativity**)

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

Einstein convention
Summation over repeated indices

Space-time invariant interval (**Special Relativity**)

$$\eta \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

Einstein convention
Summation over repeated indices

Space-time invariant interval (**Special Relativity**)

Minkowski Metric

$$\eta \equiv \mathrm{diag}(-,+,+,+)$$

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

Einstein convention
Summation over repeated indices

Space-time invariant interval (**Special Relativity**)

Minkowski Metric

$$\eta \equiv \mathrm{diag}(-,+,+,+)$$

Space-time invariant interval (**General Relativity**) $\longrightarrow$

$$\boxed{ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu}$$

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

Einstein convention
Summation over repeated indices

Space-time invariant interval (**Special Relativity**)

Minkowski Metric
$$\eta \equiv \mathrm{diag}(-,+,+,+)$$

Space-time invariant interval (**General Relativity**) $\longrightarrow$ $\boxed{ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu}$

$$g_{\mu\nu}(x) dx^\mu dx^\nu = g'_{\alpha\beta}(x') dx'^\alpha dx'^\beta$$

General Relativity Equations

$$ds^2 = -c^2 dt^2 + \sum_j dx_j dx^j \longrightarrow \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

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Space-time invariant interval (**Special Relativity**)

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Space-time invariant interval (**General Relativity**) $\longrightarrow$

$$\boxed{ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu}$$

$$g_{\mu\nu}(x) dx^\mu dx^\nu = g'_{\alpha\beta}(x') dx'^\alpha dx'^\beta$$

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{dx^\mu} \frac{\partial x'^\beta}{dx^\nu} g'_{\alpha\beta}(x')$$

General Relativity Equations

General Relativity: Generalisation of Special Relativity

General Relativity Equations

General Relativity: Generalisation of Special Relativity

* **Equivalence Principle** $\longrightarrow$ Geodesic motion

$$\boxed{\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}[g_{**}]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0}$$

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}\right)$$

Christoffel Symbol

General Relativity Equations

General Relativity: Generalisation of Special Relativity

* **Equivalence Principle** $\longrightarrow$ Geodesic motion

$$\boxed{\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}[g_{**}]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0}$$

* **Principle of Relativity** $\longrightarrow$ $x'^\mu = x'^\mu(\{x^\alpha\})$ Arbitrary change of coordinates

$$g_{\mu\nu}(x)dx^\mu dx^\nu = g'_{\alpha\beta}(x')dx'^\alpha dx'^\beta$$

;

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{dx^\mu}\frac{\partial x'^\beta}{dx^\nu}g'_{\alpha\beta}(x')$$

General Relativity Equations

General Relativity: Generalisation of Special Relativity

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$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{dx^\mu}\frac{\partial x'^\beta}{dx^\nu}g'_{\alpha\beta}(x')$$

$$;\quad G_{\mu\nu}[g_{**}] \equiv \boxed{\underbrace{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}_{\text{space-time geometry}} = \frac{1}{m_p^2}\underbrace{T_{\mu\nu}}_{\text{matter (energy/p)}}}$$

General Relativity Equations

General Relativity: Generalisation of Special Relativity

$$x'^{\mu} = x'^{\mu}(\{x^{\alpha}\})$$

Arbitrary change of coordinates

;

$$g_{\mu\nu}(x)dx^{\mu}dx^{\nu} = g'_{\alpha\beta}(x')dx'^{\alpha}dx'^{\beta}$$

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{dx^{\mu}}\frac{\partial x'^{\beta}}{dx^{\nu}}g'_{\alpha\beta}(x')$$

General Relativity Equations

General Relativity: Generalisation of Special Relativity

$$x'^{\mu} = x'^{\mu}(\{x^{\alpha}\})$$

Arbitrary change of coordinates ;

$$g_{\mu\nu}(x)dx^{\mu}dx^{\nu} = g'_{\alpha\beta}(x')dx'^{\alpha}dx'^{\beta}$$

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{dx^{\mu}}\frac{\partial x'^{\beta}}{dx^{\nu}}g'_{\alpha\beta}(x')$$

$$G_{\mu\nu}[g_{**}] \equiv \boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{m_p^2}T_{\mu\nu}} \quad ; \quad \frac{8\pi G}{c^4} \equiv \frac{1}{m_p^2} \quad ; \quad m_p = 2.44 \cdot 10^{18}\ \mathrm{GeV}$$

space-time geometry

matter (energy/p)

General Relativity Equations

General Relativity: Generalisation of Special Relativity

$$x'^{\mu} = x'^{\mu}(\{x^{\alpha}\})$$ Arbitrary change of coordinates ;

$$g_{\mu\nu}(x) dx^{\mu} dx^{\nu} = g'_{\alpha\beta}(x') dx'^{\alpha} dx'^{\beta}$$

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{dx^{\mu}} \frac{\partial x'^{\beta}}{dx^{\nu}} g'_{\alpha\beta}(x')$$

$$G_{\mu\nu}[g_{**}] \equiv \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{m_p^2} T_{\mu\nu}} \quad ; \quad \frac{8\pi G}{c^4} \equiv \frac{1}{m_p^2} \quad ; \quad m_p = 2.44 \cdot 10^{18}\ \mathrm{GeV}$$

space-time geometry

matter (energy/p)

$$R_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + \Gamma^{\mu}{}_{\lambda\mu} \Gamma^{\lambda}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\lambda\beta} \Gamma^{\lambda}{}_{\alpha\mu}$$ Ricci tensor

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \left(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda} \right) \sim (metric)^2$$ Christoffel Symbol

General Relativity Equations

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, \ldots)$$

source

2nd order, <u>non-Linear</u>

General Relativity Equations

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

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2nd order, non-Linear

source

Extremely difficult to solve !

END of digression on GENERAL RELATIVITY

)

Let’s continue with

PRIMER ON

GRAVITATIONAL WAVES

Gravitational Framework

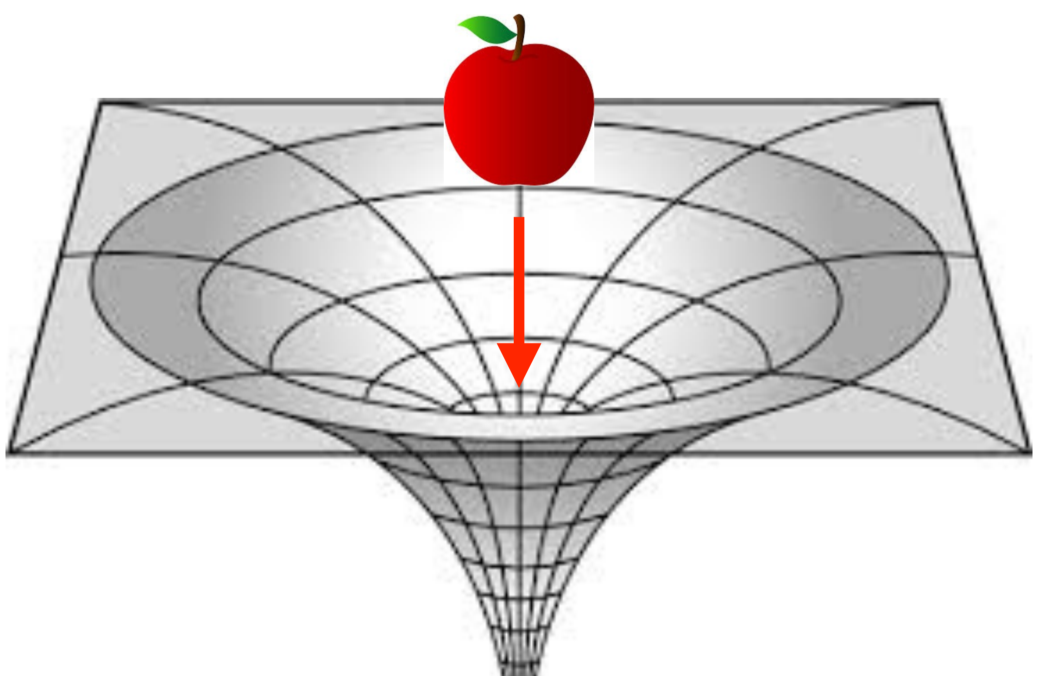
General Relativity (GR)

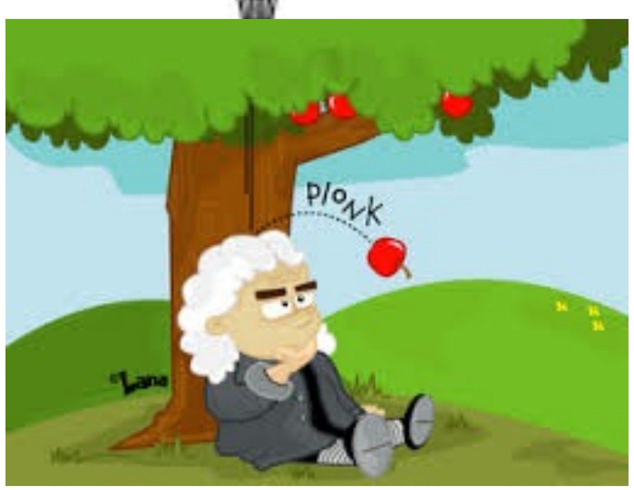
$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass





$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$\mathrm{DIFF}: \quad x^\mu \to x'^\mu(x)$$

symmetry

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, \ldots)$$

2nd order, non-Linear

source

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, ...)$$

source

2nd order, non-Linear

How do we define GWs ?

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt, Rad, Top.Defects, DarkEnergy}, ...)$$

source

expand in perturbations

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

How do we define GWs ?

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt, Rad, Top.Defects, DarkEnergy, ...})$$

expand in perturbations

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

source of GWs

How do we define GWs ?

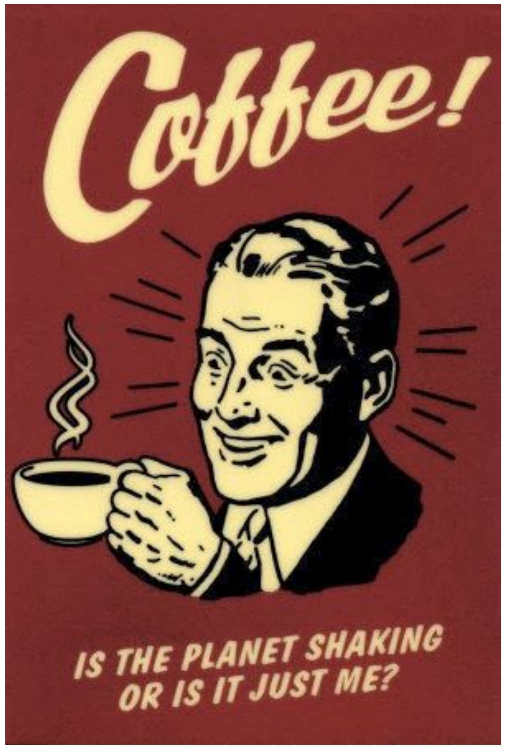
$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

Let's continue this approach…

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

Let's continue this approach…

But not before you load yourselves with coffee ('cause you are gonna need it)



Definition of GWs
1st approach

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Gravitational Wave Definition

LINEARIZED GRAVITY

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$$(\; |h_{\mu\nu}| \ll 1 \;)$$

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \underset{\text{Minkowski}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$\mathrm{DIFF}: \quad x^{\mu} \to x'^{\mu}(x)$$

symmetry **?**

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$\mathrm{DIFF}: \quad \cancel{x^\mu \to x'^\mu}(x)$$

Gravitational Wave Definition

Minkowski

1st approach to GWs $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$ **fixed frame**

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$\text{DIFF}: \quad \cancel{x^\mu \to x'^\mu(x)}$$

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$\boxed{(\ |\partial_\mu \xi_\nu(x)| \lesssim |h_{\mu\nu}|\ )}$$

residual symm.

$$\boxed{h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)}}$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

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residual symm.

$$\boxed{h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)}}$$

Notation:
$$\begin{cases} \partial_{(\mu}\xi_{\nu)} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \partial_{[\mu}\xi_{\nu]} \equiv \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \end{cases}$$

Gravitational Wave Definition

Minkowski

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Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

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Let’s expand Einstein Equations !

Gravitational Wave Definition

Minkowski

1st approach to GWs

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fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)} = -\frac{2}{m_p^2} T_{\mu\nu}$$

Gravitational Wave Definition

1st approach to GWs

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fixed frame

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$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \textbf{Einstein tensor expanded}} = -\frac{2}{m_p^2} T_{\mu\nu}$$

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

$$\Rightarrow \underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu}\bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \textbf{Einstein tensor expanded}} = -\frac{2}{m_p^2} T_{\mu\nu}$$

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residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Gravitational Wave Definition

Minkowski

1st approach to GWs

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fixed frame

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Einstein tensor expanded

residual symm.

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Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

Gravitational Wave Definition

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Einstein tensor expanded

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Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{[\mu} \xi_{\nu]}$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$\underbrace{g_{\mu\nu} = \eta_{\mu\nu}}_{} + h_{\mu\nu}(x)$$

fixed frame

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$$\underbrace{\partial^\alpha\partial_\alpha\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^\alpha\partial_{(\mu}\bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \text{Einstein tensor expanded}} = -\frac{2}{m_p^2}T_{\mu\nu}$$

residual symm.

$$\boxed{\partial^\nu\bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu\bar{h}_{\mu\nu} \neq 0$

$$\partial'^\mu\bar{h}'_{\mu\nu}(x') = \underbrace{\partial^\mu\bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box\xi_\nu$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

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Einstein tensor expanded

residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$\partial'^\mu \bar{h}'_{\mu\nu}(x') = \underbrace{\partial^\mu \bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box \xi_\nu = 0 \quad \Leftrightarrow \quad \Box \xi_\nu = f(x)$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$(\ |h_{\mu\nu}| \ll 1 \ )$

Trace-reversed

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$\mathcal{O}(h_{**})$ **Einstein tensor expanded**

residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$\boxed{\partial'^\mu \bar{h}'_{\mu\nu}(x')} = \underbrace{\partial^\mu \bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box \xi_\nu \boxed{= 0} \Leftrightarrow \Box \xi_\nu = f(x)$$

(solution always!)

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

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residual symm.

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Lorentz gauge

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

fixed frame

Trace-reversed

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residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Gravitational Wave Definition

Minkowski

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residual symm.

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Lorentz gauge

Gravitational Wave Definition

Minkowski

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$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}$$

Gravitational Wave Definition

Minkowski

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residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

(10 - 4 = 6 d.o.f.)

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Minkowski}} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Is that all ?

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Is that all ? Not really …

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

$$\left(\partial^\mu \bar{h}_{\mu\nu} = 0 \quad \rightarrow \quad \partial'^\mu \bar{h}'_{\mu\nu} = 0\right)$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} = 0$

Outside Source

Gravitational Wave Definition

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1st approach to GWs

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IF $T_{\mu\nu} = 0$

Outside Source

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

(transverse-traceless gauge)

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

(transverse-traceless gauge)

(6 - 4 = 2 d.o.f.)

Gravitational Wave Definition

Minkowski

1st approach to GWs

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(further residual gauge)

IF $T_{\mu\nu} \cancel{\neq} 0$

Inside Source !

$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

?

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

(transverse-traceless gauge)

6 - 4 = 2 d.o.f. ?

?

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IF $T_{\mu\nu} \neq 0$

Inside Source !

$$\boxed{\cancel{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}}$$

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

~~(transverse-traceless gauge)~~

6 - 4 = 2 d.o.f. ?

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$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} \cancel{\neq} 0$

Inside Source !

Cannot make $h_{*0} = 0$

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

(6 - 4 = 2 d.o.f.)

Yet there are still only 2 <u>radiative</u> *dof* !

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ?

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. $\longrightarrow$ Gravitational Waves !

can GW be 'gauged away' ? No !

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

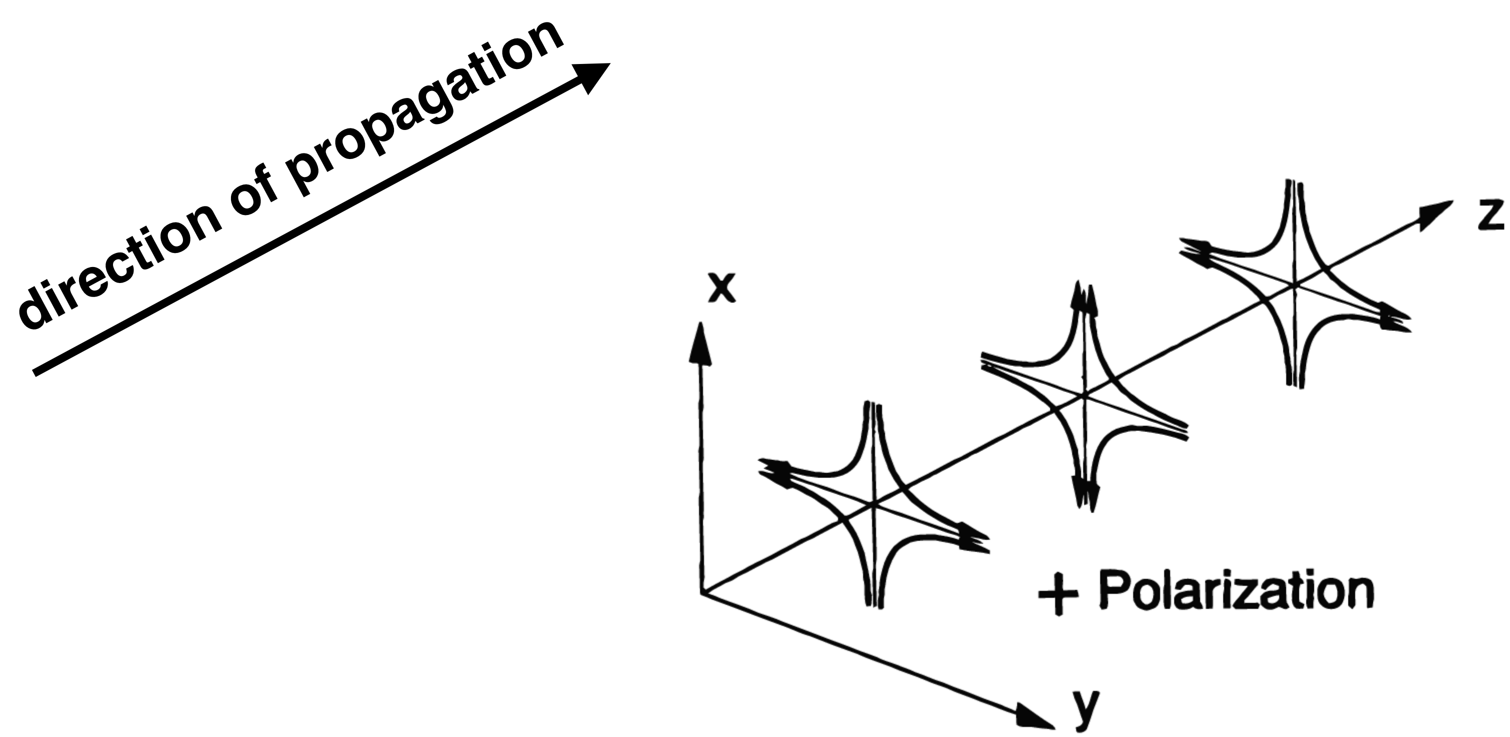
$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. ⟶ Gravitational Waves !

can GW be 'gauged away' ? No !



Transverse
(& Traceless)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

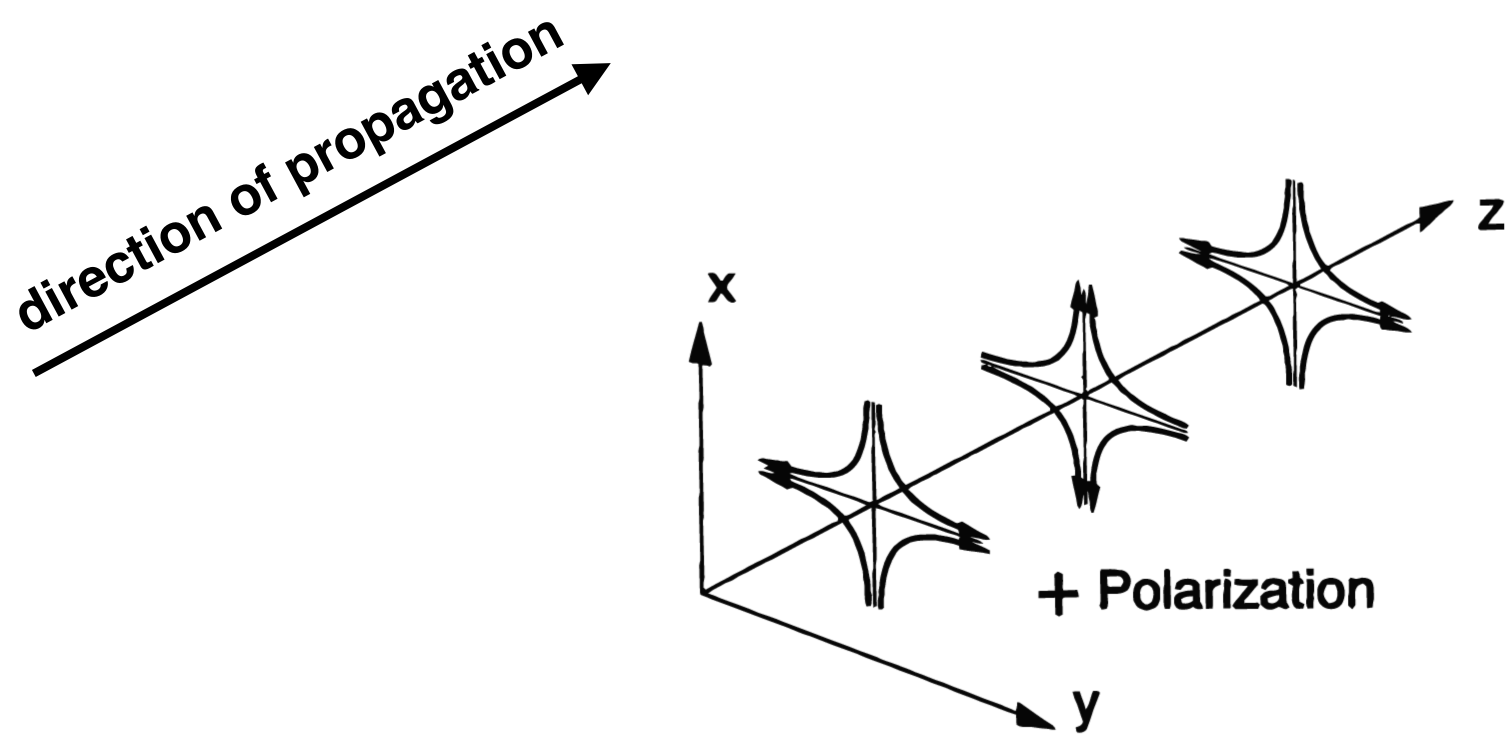
$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. ⟶ Gravitational Waves !

can GW be 'gauged away' ? No !



2 *dof* =
2 polarizations

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

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$$h_{ab}(t, \mathbf{x}) = \int_{-\infty}^{\infty} df \int d\hat{n}\, h_{ab}(f, \hat{n}) e^{-2\pi i f (t - \hat{n}\mathbf{x})}$$

transverse plane

(plane wave)

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$$h_{ab}(f, \hat{n}) = \sum_{A=+,\mathrm{x}} h_A(f, \hat{n}) \epsilon^{(A)}_{ab}(\hat{n}) = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Transverse-Traceless (2 dof)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs $\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$

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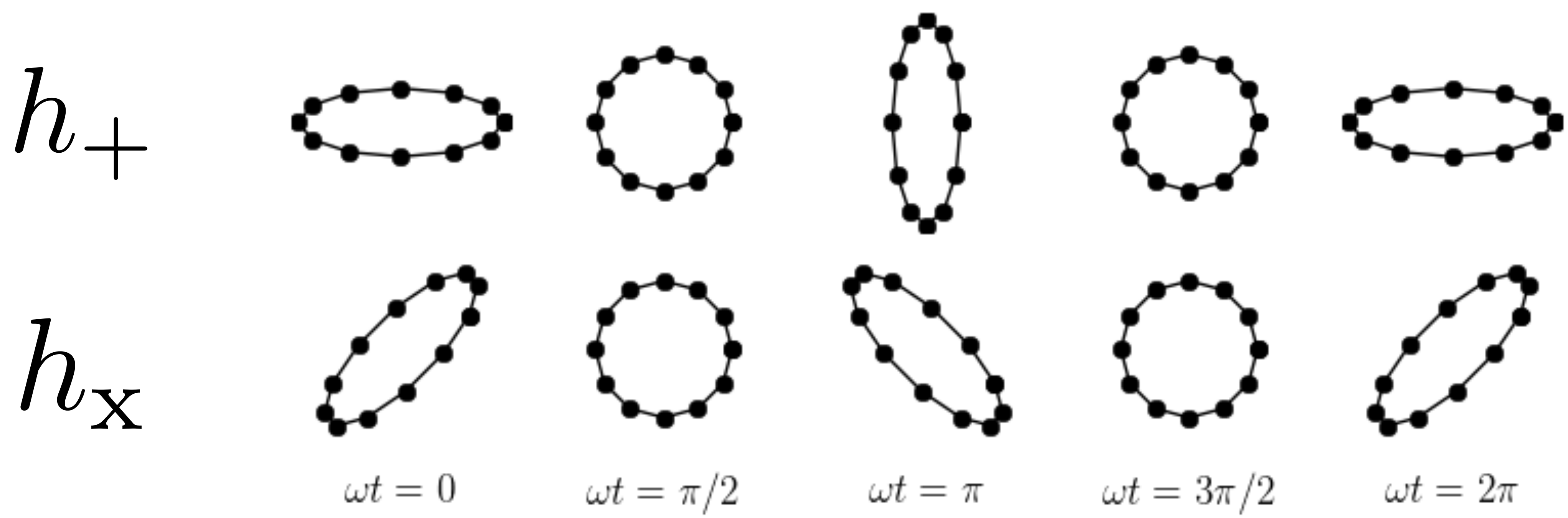

https://www.youtube.com/watch?v=U_hLM1WPDqM https://www.youtube.com/watch?v=EtL9UyRx_Us

Definition of GWs
2nd approach

Gravitational Wave Definition

Minkowski

2nd approach to GWs

(gauge invariant def.)

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

Gravitational Wave Definition

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Minkowski

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$$\delta g_{00} = -2\phi,$$

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s: scalar
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Recall lectures by David Wands

Gravitational Wave Definition

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Metric perturbations

Gravitational Wave Definition

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6 constraints for metric perturbations

6 constraints for E/p tensor components

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$$\partial^\mu T_{\mu\nu} = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \nabla^2 u = \dot{\rho} \ \ (1 \text{ constraint}), \\ \nabla^2 \sigma = \frac{3}{2}(\dot{u} - p) \ \ (1 \text{ constraint}), \\ \nabla^2 v_i = \dot{u}_i \ \ (2 \text{ constraints}). \end{array} \right\}$$

4 constraints (due to E/p conservation)

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~~10~~ degrees of freedom

Physical Constraints

$$\partial^\mu T_{\mu\nu} = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \nabla^2 u = \dot{\rho} \;\; \text{(1 constraint)}, \\ \nabla^2 \sigma = \frac{3}{2}(\dot{u} - p) \;\; \text{(1 constraint)}, \\ \nabla^2 v_i = \dot{u}_i \;\; \text{(2 constraints)}. \end{array} \right\}$$

4 constraints (due to E/p conservation)

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

10 degrees of freedom

(svt E/p-tensor components)

$$T_{00} = \rho,$$

$$T_{0i} = T_{i0} = \partial_i u + u_i,$$

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6 degrees of freedom

Physical Constraints

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6 degrees of freedom

Physical Constraints

$$\partial^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad [\ldots]$$

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6 degrees of freedom

Physical Symmetry

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

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6 degrees of freedom

Physical Symmetry

(4 *d.o.f.* spurious)

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i)$$

$$\text{with } \partial_i d_i = 0,$$

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E},$$

$$\Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E,$$

$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i,$$

with $\partial_i\Sigma_i = 0$

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6 degrees of freedom

Physical Symmetry

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$$h_{ij} \equiv h_{ij}\,, \qquad (h_{ii} = \partial_i h_{ij} = 0) \qquad (2)$$

6 gauge invariant degrees of freedom

Gravitational Wave Definition

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$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad \mathbf{(1)}$$

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6 gauge invariant degrees of freedom

$$\Downarrow$$

Gauge Invariant Einstein Tensor

$$G_{00} = -\nabla^2\Theta,$$

$$G_{0i} = -\frac{1}{2}\nabla^2\Sigma_i - \partial_i\dot{\Theta},$$

$$G_{ij} = -\frac{1}{2}\Box h_{ij} - \partial_{(i}\dot{\Sigma}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi+\Theta\right) + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi+\Theta\right) - \ddot{\Theta}\right].$$

Gravitational Wave Definition

Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad \textbf{(1)}$$

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6 gauge invariant degrees of freedom

$\Downarrow$

Gauge Invariant (perturbed) Einstein Eqs.

$$\nabla^2\Theta = -\frac{1}{m_p^2}\rho, \qquad \nabla^2\Phi = \frac{1}{2m_p^2}(\rho + 3p - 3\dot{u})$$

$$\nabla^2\Sigma_i = -\frac{2}{m_p^2}S_i, \qquad \Box h_{ij} = -\frac{2}{m_p^2}\Pi_{ij}.$$

Gravitational Wave Definition

Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E}, \qquad \textbf{(1)}$$

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6 gauge invariant degrees of freedom

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Gravitational Wave Definition

6 gauge invariant *d.o.f.*

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Gravitational Wave Definition

6 gauge invariant *d.o.f.*

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$$\boxed{h_{ij}\,, \quad (h_{ii} = \partial_i h_{ij} = 0)}$$

transverse & traceless (TT *d.o.f.*)

Gravitational Wave Definition

6 gauge invariant *d.o.f.*

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Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !

Gravitational Wave Definition

6 gauge invariant *d.o.f.*

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transverse & traceless (TT *d.o.f.*)

Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !

Gravitational Waves (GWs) are TT *d.o.f.* metric perturbations, independently of system of reference

Definition of GWs
3rd approach

Gravitational Wave Definition

3rd approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

(separation not well defined)

Gravitational Wave Definition

3rd approach to GWs

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See e.g. Maggiore's 1st Book on GWs

More subtle problem! Solution: Separation of scales !

Gravitational Wave Definition

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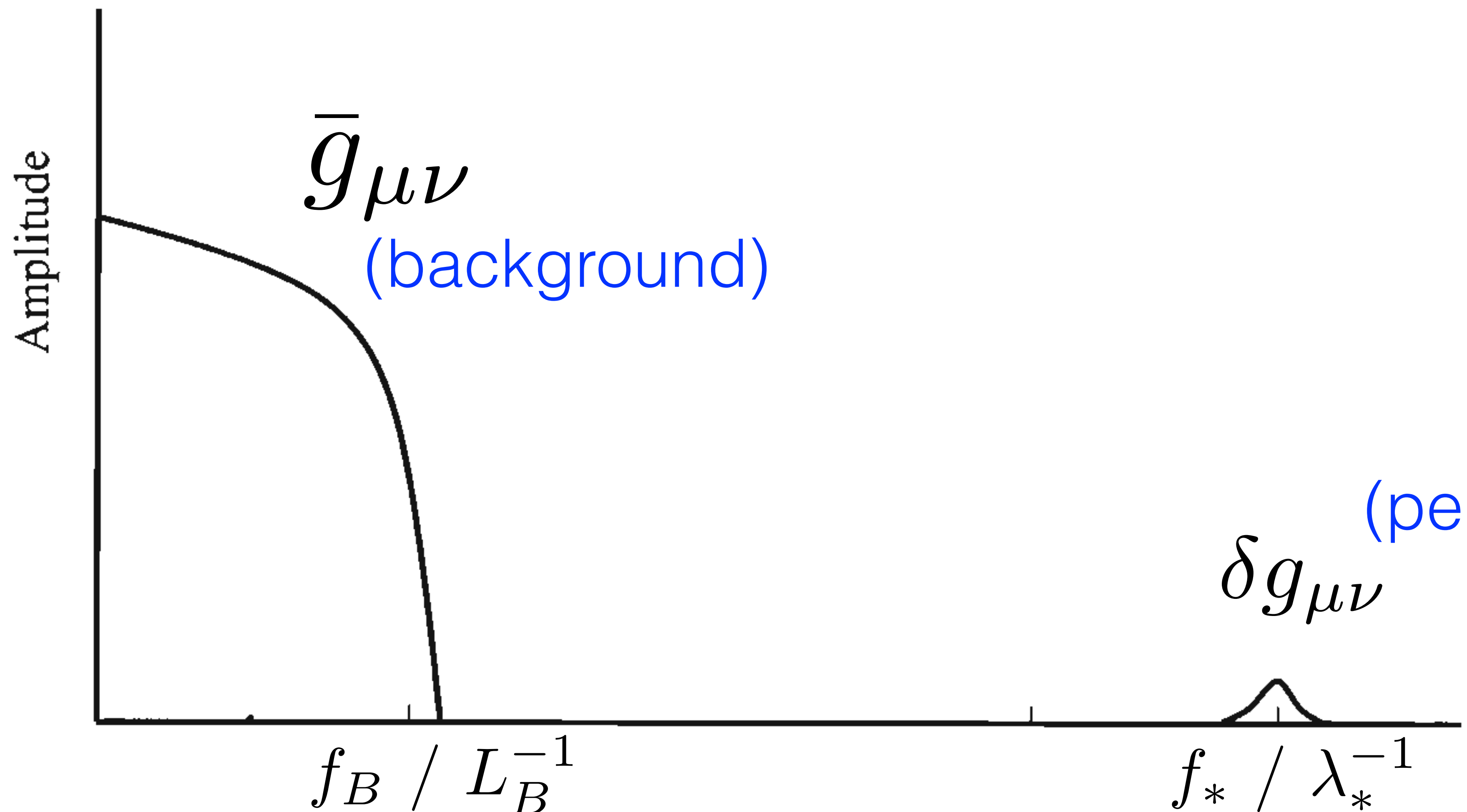

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3rd approach to GWs

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More subtle problem! Solution: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Gravitational Wave Definition

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More subtle problem! Solution: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \Longrightarrow R_{\mu\nu} = \underbrace{\bar{R}_{\mu\nu}}_{\text{(background)}} + \underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}(\delta g)} + \underbrace{R^{(2)}_{\mu\nu}}_{\mathcal{O}(\delta g^2)} + \ldots\,,$$

Gravitational Wave Definition

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Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\mathrm{Low}} + \dfrac{1}{m_p^2}\left(T_{\mu\nu} - \dfrac{1}{2}g_{\mu\nu}T\right)^{\mathrm{Low}}$

Gravitational Wave Definition

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(for a curved space-time)

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High Freq. / Short Scale: $$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$$

Gravitational Wave Definition

3rd approach to GWs

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(separation not well defined)

More subtle problem! Solution: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \longrightarrow R_{\mu\nu} = \underbrace{\bar{R}_{\mu\nu}}_{\text{(background)}} + \underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}(\delta g)} + \underbrace{R^{(2)}_{\mu\nu}}_{\mathcal{O}(\delta g^2)} + \ldots\,,$$

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Gravitational Wave Definition

Low Freq. / Long Scale: $$\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\mathrm{Low}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\mathrm{Low}}$$

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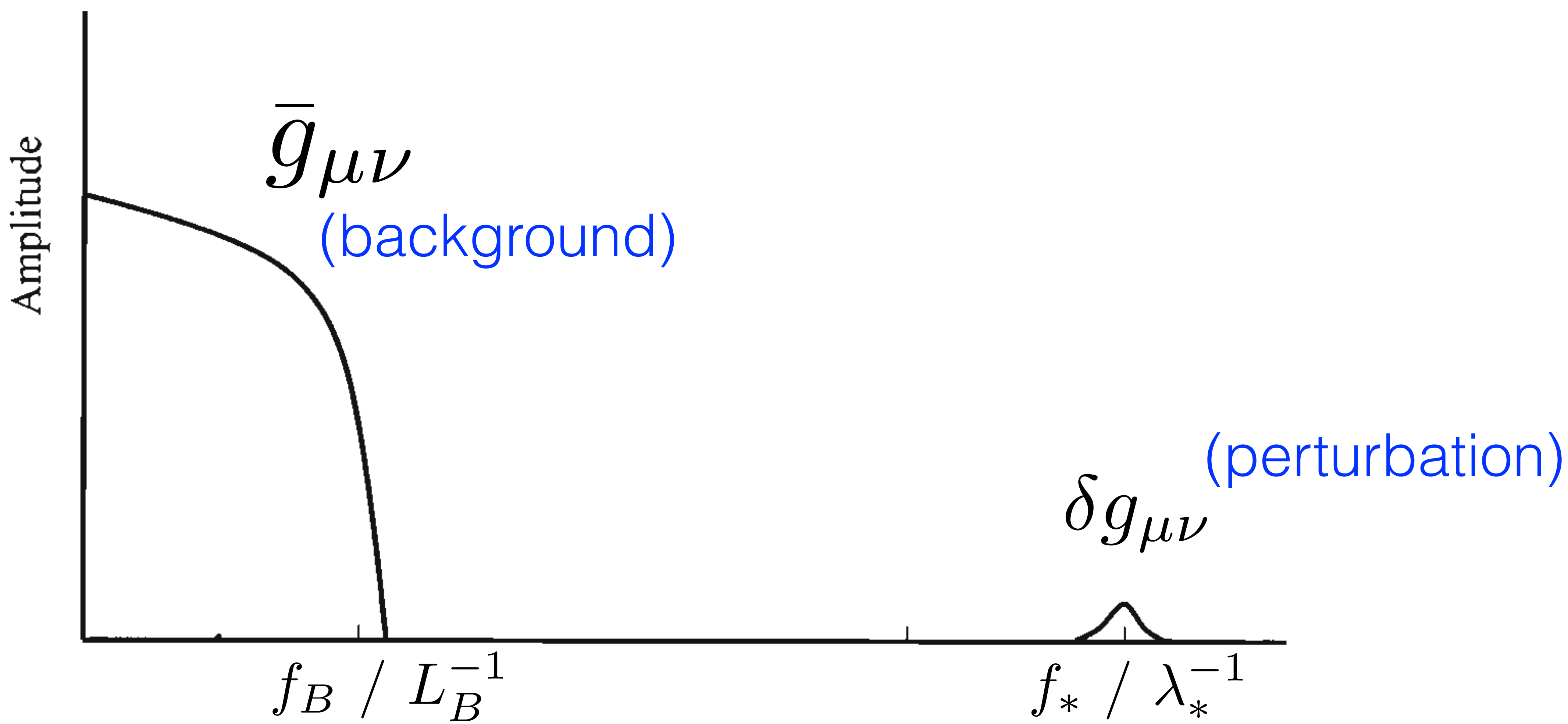

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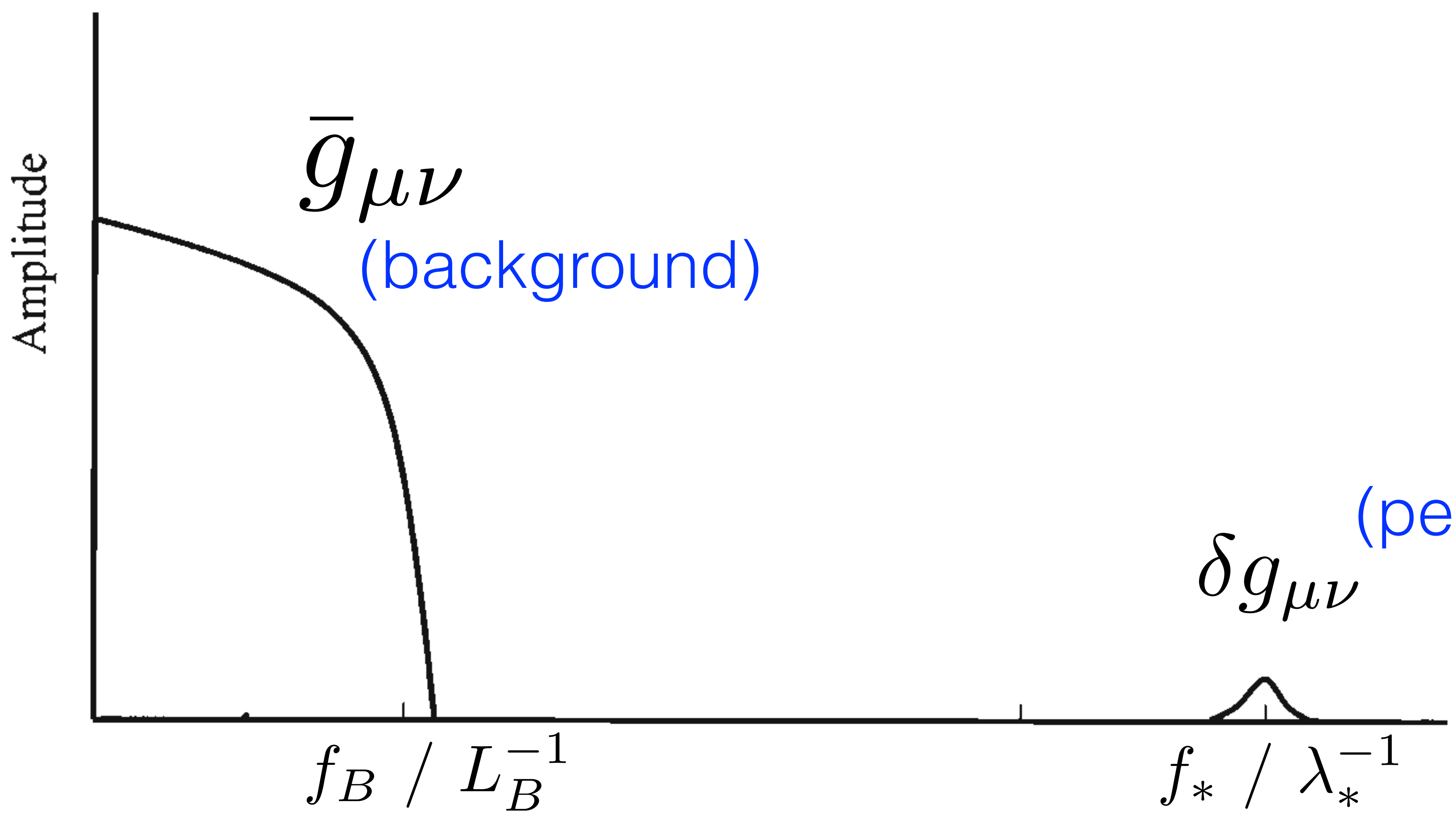
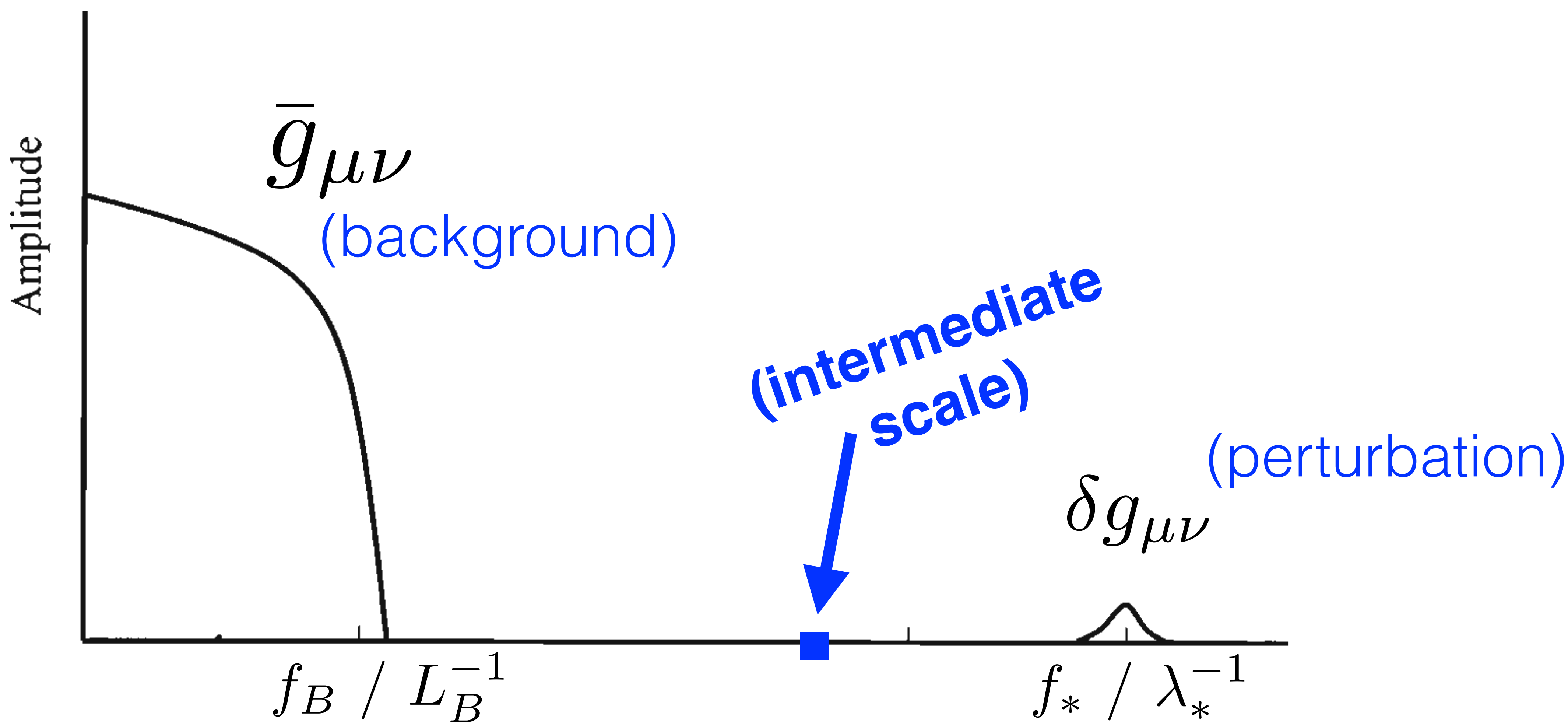
(space/time average)



Gravitational Wave Definition

Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \dfrac{1}{m_p^2} \langle T_{\mu\nu} - \dfrac{1}{2} g_{\mu\nu} T \rangle$ **(space/time average)**



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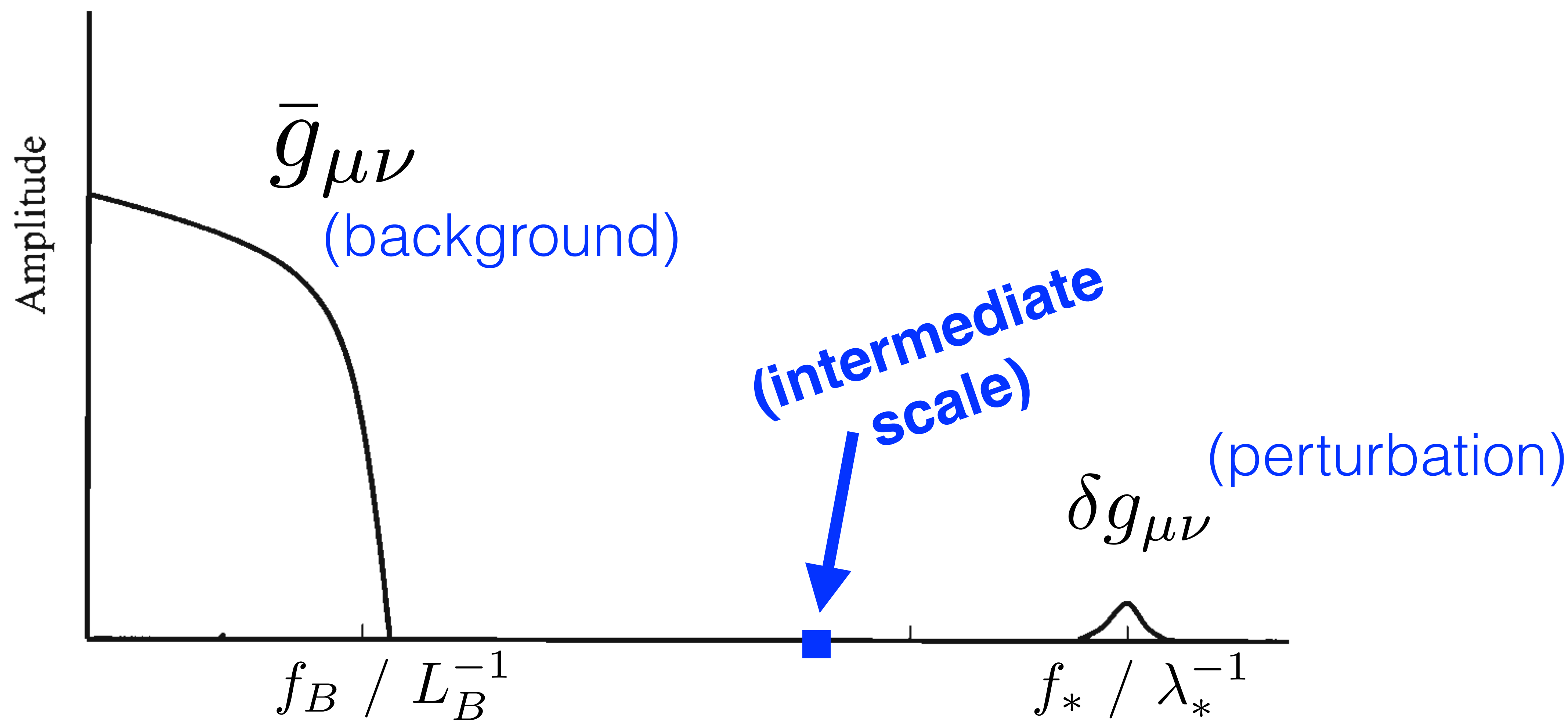
$$t_{\mu\nu} = -\frac{1}{m_p^2} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle \qquad \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle = \bar{T}^{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}$$



Gravitational Wave Definition

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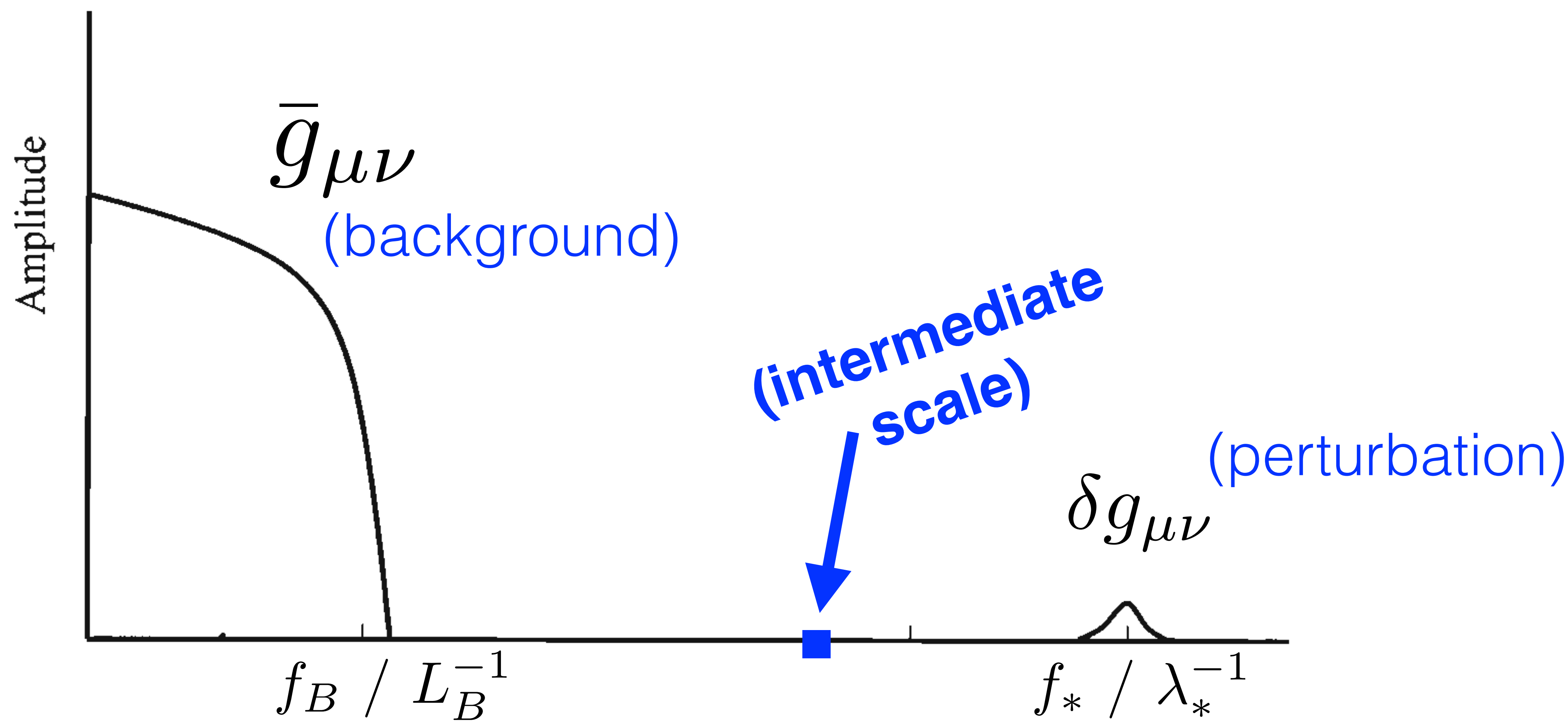

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$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle \partial_\mu \delta g_{\alpha\beta}\, \partial_\nu \delta g^{\alpha\beta}\rangle \quad\Longrightarrow\quad t_{\mu\nu} = \frac{m_p^2}{4}\langle \partial_\mu \delta g_{\alpha\beta}\, \partial_\nu \delta g^{\alpha\beta}\rangle$$

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It can be shown that only TT *dof* contribute to < … >

Gravitational Wave Definition

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$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g^{\rm TT}_{ij}\,\partial_\nu\delta g^{\rm TT}_{ij}\rangle}$$

GW energy-momentum tensor

Gravitational Wave Definition

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$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g_{ij}^{\rm TT}\,\partial_\nu\delta g_{ij}^{\rm TT}\rangle} \;\xrightarrow[(\delta g_{ij}\,\equiv\,h_{ij})]{}\; \boxed{\frac{dE}{dAdt} = \frac{m_p^2}{4}\langle\dot{h}_{ij}^{\rm TT}\,\dot{h}_{ij}^{\rm TT}\rangle}$$

GW energy-momentum tensor — GW power/area radiated

Gravitational Wave Definition

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It can be shown that only TT *dof* contribute to < … >

$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g_{ij}^{\rm TT}\,\partial_\nu\delta g_{ij}^{\rm TT}\rangle} \;\xrightarrow[(\delta g_{ij}\equiv h_{ij})]{}\; \boxed{\rho_{\rm GW} \equiv \frac{m_p^2}{4}\langle\dot{h}_{ij}^{\rm TT}\,\dot{h}_{ij}^{\rm TT}\rangle}$$

GW energy-momentum tensor

GW energy density

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{High}}$$

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$$D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\, \overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta} \,\right)$$

Lorentz gauge

Gravitational Wave Propagation

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Lorentz gauge

vacuum

$$\longrightarrow \quad \boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = 0}$$

Propagation of GWs
in curved space-time

Gravitational Wave Propagation

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Lorentz gauge

→

vacuum

$$\longrightarrow \quad \boxed{D_\alpha D^\alpha \delta g^{\mathrm{TT}}_{ij} = 0}$$

Propagation of GWs
in curved space-time
$\left(D_i \delta g^{\mathrm{TT}}_{ij} = \bar{g}^{ij}\delta g^{\mathrm{TT}}_{ij} = 0 \right)$

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\mathrm{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\mathrm{High}}$$

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Lorentz gauge

$$\longrightarrow \quad \boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = \overset{\text{matter}}{\Pi_{\mu\nu}}}$$

Creation of GWs
in curved space-time

Gravitational Wave Propagation

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Lorentz gauge

$$\longrightarrow \quad \boxed{D_\alpha D^\alpha \delta g^{\mathrm{TT}}_{\mu\nu} = \Pi^{\mathrm{TT}}_{\mu\nu}}$$

matter

Creation of GWs
in curved space-time
TT dof = truly radiative !
[no gauge choice]

Definition of GWs

* 1st approach: Lin Grav in Minkowski ✓
* 2nd approach: SVT decomp. ✓
* 3rd approach: General backgrounds

Some perspective

GW Propagation/Creation in Cosmology

FLRW: $ds^2 = a^2(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j),$ $\qquad \mathrm{TT}: \begin{cases} h_{ii} = 0 \\ h_{ij},_j = 0 \end{cases}$

(conformal time)

GW Propagation/Creation in Cosmology

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(conformal time)

Creation/Propagation GWs in FLRW

Eom: $\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi^{\mathrm{TT}}_{ij},}$

Anisotropic Stress

$\boxed{\Pi_{ij} = T_{ij} - \langle T_{ij} \rangle_{\text{FLRW}}}$

GW Propagation/Creation in Cosmology

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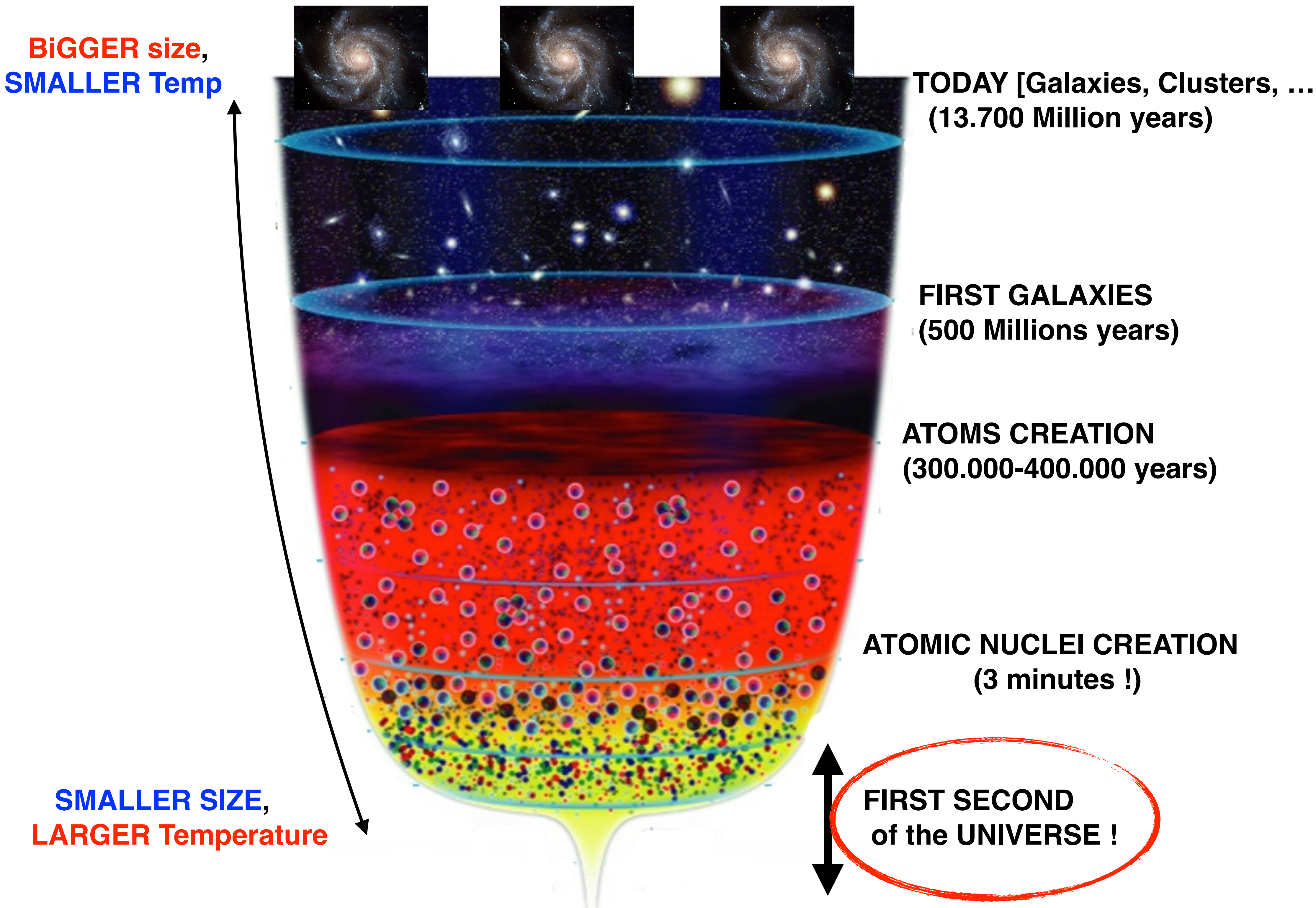
GW Source(s): (SCALARS , VECTOR , FERMIONS)

$\Pi^{TT}_{ij} \propto \{\partial_i\chi^a\partial_j\chi^a\}^{TT}, \quad \{E_iE_j + B_iB_j\}^{TT}, \quad \{\bar\psi\gamma_iD_j\psi\}^{TT}$

Cosmic History

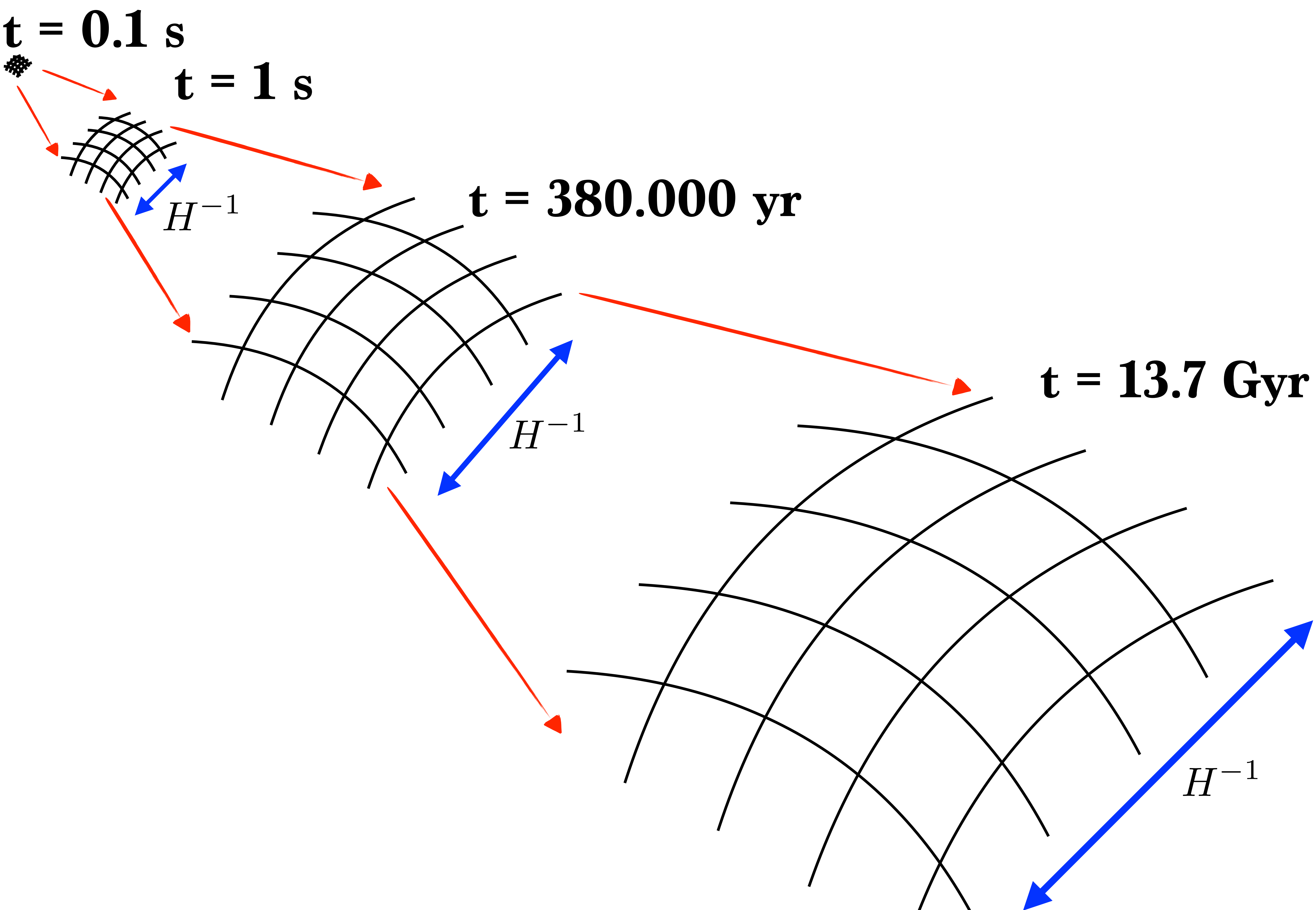


To Be Continued …

BACK SLIDES

FLRW

Expanding Universe



Expanding Universe

H & I

$$T^{\mu}_{\nu} \equiv \mathrm{diag}(-\rho, p, p, p)$$

$$\Downarrow$$

$$m_p^2 G^{\mu}_{\nu}\left[g_{**}^{(FRW)}\right] = T^{\mu}_{\nu}$$

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$$\Downarrow$$

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$$\Rightarrow$$

Friedmann Equations

$$\boxed{\begin{aligned} \frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{\rho}{6m_p^2}(1+3w) \quad \text{(I)} \\ H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 &= \frac{\rho}{3m_p^2} - \frac{K}{a^2} \quad \text{(II)} \end{aligned}}$$

$$\left(w \equiv \frac{p}{\rho}\right)$$ Equation of State (EoS)

Expanding Universe

H & I

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(I)

(II)

$$\left(w \equiv \frac{p}{\rho} \right)$$ Equation of State (EoS)

(I) + (II) $\Rightarrow$

$$\boxed{\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt}(1+w)}$$

(III)

Expanding Universe

Friedmann Equations

UNIVERSE: 1) GR + 2) H & I $\Rightarrow$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{\rho}{6m_p^2}(1+3w) \qquad \text{(I)}$$

$$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \qquad \text{(II)}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt}(1+w) \qquad \text{(III)}$$

$$\left(w \equiv \frac{p}{\rho}\right)$$

Equation of State (EoS)

Expanding Universe

(II) $H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2}$ $\longrightarrow$ $\boxed{\rho_c \equiv 3m_p^2 H^2}$

Critical density $(\rho = \rho_c \Leftrightarrow K = 0)$

Expanding Universe

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$$\rho = \sum_i \rho_i \;\; ; \;\; \Omega_i \equiv \frac{\rho_i}{\rho_c} \Rightarrow \Omega \equiv \frac{\rho}{\rho_c} = \sum_i \Omega_i \Rightarrow \boxed{\Omega - 1 \equiv \frac{k}{a^2 H^2}}$$

Cosmic Sum

Expanding Universe

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Cosmic Sum

$$\begin{cases} \Omega > 1 \Rightarrow \mathrm{Close}(k > 0) \\ \Omega = 1 \Rightarrow \mathrm{Flat}(k = 0) \\ \Omega < 1 \Rightarrow \mathrm{Open}(k < 0) \end{cases}$$

one-to-one correlation

Expanding Universe

(II) $H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \longrightarrow \boxed{\rho_c \equiv 3m_p^2 H^2}$

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Cosmic Sum

(III)

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt}(1+w) \Rightarrow \rho \propto e^{-3\int \frac{da}{a}(1+w)} = \begin{cases} 1/a^3 & , \mathrm{Mat.}(w=0) \\ 1/a^4 & , \mathrm{Rad.}(w=1/3) \\ \mathrm{const.} & , \mathrm{C.C.}(w=-1) \end{cases}$$

Expanding Universe

(II) $$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{\rho}{3m_p^2} - \frac{K}{a^2} \longrightarrow \boxed{\rho_c \equiv 3m_p^2 H^2}$$

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Cosmic Sum

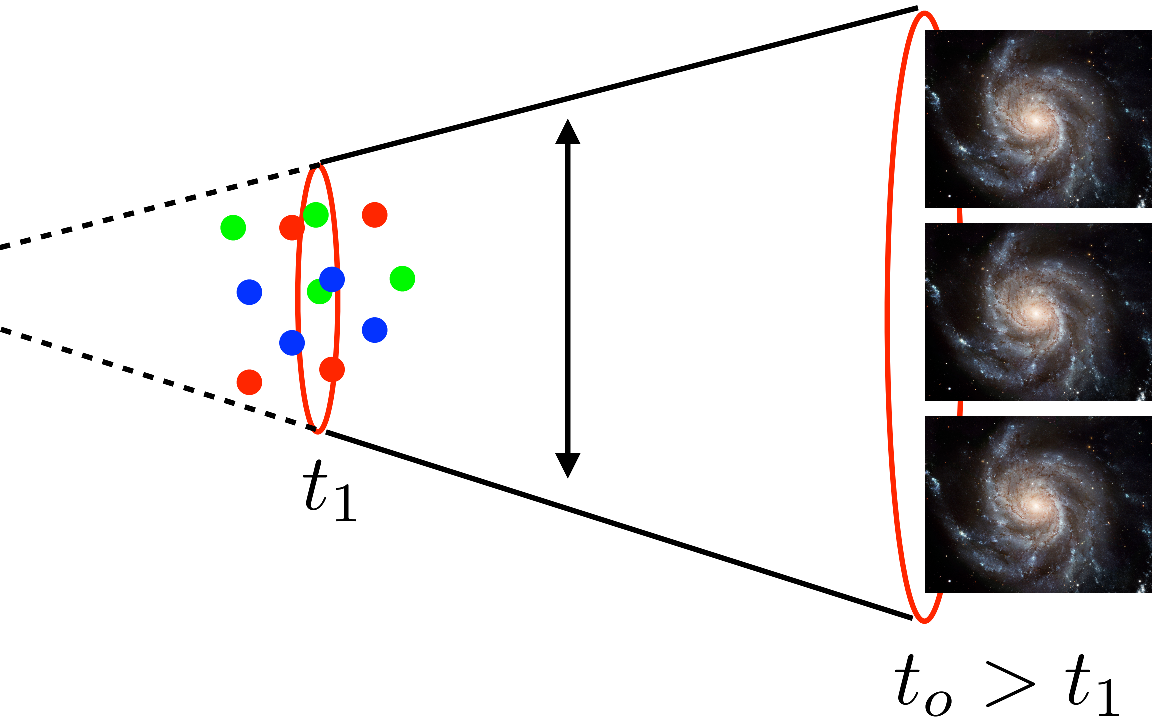
(III) + (II) :

$$H^2(a) = H_o^2 \left\{ \Omega_{\mathrm{R}}^{(o)} \left(\frac{a_o}{a}\right)^4 + \Omega_{\mathrm{M}}^{(o)} \left(\frac{a_o}{a}\right)^3 + \Omega_{\mathrm{k}}^{(o)} \left(\frac{a_o}{a}\right)^2 + \Omega_{\mathrm{DE}}^{(o)} e^{-3\int \frac{da}{a}(1+w)} \right\}$$

$$\equiv H_o^2 E^2(a)$$

$$E(a) \equiv \sqrt{\Omega_{\mathrm{R}}^{(o)} \left(\frac{a_o}{a}\right)^4 + \Omega_{\mathrm{M}}^{(o)} \left(\frac{a_o}{a}\right)^3 + \Omega_{\mathrm{k}}^{(o)} \left(\frac{a_o}{a}\right)^2 + \Omega_{\mathrm{DE}}^{(o)} e^{-3\int \frac{da}{a}(1+w)}} \qquad \Omega_{\mathrm{k}}^{(o)} \equiv -\frac{k}{a_o^2 H_o^2}$$

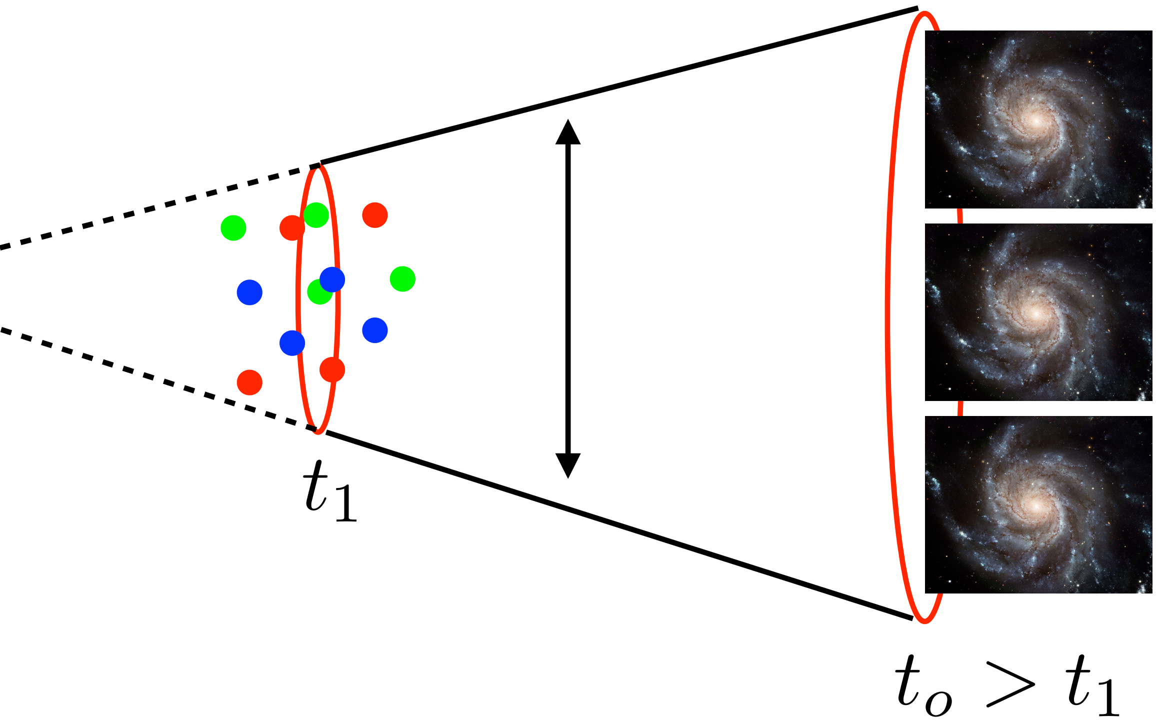
Expansion History



Past: particle ensemble

Statistical Mechanics

Expansion History

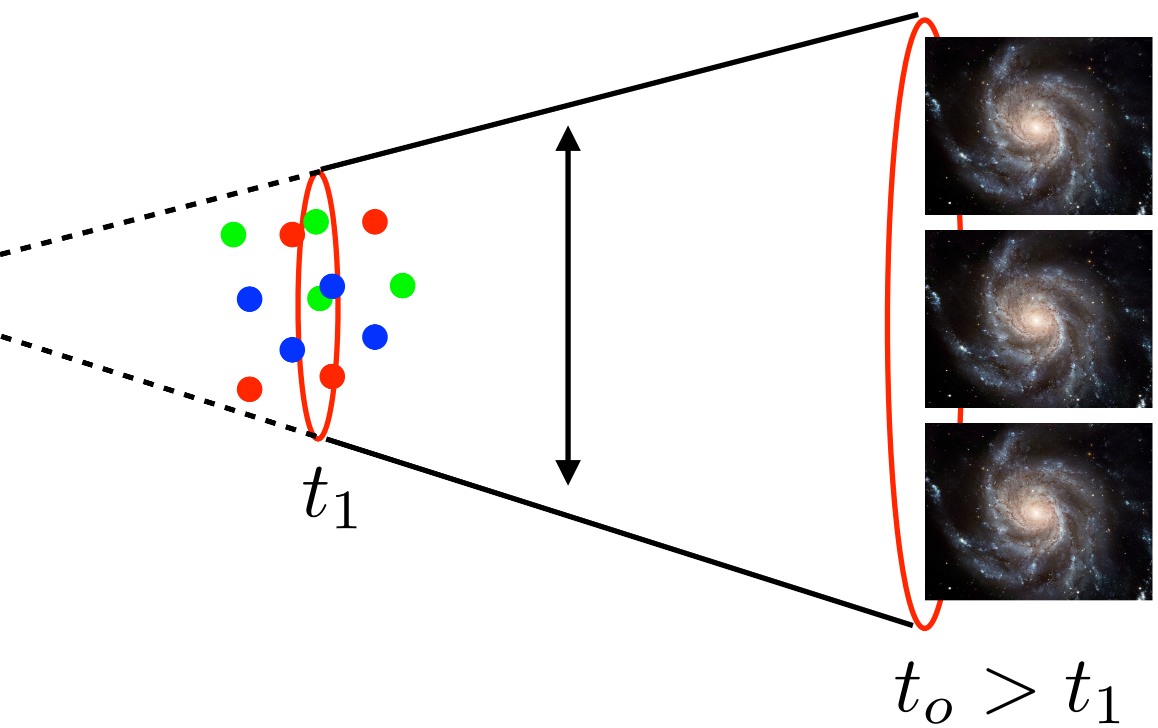


Past: particle ensemble

Statistical Mechanics

$$\text{(III)} \quad \frac{d\rho}{dt} + 3H(\rho + p) = 0 \longrightarrow \quad \frac{dU}{dt} + p\frac{dV}{dt} = 0\,, \qquad \begin{cases} U = a^3\rho\,, \\ V = a^3 \end{cases}$$

Expansion History



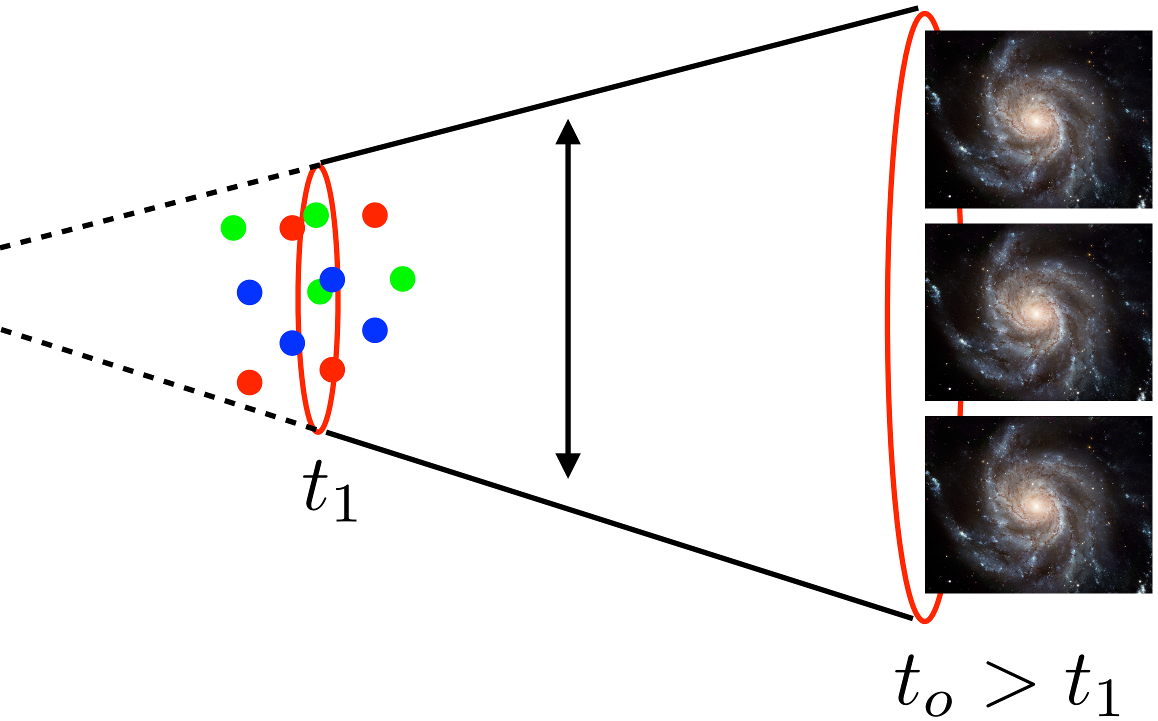
Past: particle ensemble

Statistical Mechanics

$$\text{(III)}\quad \frac{d\rho}{dt} + 3H(\rho + p) = 0 \longrightarrow \quad \frac{dU}{dt} + p\frac{dV}{dt} = 0\,, \quad \begin{cases} U = a^3\rho\,, \\ V = a^3 \end{cases} \quad \Rightarrow$$

$$\Rightarrow \quad \begin{cases} \frac{dU}{dt} + p\frac{dV}{dt} = T\frac{dS}{dt}\,, \longrightarrow \textbf{Thermal Eq.} \\ \frac{dS}{dt} = 0\,, \longrightarrow \textbf{Adiabatic Exp.} \end{cases}$$

Expansion History

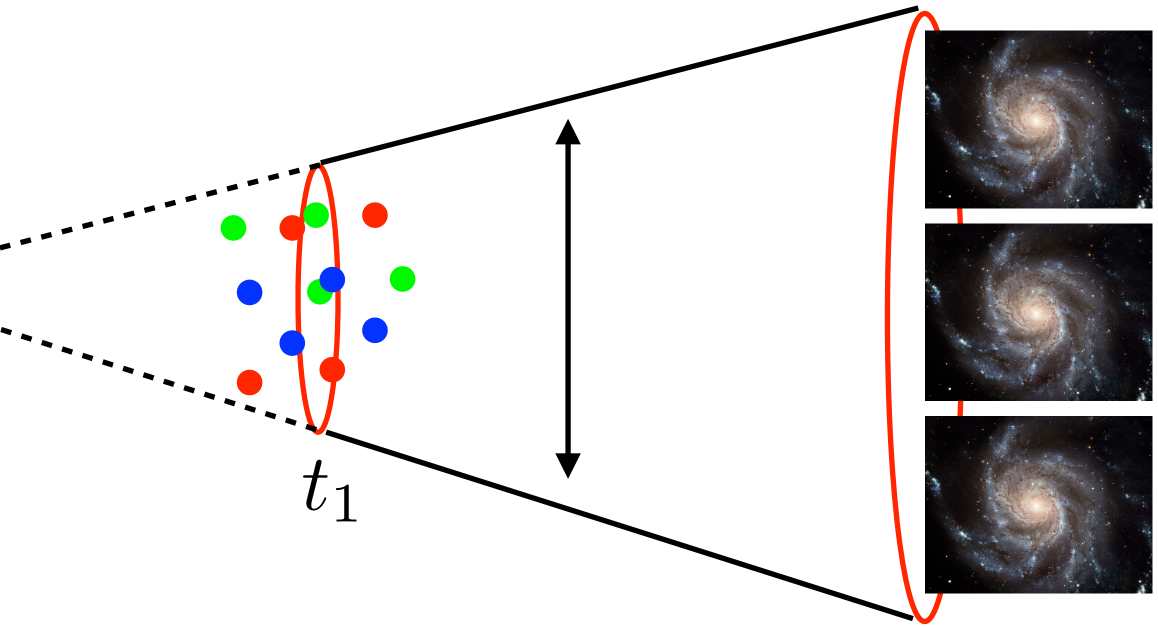


Thermal Eq. **(densities)**

$$
\begin{cases}
n = g_* \int d\vec{p}\, f(\vec{p})\,, & \textbf{number} \\
\rho = g_* \int d\vec{p}\, E(\vec{p}) f(\vec{p})\,, & \textbf{energy} \\
p = g_* \int d\vec{p}\, \frac{|\vec{p}|^2}{3E(\vec{p})} f(\vec{p})\,, & \textbf{pressure}
\end{cases}
$$

dof *Dispersion relation* *Statistical Distribution*

Expansion History



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dof *Dispersion relation* *Statistical Distribution*

Bose-Einstein / Fermi-Dirac: $$f(\vec{p}) = \left(e^{E(\vec{p})/T} \pm 1\right)^{-1}\,, \begin{cases} F(+) & \textbf{[fermions]} \\ B(-) & \textbf{[bosons]} \end{cases}$$

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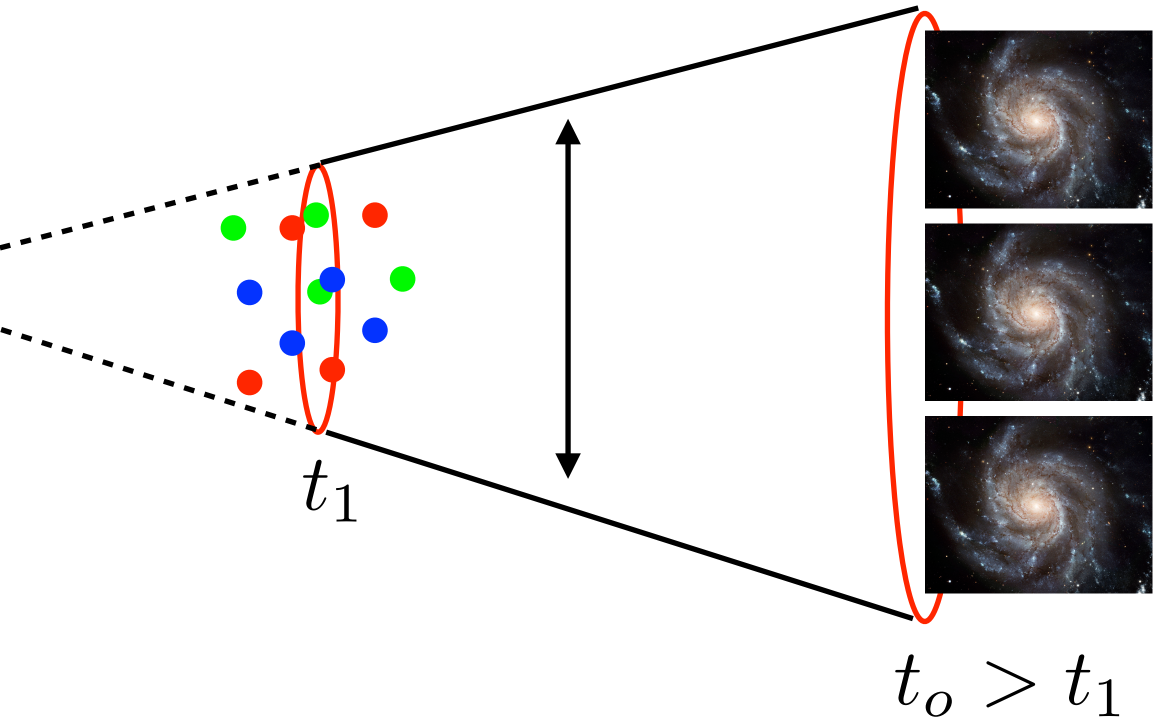

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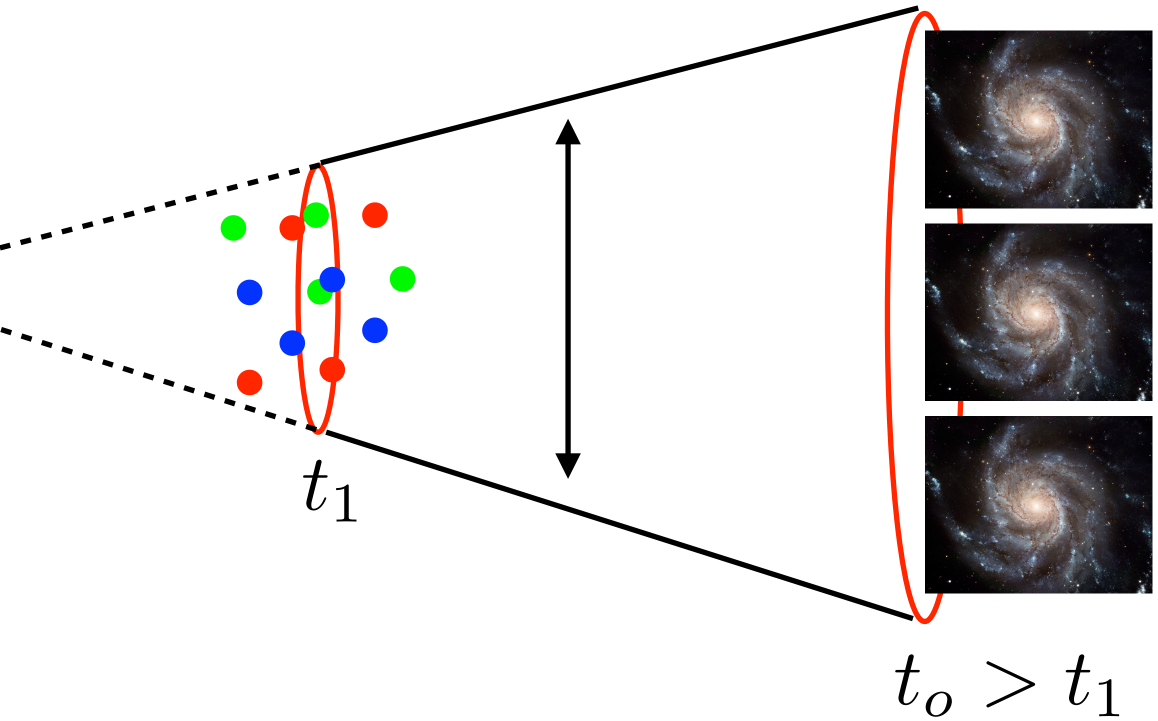
$$\frac{\rho_R \propto 1/a^4}{\rho_M \propto 1/a^3} \propto 1/a\,, \quad \Rightarrow \quad z \geq z_{\rm EQ}\ (t \leq t_{\rm EQ})\,, \quad \rho_R > \rho_M$$

Expansion History



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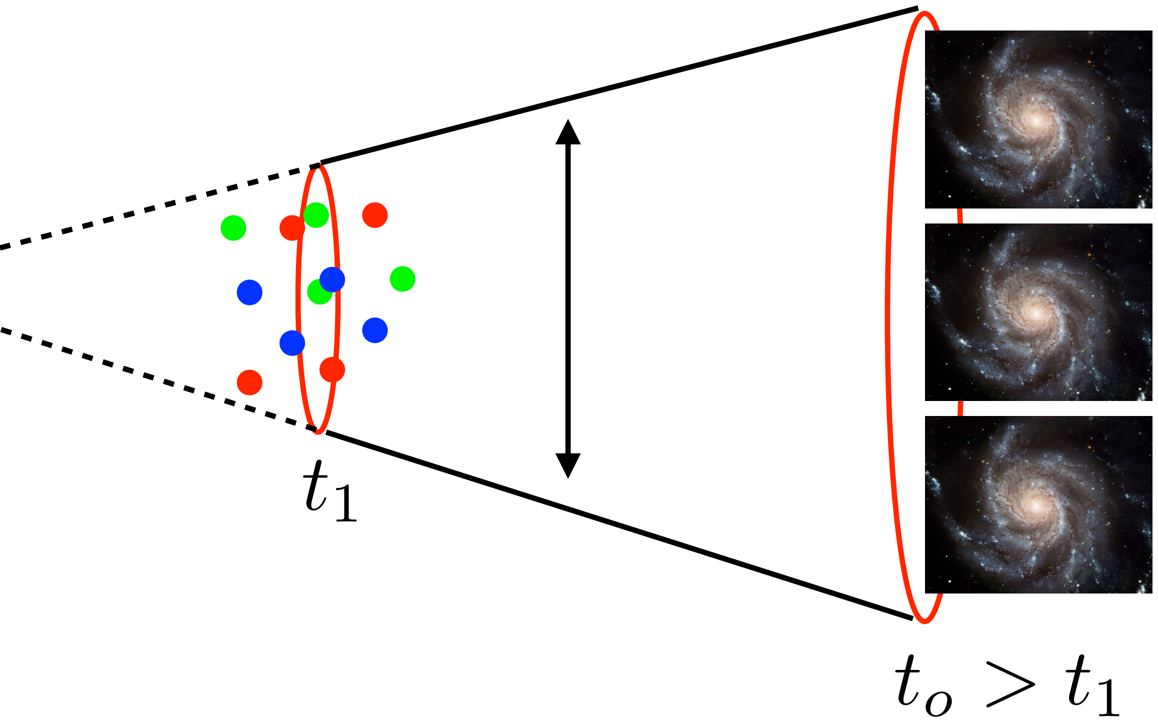
Past: Radiation Domination ! $\boxed{1 + z_{\mathrm{EQ}} = \Omega_{\mathrm{M}}^{(o)}/\Omega_{\mathrm{Rad}}^{(o)} \sim 3400}$

Expansion History



Thermal Eq.

(densities)

$$\begin{cases} n = g_* \int d\vec{p}\, f(\vec{p})\,, & \textbf{number} \\ \rho = g_* \int d\vec{p}\, E(\vec{p}) f(\vec{p})\,, & \textbf{energy} \\ p = \underbrace{g_*}_{\text{dof}} \int d\vec{p}\, \underbrace{\frac{|\vec{p}|^2}{3E(\vec{p})}}_{\text{Dispersion relation}} \underbrace{f(\vec{p})}_{\text{Statistical Distribution}}\,, & \textbf{pressure} \end{cases}$$

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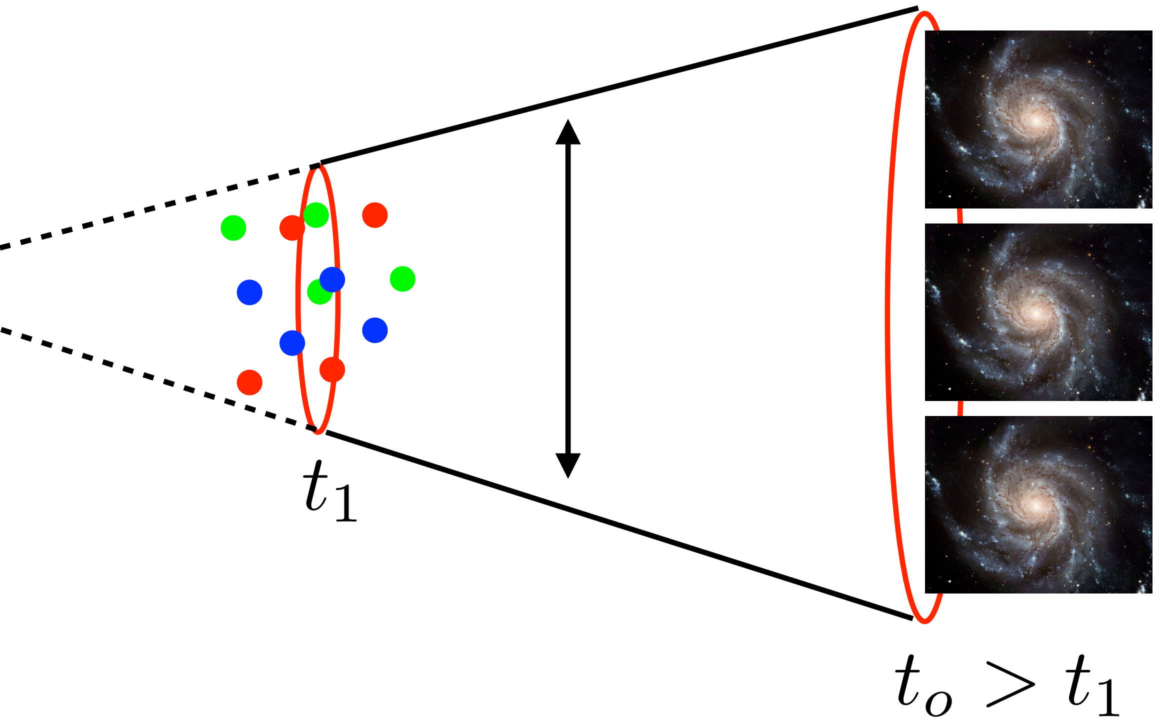
Past: Radiation Domination (RD)

$$\rho_R^{(i)} = f_i g_*^{(i)} \frac{\pi^2}{30} T_i^4\,, \quad f_i = \begin{cases} 1, & \mathrm{B} \\ \frac{7}{8}, & \mathrm{F} \end{cases}$$

Expansion History



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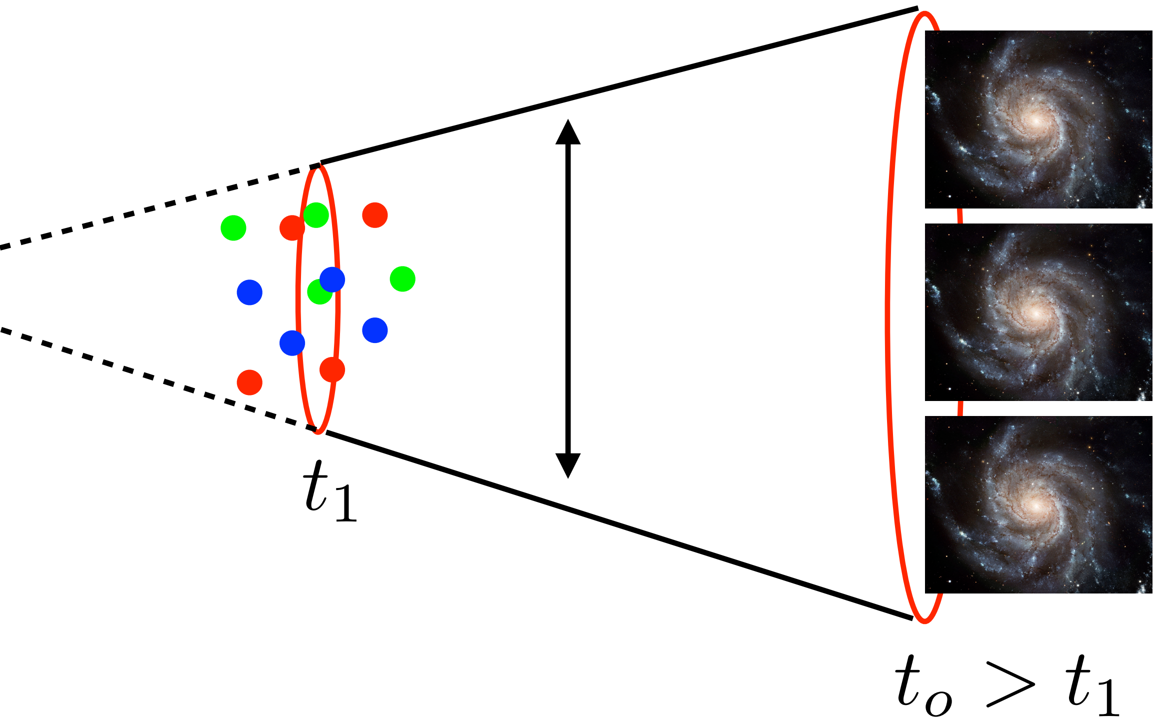

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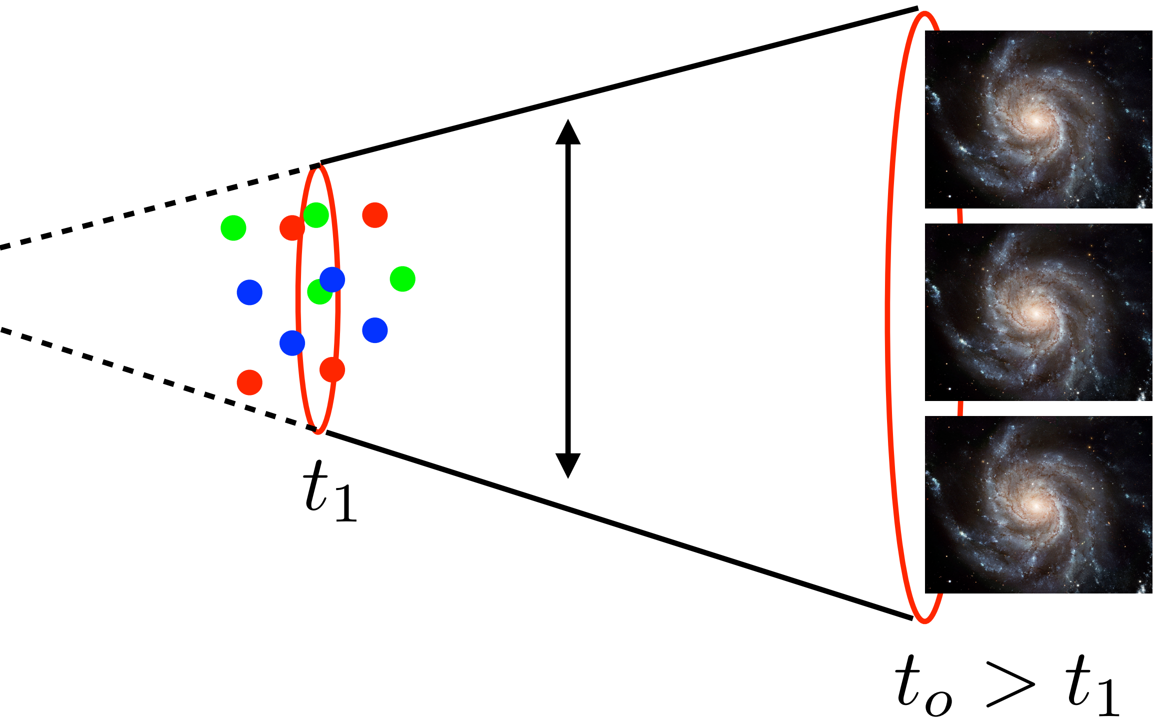
$$\boxed{\rho_R = \sum_i \rho_R^{(i)} \equiv g_*(T) \frac{\pi^2}{30} T^4}$$

$$g_*(T) \equiv \sum_i g_{*,i}^{(B)} \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_i g_{*,i}^{(F)} \left(\frac{T_i}{T}\right)^4$$

Expansion History



Adiabatic Exp:

$$S = \frac{a^3(\rho + p)}{T} = const.$$

$$\Downarrow$$

$$\boxed{a^3 T^3 g_*^{(s)}(T) = const.}$$

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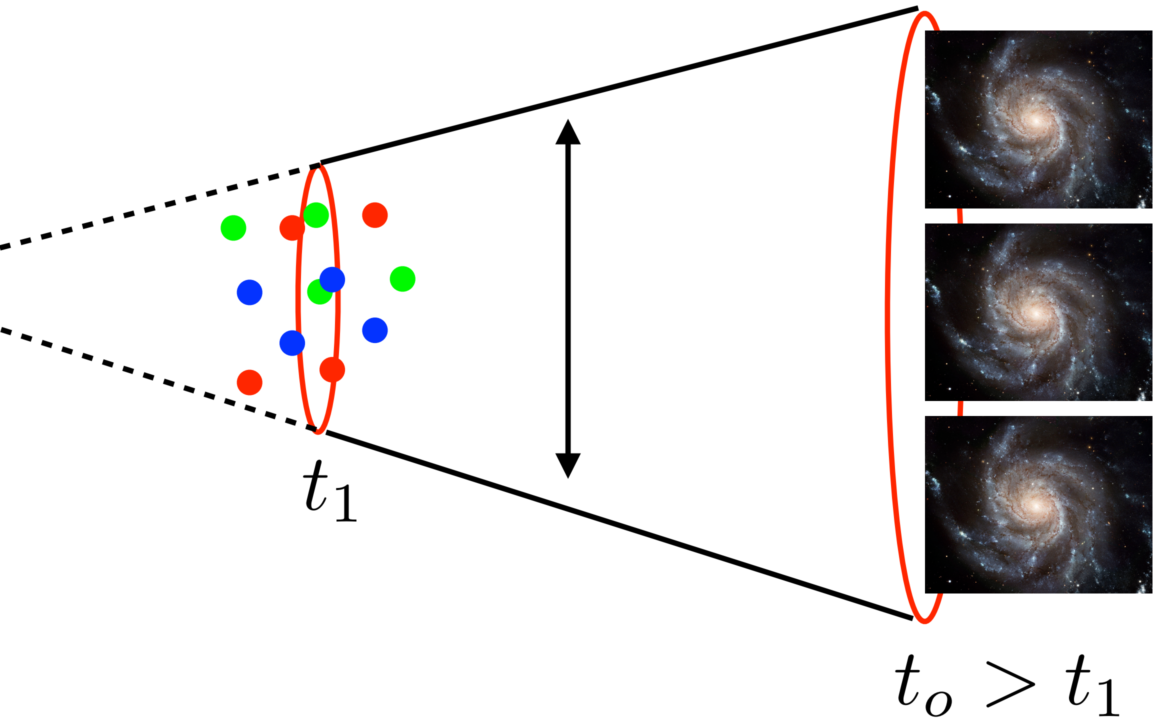

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When do $g_*(T), g_*^{(s)}(T)$ change ?

1) Species Decoupling , $T \to T_i$,

2) Mass threshold , $T < 2m_i$,