Actions on δ-median spaces

Indira CHATTERJI

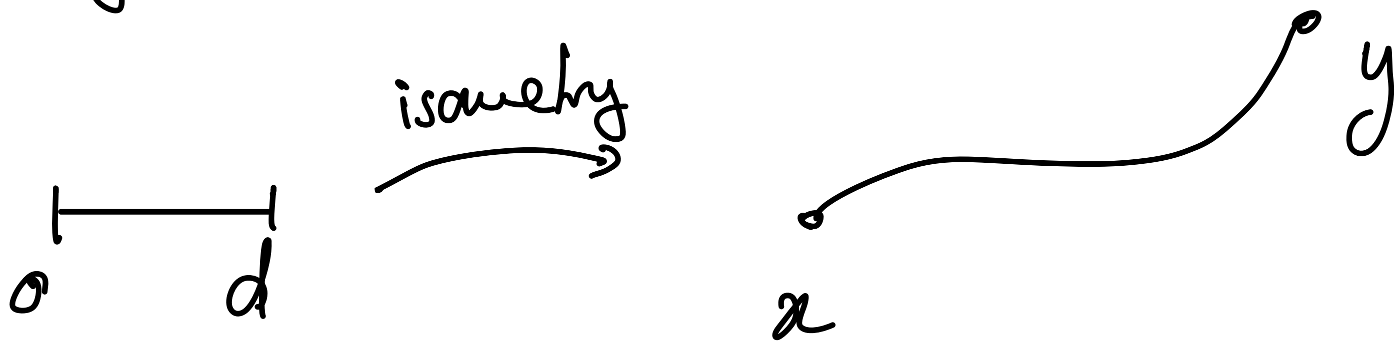
Plan for the talk:

- δ-median spaces : definitions & examples
- a conjecture & evidences
- some known cases

δ-median spaces

(X,d) geodesic metric space :

γ :



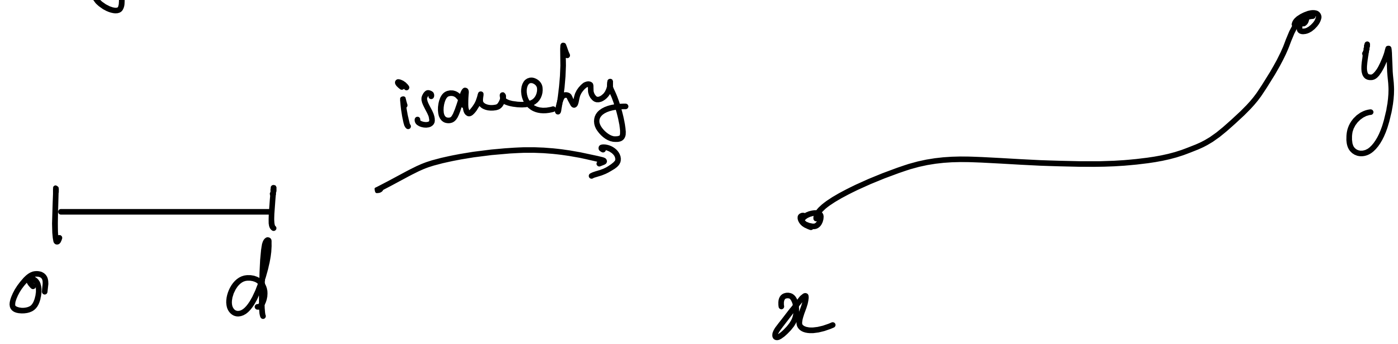
$$I(x,y) = \{t \in X \mid d(x,t) + d(t,y) = d(x,y)\}$$

the interval between x and y

δ-median spaces

(X,d) geodesic metric space :

γ :



$\delta \geq 0$

$$I_\delta(x,y) = \{ t \in X \mid d(x,t) + d(t,y) \leq d(x,y) + \delta \}$$

the δ-interval between x and y

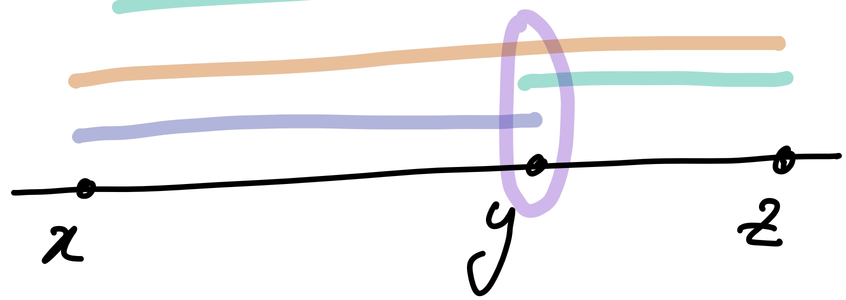
<u>S - median spaces</u>: (X,d) geodesic metric space is called <u>median</u> if for any $x,y,z \in X$

$$I(x,y) \cap I(y,z) \cap I(z,x) = \{pt\}$$

Median spaces: (X,d) geodesic metric space is called **median** if for any $x,y,z \in X$

$$I(x,y) \cap I(y,z) \cap I(z,x) = \{pt\}$$

Ex: $\mathbb{R}$

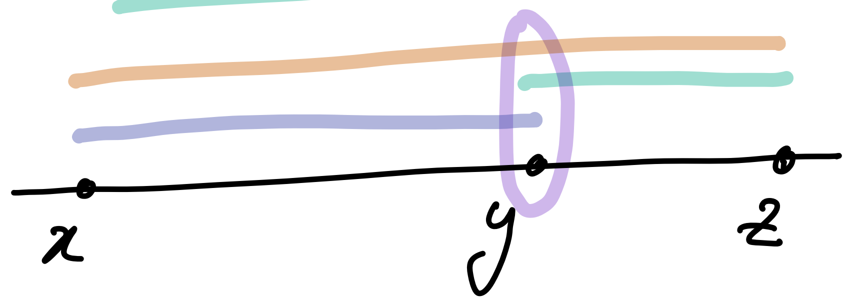


S-median spaces: (X,d) geodesic metric space

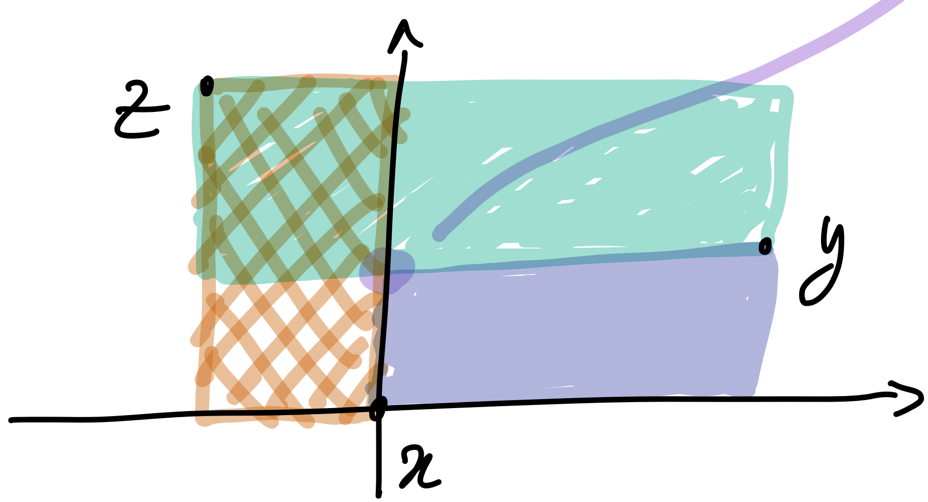
is called **median** if for any $x,y,z \in X$

$$I(x,y)\cap I(y,z)\cap I(z,x)=\{pt\}$$

Ex: $\mathbb{R}$



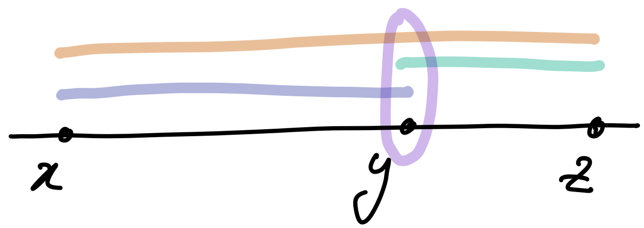
$\mathbb{R}^2$

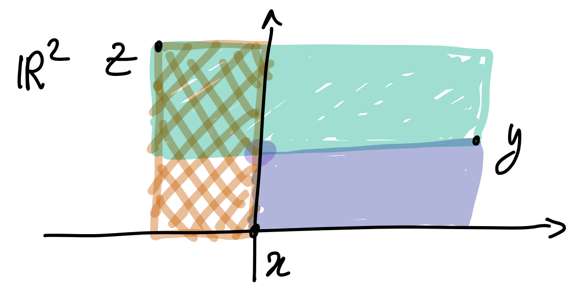


S-median spaces: (X,d) geodesic metric space is called **median** if for any $x,y,z \in X$

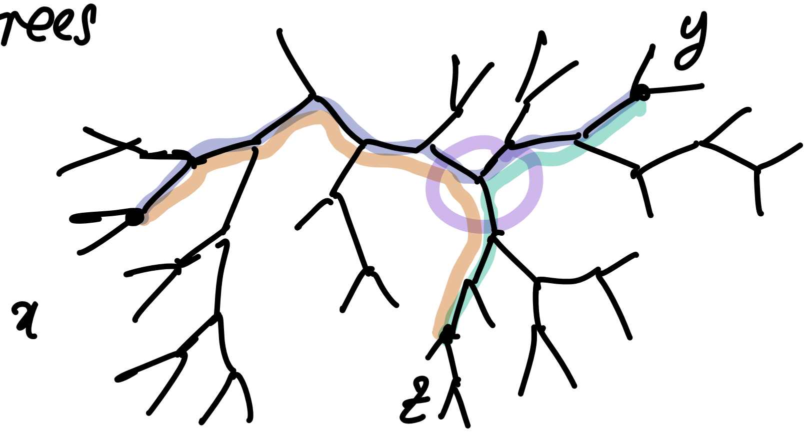
$$I(x,y) \cap I(y,z) \cap I(z,x) = \{pt\}$$

Ex: $\mathbb{R}$





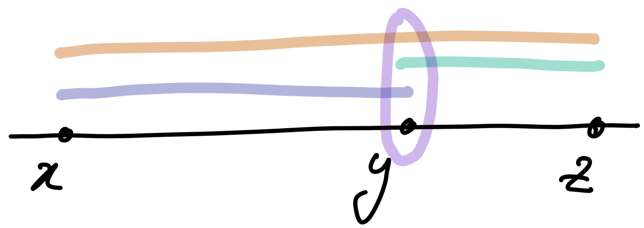
trees



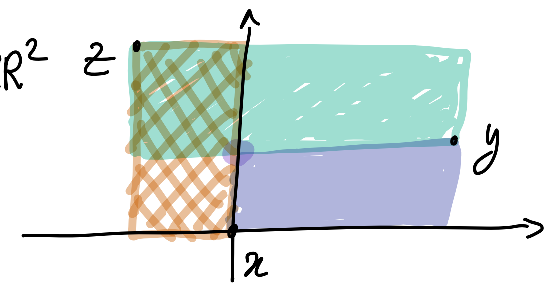
<u>S-median spaces</u>: (X,d) geodesic metric space is called <u>median</u> if for any $x,y,z \in X$

$$I(x,y) \cap I(y,z) \cap I(z,x) = \{pt\}$$

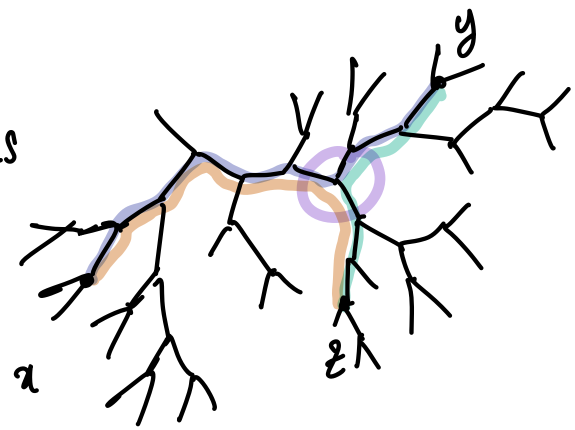
<u>Ex</u>: $\mathbb{R}$



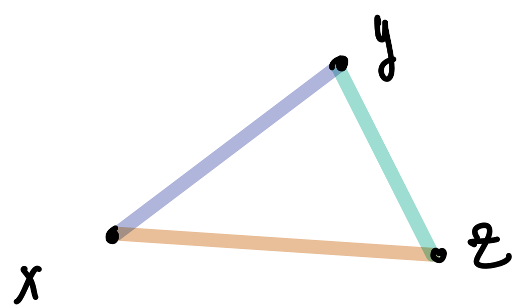
$\mathbb{R}^2$



trees



<u>NOT</u> $(\mathbb{R}^2, \text{euclidean})$



S-median spaces: (X,d) geodesic metric space is called median if for any $x,y,z \in X$

$$I(x,y) \cap I(y,z) \cap I(z,x) = \{pt\}$$

δ-median spaces: (X,d) geodesic metric space

$\delta \geq 0$ is called δ-median if for any $x,y,z \in X$

$$d_H\left(I_\delta(x,y) \cap I_\delta(y,z) \cap I_\delta(z,x),\ \{pt\}\right) \leq C = C(\delta)$$

$$I_\delta(x,y) = \{t \in X \mid d(x,t) + d(t,y) \leq d(x,y) + \delta\}$$

the δ-interval between x and y

δ-median spaces: (X,d) geodesic metric space

$\delta \geq 0$ is called **δ-median** if for any $x,y,z \in X$

$$d_H\left(I_\delta(x,y) \cap I_\delta(y,z) \cap I_\delta(z,x), \{pt\}\right) \leq C = C(\delta)$$

Examples: • median spaces are 0-median

• symmetric spaces associated to rank one Lie groups

$$I_\delta(x,y) = \{t \in X \mid d(x,t) + d(t,y) \leq d(x,y) + \delta\}$$

the δ-interval between x and y

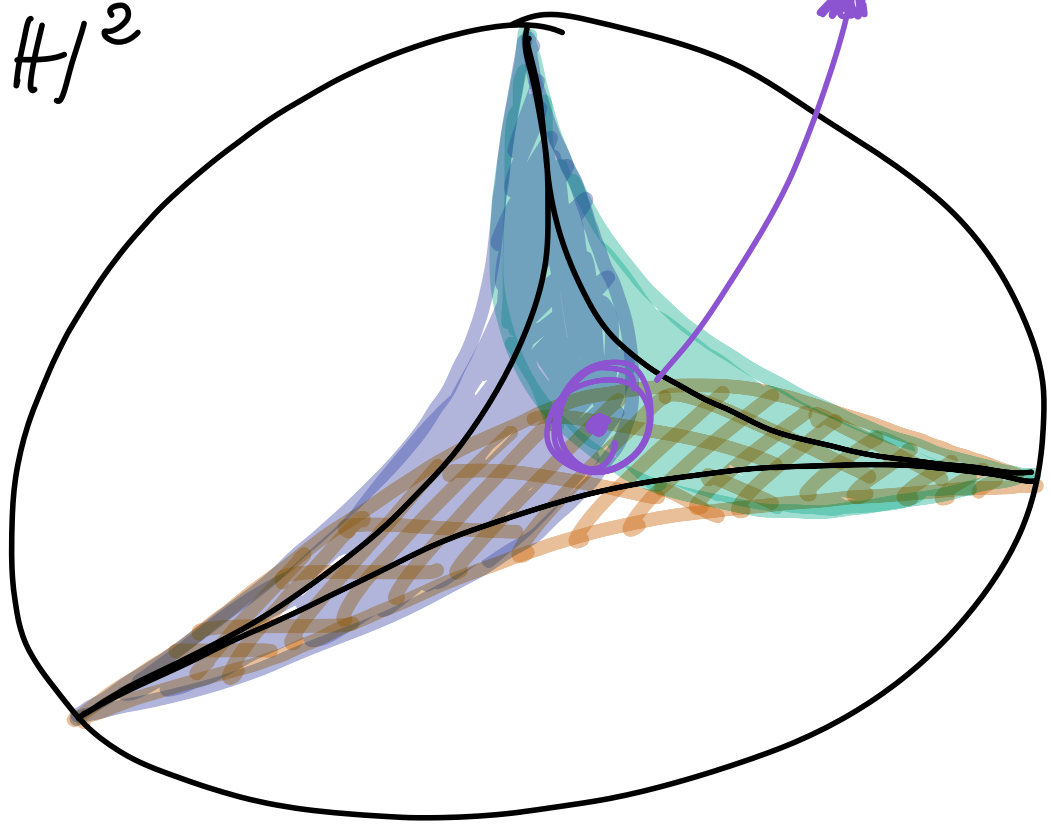
δ-median spaces: (X,d) geodesic metric space

$\delta \geq 0$ is called **δ-median** if for any $x, y, z \in X$

$$d_H\left(I_\delta(x,y) \cap I_\delta(y,z) \cap I_\delta(z,x), \{pt\} \right) \leq C = C(\delta)$$

Examples: $\mathbb{H}^2$

(or any Gromov hyperbolic space)



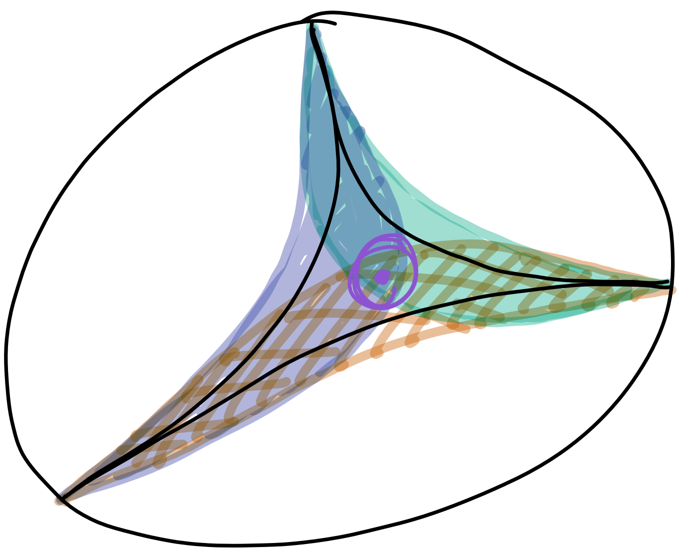
δ-median spaces : (X,d) geodesic metric space

$\delta \geq 0$ is called δ-median if for any $x,y,z \in X$

$$d_H\left(I_\delta(x,y) \cap I_\delta(y,z) \cap I_\delta(z,x), \{pt\}\right) \leq C = C(\delta)$$

Examples : $\mathbb{H}^2$, $(\mathbb{H}^2 \times \mathbb{H}^2, d = d_{\mathbb{H}^2} + d_{\mathbb{H}^2})$

products of $\mathbb{H}^n$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{H}}$



NOT : symmetric spaces associated to higher rank Lie groups.

Plan for the talk:

- δ-median spaces: definitions & examples
- a conjecture & evidences
- some known cases

<u>Conjecture</u>: Let G be a finitely generated group

G acts properly by isometries on a uniformly locally finite δ-median space.

$$\Longleftrightarrow$$

G acts properly by affine isometries on some ℓ^p space.

probably equivalent

<u>Conjecture</u>: Let G be a finitely generated group

has bounded orbits for actions

G ~~acts properly~~ by isometries on a uniformly

locally finite δ-median space.

$$\Longleftrightarrow$$

any action of G

~~G acts properly~~ by affine isometries on some

ℓ^p space has a fixed point. (strong property (T))

Conjecture: Let G be a finitely generated group

G acts properly by isometries on a uniformly locally finite δ-median space.

↳ balls intersected with uniform nets have uniformly bounded cardinality

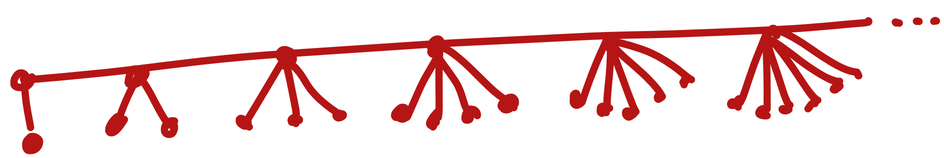
$\Longleftrightarrow$

G acts properly by affine isometries on some ℓ^p space.

__Conjecture__: Let G be a finitely generated group

G acts properly by isometries on a uniformly locally finite δ-median space.

↳ not that:



$$\iff$$

G acts properly by affine isometries on some ℓ^p space.

Plan for the talk:

- δ-median spaces : definitions & examples
- a conjecture & evidences
- some known cases

Status of that conjecture, evidences:

Yu, Alvarez-Lafforgue, Pansu, Bourdon (98 -...- 2016)

If G acts properly by isometries on a uniformly locally finite δ-hyperbolic space, then G acts on $\ell^p(G \times G)$ for p large enough.

Status of that conjecture, evidences:

Yu, Alvarez-Lafforgue, Pansu, Bourdon (98 -...- 2016)
If G acts properly by isometries on a uniformly locally finite δ-hyperbolic space, then G acts on $\ell^p(G\times G)$ for p large enough.

Chatterji - Drutu - Haglund (2007)
G acts properly on a median space $\Leftrightarrow$
G acts properly on a Hilbert space

Status of that conjecture, evidences:

Yu, Alvarez-Lafforgue, Pansu, Bourdon (98 - ... - 2016)
If G acts properly by isometries on a uniformly locally finite δ-hyperbolic space, then G acts on $\ell^p(G \times G)$ for p large enough.

Chatterji - Drutu - Haglund (2007)
G acts properly on a median space $\Longleftrightarrow$
G acts properly on a Hilbert space
G acts properly

Chatterji - Drutu - Haglund (2007)
Any action of G by isometries on a median space has a bounded orbit
$\Longleftrightarrow$ G has Kazhdan property (T).

Status of that conjecture, evidences:

Yu, Alvarez-Lafforgue, Pansu, Bourdon (98 –…– 2016)
If G acts properly by isometries on a uniformly locally finite δ-hyperbolic space, then G acts on $\ell^p(G\times G)$ for p large enough.

Chatterji - Drutu - Haglund (2007)
G acts properly on a median space $\iff$ G acts properly on a Hilbert space

Any action of G by isometries on a median space has a bounded orbit $\iff$ G has Kazhdan property (T).

Haettel: Lattices in higher rank simple Lie groups cannot act on a δ-median space without a bounded orbit.

Bader - Furman - Gelander - Monod: Same lattices cannot act on an ℓ^p space without fix point.

Status of that conjecture, evidences:

Yu, Alvarez-Lafforgue, Pansu, Bourdon (98 –…– 2016)
If G acts properly by isometries on a uniformly locally finite δ-hyperbolic space, then G acts on $\ell^p(G\times G)$ for p large enough.

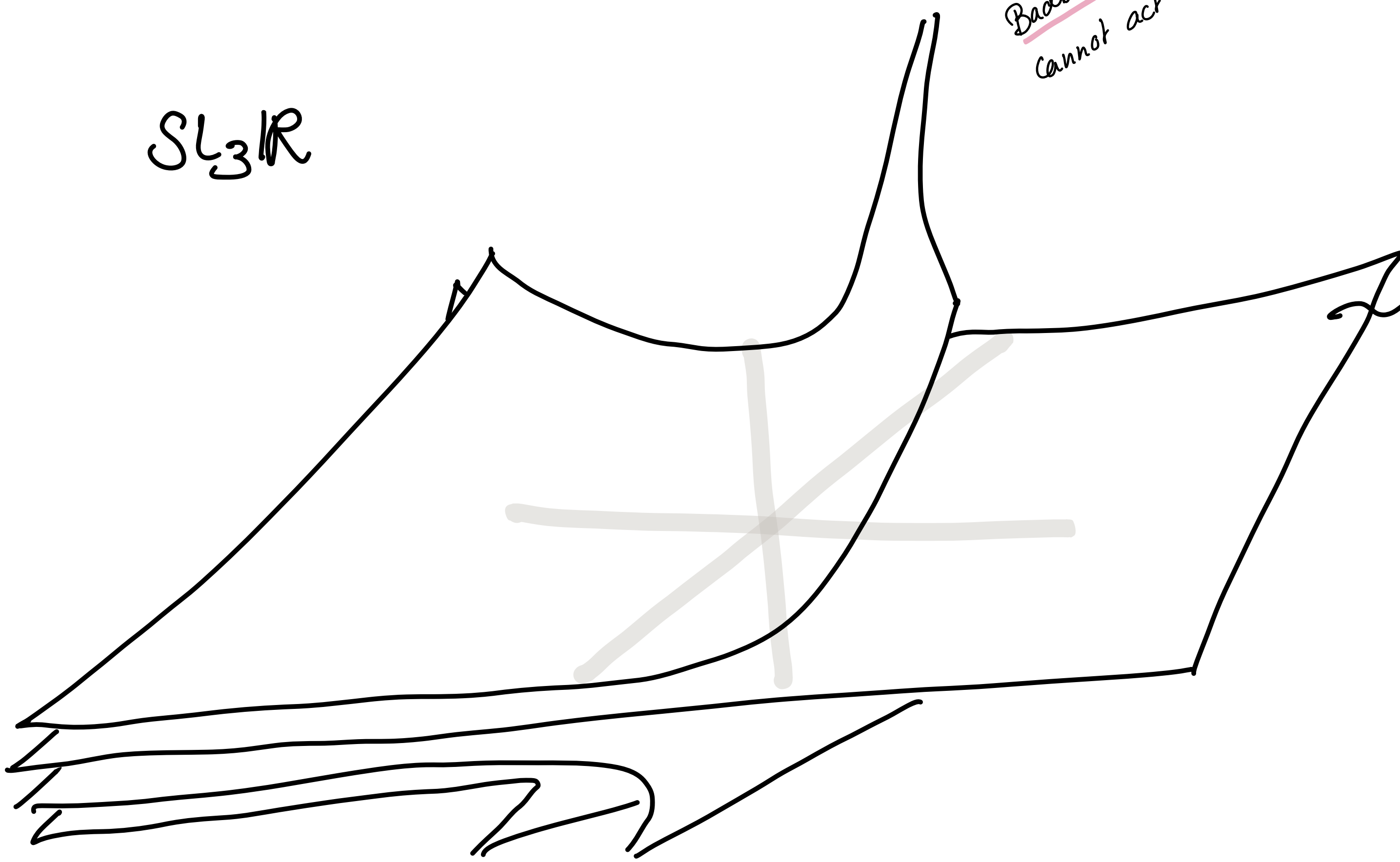
Chatterji - Druţu - Haglund (2007)
G acts properly on a median space $\Leftrightarrow$ G acts properly on a Hilbert space

Any action of G by isometries on a median space has a bounded orbit
$\Leftrightarrow$ G has Kazhdan property (T).

Haettel: Lattices in higher rank simple Lie groups cannot act on a δ-median space without a bounded orbit.

Bader - Furman - Gelander - Monod: Same lattices cannot act on an ℓ^p space without fix point.

$SL_3\mathbb{R}$



$\curvearrowright AK$

Status of that conjecture, evidences:

Yu, Alvarez-Lafforgue, Pansu, Bourdon (98 –...– 2016)
If G acts properly by isometries on a uniformly locally finite δ-hyperbolic space, then G acts on $\ell^p(G \times G)$ for p large enough.

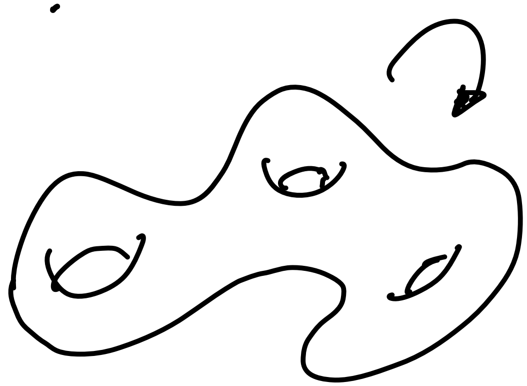
Chatterji - Drutu - Haglund (2007)
G acts properly on a median space $\Leftrightarrow$ G acts properly on a Hilbert space

Any action of G by isometries on a median space has a bounded orbit
$\Leftrightarrow$ G has Kazhdan property (T).

Haettel: Lattices in higher rank simple Lie groups cannot act on a δ-median space without a bounded orbit.

Bader - Furman - Gelander - Monod: Same lattices cannot act on an ℓ^p space without fix point.

<u>Open</u>: Do mapping class groups have property (T)? Act properly on a Hilbert space? On some ℓ^p-space?



Status of that conjecture, evidences:

Yu, Alvarez-Lafforgue, Pansu, Bourdon (98 - ... - 2016)
If G acts properly by isometries on a uniformly locally finite δ-hyperbolic space, then G acts on $\ell^p(G \times G)$ for p large enough.

Chatterji - Drutu - Haglund (2007)
G acts properly on a median space $\Leftrightarrow$ G acts properly on a Hilbert space

Any action of G by isometries on a median space has a bounded orbit
$\Leftrightarrow$ G has Kazhdan property (T).

Haettel: Lattices in higher rank simple Lie groups cannot act on a δ-median space without a bounded orbit.

Bader - Furman - Gelander - Monod: Same lattices cannot act on an ℓ^p space without fix point.

Open: Do mapping class groups have property (T)? Act properly on a Hilbert space? On some ℓ^p-space?

Known (Bestvina - Bromberg - Fujiwara, Petyt): Mapping class groups act properly by isometries on a uniformly locally finite δ-median space.