Dynamical Systems in Neuroscience

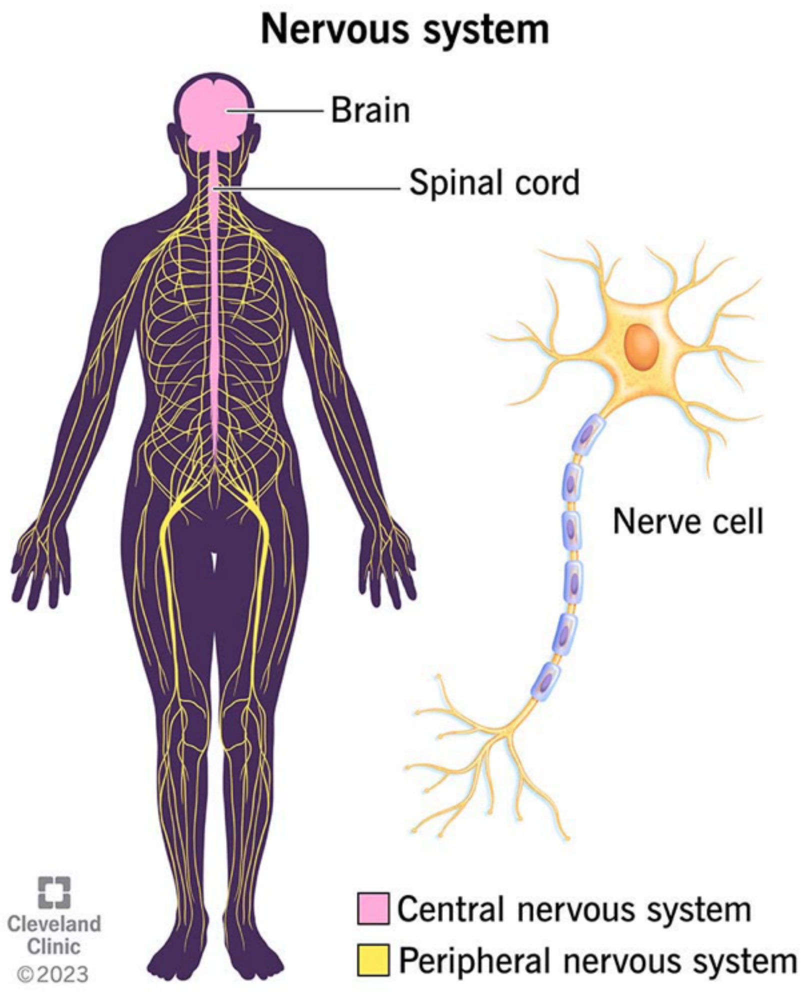
ICTS Summer Course 2026

Textbooks

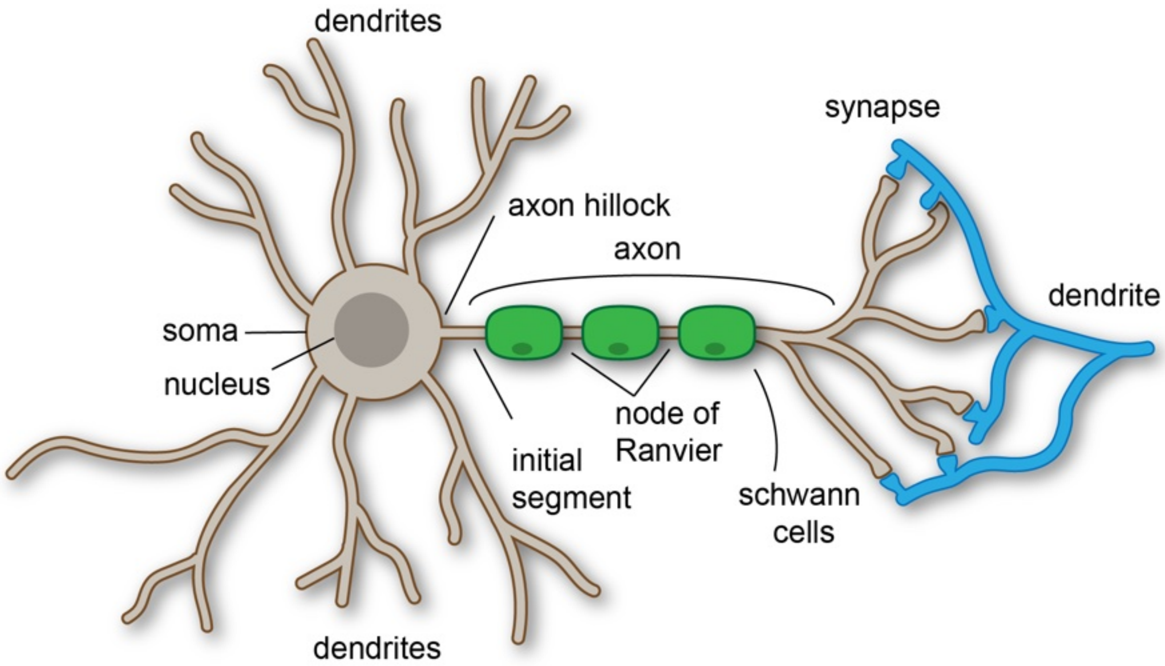
- Nonlinear Dynamics and Chaos - Steven Strogatz
- Chaos in Dynamical Systems - Edward Ott
- Dynamical Systems in Neuroscience - Eugene M. Izhikevich
- Mathematical Foundations of Neuroscience - Bard Ermentrout and David Terman
- Neuronal Dynamics - Gerstner, Kistler, Naud and Paninski

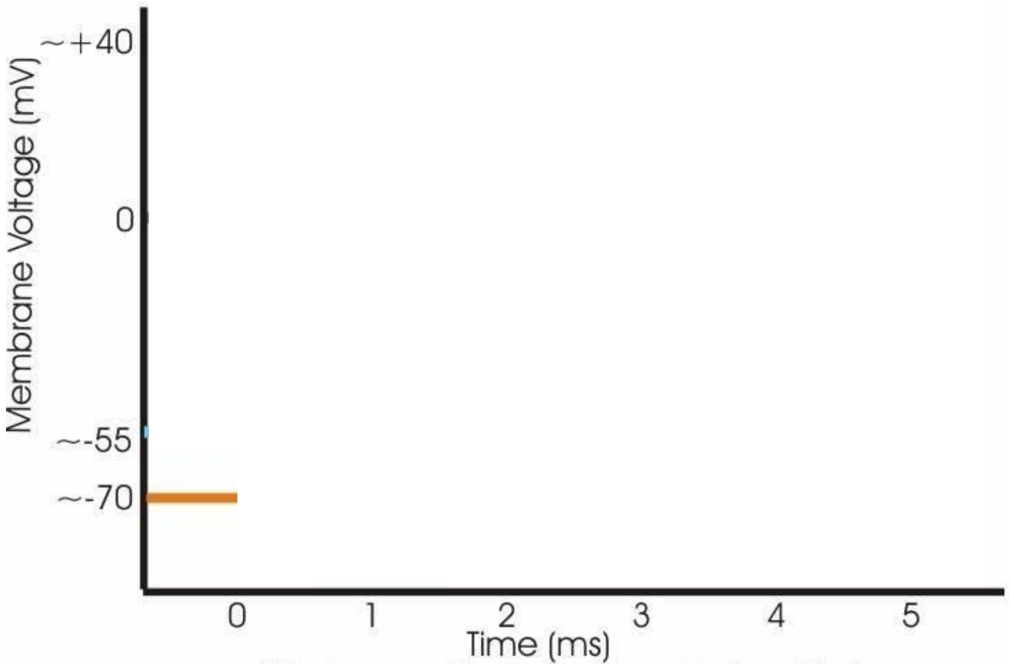
What is neuroscience?

Study of the nervous system



What is a neuron?





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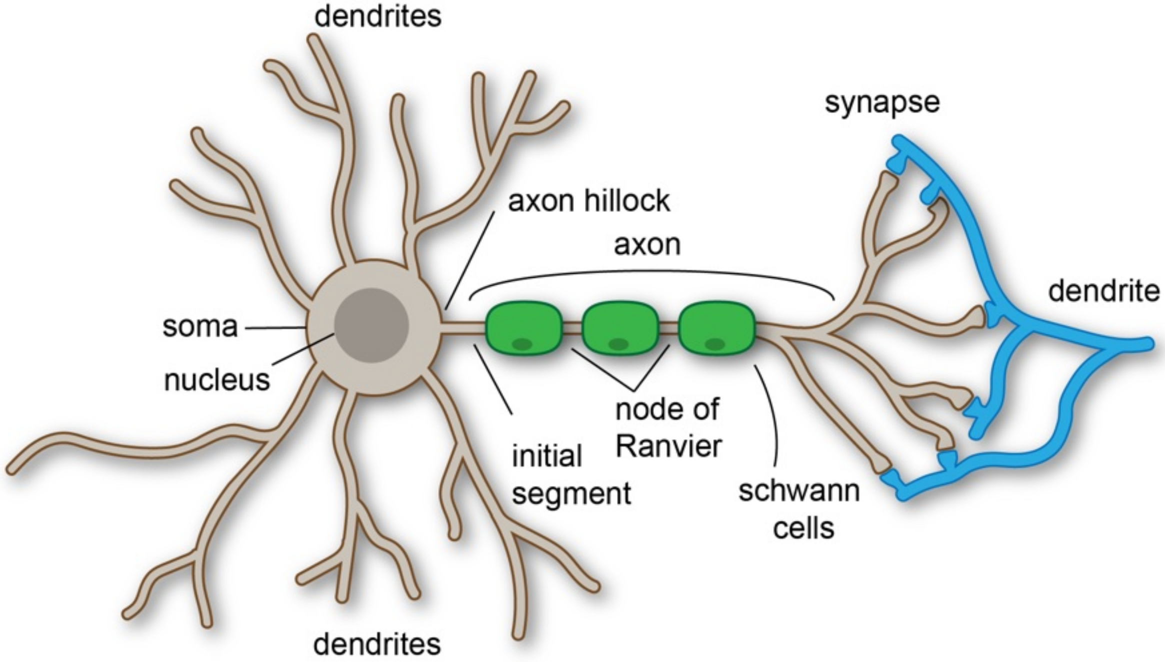
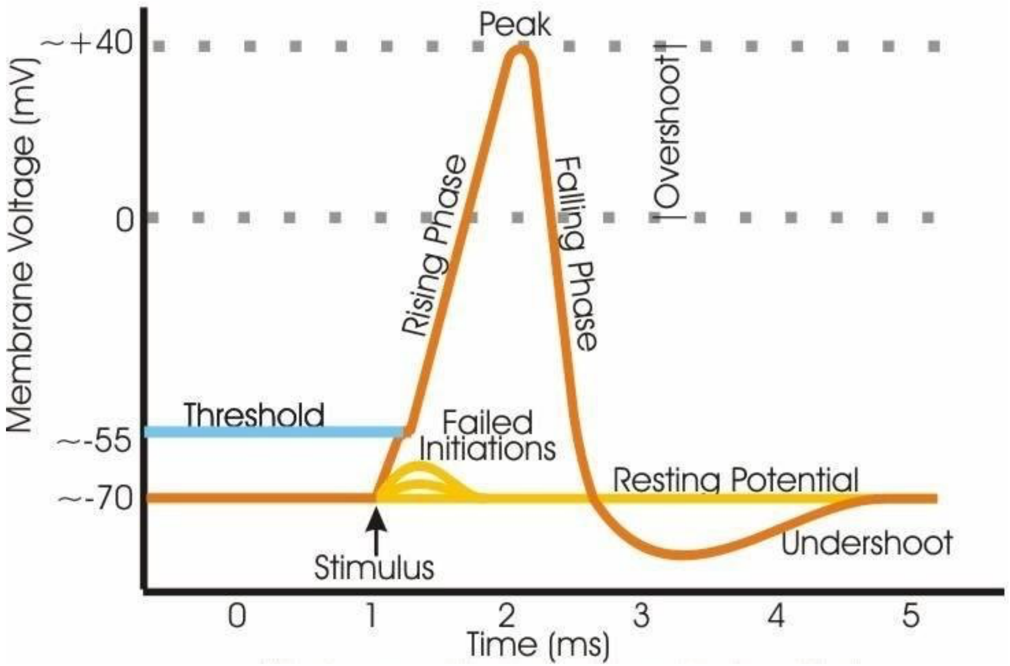
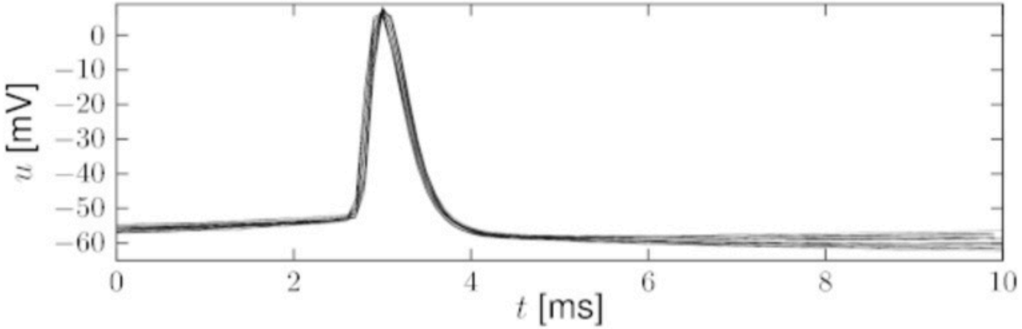
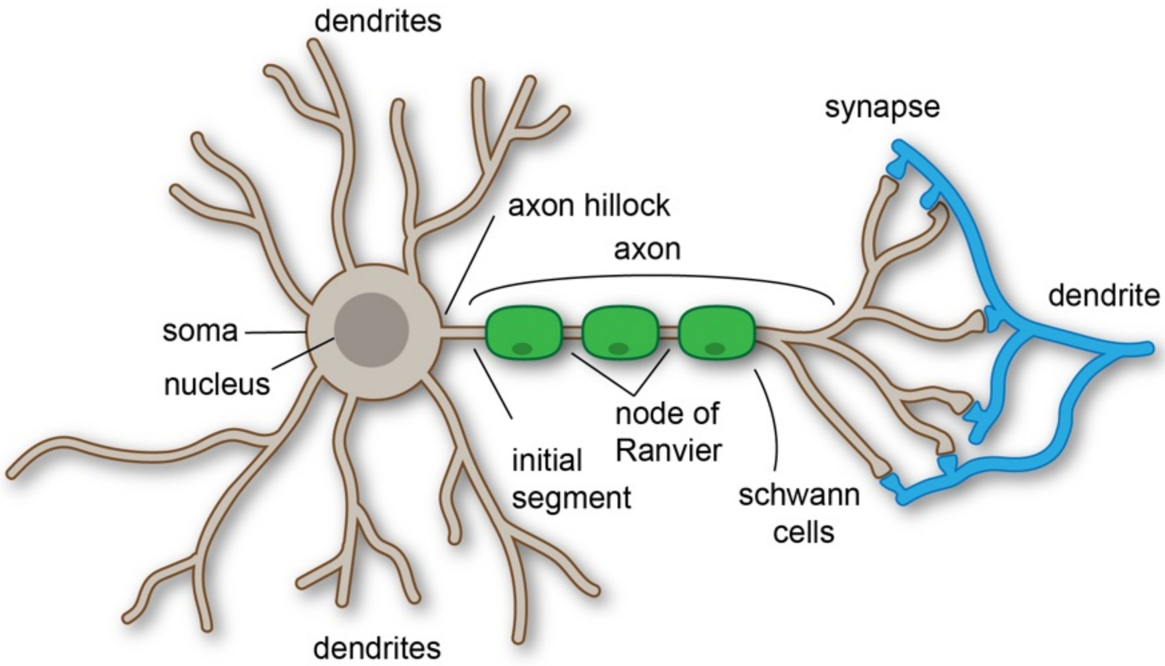


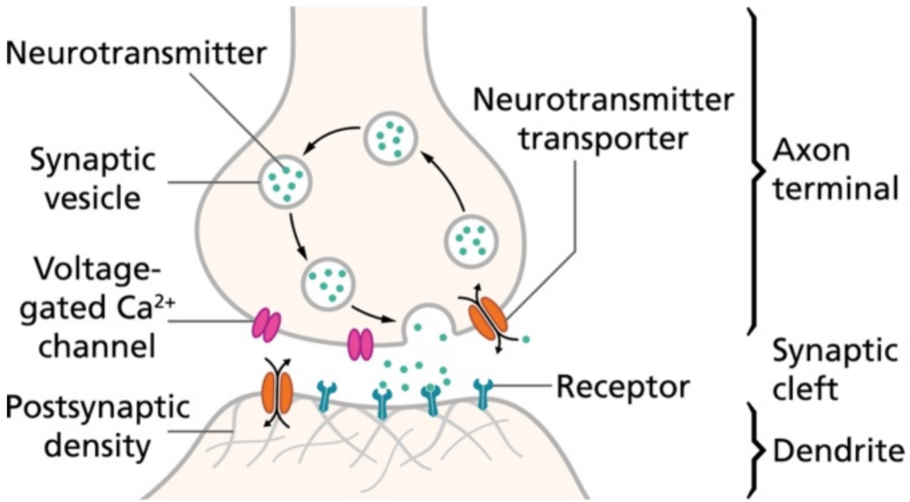




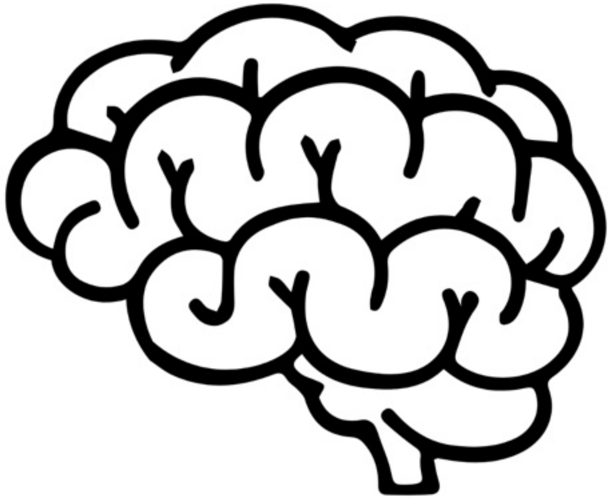
GKND Fig. 1.3

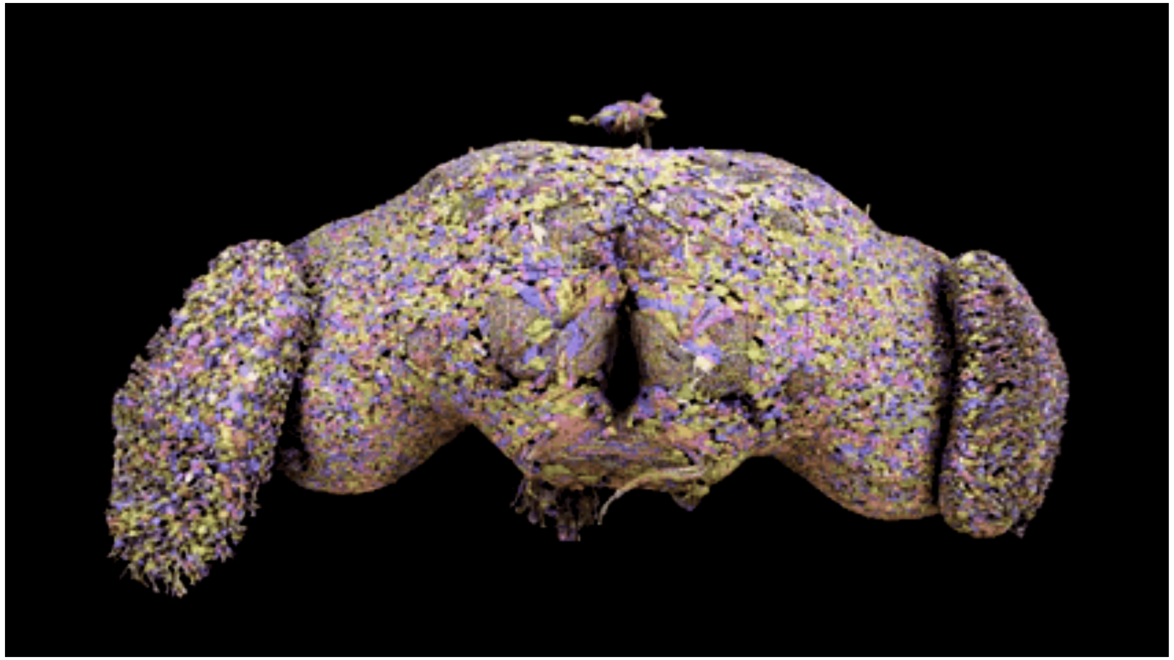
What is a neuron?





Neurons to neural circuits

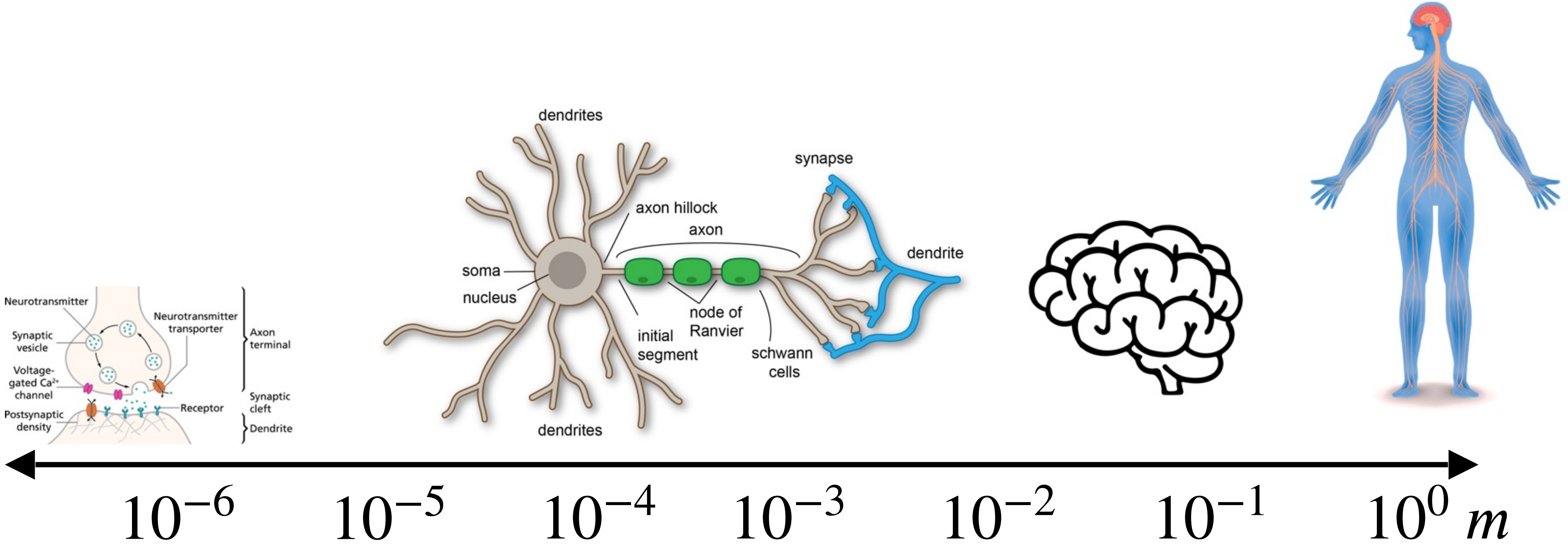




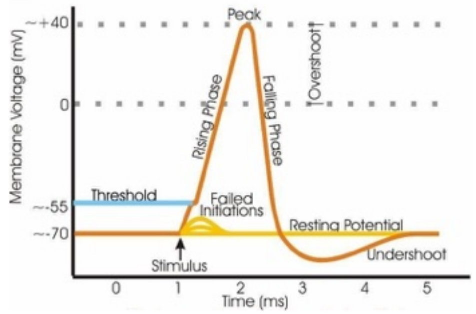
10^{11} neurons

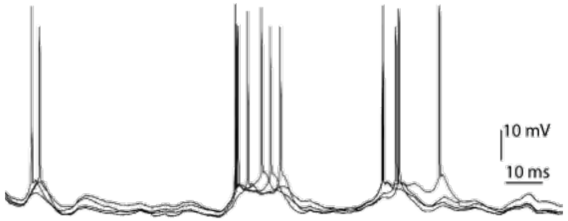
10^{14} synapses

Relevant lengthscales



Relevant timescales





Synaptic plasticity

Sensory perception

Planning, reasoning

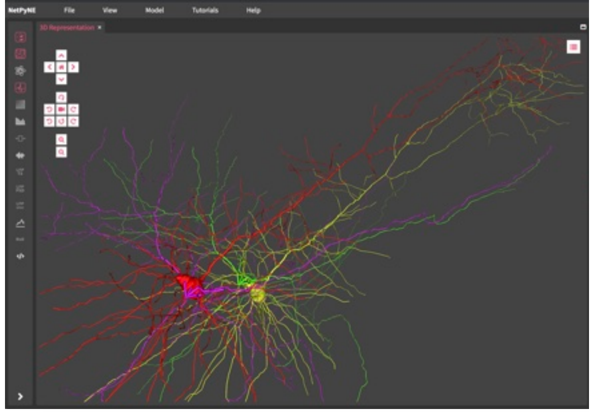
Long-term memory, learning

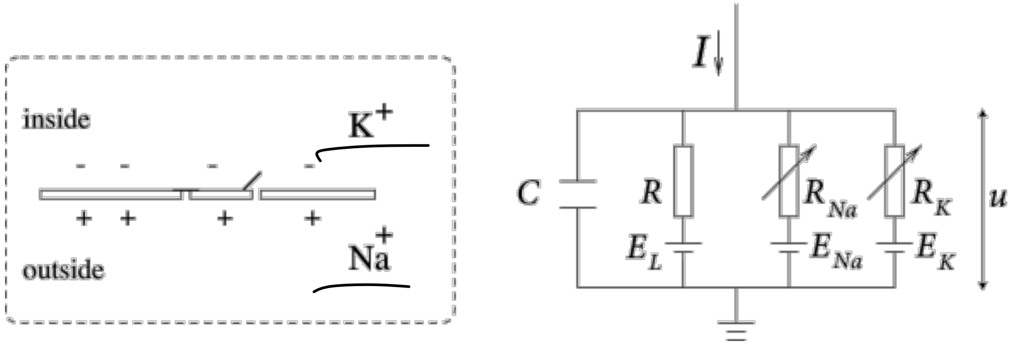
Short-term memory

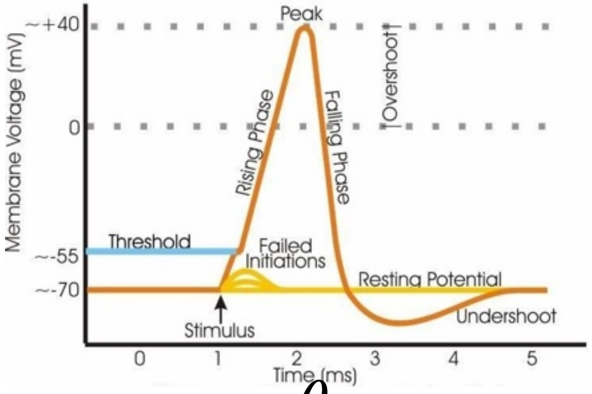
10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} $10^{5}s$

Different levels of abstraction

- Model detailed biochemistry and biophysics
- Model individual spikes







θ

$$I = C_m \frac{\mathrm{d}V_m}{\mathrm{d}t} + \bar{g}_\mathrm{K} n^4 (V_m - V_K) + \bar{g}_\mathrm{Na} m^3 h (V_m - V_{Na}) + \bar{g}_l (V_m - V_l),$$

$$\frac{dn}{dt} = \alpha_n(V_m)(1-n) - \beta_n(V_m)n$$

$$\frac{dm}{dt} = \alpha_m(V_m)(1-m) - \beta_m(V_m)m$$

$$\frac{dh}{dt} = \alpha_h(V_m)(1-h) - \beta_h(V_m)h$$

Hodgkin - Huxley

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta) I(t),$$

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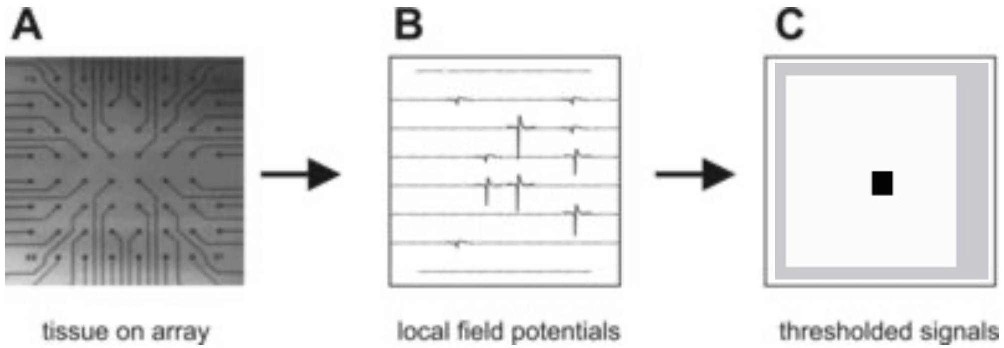
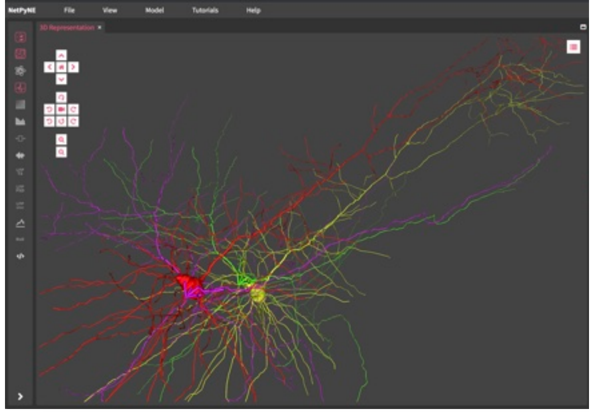
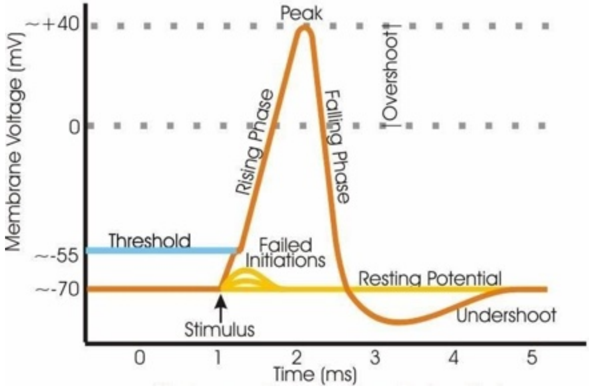
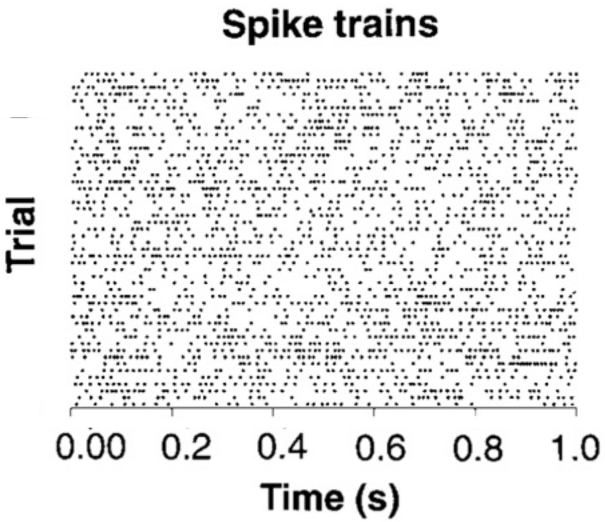


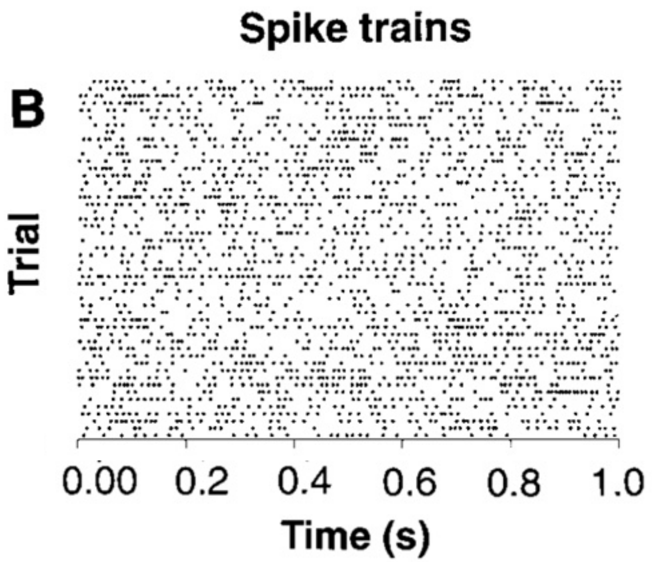


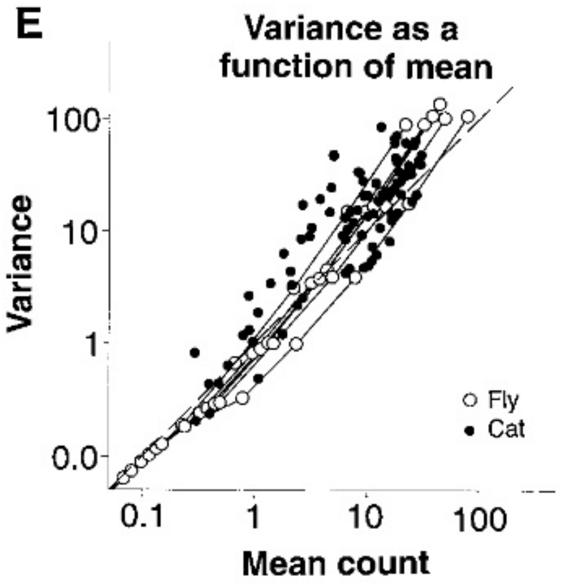




Different levels of abstraction

- Neuron spiking can seem stochastic
- Poisson spiking model: neuron action potential spike distribution





Ruyter van Steveninck et al 1997

Different levels of abstraction

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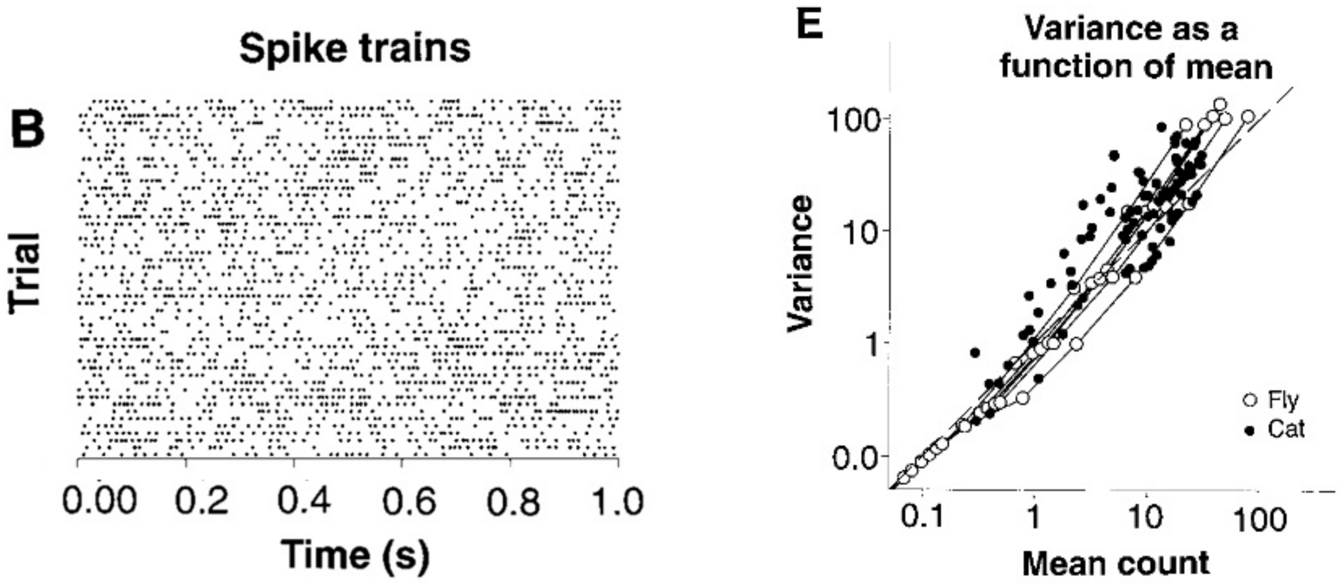

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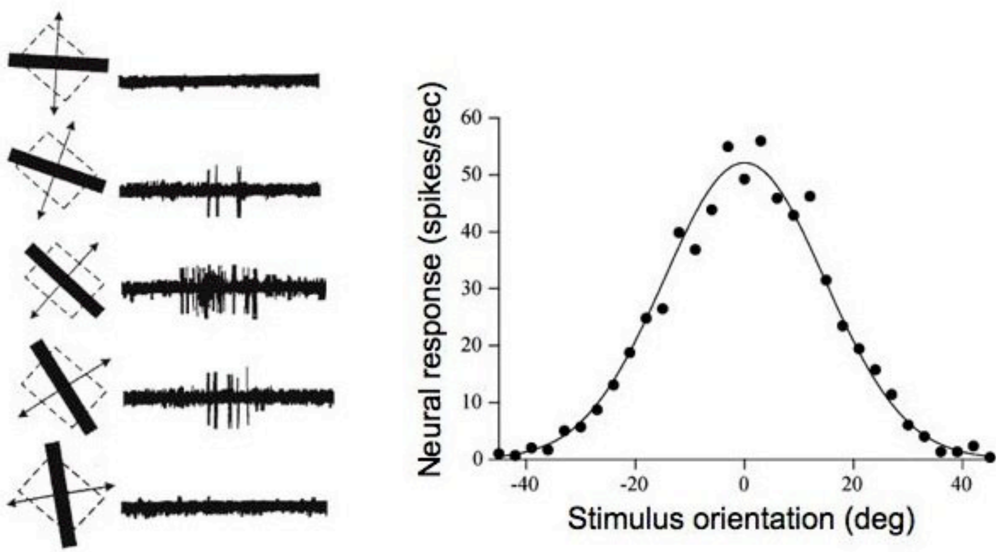

Hubel & Wiesel 1968

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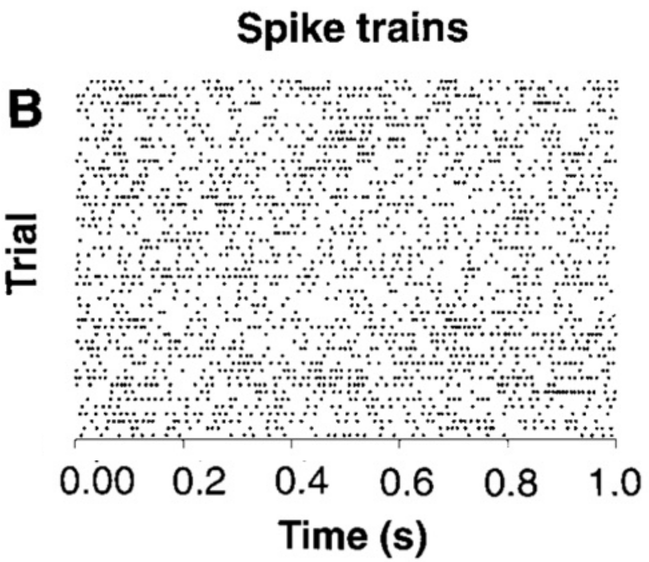

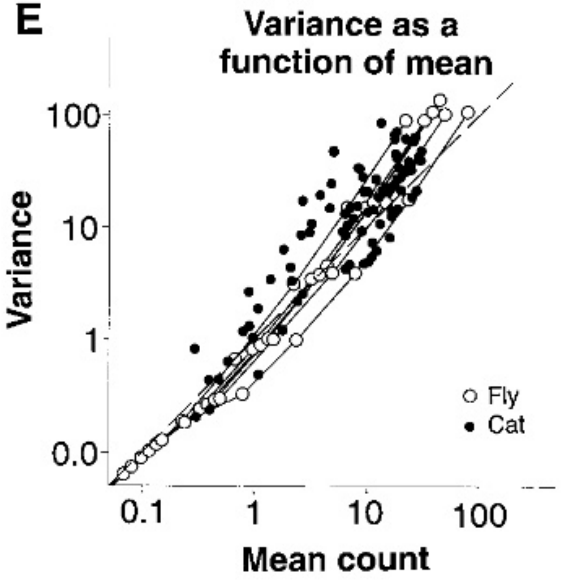

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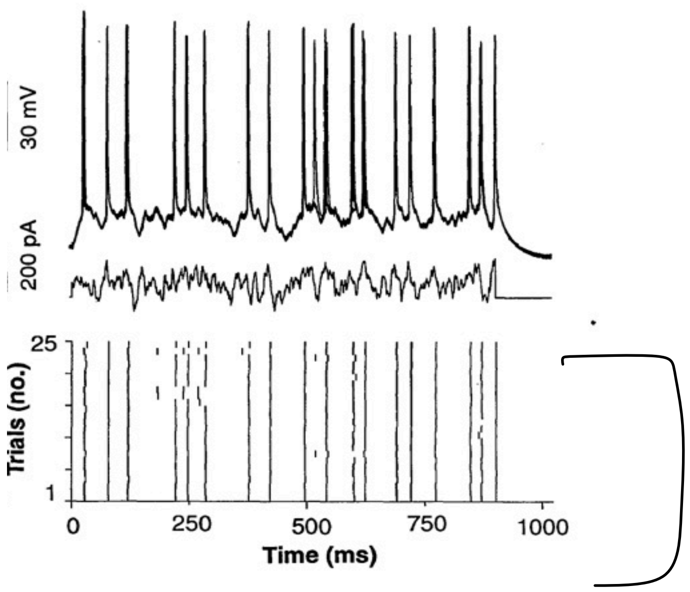

Mainen and Sejnowski 1995

Different levels of abstraction

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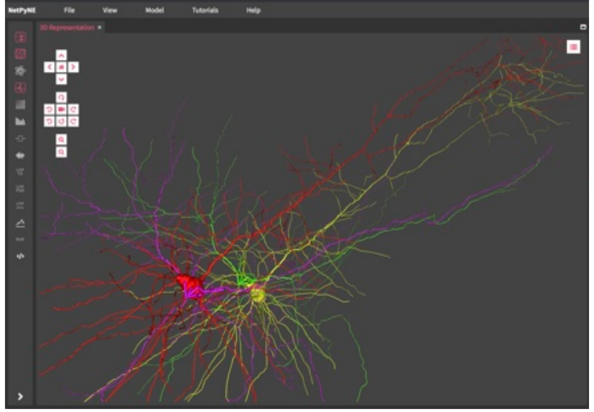
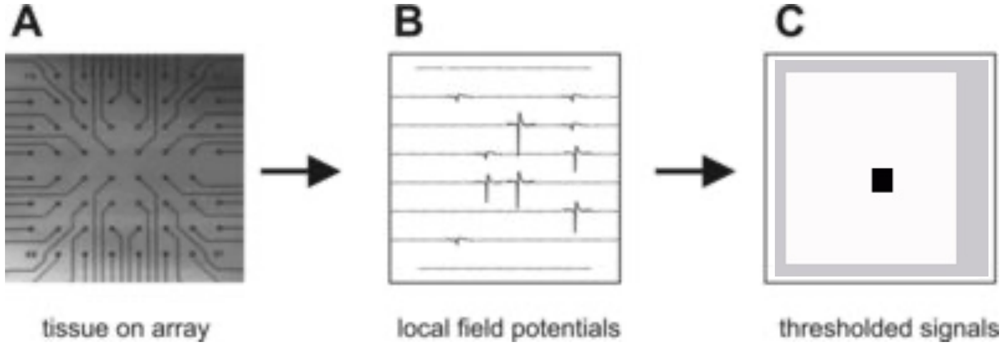
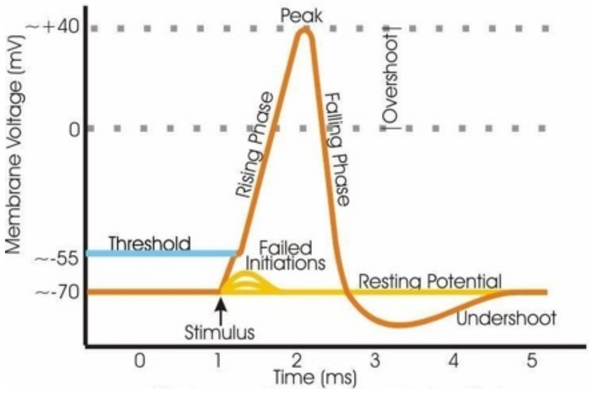
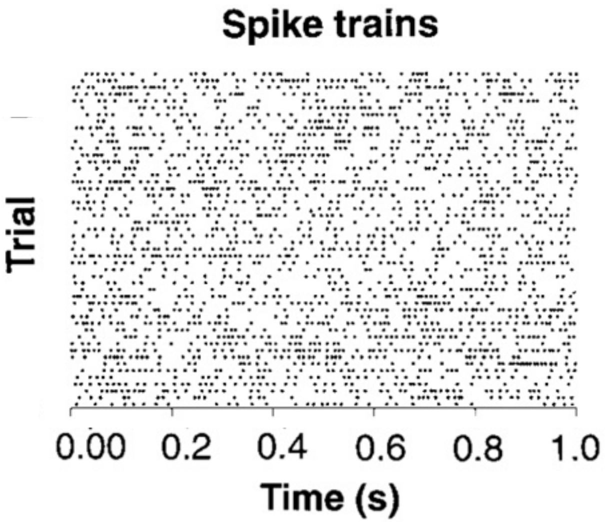
- Model individual spikes
- Model binary neurons
- Model firing rate dynamics





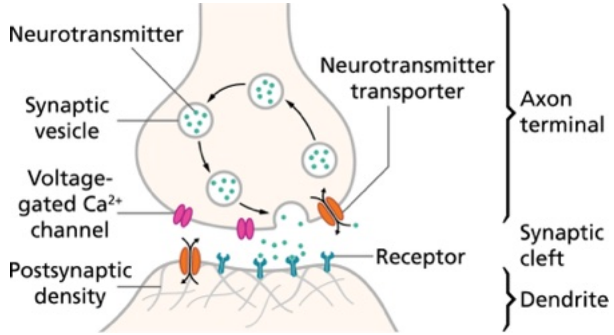


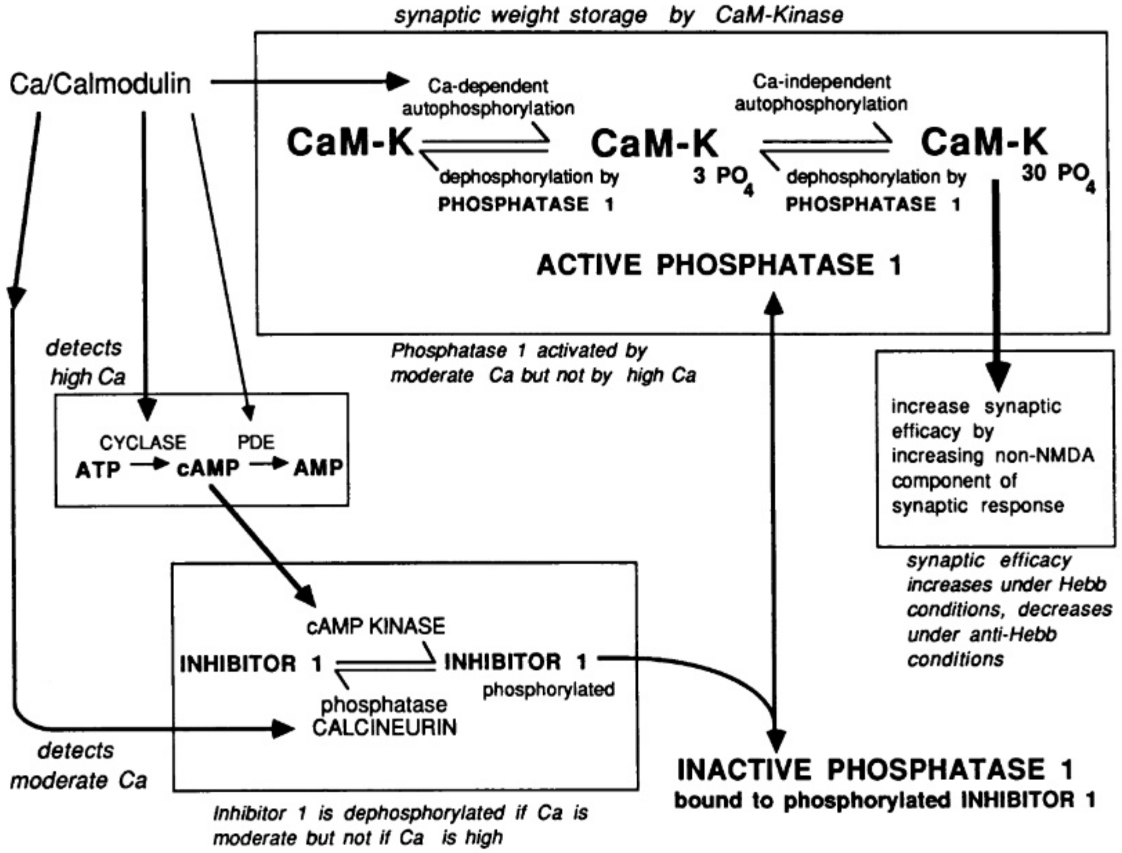


$$\frac{\partial s_i}{\partial t} + \frac{s_i}{\tau} = \phi\left[\sum_j W_{ij} s_j + B_i\right]$$

Dynamics of the circuits themselves

Neural plasticity modeled at various levels of abstraction





Proc. Natl. Acad. Sci. USA
Vol. 86, pp. 9574–9578, December 1989
Neurobiology

A mechanism for the Hebb and the anti-Hebb processes underlying learning and memory

(calmodulin/calcium/calmodulin-dependent protein kinase II/protein phosphatase 1/adenylate cyclase/calcineurin)

John Lisman
Department of Biology, Brandeis University, Waltham, MA 02254

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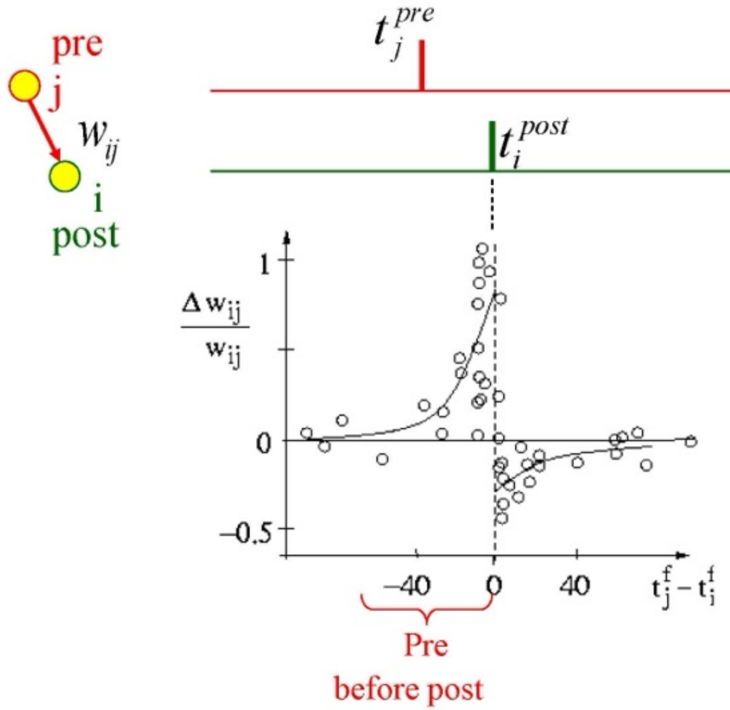

Bi & Poo (1998)

The Spike-Timing Dependence of Plasticity

Daniel E. Feldman[1,*]
[1]Department of Molecular and Cell Biology, and Helen Wills Neuroscience Institute, University of California, Berkeley, Berkeley, CA 94720-3200, USA
*Correspondence: dfeldman@berkeley.edu
http://dx.doi.org/10.1016/j.neuron.2012.08.001

Timescales of plasticity

- STDP ~10 milliseconds
- BTSP ~1-10 seconds
- LTP, LTD ~ 1-1000 minutes
- Neurodevelopment ~ hours to months
- Evolutionary?

Nonlinear dynamical systems

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- Broad goal:
  Understand and analyze nonlinear dynamics **without** solving the entire system

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$\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$

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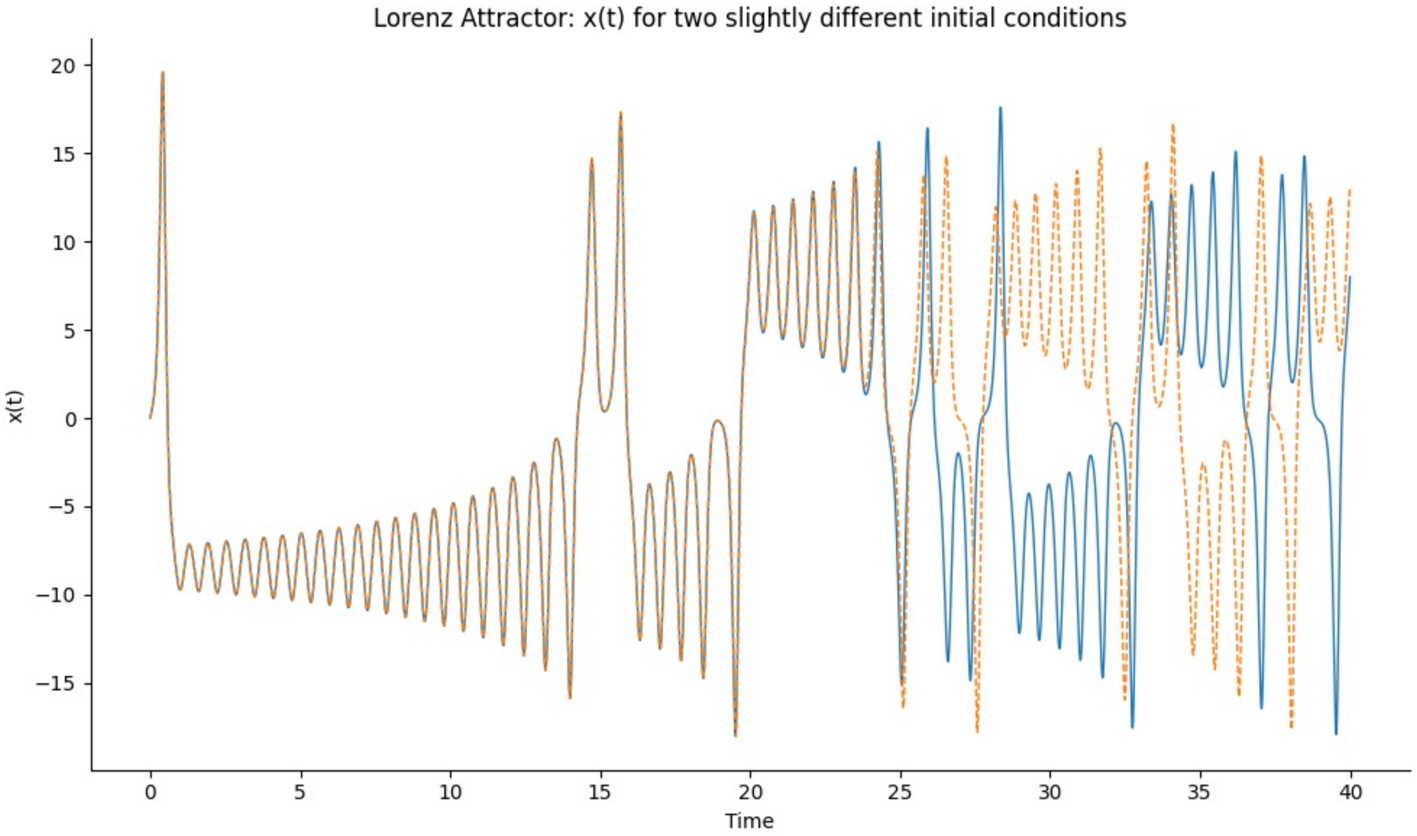

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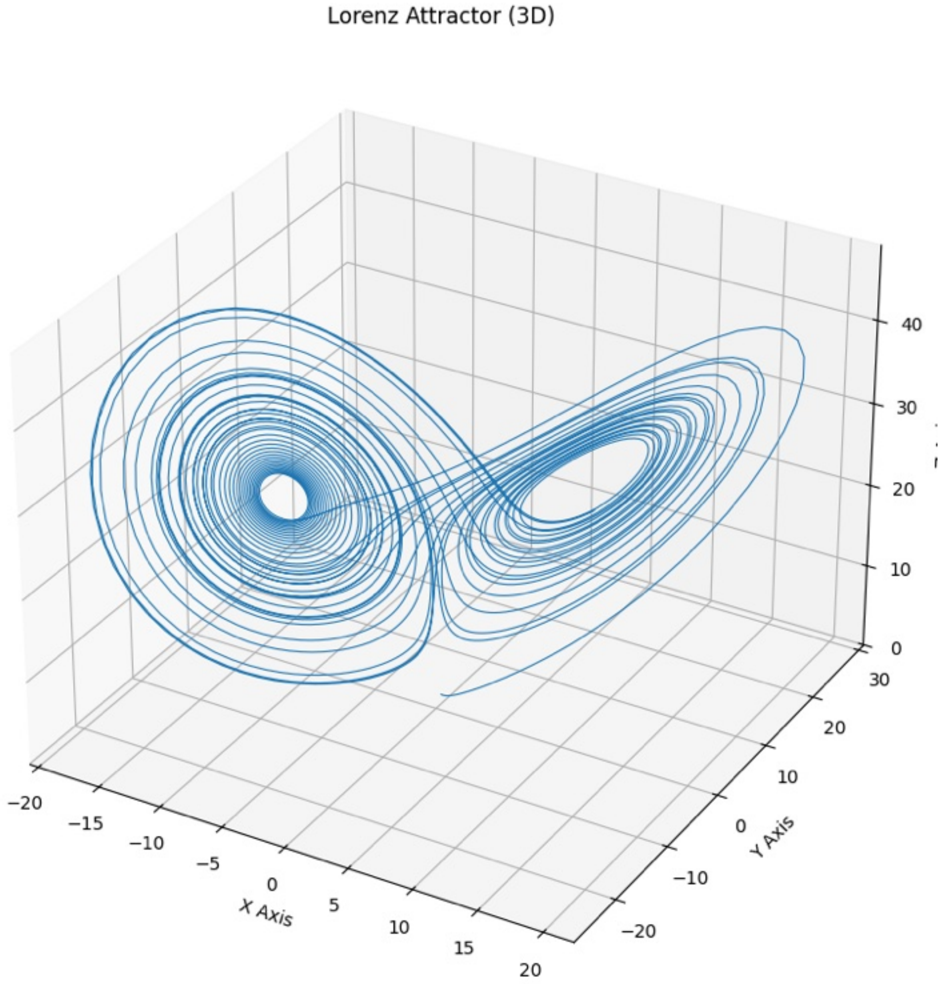

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Edward Lorenz

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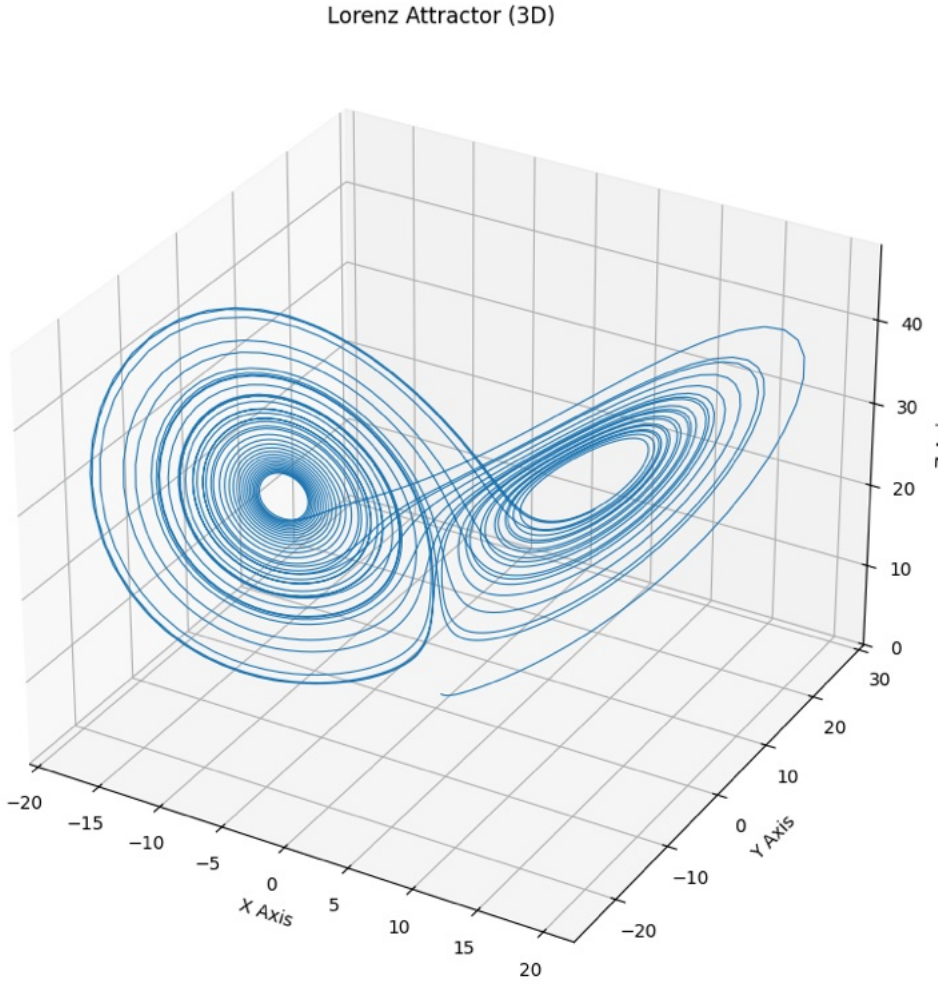

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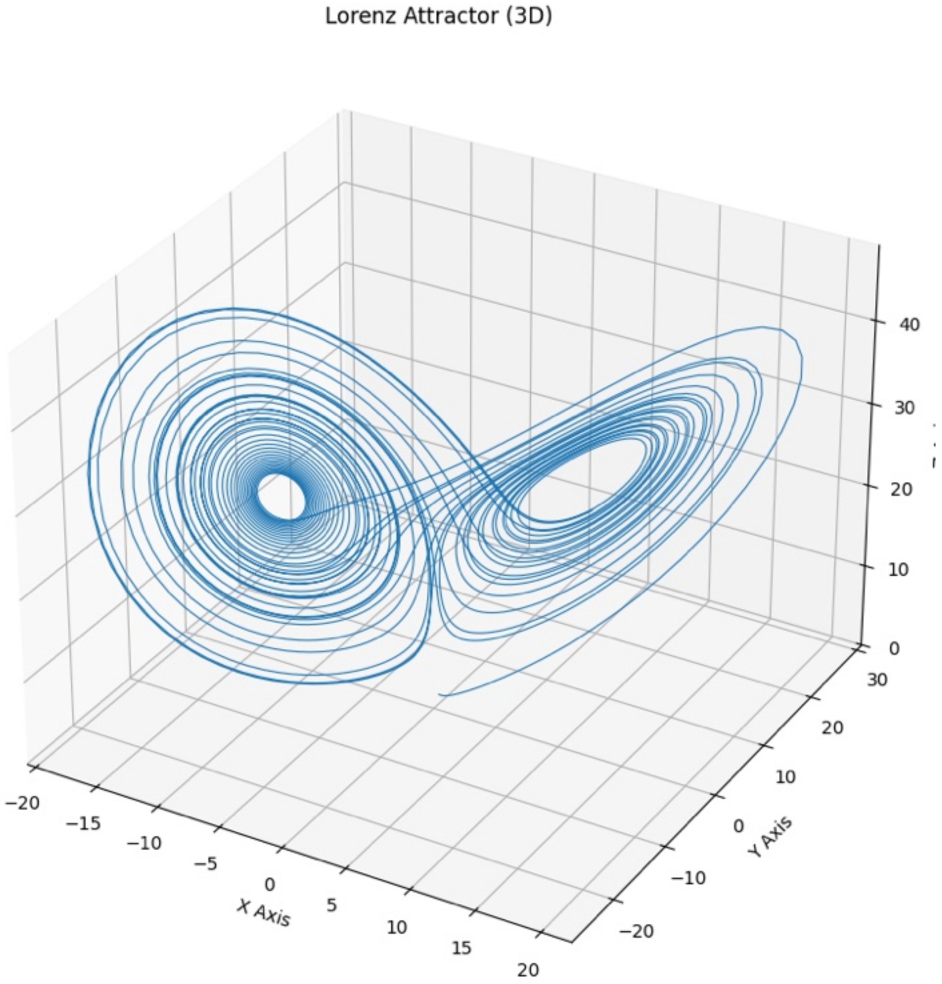
Edward Lorenz

Ellen Fetter

Margaret Hamilton



What is a Dynamical System?

$$\dot{\bar{x}} = F(\bar{x})$$ Autonomous D.S.

$$\dot{\bar{x}} = A\bar{x}$$

$$\dot{x}_1 = F_1(x_1, \dots, x_n)$$
$$\dot{x}_2 = F_2(\quad)$$
$$\vdots$$
$$\dot{x}_n = F_n(x_1, \dots, x_n)$$

1) Time explicit ←
2) Higher order Deriv. ←
3) Implicit Dyn ←
4) Neural Fields
5) Discrete systems

Non. Aut. D.S. → $\dot{\bar{x}} = F(\bar{x}, t)$

$$\bar{y} = \begin{pmatrix} \bar{x} \\ t \end{pmatrix} \qquad \dot{\bar{y}} = \begin{pmatrix} F(\bar{x}) \\ 1 \end{pmatrix}$$

$$\dot{t} = 1$$

$$\ddot{x} = -x$$
$$y = \dot{x}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$\dot{x} = F(x) \leftarrow$

$\left[\dot{x} = F(x,t) \right]$

F is cts

DF or F' is cts

Implicit

$\left[F(\dot{x}, x, t) = 0 \right]$

$\dot{\bar{x}} = A\bar{x}$ (non-normal dynamics $\leftarrow$ V. V. Intresting)

$\bar{x}(t) = c_1 e^{\lambda_1 t}\hat{e}_1 + c_2 e^{\lambda_2 t}\hat{e}_2 + \dots$

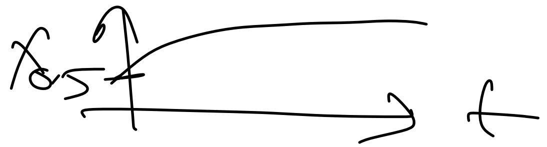
1D Flows

$x(0) = 0.5$

$\dot{x} = F(x) = -x(x-1)(x-2)$

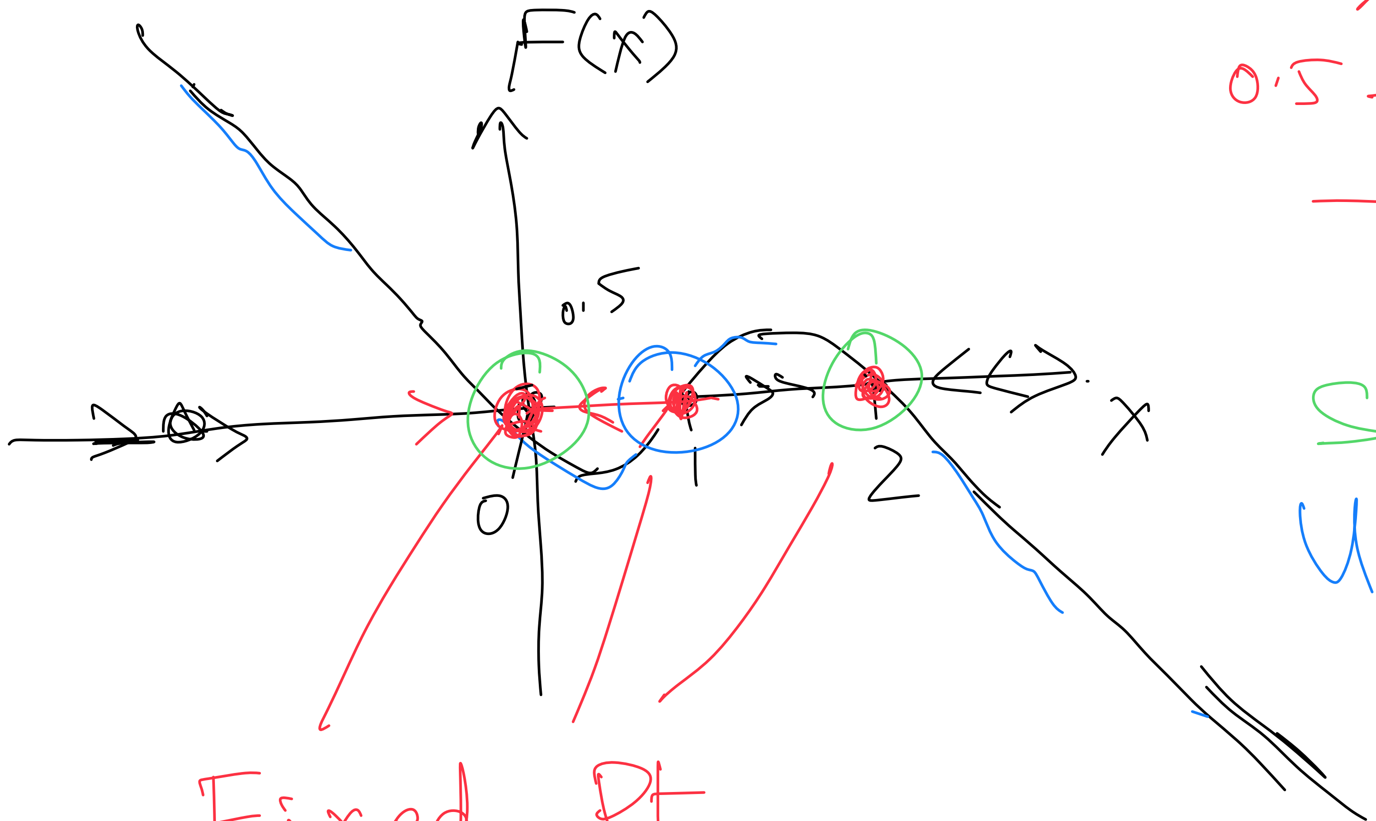
$$-\frac{dx}{x(x-1)(x-2)} = dt \longrightarrow t = -\frac{1}{2}\ln x + \ln(x-1) - \frac{1}{2}\ln(x-2) + C$$

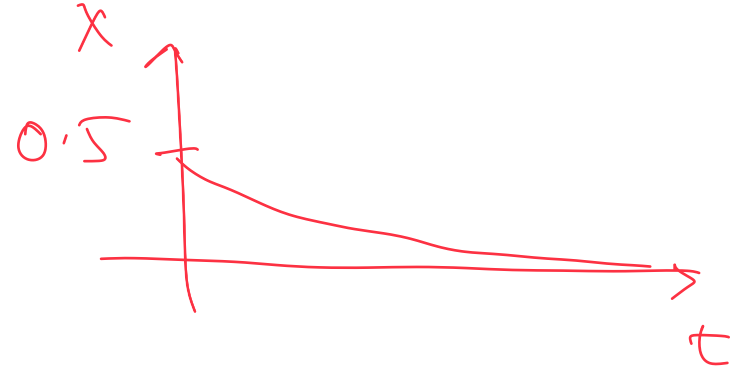
$x = \_\_\_\_$



$$\dot{x} = -x(x-1)(x-2) + \varepsilon \sin x$$

$$= F(x)$$





Stable → $F' < 0$

Unstable → $F' > 0$

Fixed Pt

→ $F(x_*) = 0$

$$x = x_* + \varepsilon$$

$$\dot{x} = F(x)$$

$$\dot{x}_* + \dot{\varepsilon} = F(x_* + \varepsilon)$$

$$\dot{\varepsilon} = \cancel{F(x_*)} + F'(x_*) \cdot \varepsilon$$

$$\dot{\varepsilon} = F'(x_*)\, \varepsilon$$

$$\dot{x} = F(x)$$

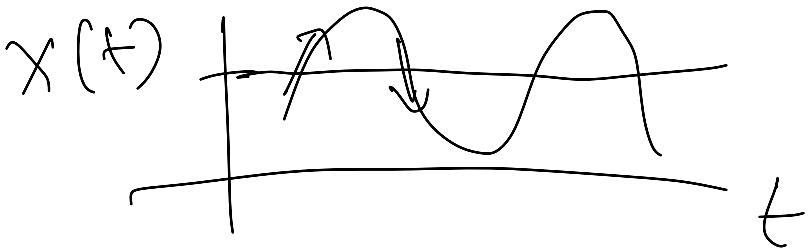
$$x(0) = x_0$$

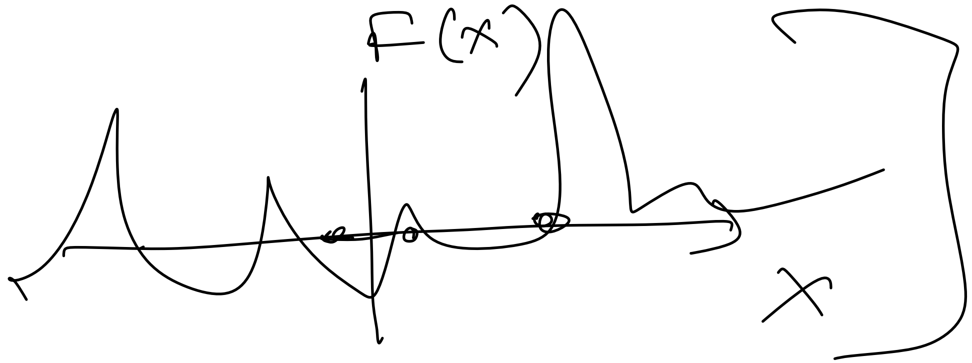
What happens as $t \to \infty$

1) $x(t) \to x^*$ (goes to a F.P.) ← can be multiple F.P. depending on init. cond.

2) $x(t) \to \infty$ or $x(t) \to -\infty$

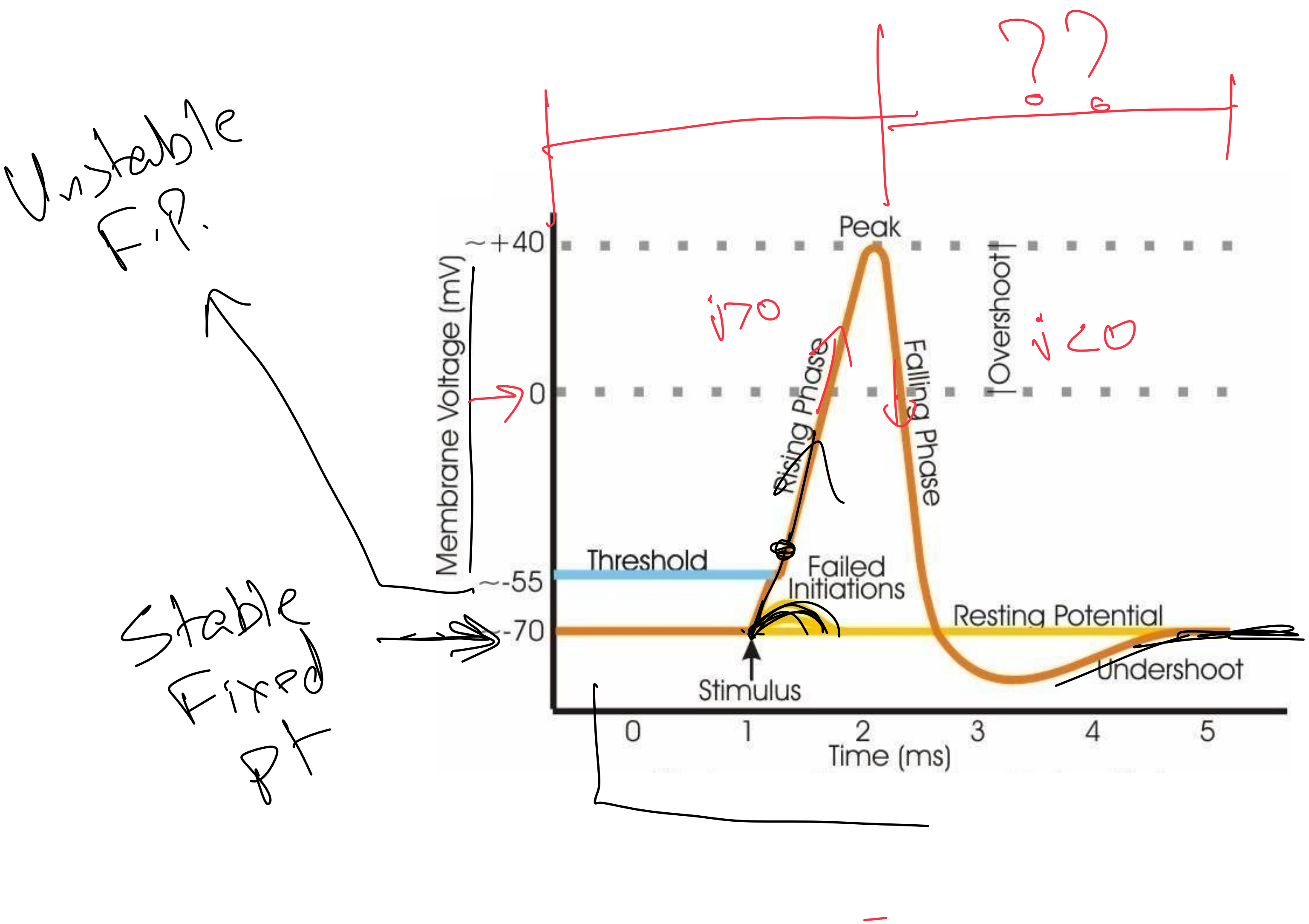
~~Periodic $x(t+T) = x(t)$~~ ✗





1D Flow

↳ cannot have periodicity

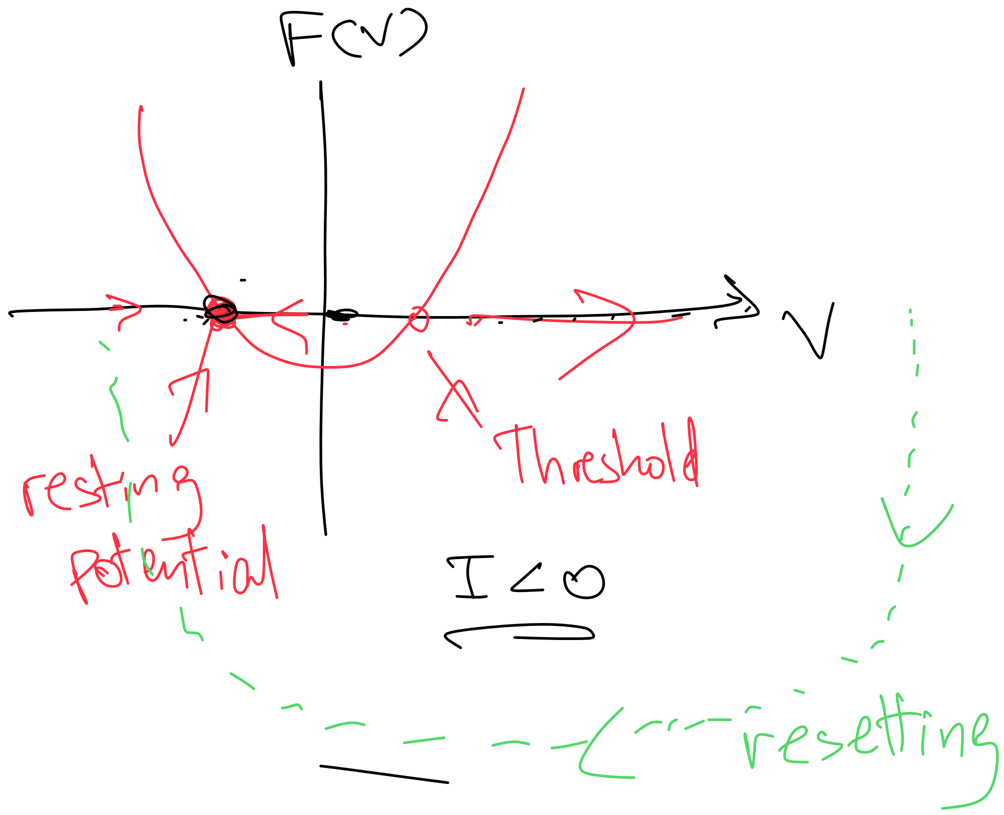


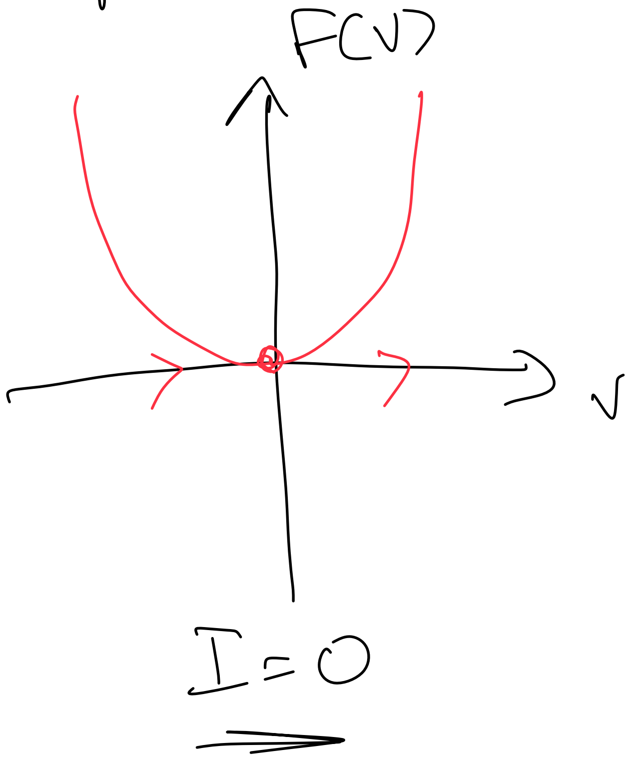
Unstable F.P.
Stable Fixed pt
??
i>0
i<0
Peak
~+40
0
~-55
-70
Membrane Voltage (mV)
Threshold
Failed Initiations
Rising Phase
Falling Phase
Overshoot
Resting Potential
Undershoot
Stimulus
0
1
2
3
4
5
Time (ms)

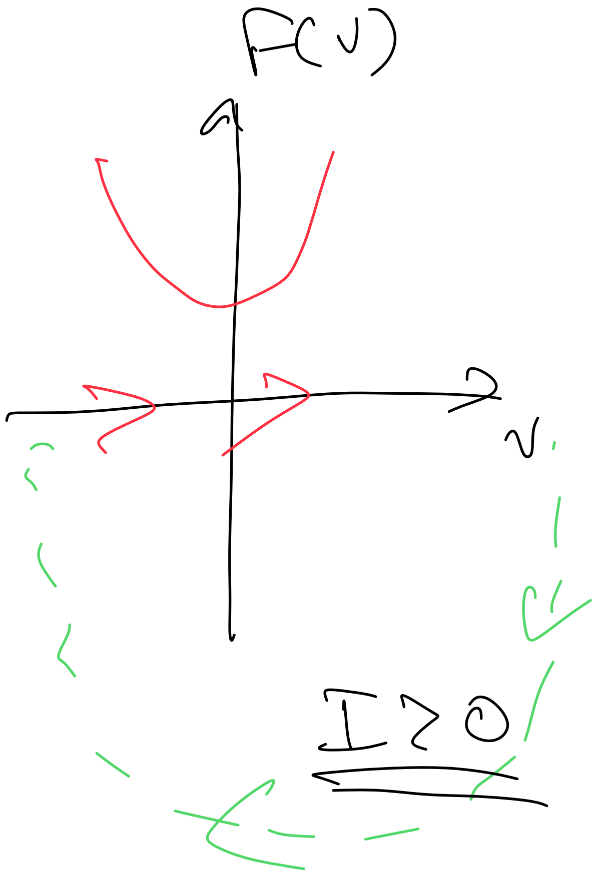
Quadratic Integrate and Fire Neuron

GKND § 5.3; Iz. § 3.3.8

$$\dot{V} = F(V) = V^2 + I \rightarrow \text{Inputs}$$







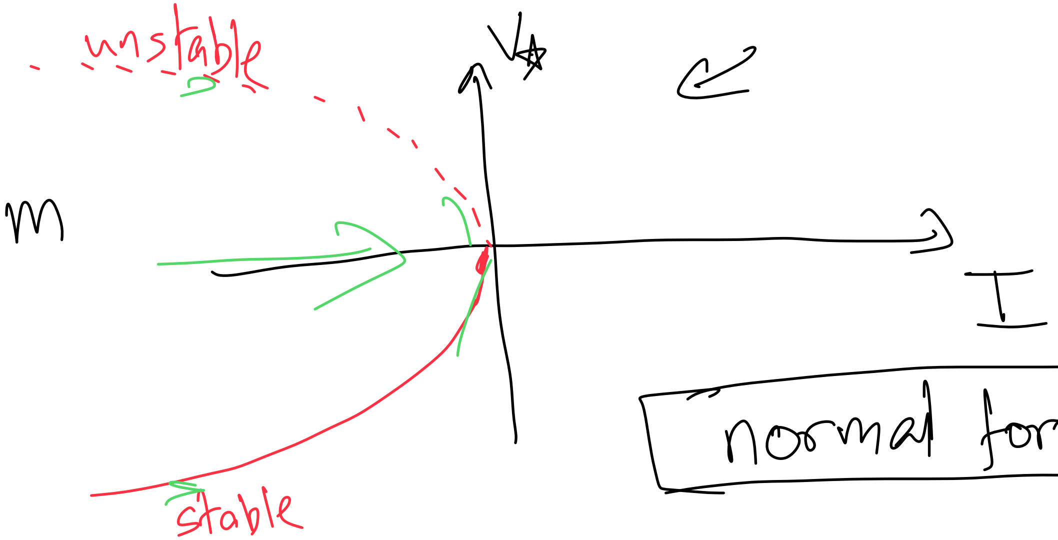
F.P. $\rightarrow V_{\pm} = \pm\sqrt{-I}$

Kind of like an extra dim

No F.P

QIF

Bifurcat^n Diagram



$$\dot{v} = v^2 + I$$

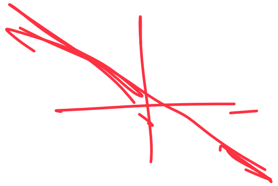
"Saddle Node Bifurcat^n"

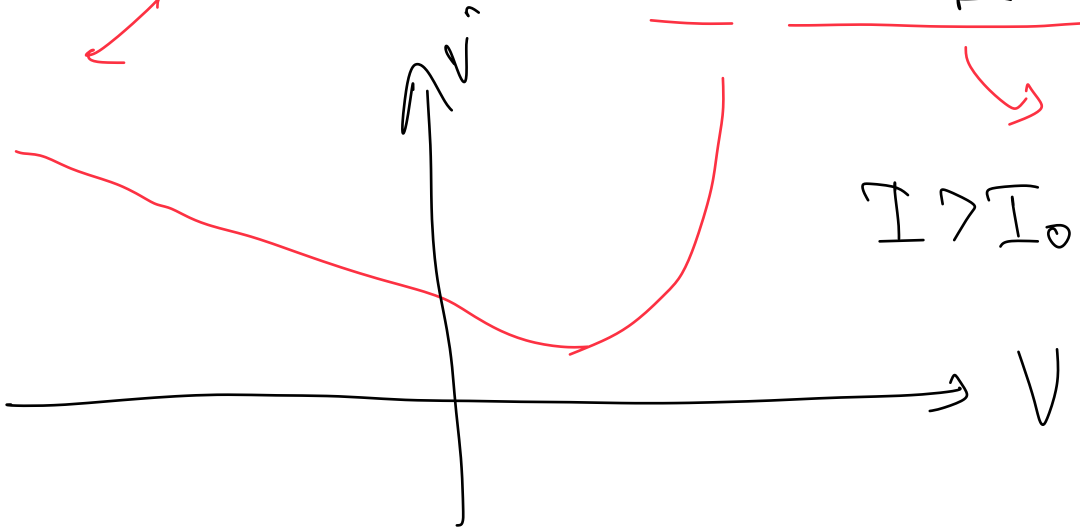
"normal form of an S.N. Bif."

Exponential Int. & Fire Neuron (EIF)

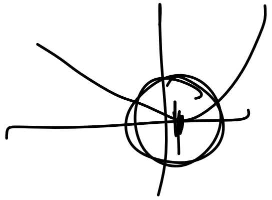
GKND § 5.2

$$\dot{v} = -(v - v_0) + k \exp\left(\frac{v - v_0}{k}\right) + I$$

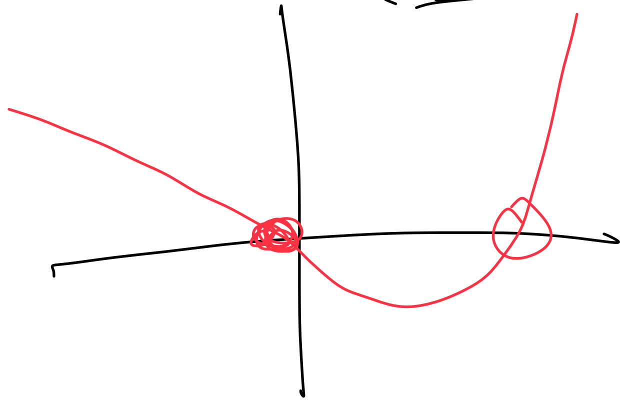




I > I$_0$



I < I$_0$

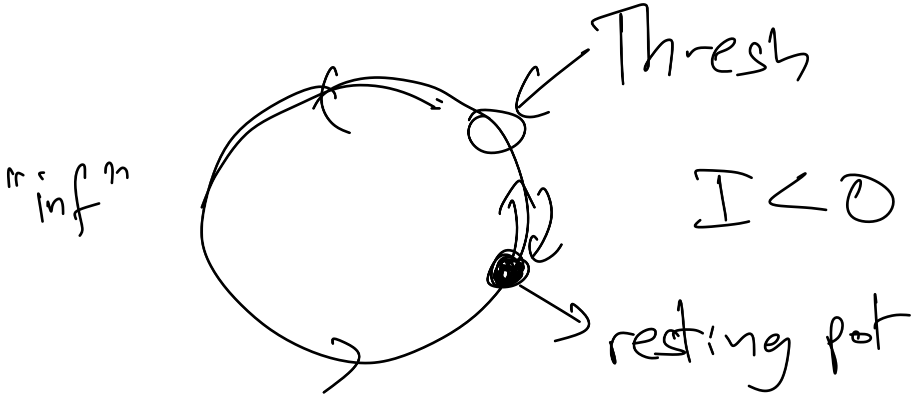


~~$\dot{x}$~~ $\dot{\theta} = F(\theta)$

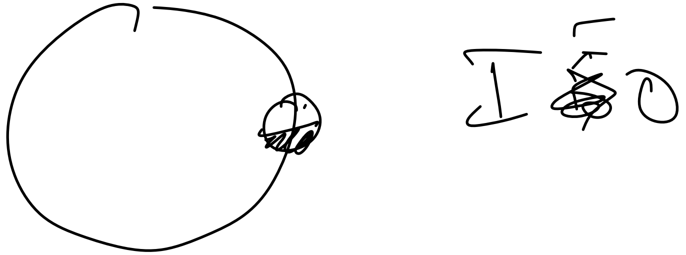
Theta Neuron Model

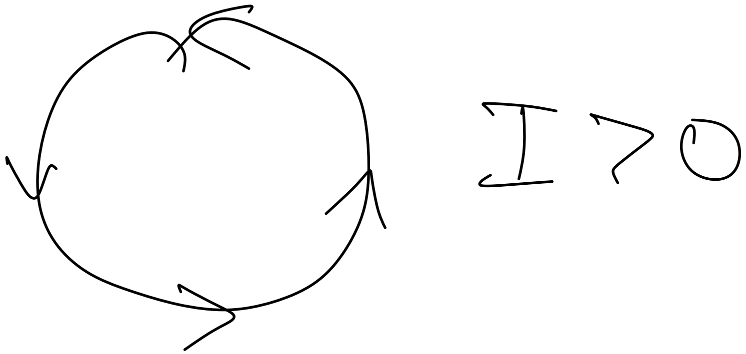
$$\dot{\theta} = 1 - \cos\theta + (1 + \cos\theta) I$$

Iz. § 3.38



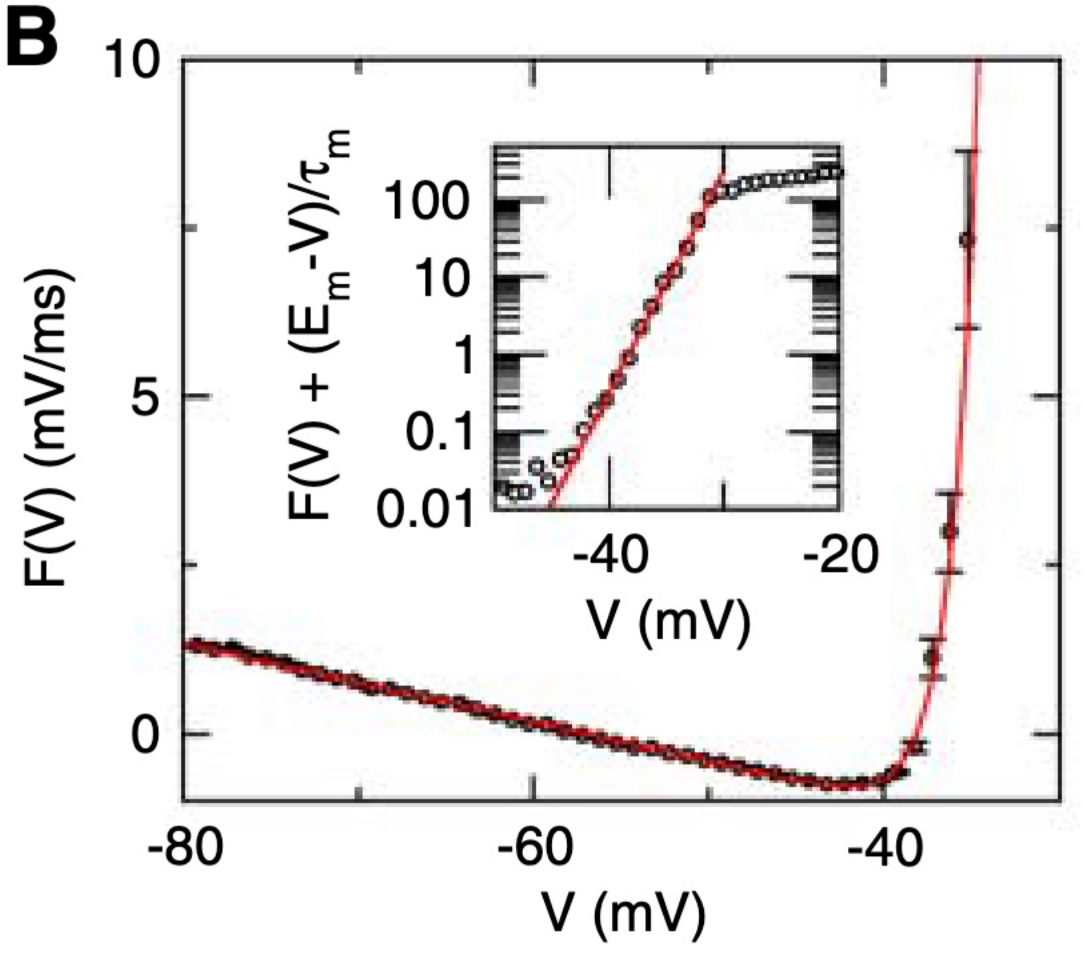
Saddle
Node Bif
on an
Invariant
Cycle
(SNIC)



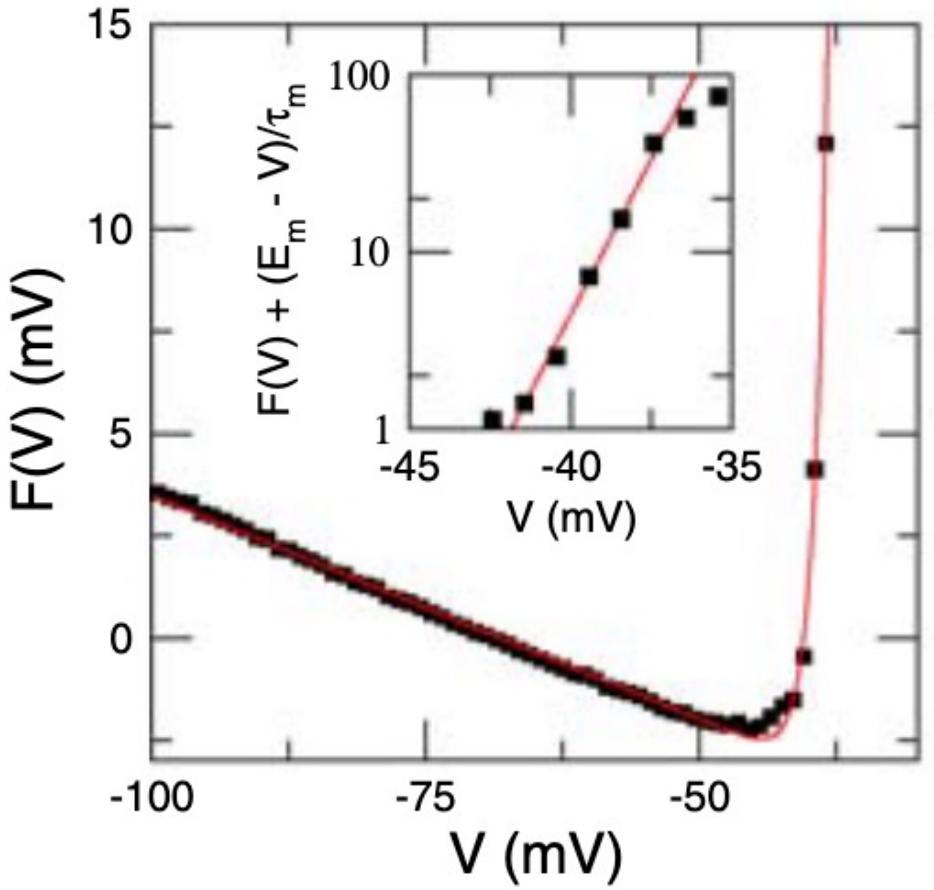


Expoential Integrate and Fire neuron

Badel et al. Biol Cybern (2008)



Pyramidal cell



Interneuron

Transcritical Bifurcation

Avalanche Dyn

$a \to$ fraction of active neurons

$a \in [0, 1] \longrightarrow a = 0$ is an absorbing boundary condition

$\uparrow$

$a = 0$ is a F.P $\leftarrow$ unstable

$$\dot{a} = -a + \lambda a(1-a)$$

$\longrightarrow$ growth $\propto$ active frac
$\propto$ available frac

$$\frac{da}{dt} = a(\lambda - 1) - \lambda a^2$$

$$\frac{da}{\lambda dt} = a\frac{(\lambda - 1)}{\lambda} - a^2$$

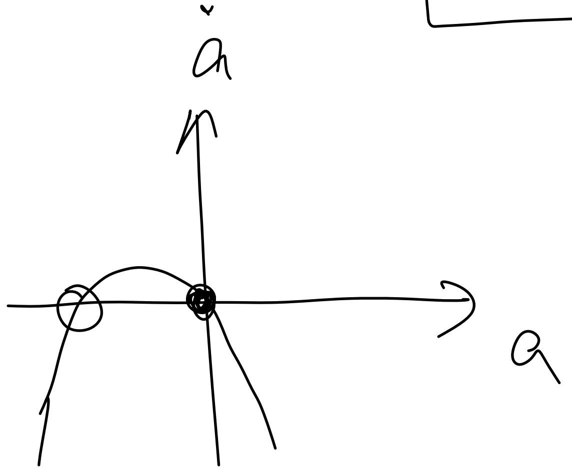
$$\tau = \lambda t$$

$$\dot{a} = a\left(\frac{\lambda - 1}{\lambda}\right) - a^2$$

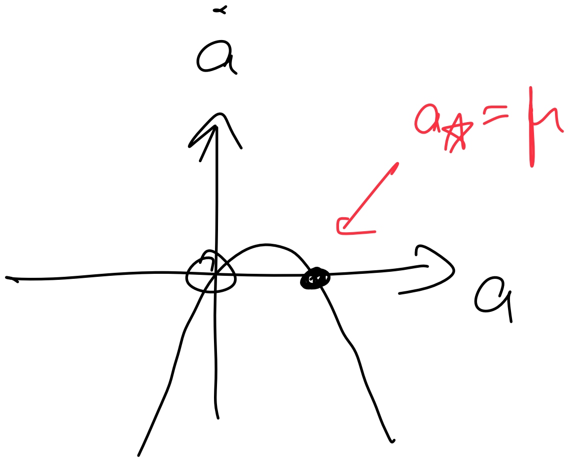
$$\dot{a} = a\underbrace{\left(\frac{\lambda - 1}{\lambda}\right)}_{\mu} - a^2$$

$\lambda \approx 1 + \mu$

$$\boxed{\dot{a} = \mu a - a^2}$$

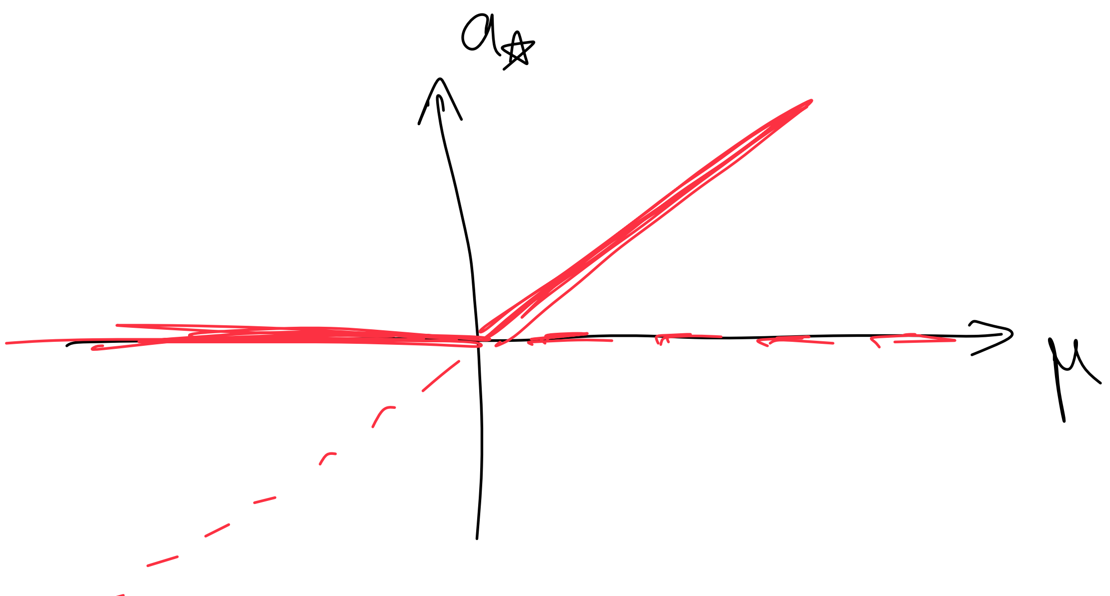


$\mu < 0$

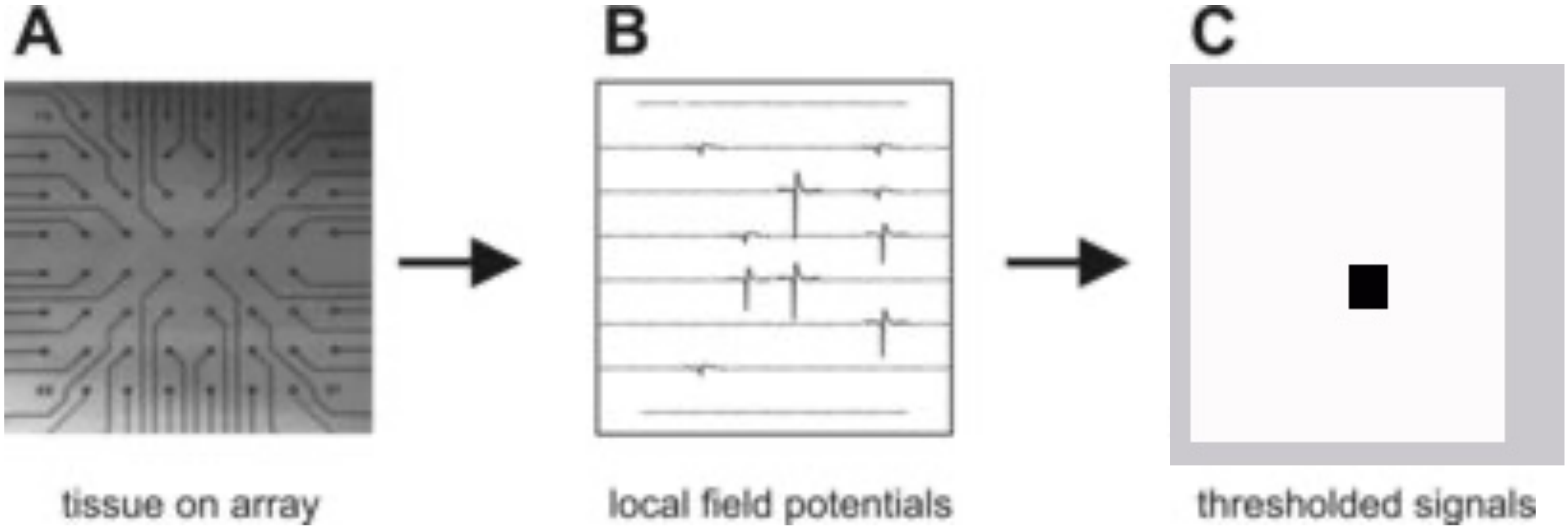


$\mu > 0$

Transcritical Bifurcatⁿ



Avalanche Dynamics



J. Beggs (2007)

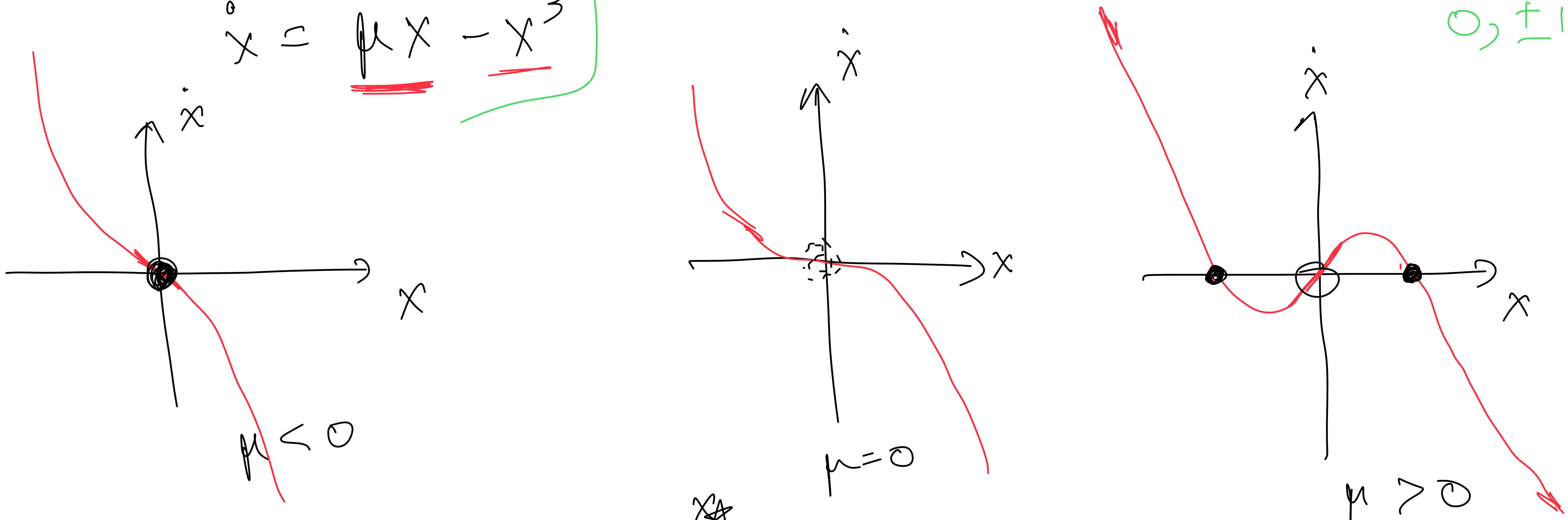
Pitchfork Bifurcation

$$\dot{x} = \mu x - x^3$$

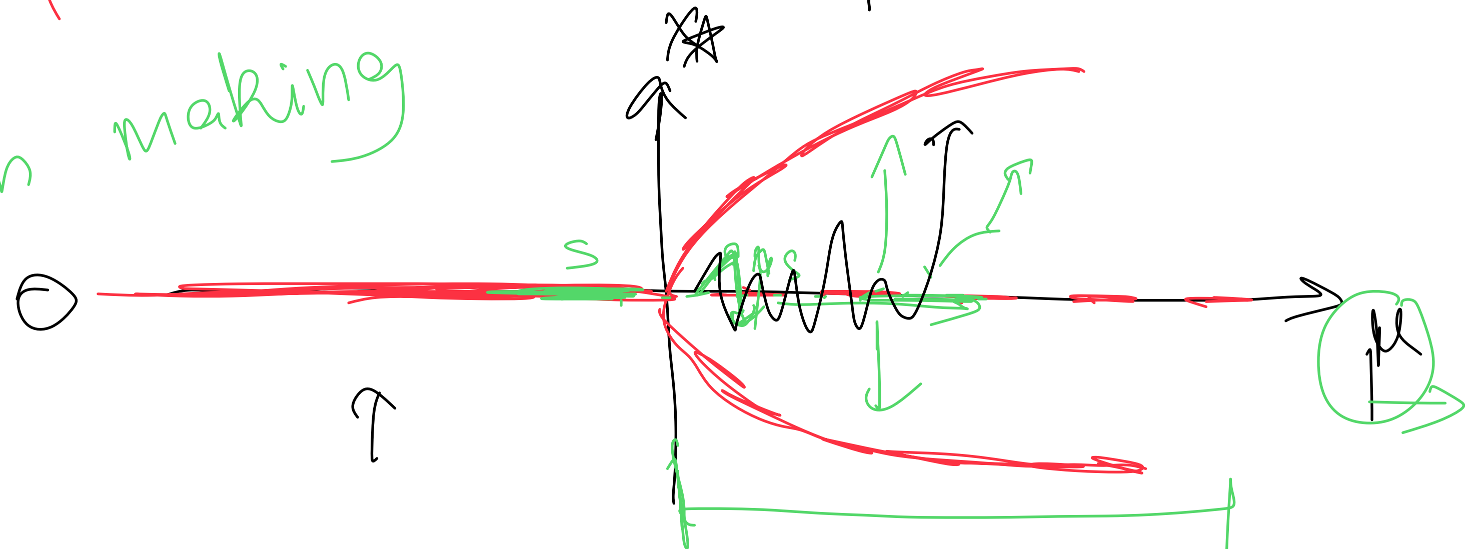
$\mu x - x^5$

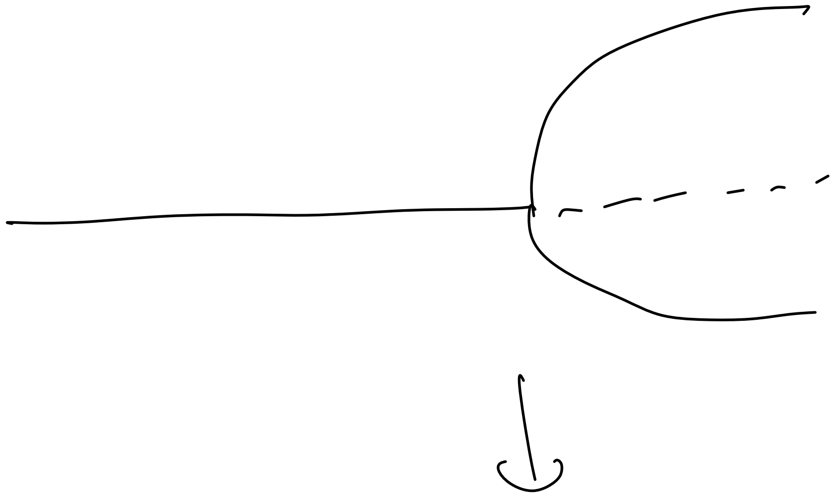
$+x - x^5$

$0, \pm 1$

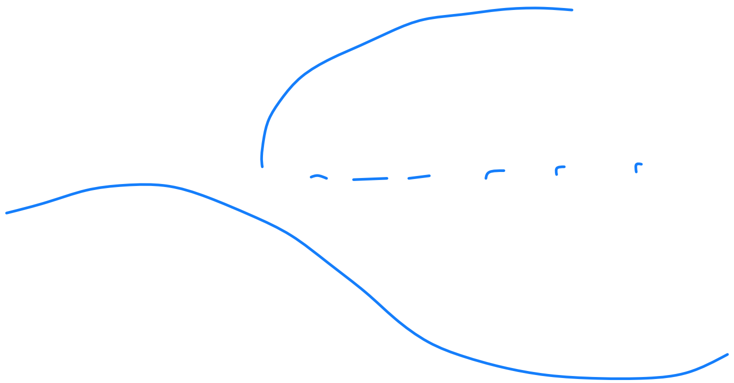


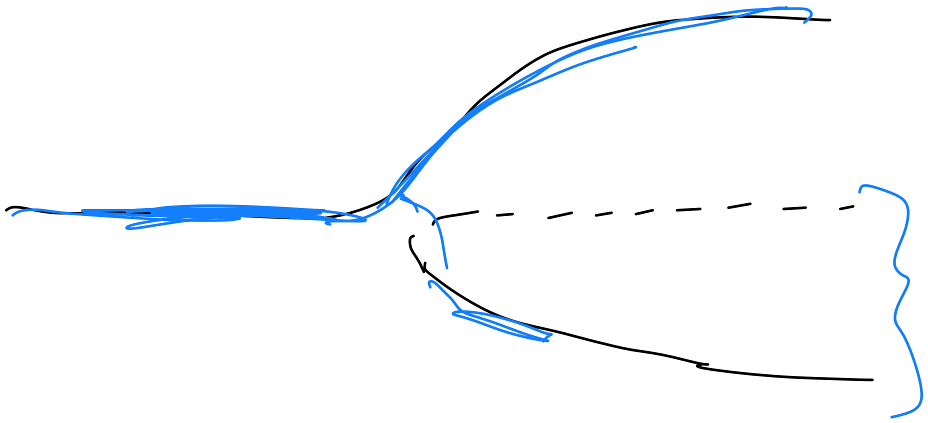
2-choice decision making





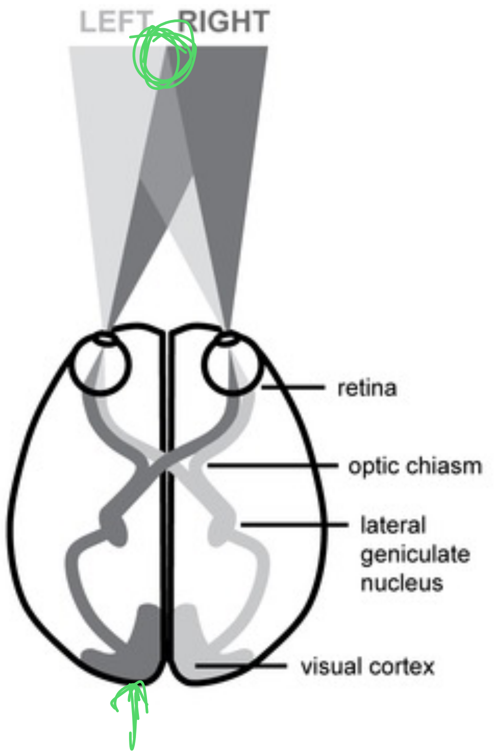
$$\dot{x} = \mu x - x^3 + f(x)$$



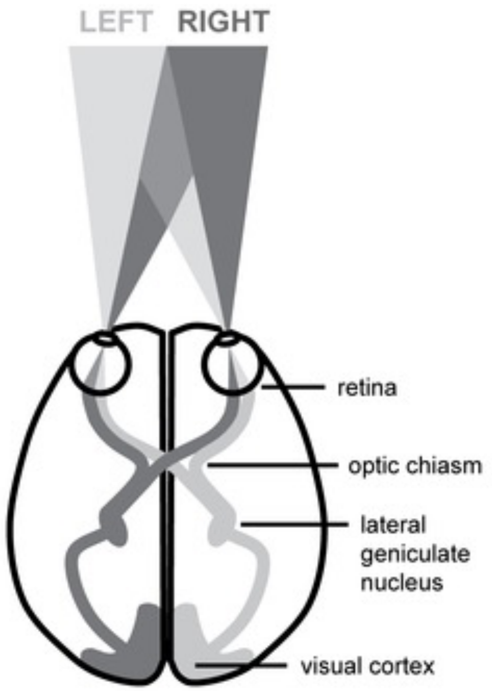


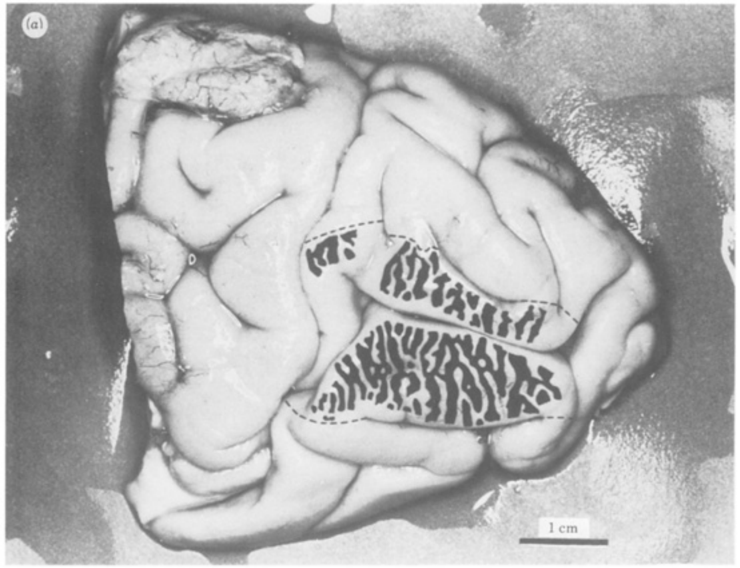
S.N. Bif.

Ocular dominance columns

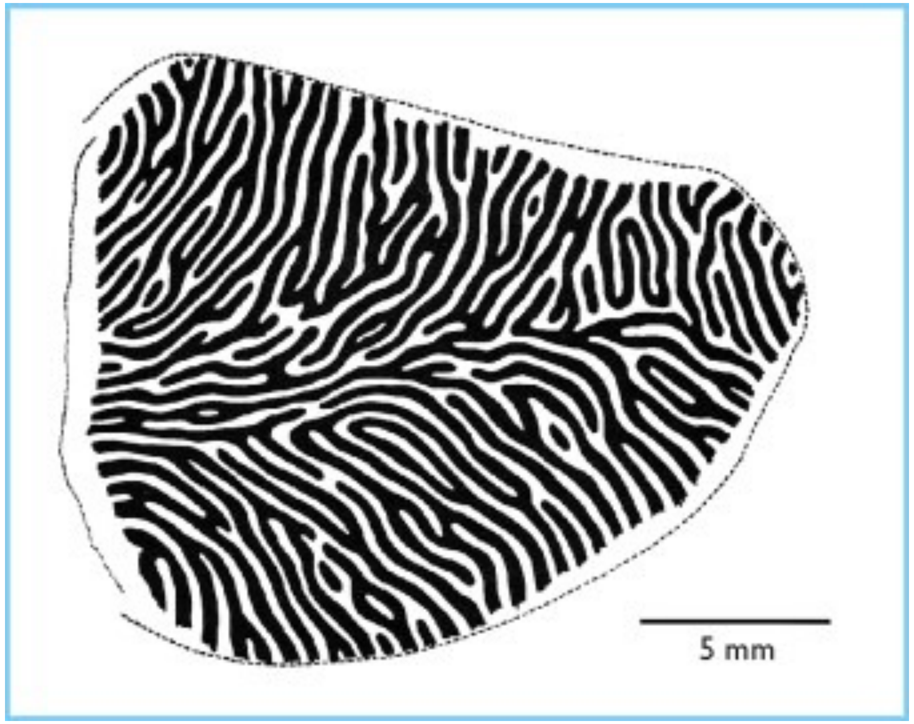


Ocular dominance columns



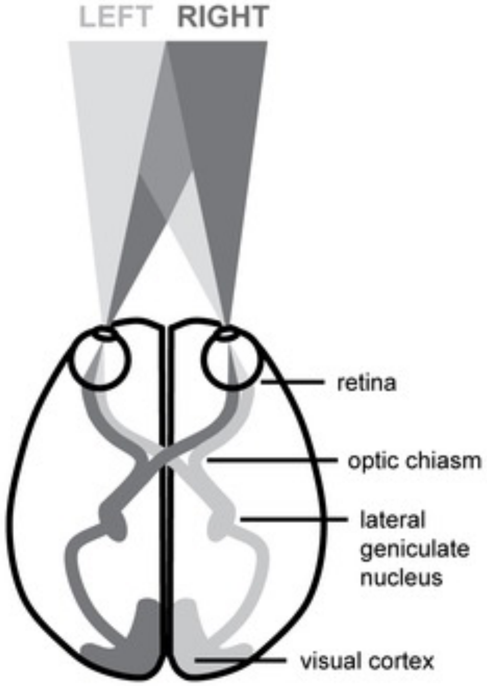


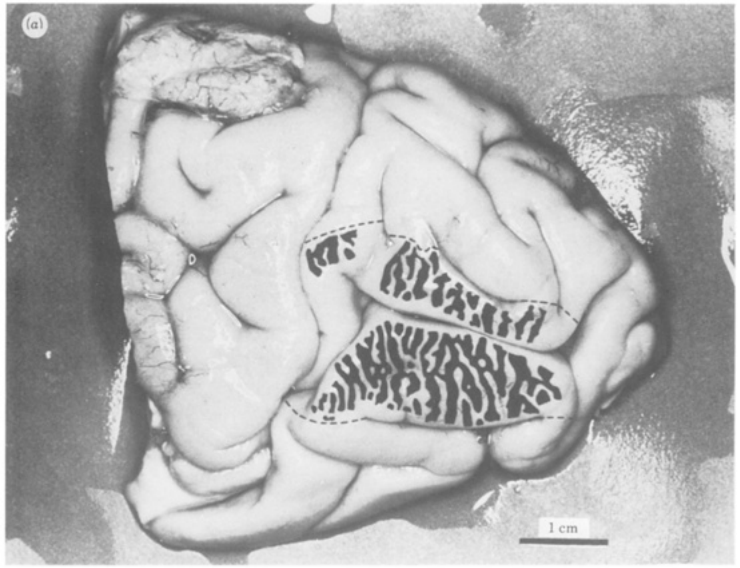
Horton & Hedley-Whyte (1984)



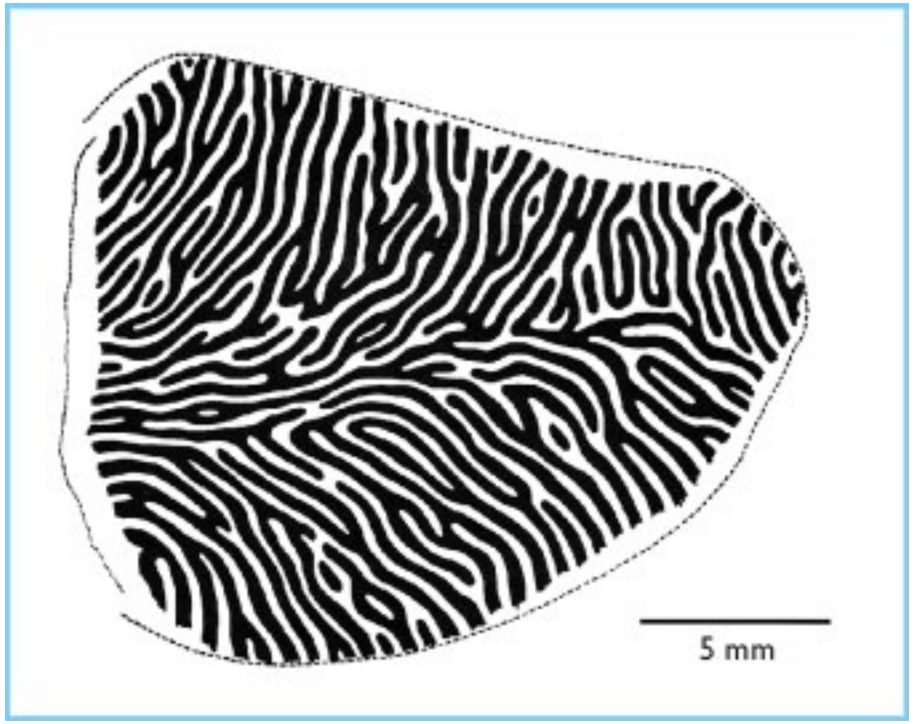
Hubel & Wiesel 1977

Ocular dominance columns





Horton & Hedley-Whyte (1984)



Hubel & Wiesel 1977

Proc. Natl. Acad. Sci. USA
Vol. 94, pp. 9944–9949, September 1997
Neurobiology

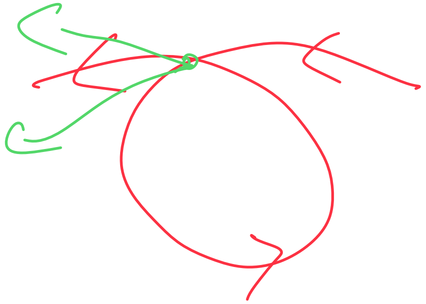
A model of ocular dominance column development by competition for trophic factor

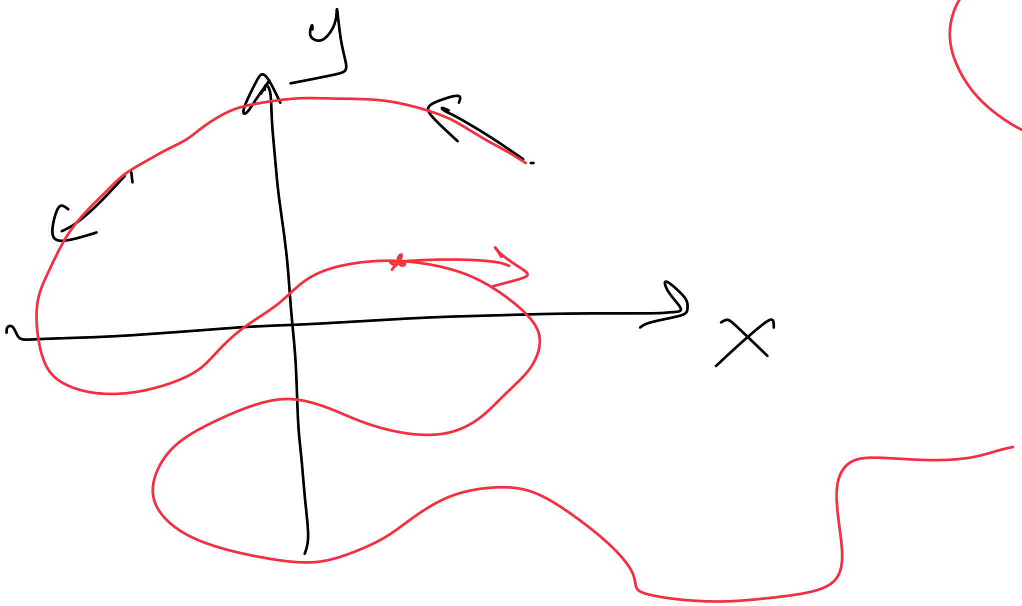
ANTHONY E. HARRIS*†‡§, G. BARD ERMENTROUT†¶, AND STEVEN L. SMALL*†‡

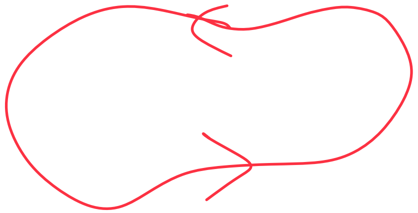
*Intelligent Systems Program, †Center for the Neural Basis of Cognition, and Departments of ‡Neurology and ¶Mathematics and Statistics, University of Pittsburgh, Pittsburgh, PA 15261

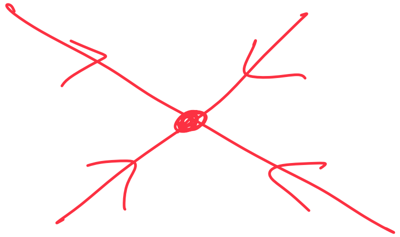
2D Flows

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} \qquad \dot{\bar{x}} = \bar{F}(\bar{x})$$









1) Go to infinity
2) Go to a F.P.
3) Go to a periodic Orbit

→ Poincare-Bendixson Theorem ←

$$\dot{\bar{x}} = F(\bar{x})$$

$$\dot{\bar{x}} = 0 \Rightarrow F(\bar{x}_*) = 0$$

Jacobian

$$\bar{x} = \bar{x}_* + \bar{\varepsilon}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>eigen values</u> $\lambda_1, \lambda_2 \rightarrow$ stable. F.P $\Longleftrightarrow Re[\lambda_1] < 0$, $Re[\lambda_2] < 0$

unstable $\rightarrow$ all other cases

$$F(x_*) = 0$$

$F' > 0$ unstable

$F' < 0$ stable.

$$x = x_* + \varepsilon$$

$$\dot{\varepsilon} = F' \cdot \varepsilon$$

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$Tr[J] = T$$

$$Det[J] = \Delta$$

$$\det[\lambda \mathbb{1} - J) = 0$$

$$\det \begin{bmatrix} \lambda - a & b \\ c & \lambda - d \end{bmatrix} = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$\lambda^2 - Tr[J]\lambda + Det[J] = 0$$

$$\lambda^2 - T\lambda + \Delta = 0$$

$$\hookrightarrow \boxed{\lambda = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}}$$

Elements of Applied Bifurcat^n Theory

25/5/26

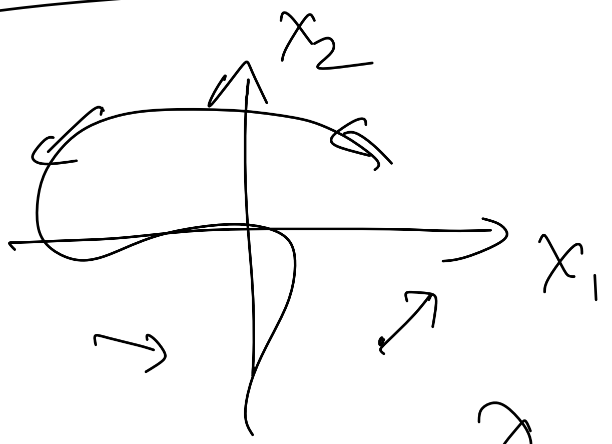
– TextBook

Kuznetson

2D FLOWS

$$\dot{\bar{x}} = \bar{F}(\bar{x})$$

$\bar{x}_\star \rightarrow \bar{F}(x_\star) = 0$



$$\bar{x} = \bar{x}_\star + \begin{pmatrix} x \\ y \end{pmatrix}$$

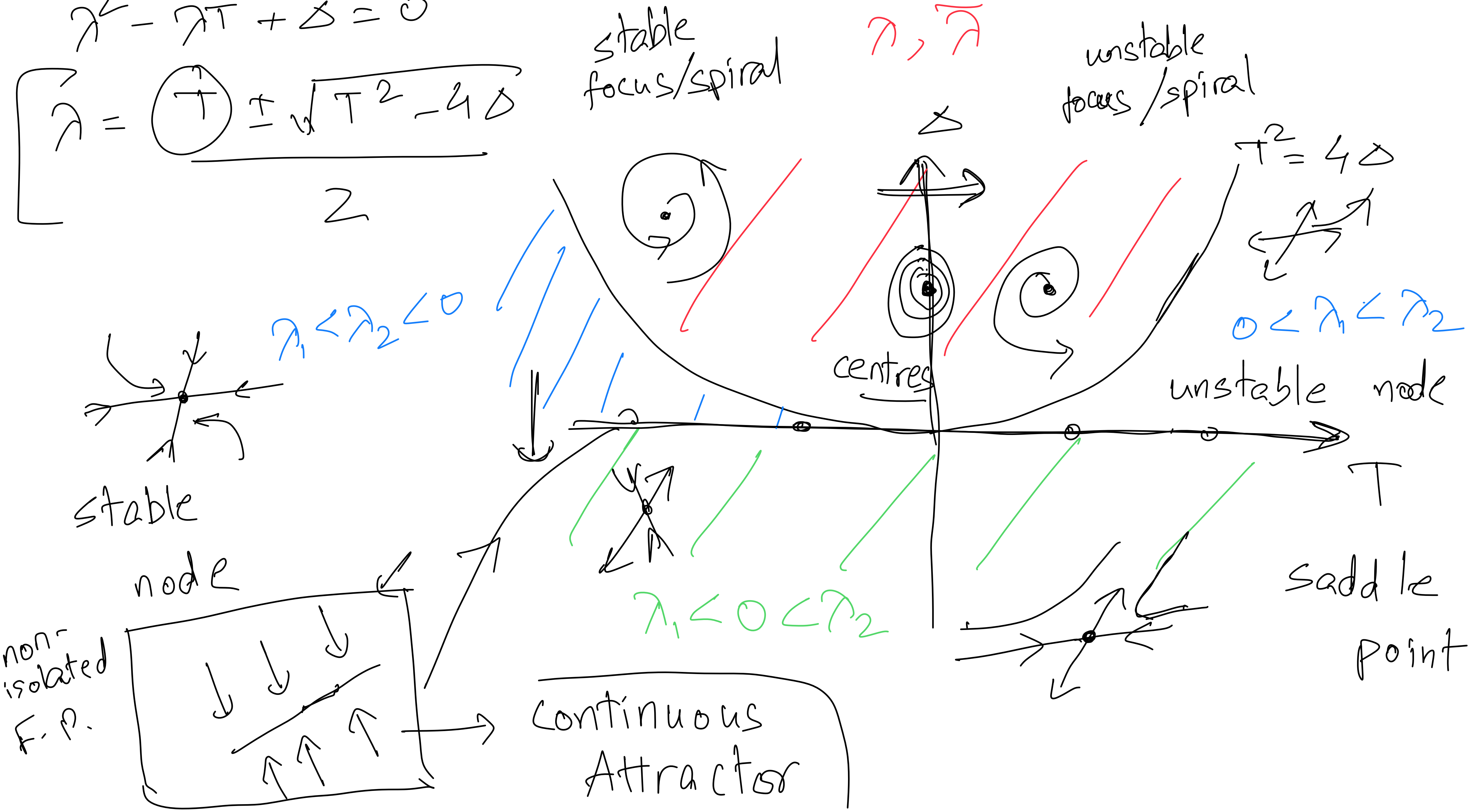
λ_1, λ_2

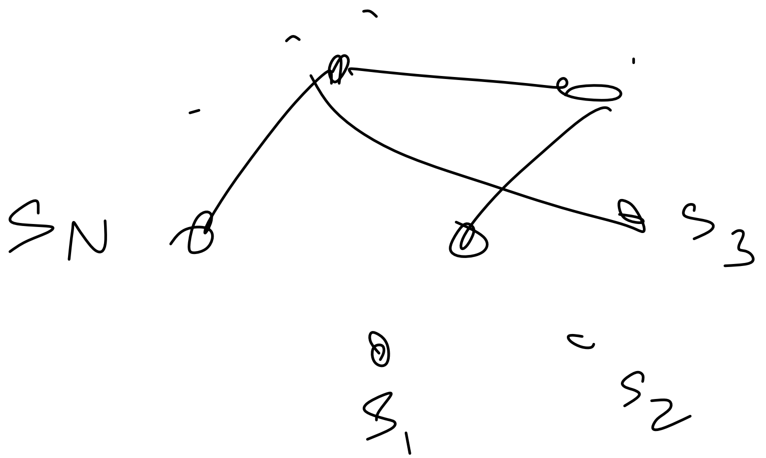
$$\lambda^2 - \lambda T + \Delta = 0$$

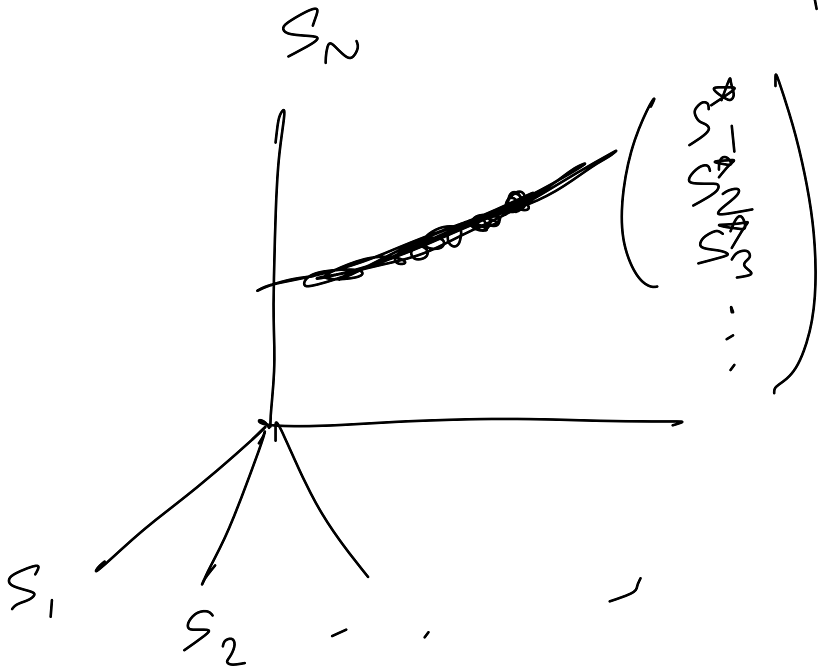
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda^2 - \lambda T + \Delta = 0$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}$$







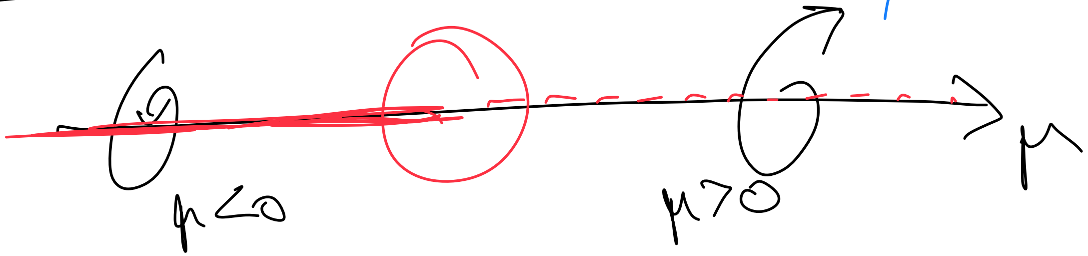
cts. attr

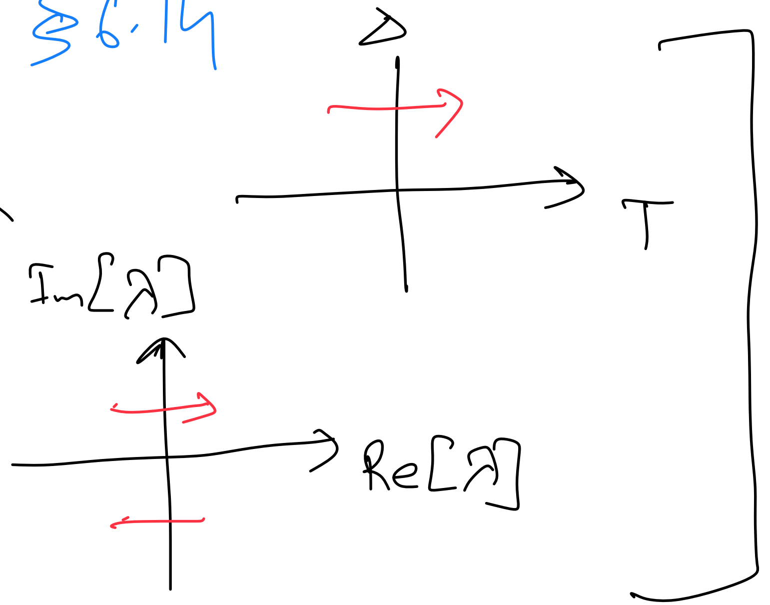
repr.

cts variable

Hopf Bifurcation

Strogatz §8.2 / Iz. §6.1.4





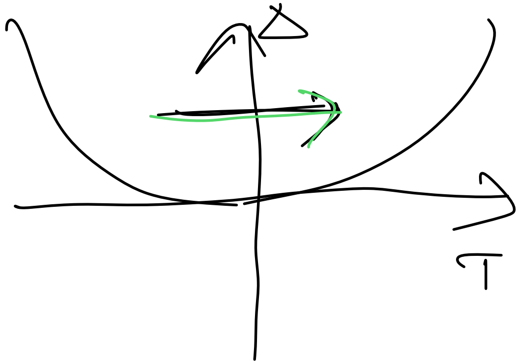
Normal form

$$\begin{cases} \dot{x} = \mu x + \omega y + (ax - by)(x^2+y^2) \\ \dot{y} = -\omega x + \mu y + (bx + ay)(x^2+y^2) \end{cases} \quad \mu, \omega, a, b \in \mathbb{R}$$

$(x,y) = (0,0)$ is a F.P.

$$J = \begin{pmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} \mu & \omega \\ -\omega & \mu \end{pmatrix}$$

$$J\Big|_{(0,0)} = \begin{pmatrix} \mu & \omega \\ -\omega & \mu \end{pmatrix} \rightarrow \begin{array}{l} T = 2\mu \\ \Delta = \mu^2 + \omega^2 \end{array}$$



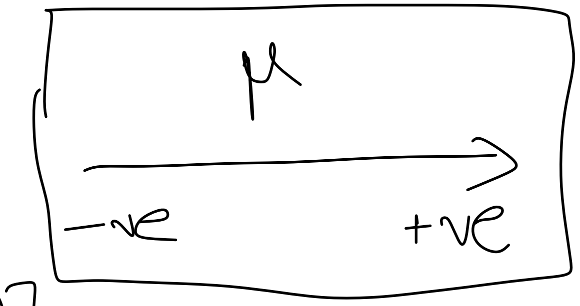
$$\Delta > 0$$

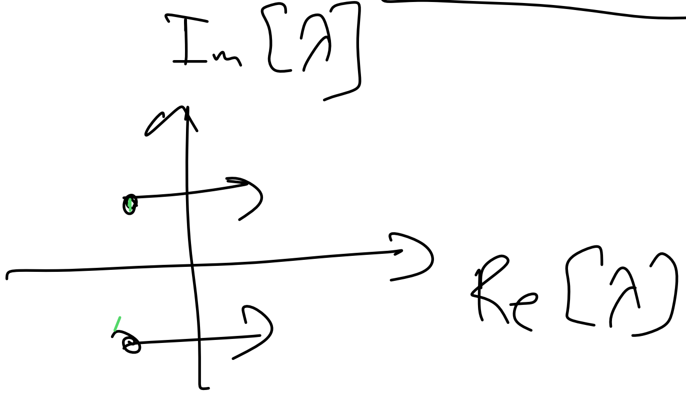
$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}$$

$$= \frac{2\mu \pm \sqrt{4\mu^2 - 4\mu^2 - 4\omega^2}}{2}$$

$$= \mu \pm \sqrt{-\omega^2}$$

$$= \mu \pm i\omega$$





$$(x, y) \longrightarrow (r, \theta)$$

$$r^2 = x^2 + y^2$$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \mu r + a r^3$$

$$\theta = \tan^{-1}(y/x)$$

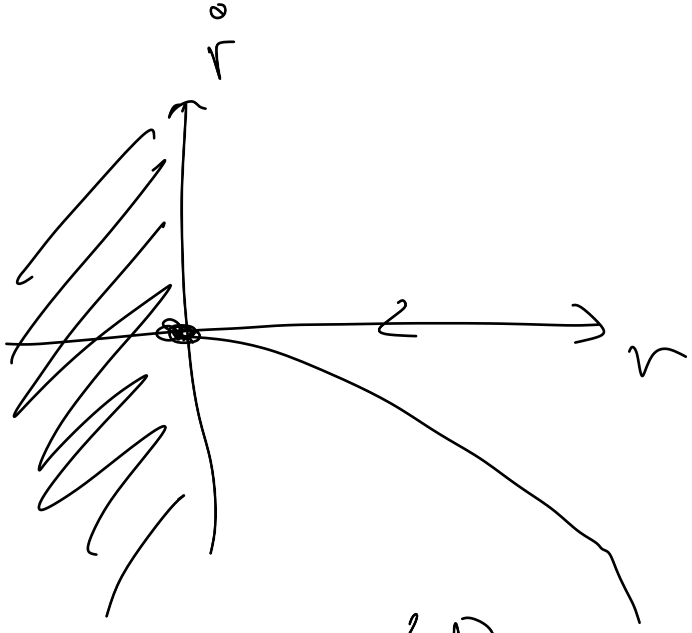
$$\dot{\theta} = \frac{1}{1 + y^2/x^2}\left(\frac{x\dot{y} - y\dot{x}}{x^2}\right) = \frac{x\dot{y} - y\dot{x}}{r^2} = \omega + b r^2$$

$$\dot r = \mu r + a r^3$$

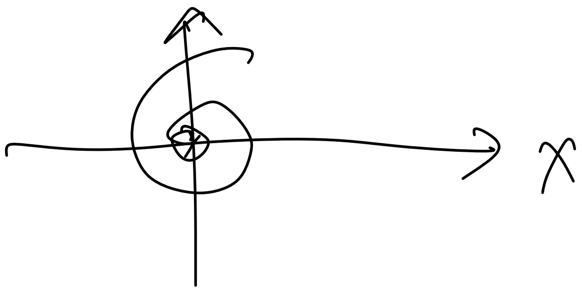
$$\dot\theta = \omega + b r^2$$

$a < 0$

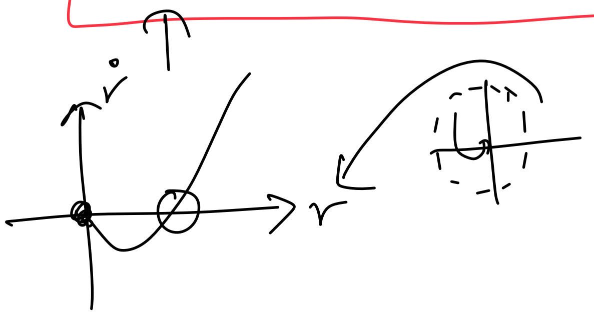
Supercritical Hopf Bif.



$\mu < 0$

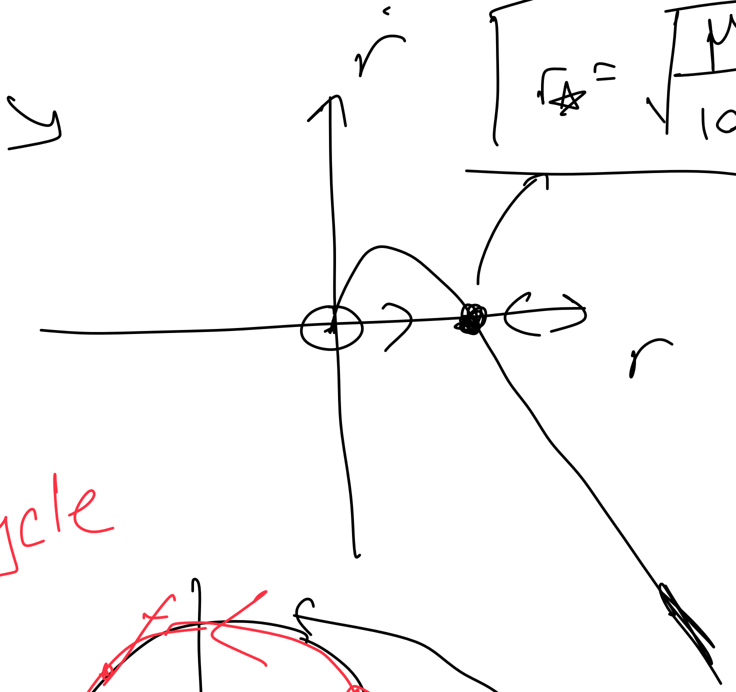


$a > 0 \rightarrow$ Subcritical Hopf

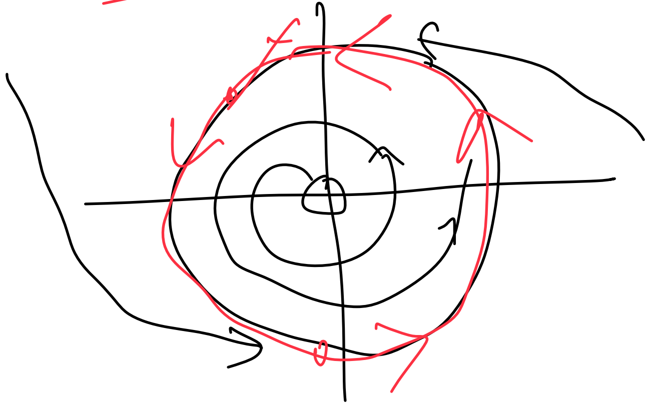


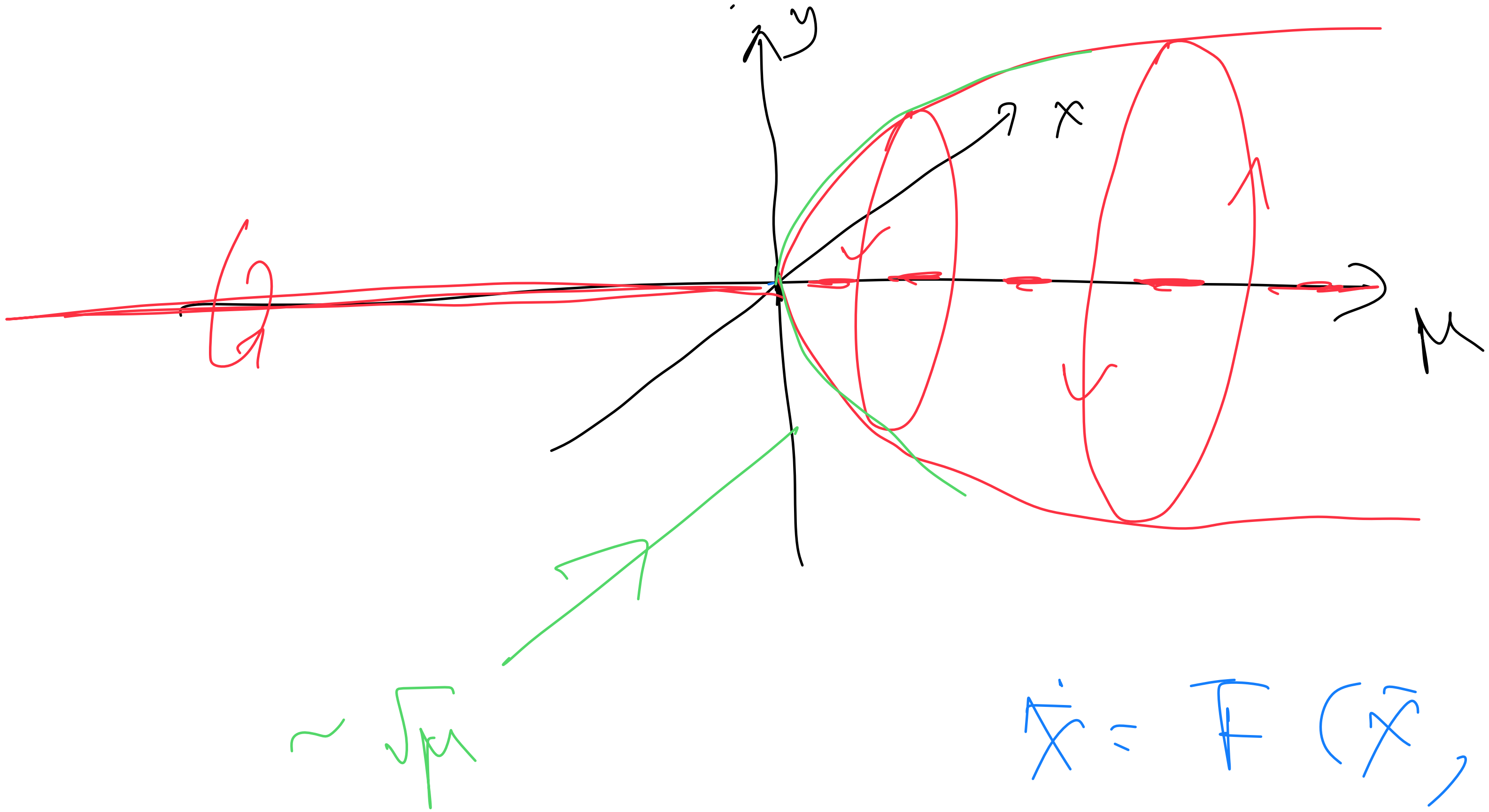
$\mu > 0$

$$r_* = \sqrt{\frac{\mu}{|a|}}$$



Limit cycle





Bifurcat'n dig of Hopf

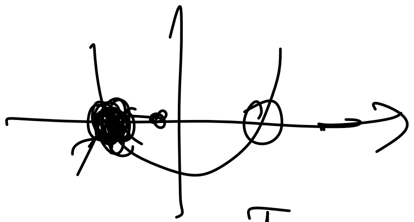
$$\dot{\bar{x}} = \bar{F}(\bar{x}, \bar{\mu})$$

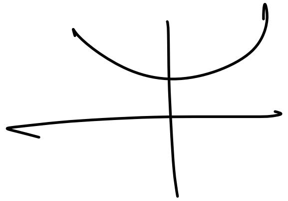
$$|||$$

$$\dot{\bar{y}} = \bar{F}(\bar{x}, \bar{\mu} + \bar{\varepsilon})$$

FitzHugh–Nagumo Model (FHN) → Bonhoffer van der Pol oscillator

Q) IF → $\dot{v} = v^2 + I$ →



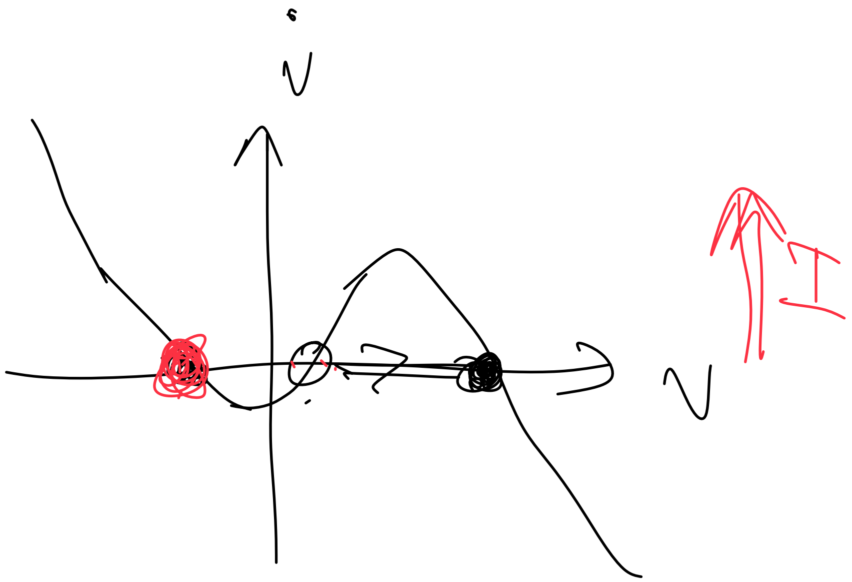


v → threshold for firing ←

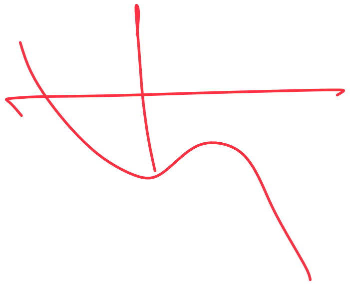
→ Reset v

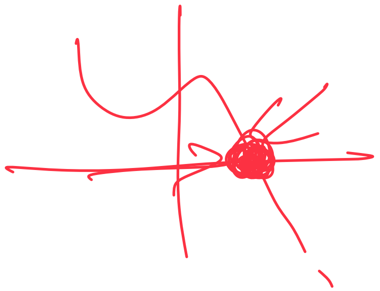
→ v does not go to ∞ ←

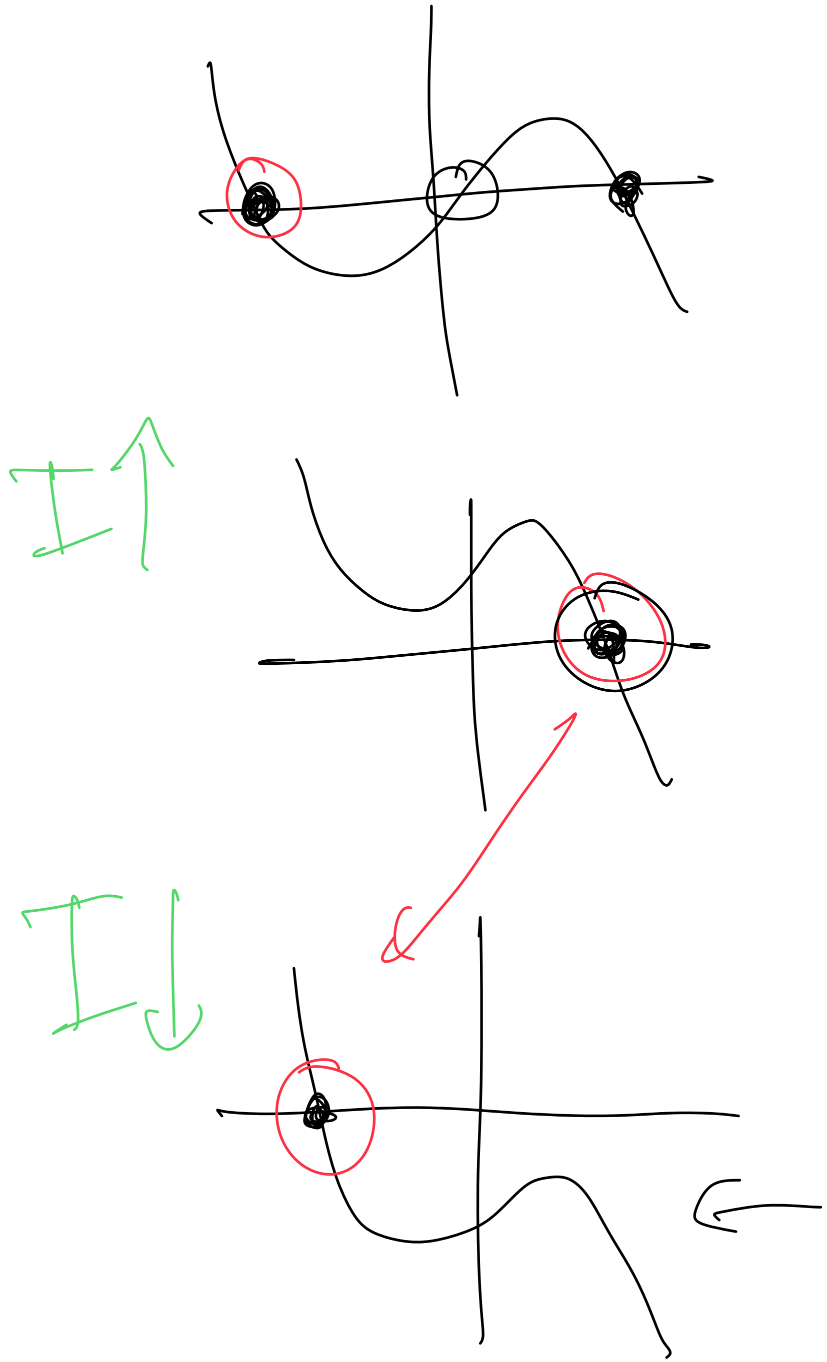
$$\dot{v} = v - \frac{v^3}{3} + I$$

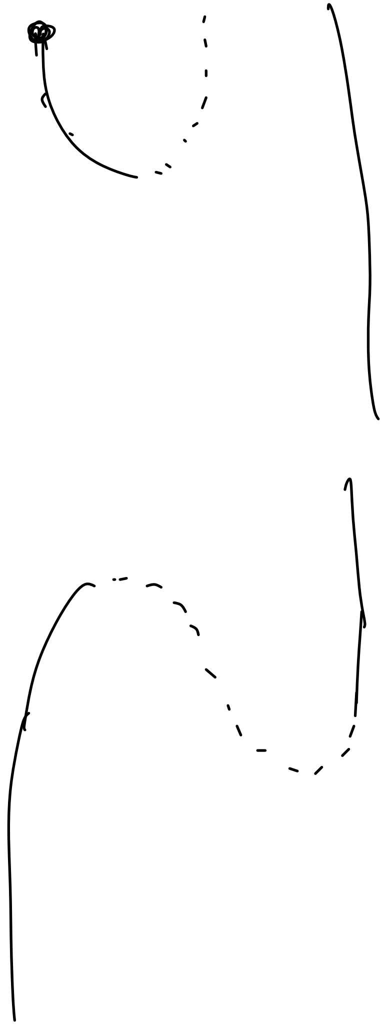


I↓









$$\dot{v} = v - \frac{v^3}{3} + I - w$$

w? → if v is high ⟸
w inc.

→ reset if v ⟸
is small

$$\dot{w} = \tau(v - bw + a)$$

$$\tau << 1$$

FHN

$\tau \ll 1$

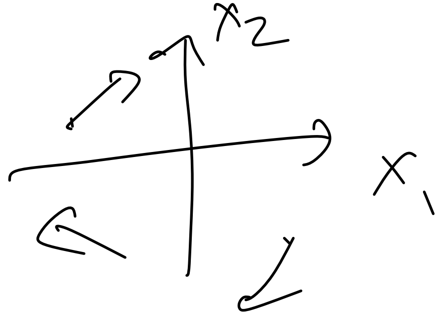
$0 < b < 1$

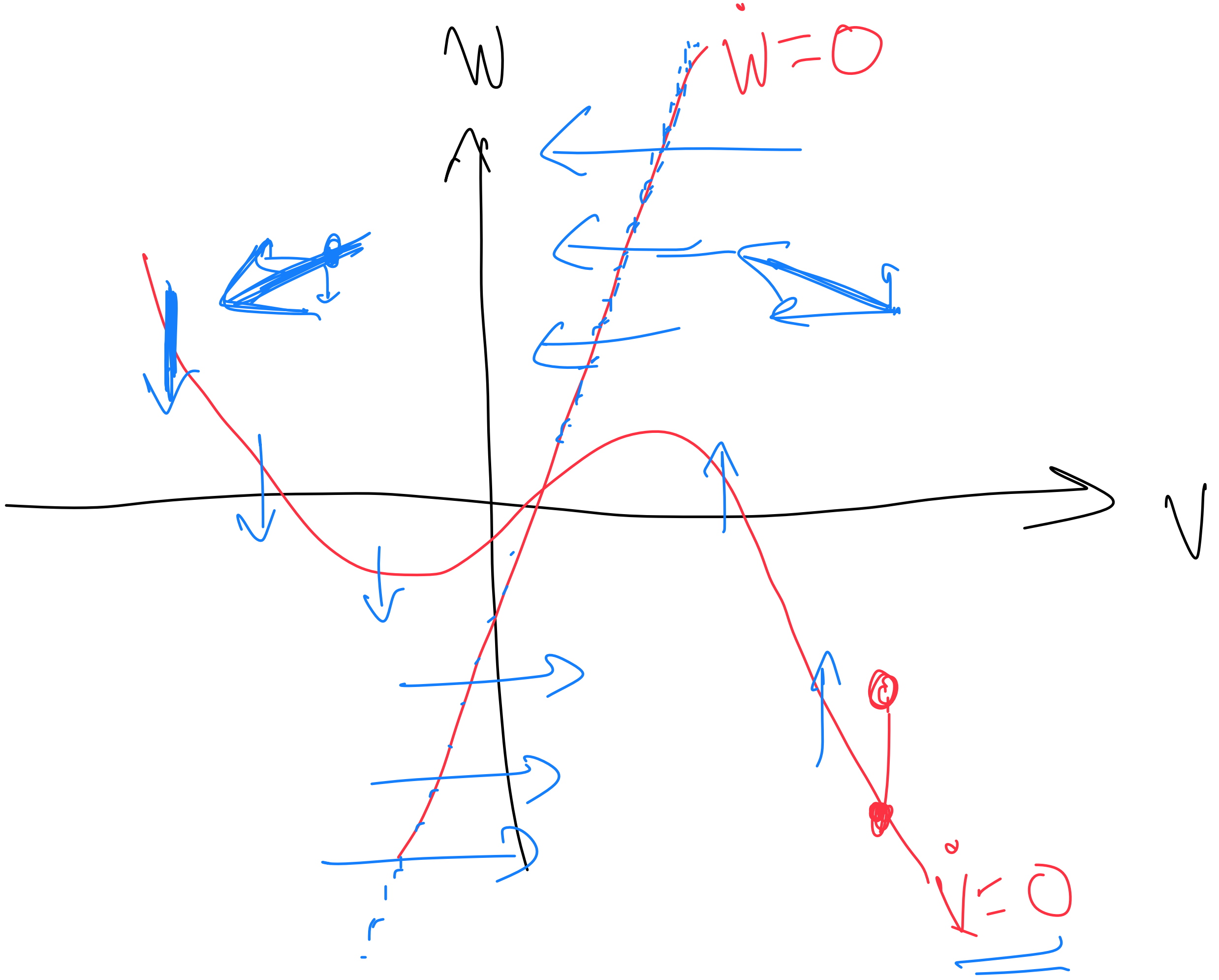
$$\dot{v} = v - \frac{v^3}{3} - w + I$$

$$\dot{w} = \tau(v - bw + a)$$

29-5-2026

$$\dot{x} = F(x)$$





null clines

$\dot{v} = 0$

$$v - \frac{v^3}{3} - w + I = 0$$

$$w = v - \frac{v^3}{3} + I$$

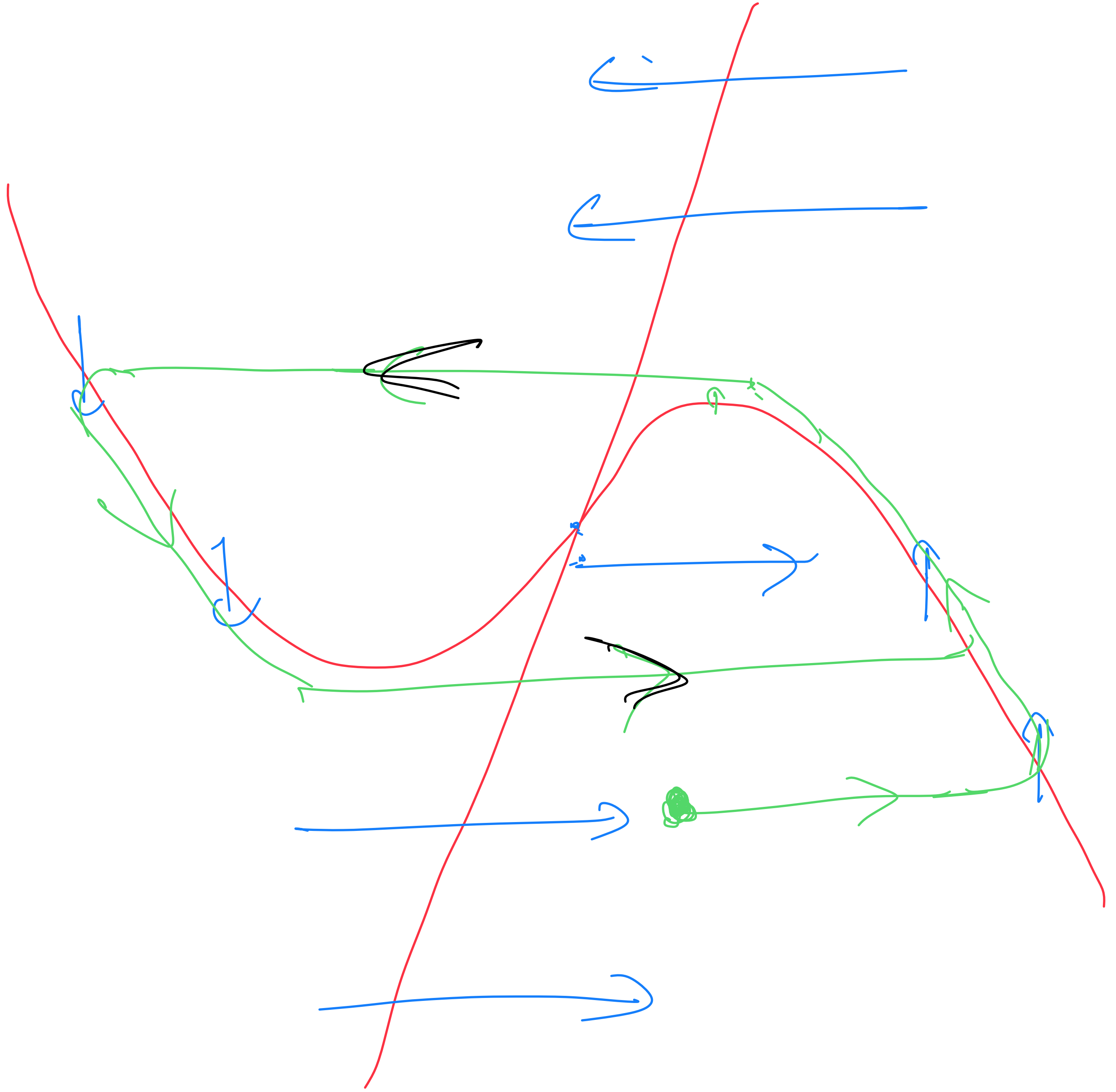
$\dot{w} = 0$

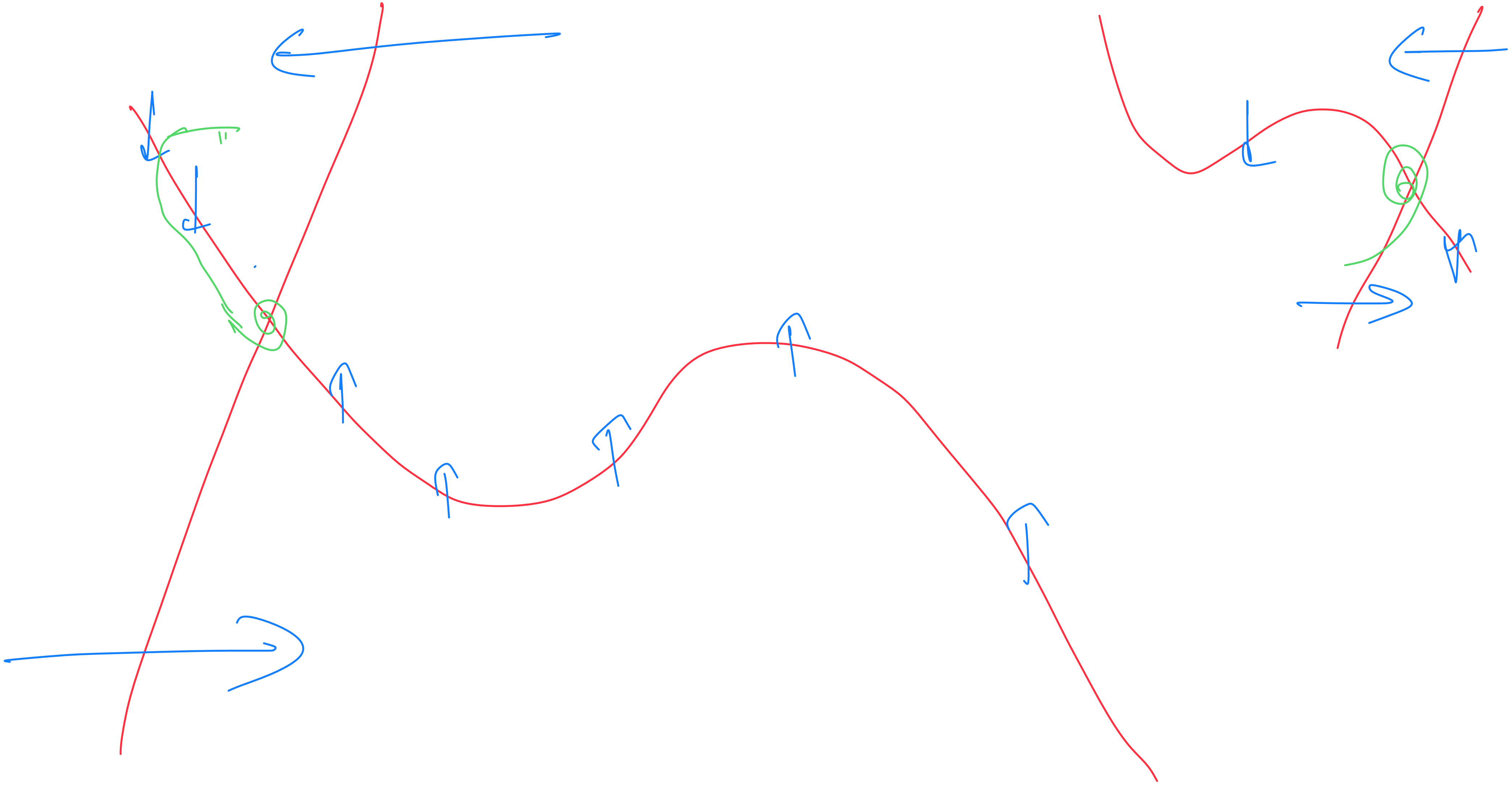
$$v - bw + a = 0$$

$$w = \frac{v}{b} + \frac{a}{b}$$

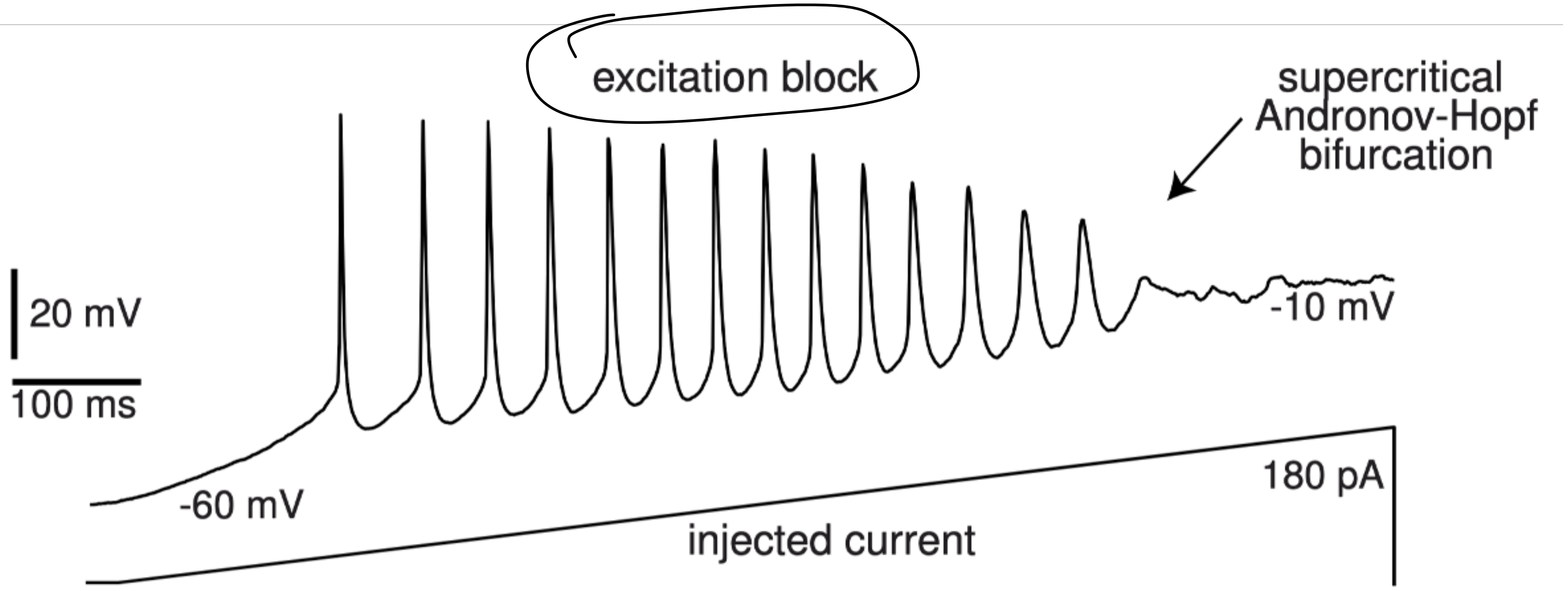
Iz. § 4.2.6

Strogatz § 7.5

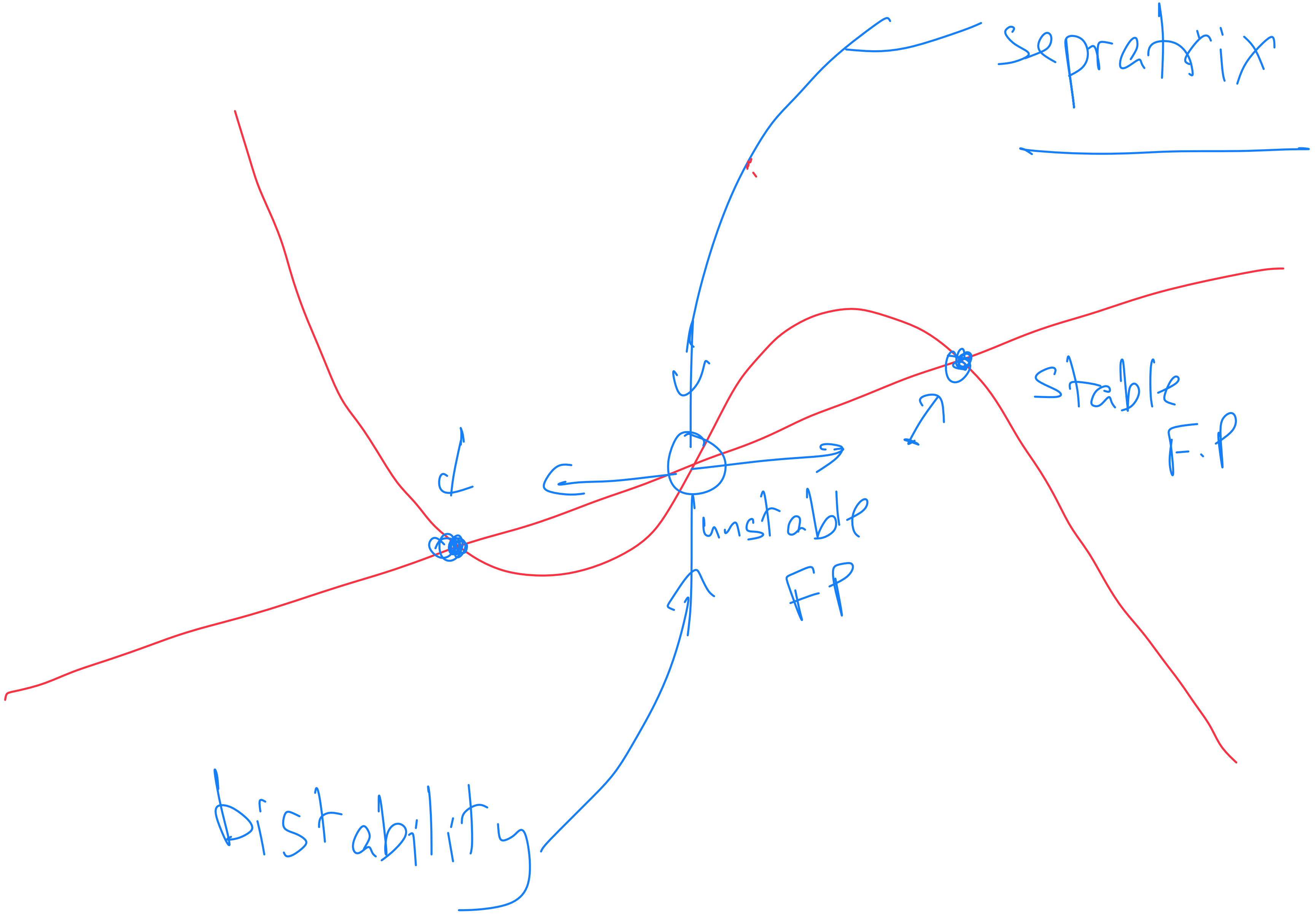




Iz. → Ch 4


excitation block
supercritical
Andronov-Hopf
bifurcation
20 mV
100 ms
-10 mV
-60 mV
180 pA
injected current

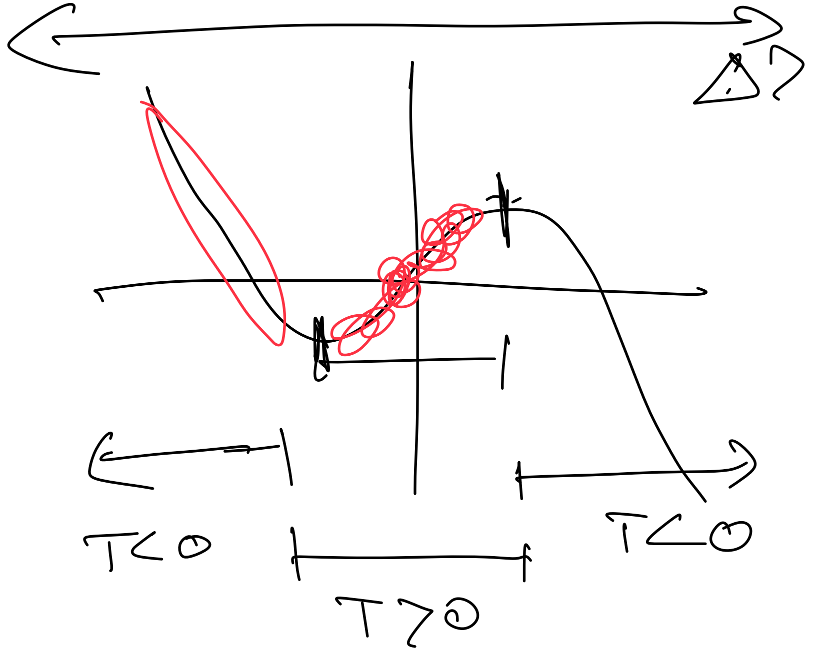
$b > 1$



$\tau << 1$
$0 < b < 1$

$$\dot{V} = \frac{V - \frac{V^3}{3} - W + I}{}$$

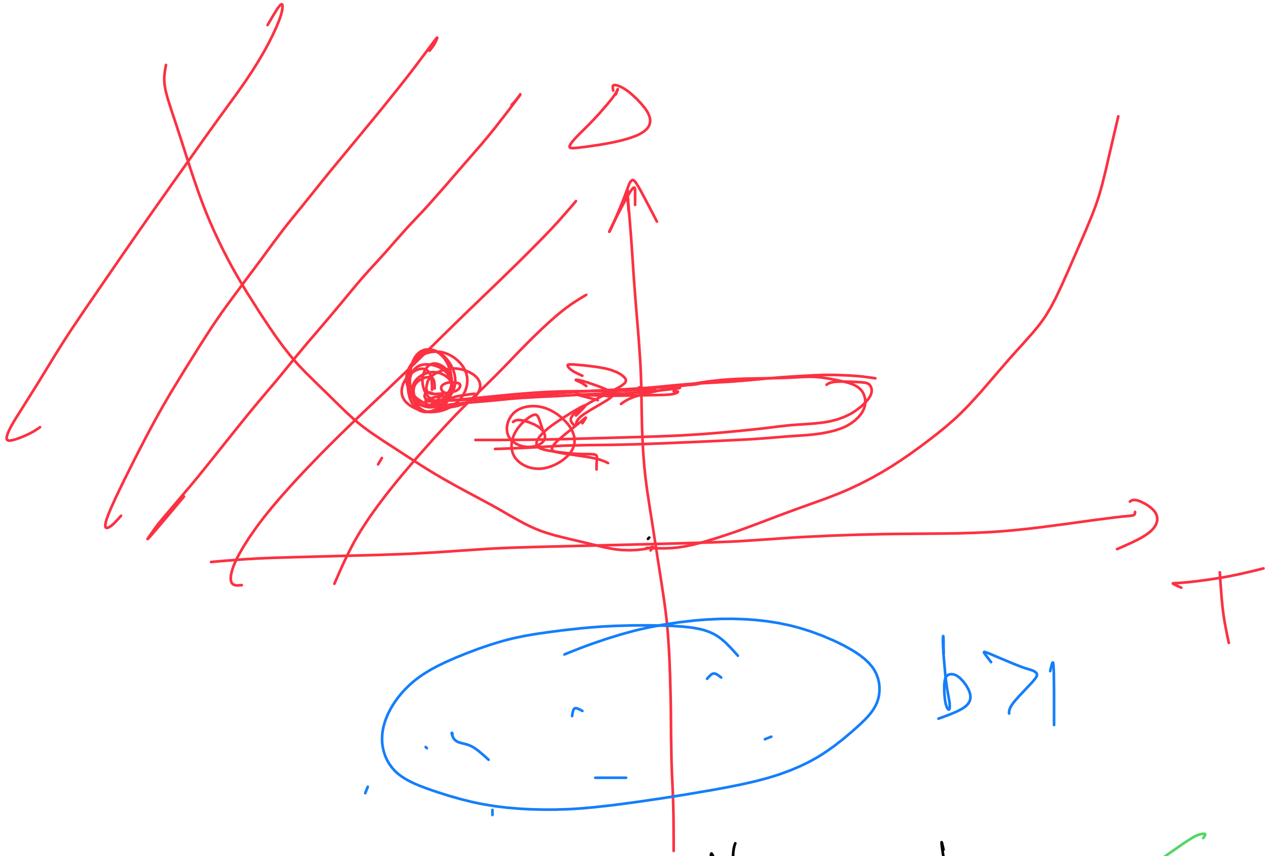
$$\dot{W} = \tau(V + a - bW)$$

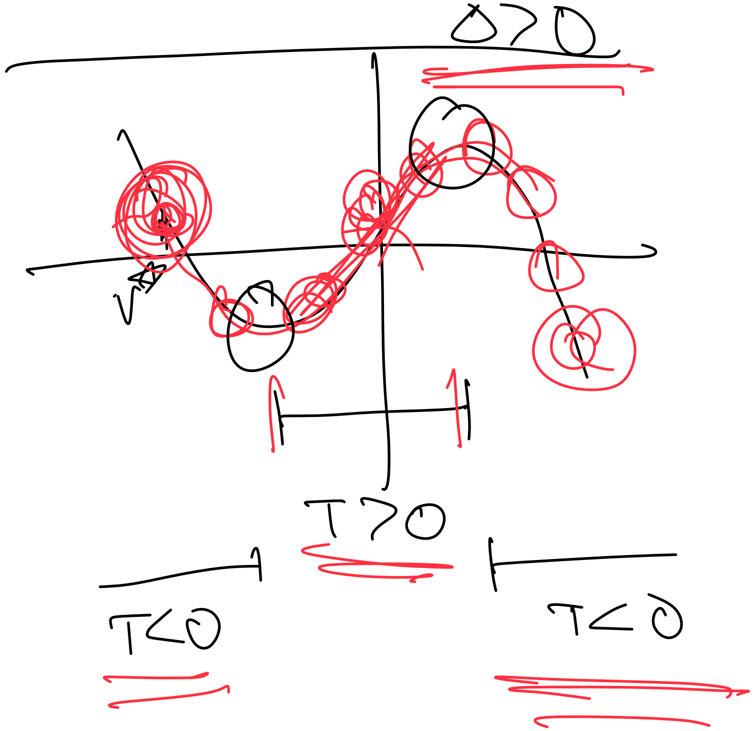


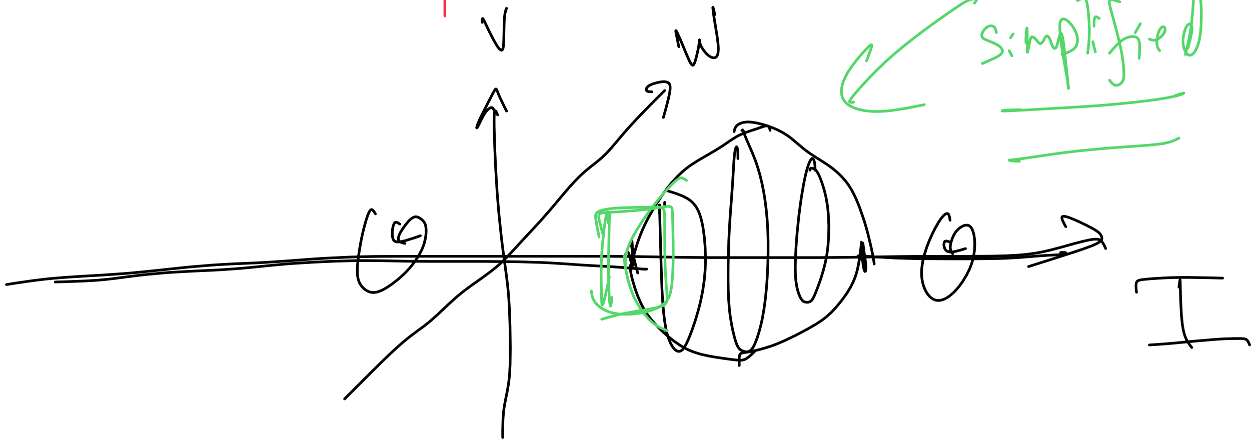
$$J = \begin{pmatrix} 1 - V^2 & -1 \\ \tau & -\tau b \end{pmatrix}\Bigg|_{V^*, W^*}$$

$$T = 1 - V^2 - \tau b \longrightarrow \begin{cases} T < 0 & \text{if } V < -1 \text{ or } V > +1 \\ T > 0 & \text{if } -1 < V < 1 \end{cases}$$

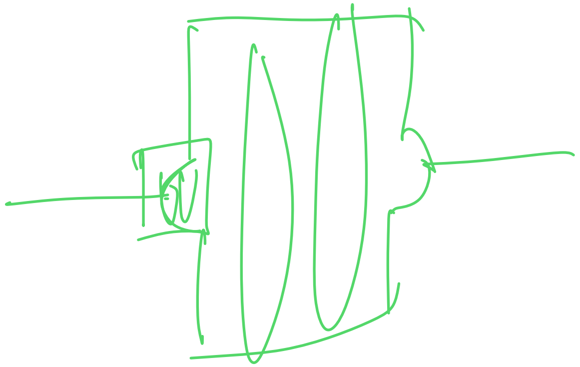
$$\Delta = (1 - V^2)(-\tau b) + \tau$$
$$\tau(1 - b - bV^2) = \tau\left(1 - b(1 - V^2)\right) > 0$$







simplified



Limit cycles — Perturbatⁿs

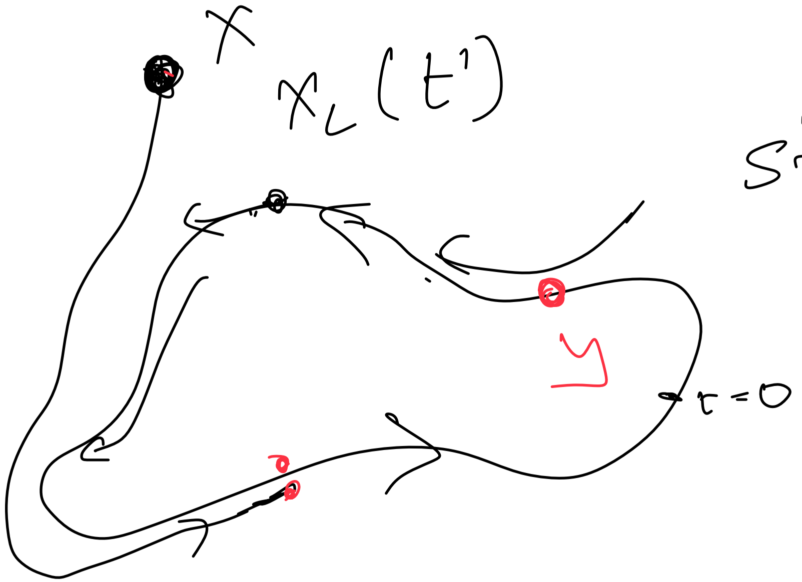
"Phase Reductⁿ"

E&T Ch 9, Iz. Ch. 10

$$\dot{\bar{x}} = \bar{F}(\bar{x})$$

$$\bar{X}_L(t+T) = \bar{X}_L(t)$$

$$\Omega = 2\pi / T$$

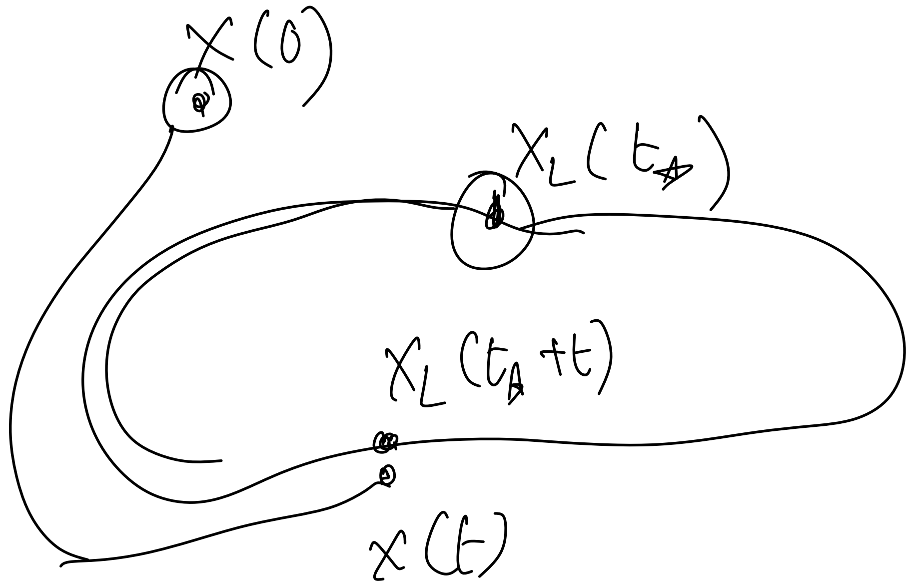


$$\boxed{\Theta(x_L(t)) = \Omega t \mod 2\pi}$$

Extend definitⁿ of Θ to all $\bar{x}$

If $\lim_{t \to \infty} \| x_L(t_\Delta + t) - \underline{x(t)} \| = 0$]

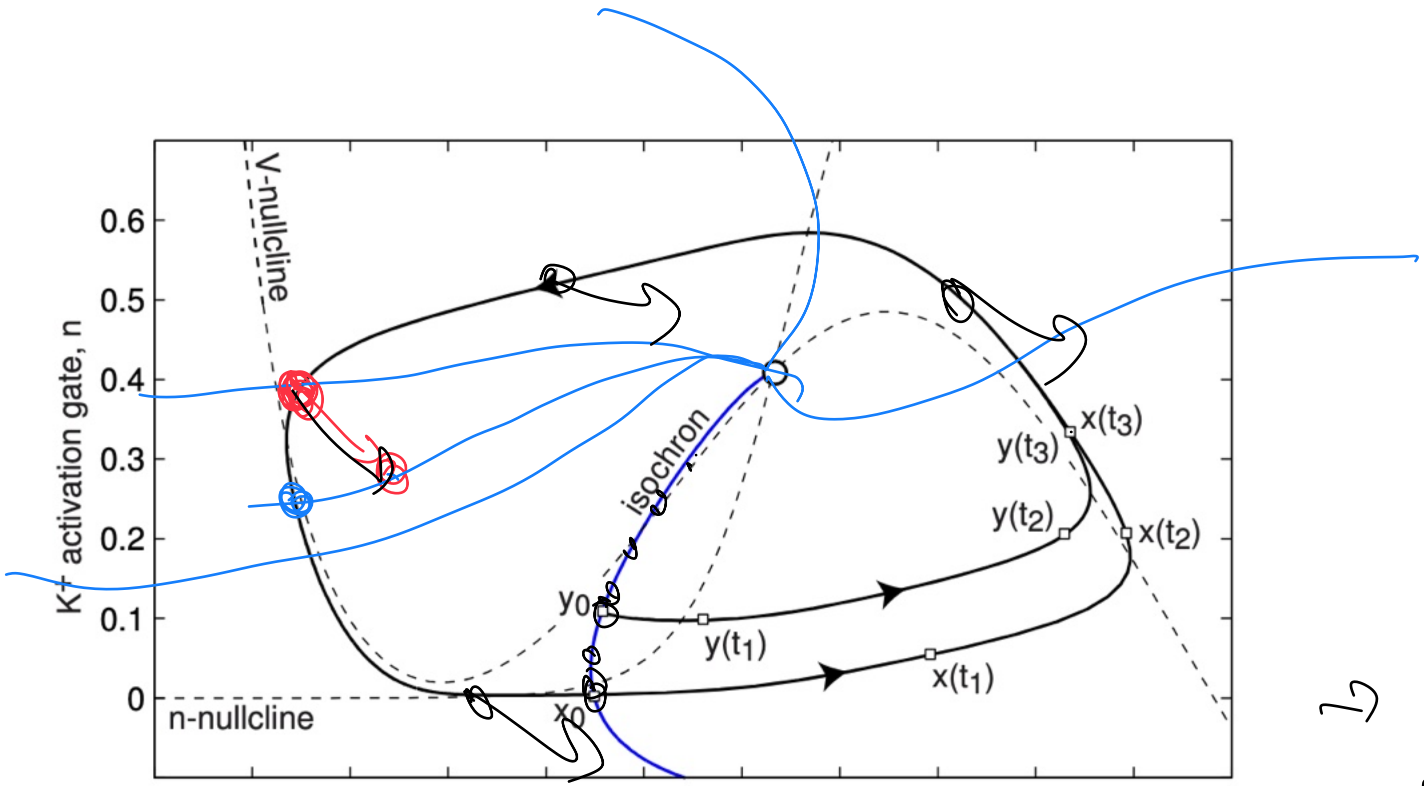
then $\Theta(x(0)) = \Theta(x_L(t_\Delta))$



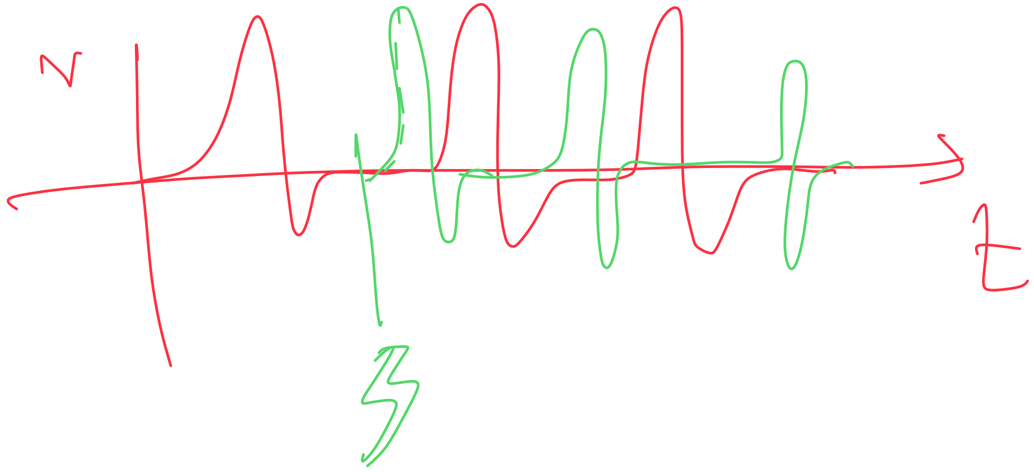
Isochrons ($I_{Na,p}+I_K$)

Isochron → level set of Θ

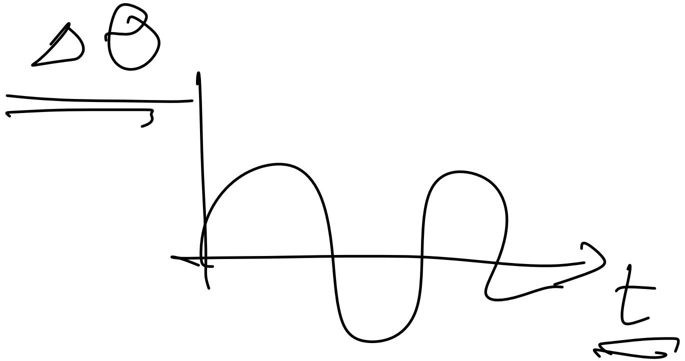
$\Theta(\vec{x}) = C$



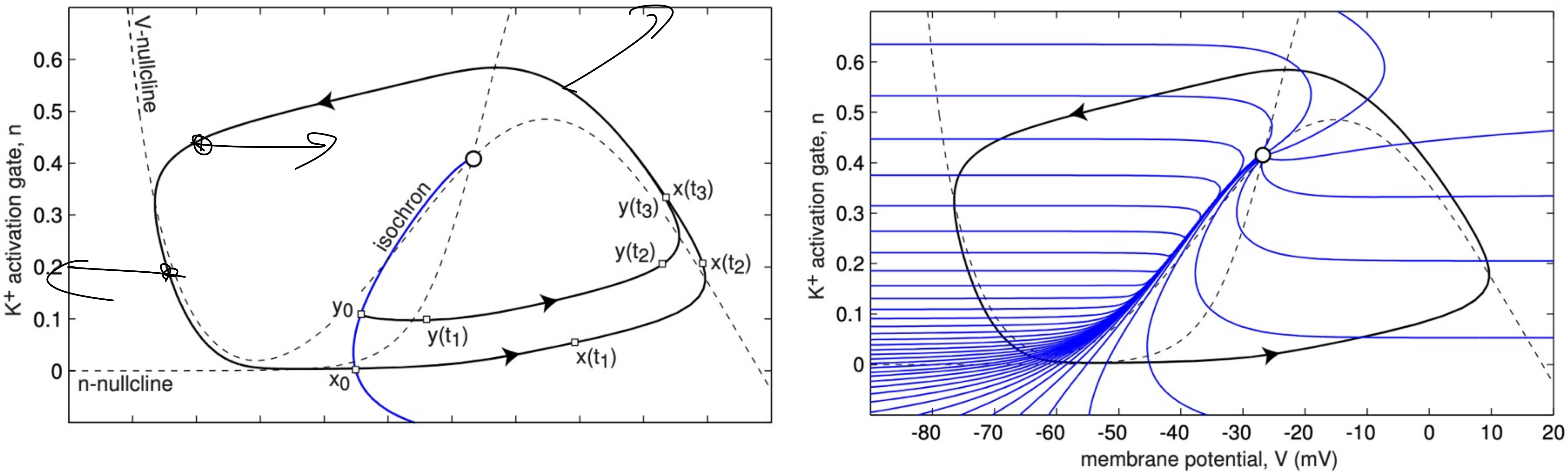
Izhikevich Fig. 10.2



→ Phase Response Curve

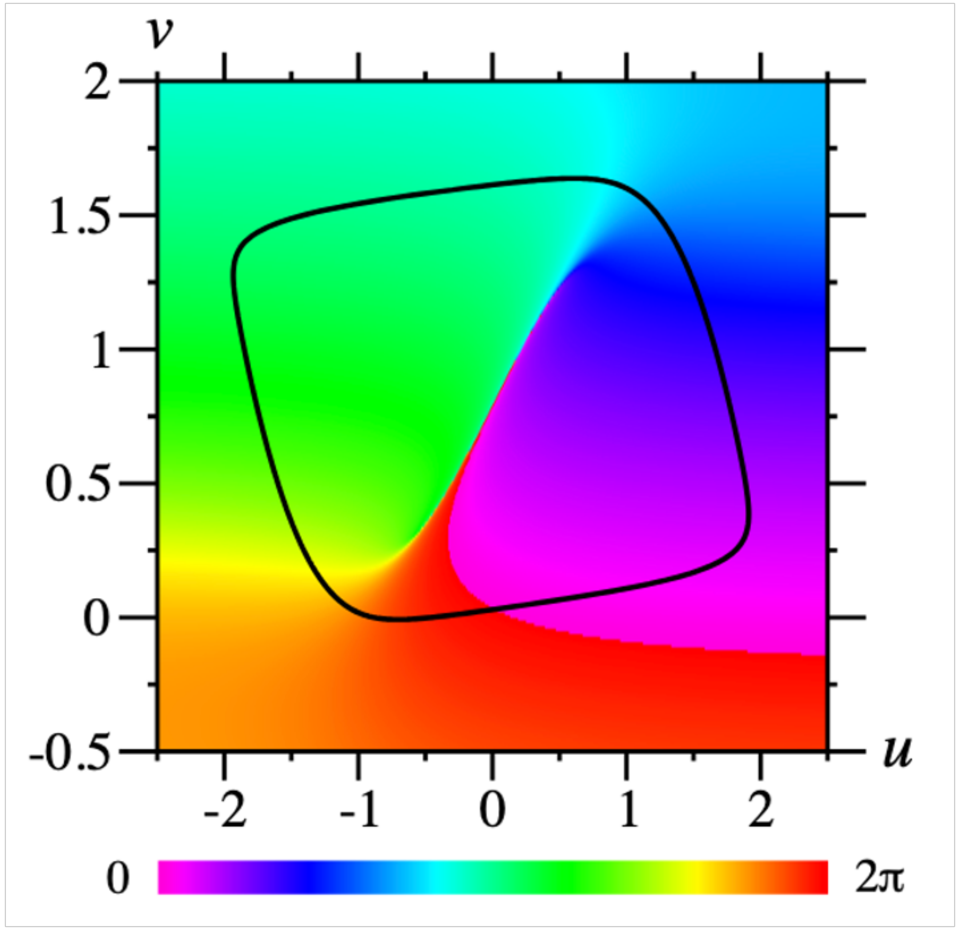


Isochrons ($I_{Na,p}$+I_K)



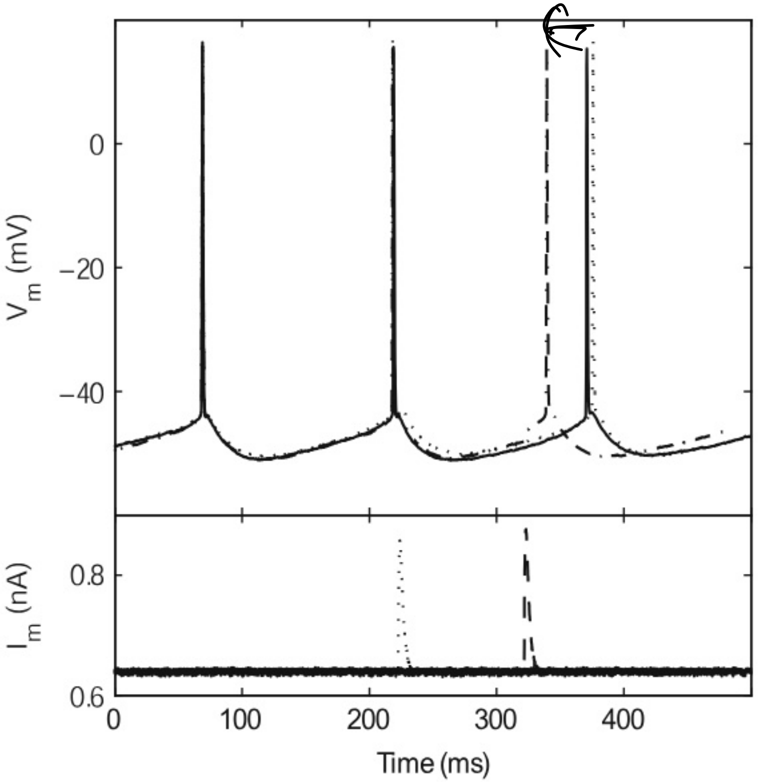
Izhikevich Fig. 10.2

Isochrons (FHN)



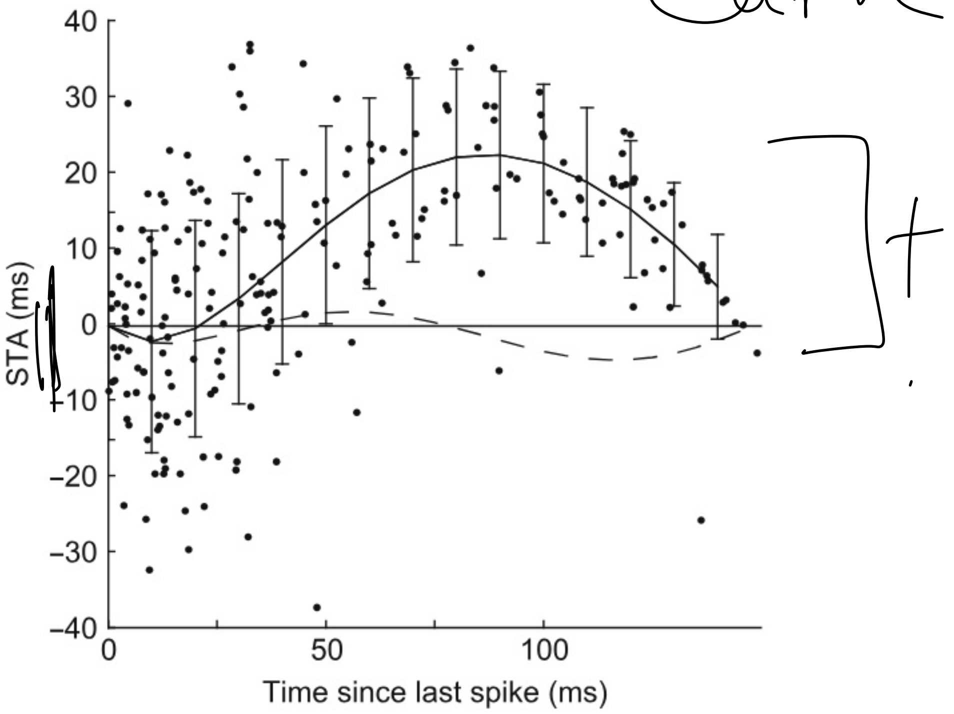
Nakao, H. (2016) ‘Phase reduction approach to synchronisation of nonlinear oscillators’, *Contemporary Physics*, 57(2), pp. 188–214.

PRC in neurons



Netoff et al (2005)

Phase Response Curve



$$\frac{d}{dt}\theta(x) = \Omega$$

$$\theta(x_L(t)) = \Omega t \bmod 2\pi$$

$$\dot{\bar{x}} = \bar{F}(\bar{x})$$

$$\nabla\theta \cdot \dot{\bar{x}} = \Omega$$

$$\bar{\nabla}\theta \cdot \bar{F} = \Omega$$

Let

$$Z(\theta) = \nabla\theta$$

$$\dot{\bar{x}} = F(\bar{x}) + \varepsilon\bar{p}$$

$$\bar{\nabla}\theta \cdot \dot{\bar{x}} = \underbrace{\bar{\nabla}\theta \cdot \bar{F}} + \varepsilon\,\bar{\nabla}\theta \cdot \bar{p}$$

$$\frac{d\theta}{dt} = \Omega + \varepsilon\,\underbrace{\bar{\nabla}\theta} \cdot \bar{p}$$

$\boxed{\bar{Z}(\bar{\theta})}$ ← Phase Response Curve

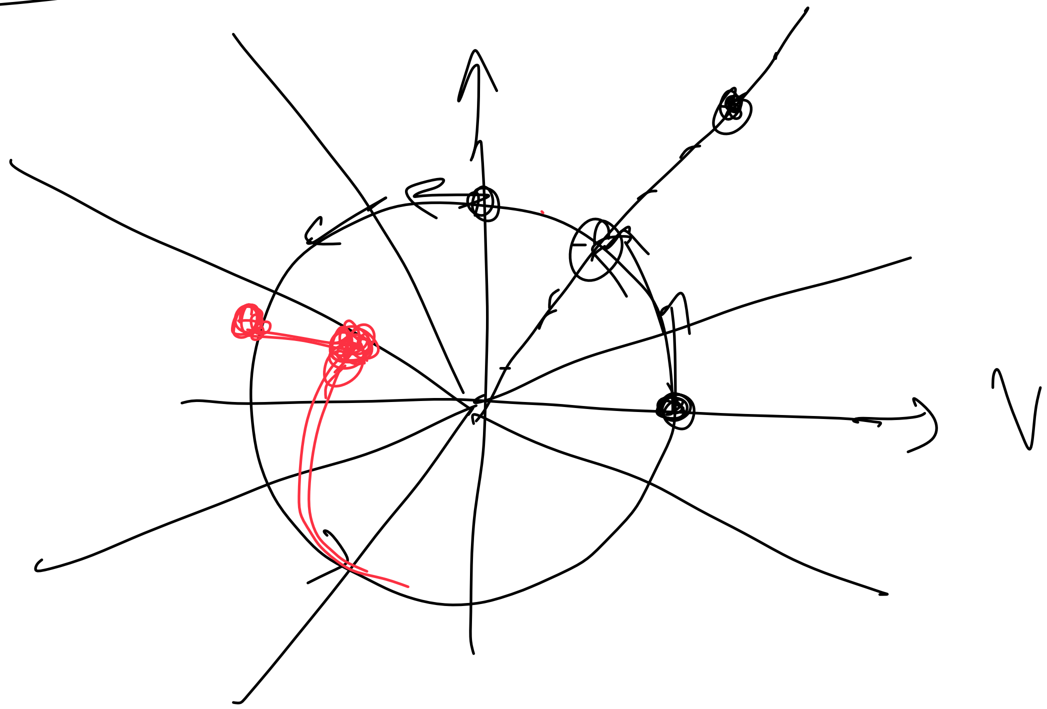
$$\frac{d\theta}{dt} = \Omega + \bar{Z}(\theta) \cdot \bar{p}$$

$$\frac{d\theta}{dt} = \Omega$$

$\bar{Z}(\theta) \cdot \hat{v}$

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\phi} = \omega \end{cases}$$

ϕ



$$\bar{Z}(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

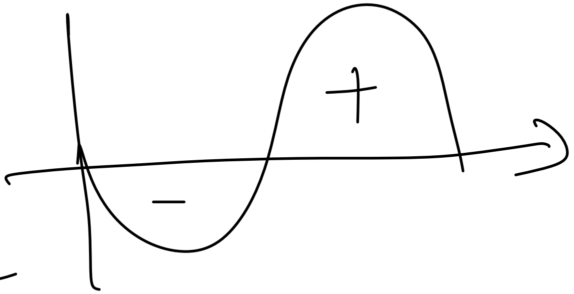
$\bar{Z}(\theta) \cdot \hat{v} = -\sin\theta$

$\hat{\theta}$

Hopf. oscillator

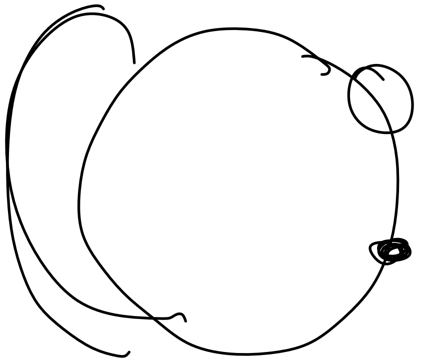
PRC

Type II Neuron/PRC



Biphasic PRC

Sync. easily



SNIC

Type I PRC

Neuron

01-06-2026

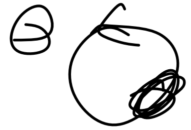
$$\dot{\bar{x}} = \bar{F}(\bar{x}) + \varepsilon \bar{p} \qquad \varepsilon \ll 1$$

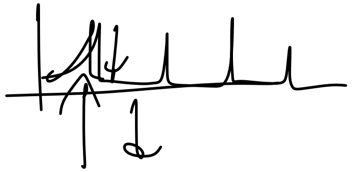
$$\rightarrow \frac{d\theta}{dt} = \Omega + \bar{Z}(\theta) \cdot (\varepsilon \bar{p}) \quad \leftarrow Z(\theta) = \bar{\nabla}\theta \quad \leftarrow \theta(\bar{x})$$

$$\dot{x}_i = F(x_i) + \varepsilon\, p_i(x_1, \ldots, x_N)$$

$$= F(x_i) + \varepsilon \left[\sum_j g_{ij}(x_i, x_j) \right]$$

$$\dot{\theta}_i = \Omega + \varepsilon \underbrace{\sum Z_i(\theta_i)\, g_{ij}\left(x_i(\theta_i), x_j(\theta_j)\right)}_{\theta_{ij}(\theta_i, \theta_j)}$$





$$\left[\ \dot{\theta}_i = \Omega + \varepsilon \sum Q_{ij}(\theta_i, \theta_j)\right.$$

$$\left. Q_{ij}(\theta_i, \theta_j) = Q_i(\theta_i, \theta_i + \Delta\theta)\ \right]$$

$$H_{ij}(\Delta\theta) = \frac{1}{2\pi}\int_0^{2\pi} Q_{ij}(\theta, \theta + \Delta\theta)\, d\theta$$ ← average over fast timescale

$$\dot{\theta}_i = \Omega + \varepsilon \sum H_{ij}(\theta_j - \theta_i)$$ ← Kuramoto-Daido form

$$H_{ij}(\Delta\theta) = \sum a_n^{ij} \sin(\Delta\theta) + b_n^{ij} \cos(\Delta\theta)$$

$$= \underline{c^{ij}} \sin(\theta_j - \theta_i + \alpha^{ij})$$ ← Kuramoto-Sakaguchi

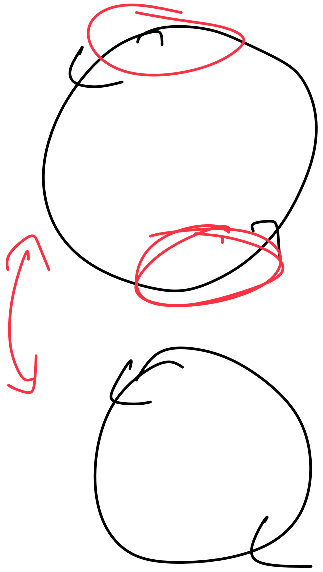
if $g_{ij}(x_i, x_j) = g_{ij}(x_i - x_j)$ ← Diffusive coupling → $H_{ij}(0) = 0$

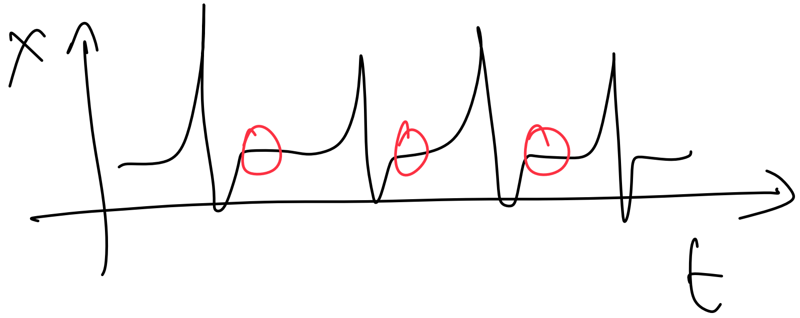
$$H_{ij} = \sin(\theta_j - \theta_i)$$

$$\dot{\theta}_i = \Omega + \varepsilon \sum_j c^{ij} \sin(\theta_j - \theta_i) \quad \leftarrow \text{Kuramoto model}$$

↑ Exc. approx ~~Exact~~ if L.C. were near a Hopf. Bif^n

$$\dot{\theta} = \Omega$$





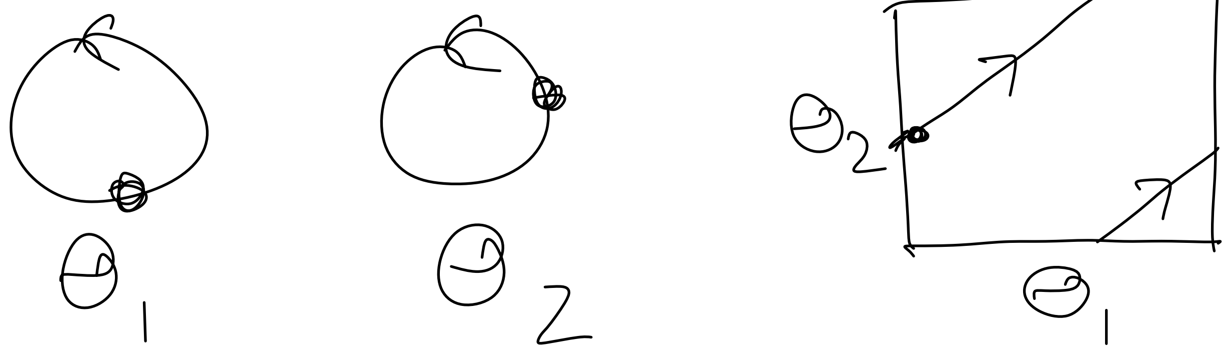
$$\dot{\theta}_1 = \omega + \underline{K_1} \sin(\theta_2 - \theta_1)$$

$$\dot{\theta}_2 = \omega + \underline{K_2} \sin(\theta_1 - \theta_2)$$

If $K_1 = K_2 = 0$

$$\dot{\theta}_1 = \omega$$

$$\dot{\theta}_2 = \omega$$

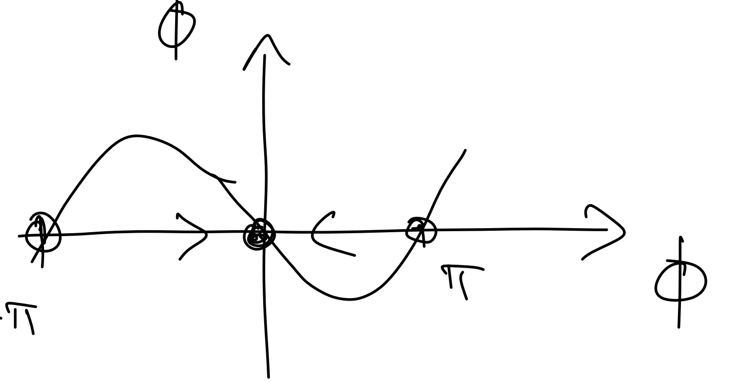


$$\phi = \theta_1 - \theta_2$$

$$\dot{\phi} = K_1 \sin(\theta_2 - \theta_1) - K_2 \sin(\theta_1 - \theta_2)$$

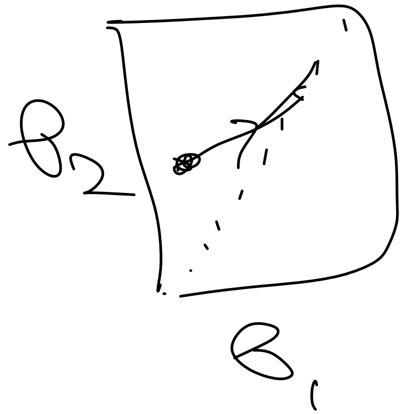
$$\dot{\phi} = (K_1 + K_2) \sin(\theta_2 - \theta_1) = -(K_1 + K_2) \sin\phi$$

$$\dot\phi = -K\sin\phi$$



$\phi \to 0$

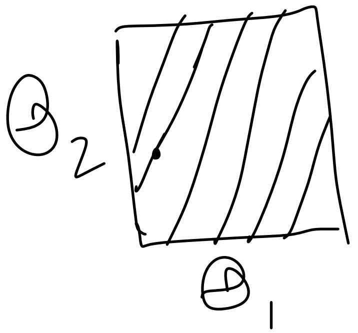
$\theta_1 - \theta_2 \to 0 \to$ system synchronizes



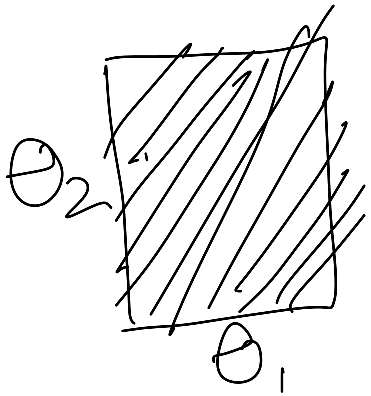
$$\dot\theta_i = \omega_i + K\sum \sin(\theta_j - \theta_i)$$

$$\dot\theta_1 = \omega_1$$
$$\dot\theta_2 = \omega_2$$

if $\omega_1/\omega_2 \in \mathbb{Q}$



$\omega_1/\omega_2 \notin \mathbb{Q}$



~~FP~~

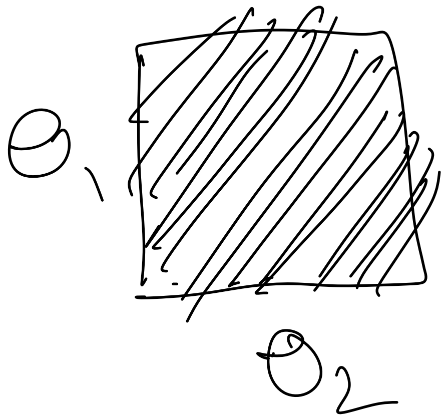
~~P.O.~~

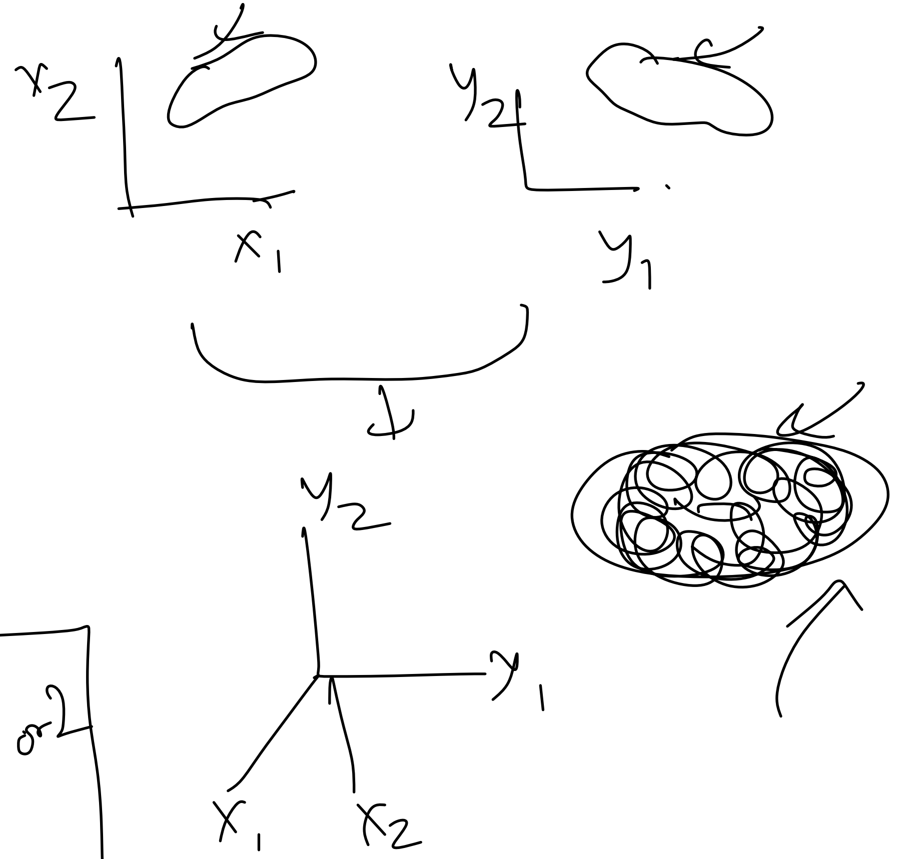
Quasiperiodic attractor

$\dot\theta_1 = \omega_1$

$\dot\theta_2 = \omega_2$

$\longleftarrow$ $\dot{\bar x} = \bar F(\bar x)$ $\quad \bar x \in \mathbb{R}^2$





Strogatz § 8.6 Ed. 1 or 2

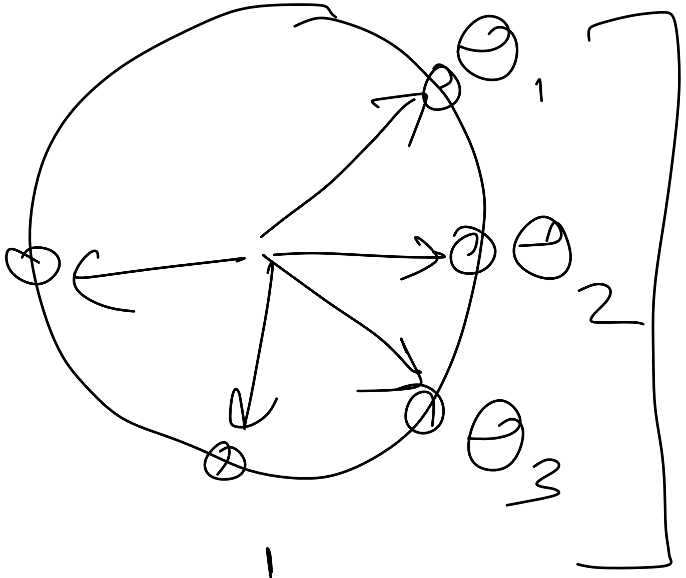
" § 13 Ed 3

for 2 osc. coupled.

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum \sin(\theta_j - \theta_i)$$

if $\theta_j = \theta_i \pm \varepsilon$

$\rightarrow \theta_i$ goes faster / slower

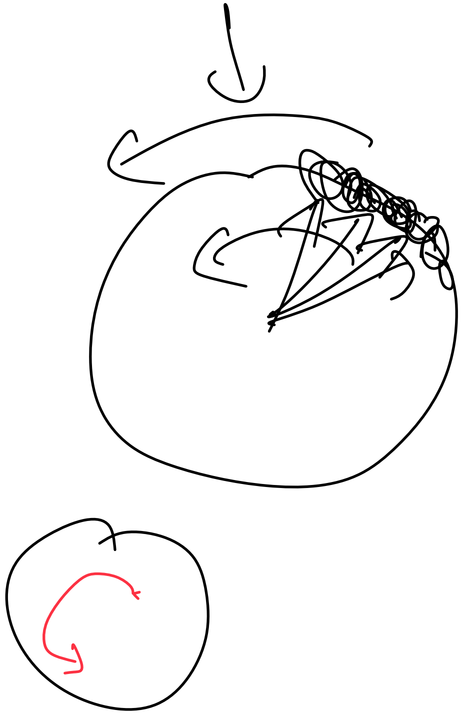


Order parameter

$$z = \frac{1}{N}\sum_i e^{i\theta_i}$$

$|z| \rightarrow 0 \Rightarrow$ async.

$\rightarrow 1 \Rightarrow$ sync.



$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i)$$

$$= \omega_i + \frac{K}{N}\sum_j \mathrm{Im}\left[e^{i(\theta_j - \theta_i)}\right]$$

$$= \omega_i + K\,\mathrm{Im}\left[\frac{1}{N}\sum_j e^{i\theta_j} \cdot e^{-i\theta_i}\right]$$

$\rightarrow z = re^{i\psi}$

$$\dot{\theta}_i = \omega_i + K\,\mathrm{Im}\left[z \cdot e^{-i\theta_i}\right]$$

$$= \omega_i + Kr\,\mathrm{Im}\left[e^{i\psi} \cdot e^{-i\theta_i}\right]$$

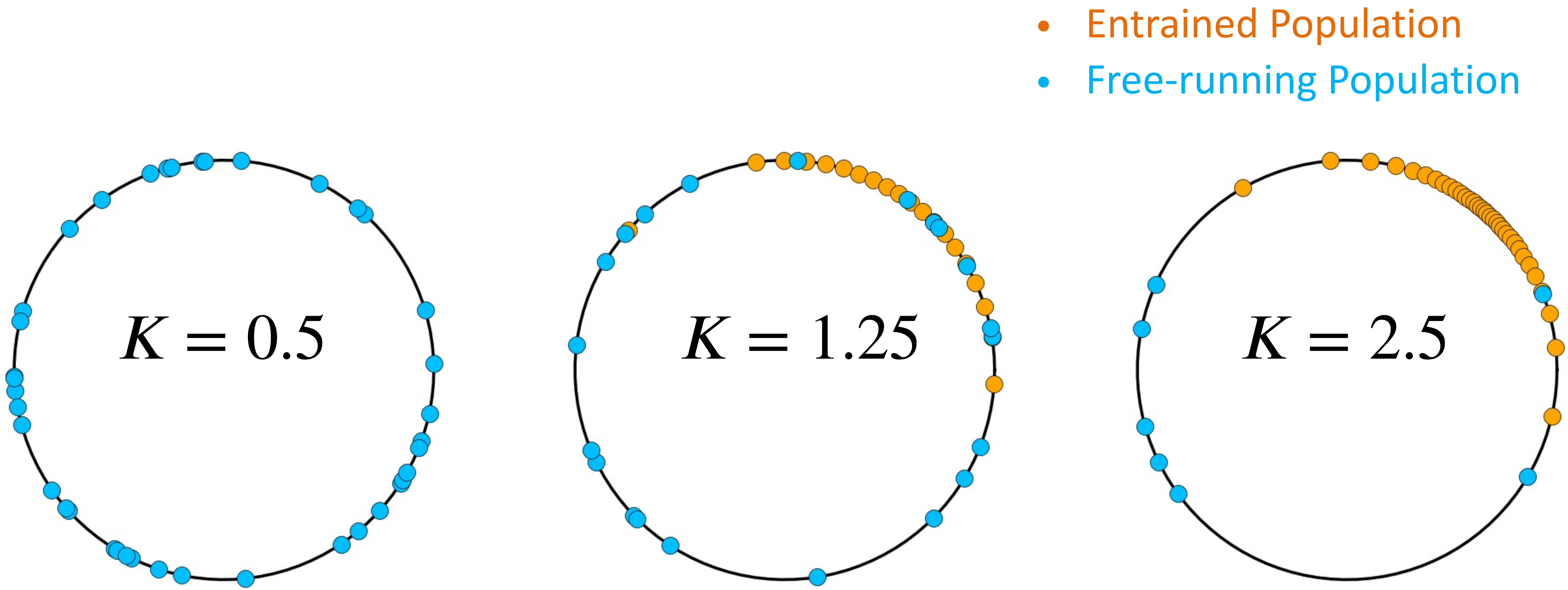
$$\dot{\theta}_i = \underbrace{\omega_i}_{\text{oppose synch.}} + \underbrace{Kr\sin(\psi - \theta_i)}_{\text{promotes synch.}}$$

Strogatz Ch 13

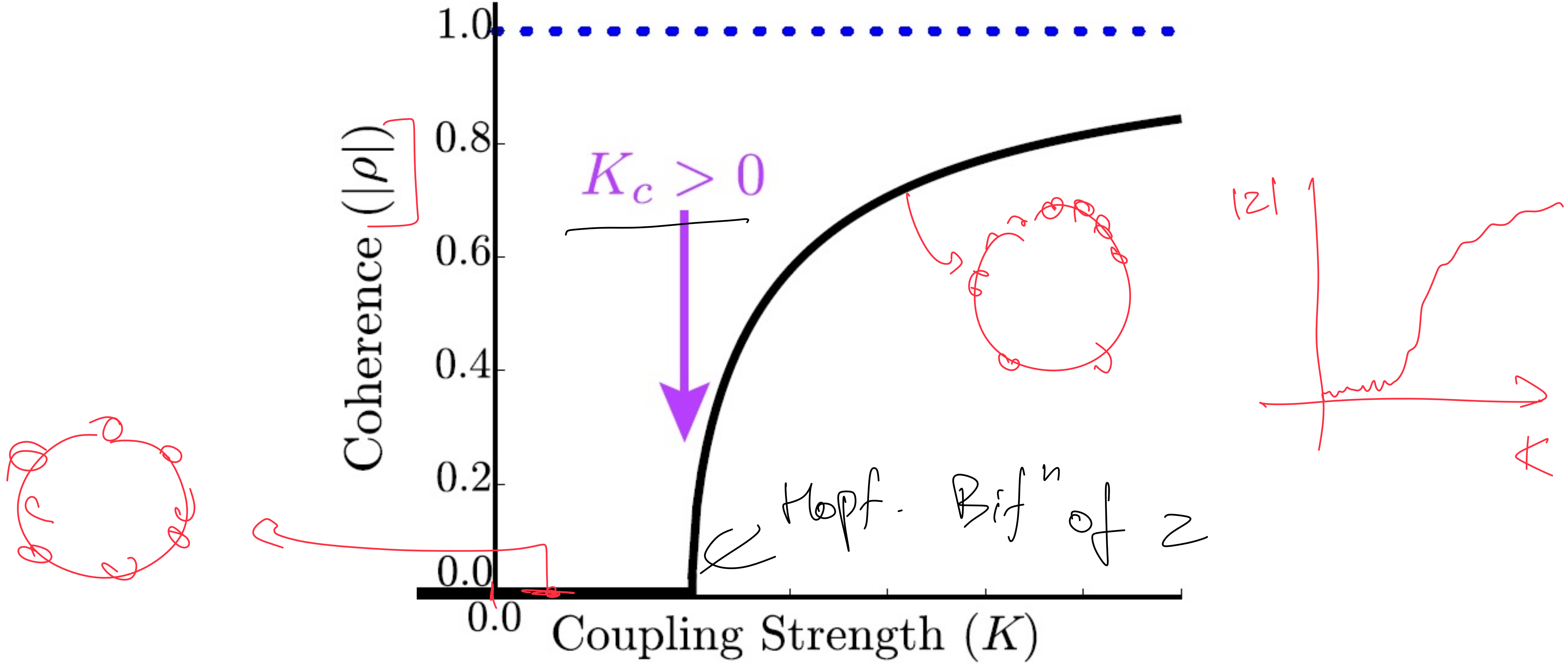
[Ott § 6.5

ω_i

$\omega_i \rightarrow \omega_i - \langle \omega_i \rangle$



$\rho = Z$



1.0
0.8
0.6
0.4
0.2
0.0
Coherence ($|\rho|$)
$K_c > 0$
Hopf. Bif^n of Z
|Z|
K
0.0
Coupling Strength (K)

1D - Flows → 2D Flows → 2D Flows on perodic veriable

1D - Flows ↓ F.P., $\pm\infty$

2D Flows ↓ F.P., $\pm\infty$, P.O.

2D Flows on perodic veriable: F.P., P.O., Q.A.

$$\dot{\bar{x}} = \bar{F}(\bar{x})$$

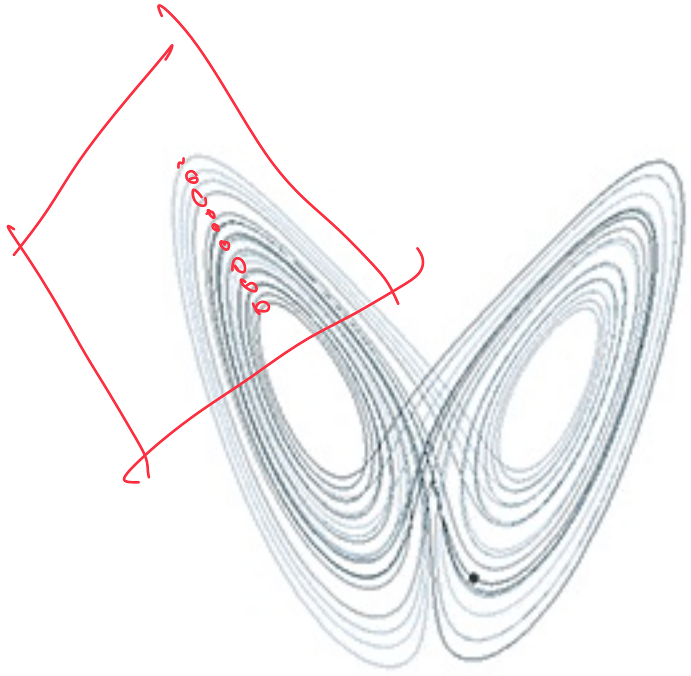
Chaotic attractor → Need 3 or more dim.

↳ If Flows

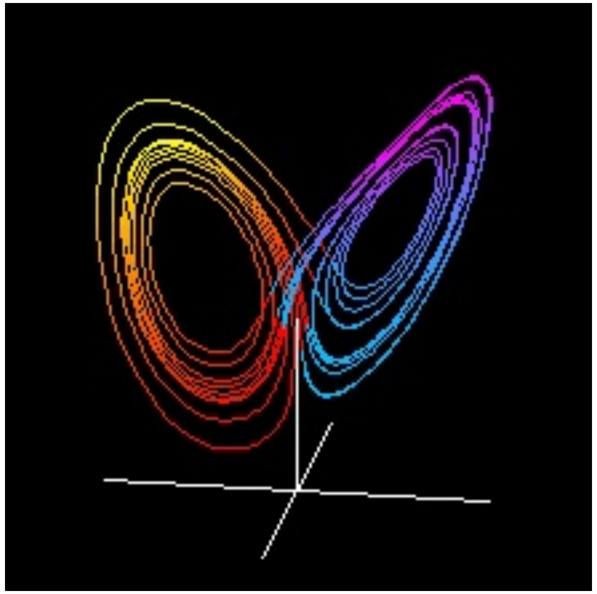
→ 1D Map → enough for chaos

Lorenz Model

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$



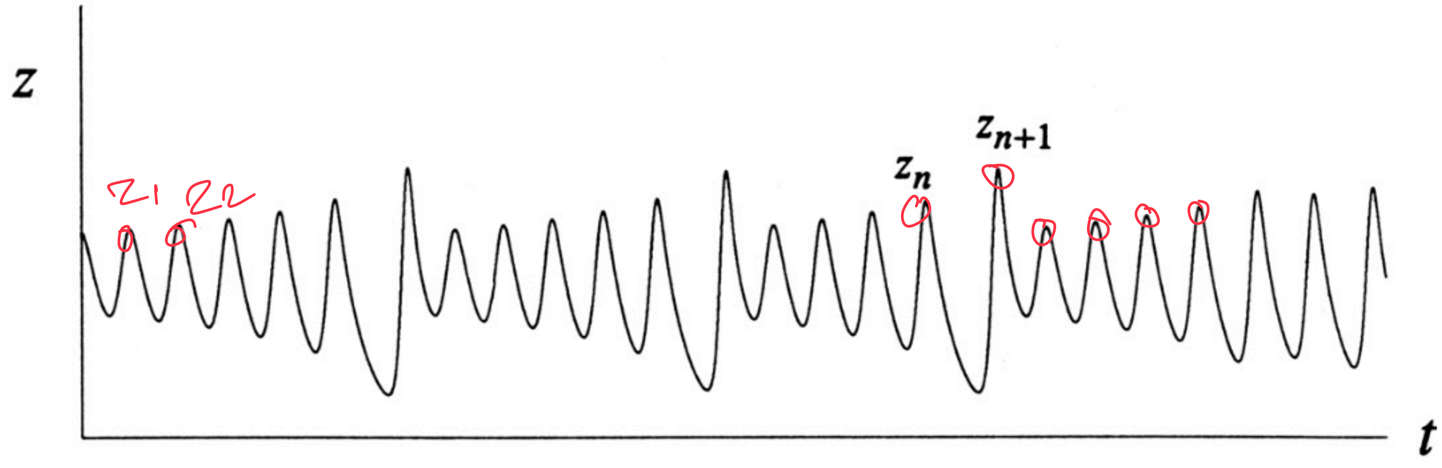
Wikipedia



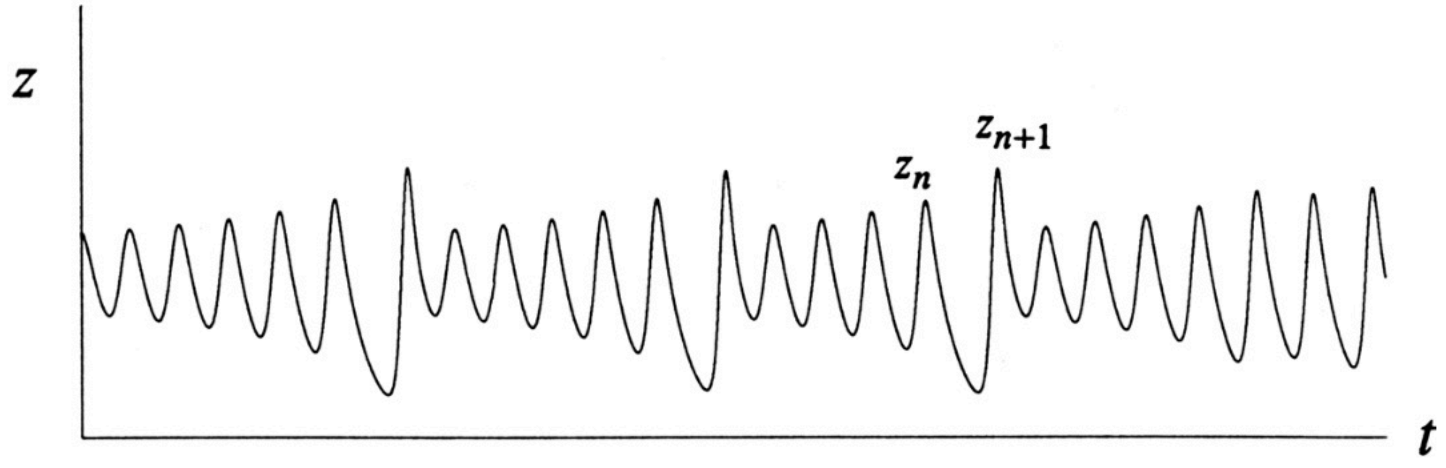
Source

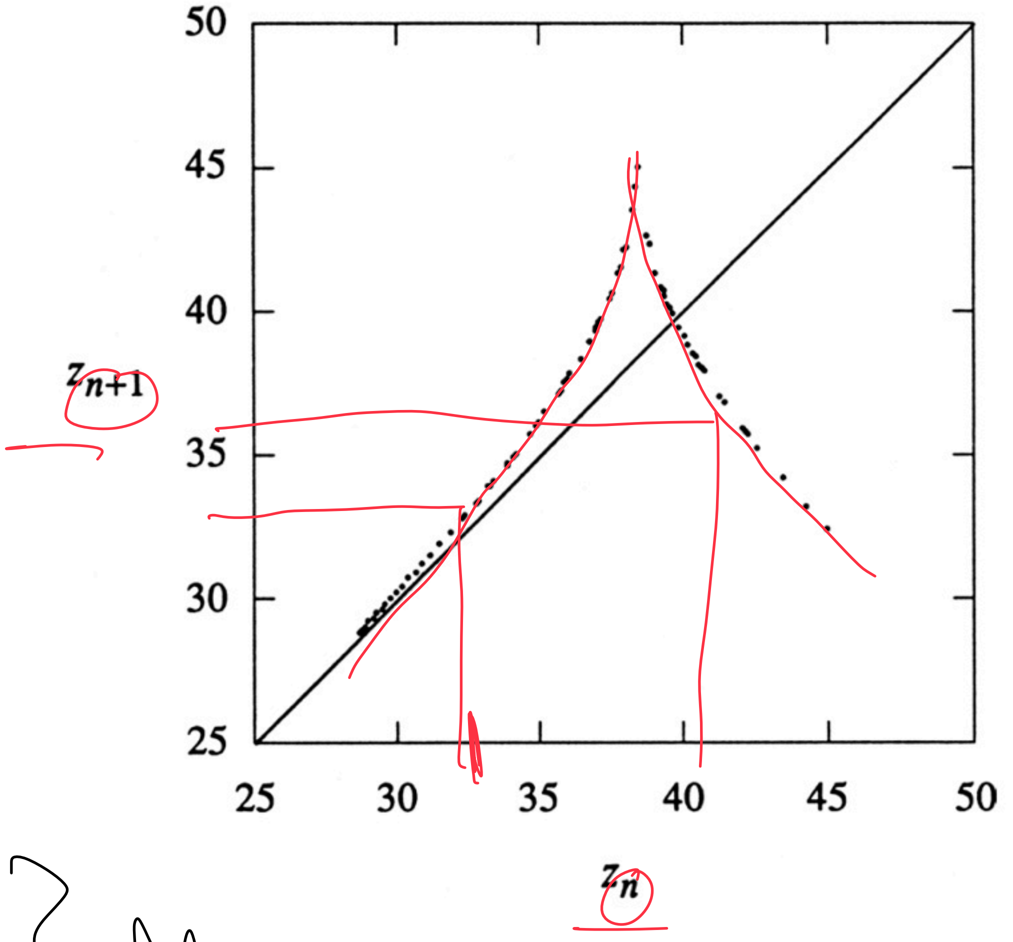
See also: https://itp.uni-frankfurt.de/~gros/Vorlesungen/SO/simulation_example/

Lorenz Model z return map



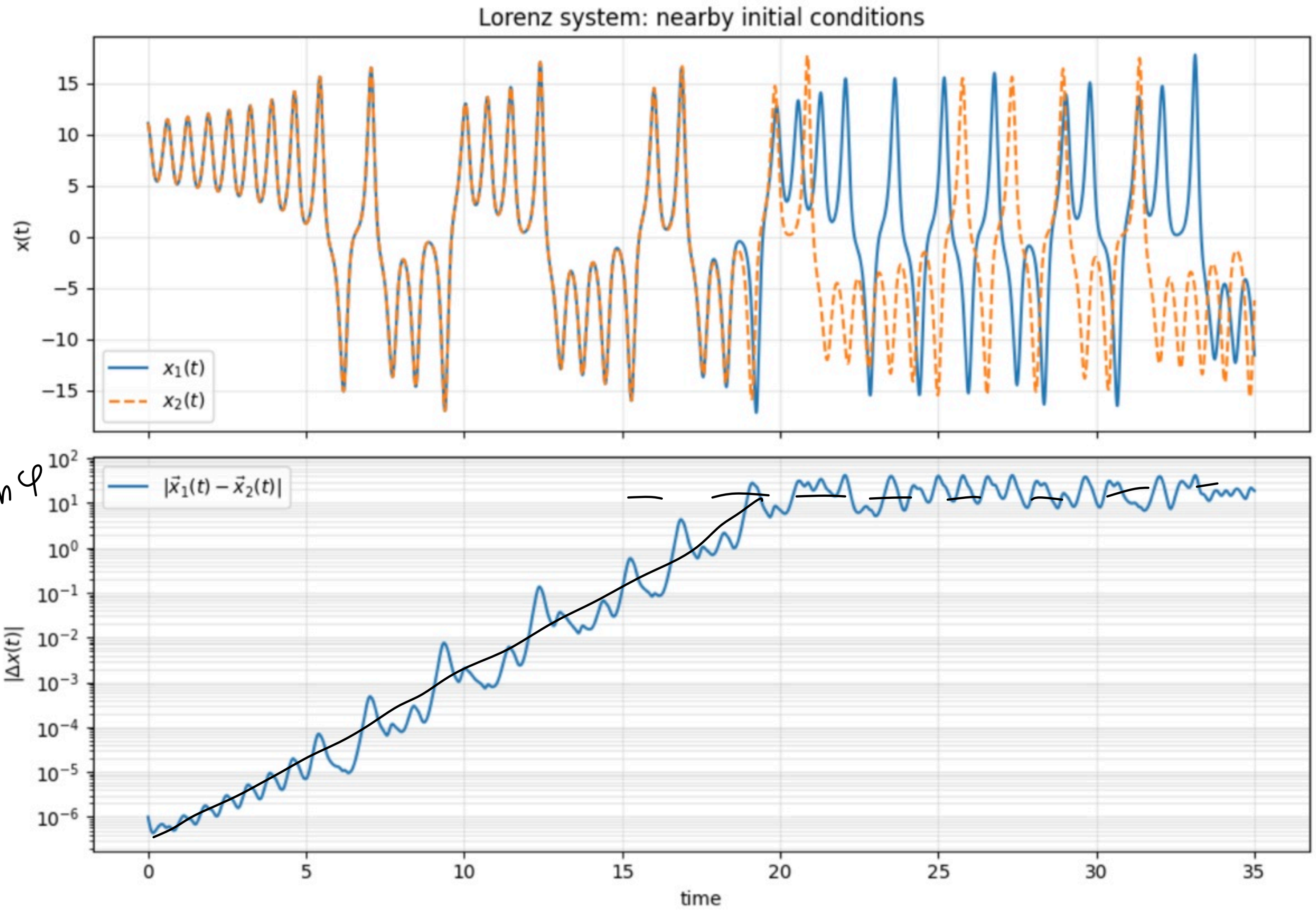
Lorenz Model z return map



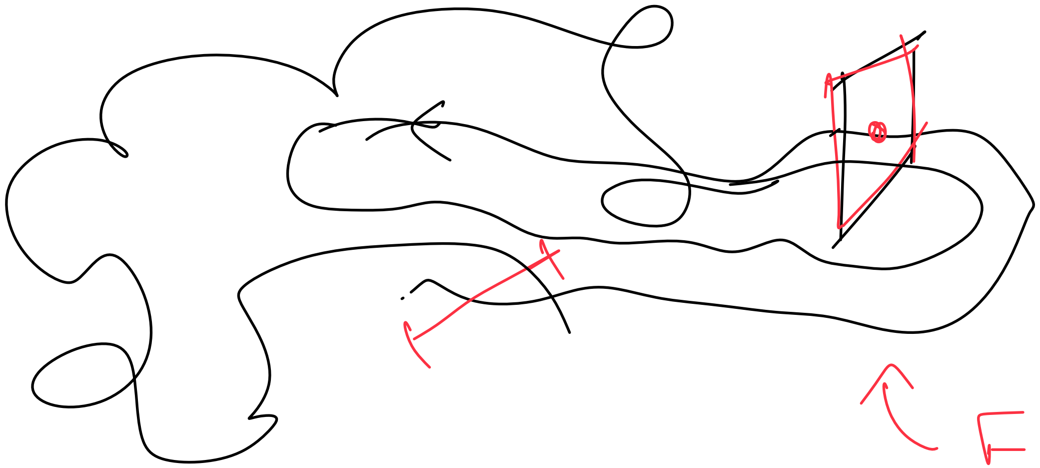


Dynamical System
in discrete time } Map

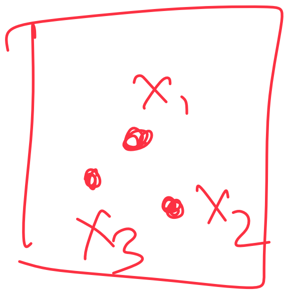
Lorenz Model



Chaos
⇓
Exp. divergence
but
bounded



Flow

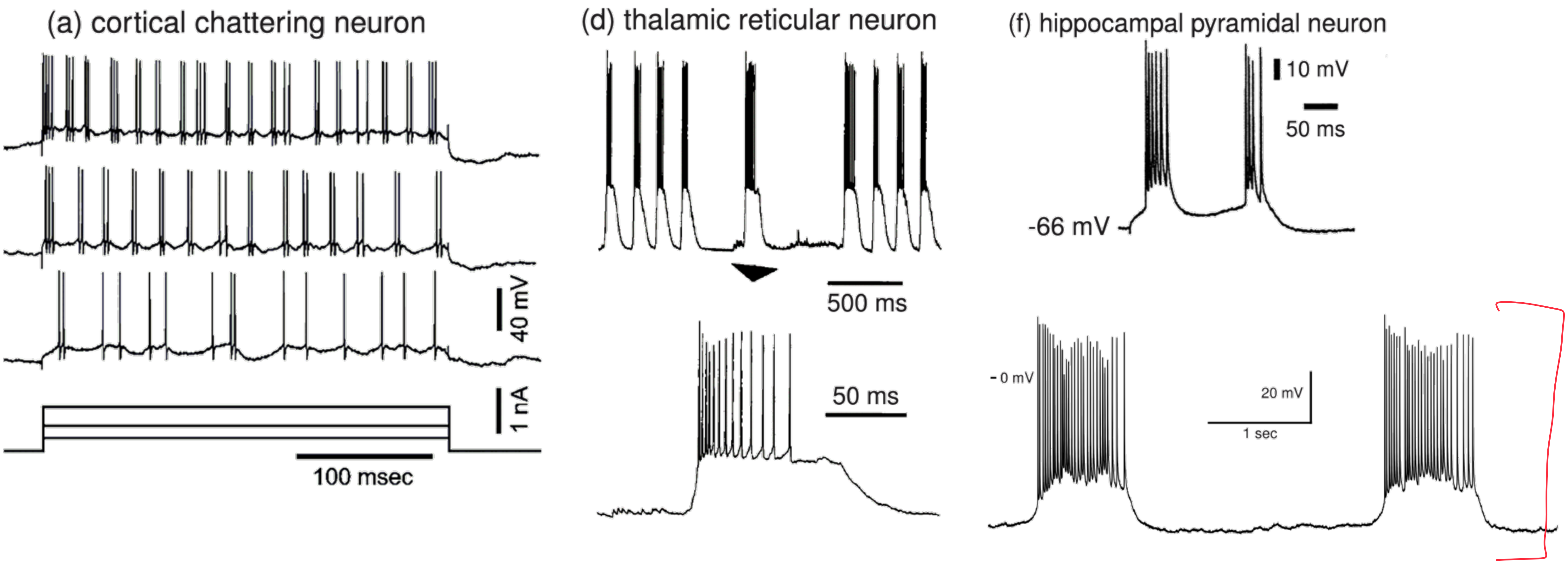


Map

Poincaré Section

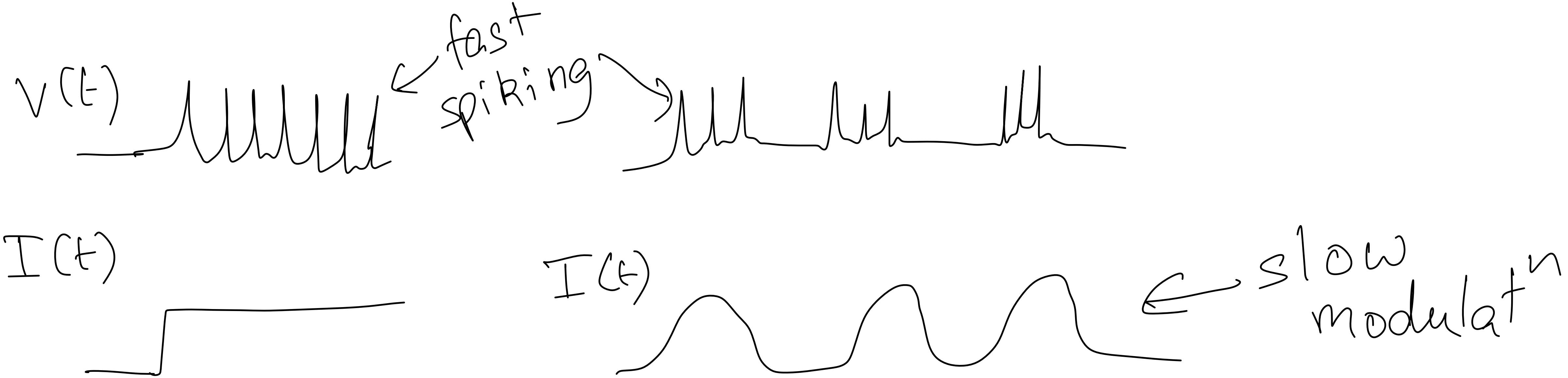
Bursting neurons

E&T Ch 6 + Iz. Ch 9



Iz. Fig. 9.1

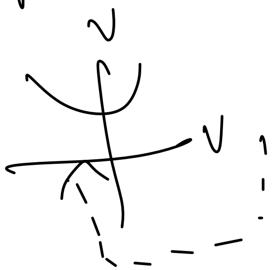
Iz. Fig. 9.27

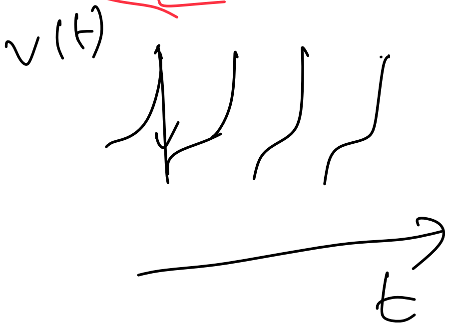


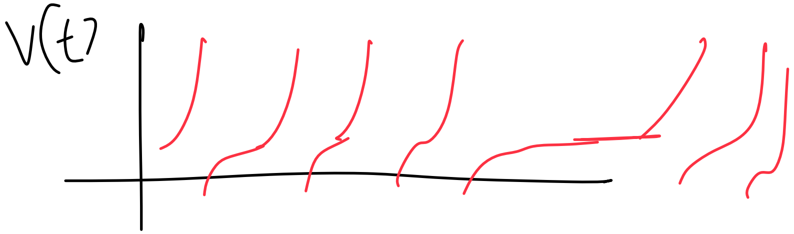
$\mu \ll 1$

$$\begin{cases} \dot{v} = v^2 + I - u \\ \dot{u} = -\mu u \end{cases} \longrightarrow$$

Resetting
if $v \to \infty$
reset $v \to 1$
$u \to u + d$





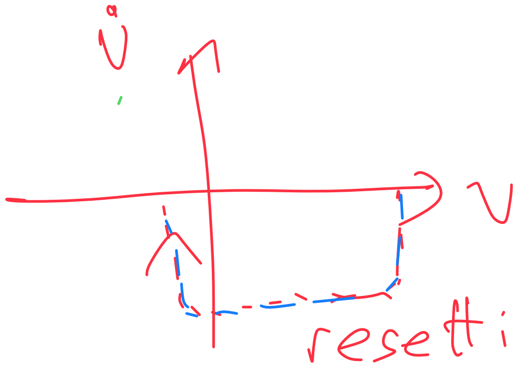


$$\begin{cases} \dot v = v^2 + I - u \\ \dot u = -\mu u \end{cases}$$

$\mu \ll 1$

→ 3d Flow

@ $v \to 1$ reset $u \to u + d$



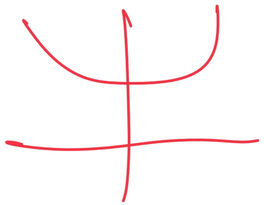
resetting → +1 dim

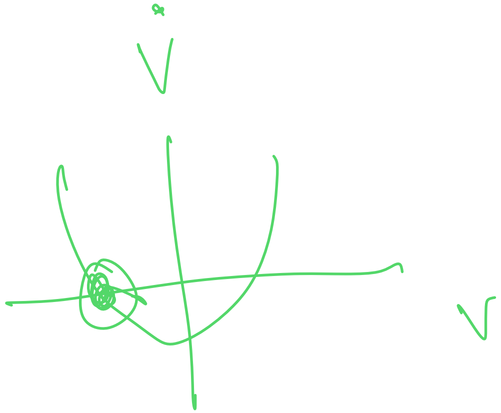
Fold-Homoclinic

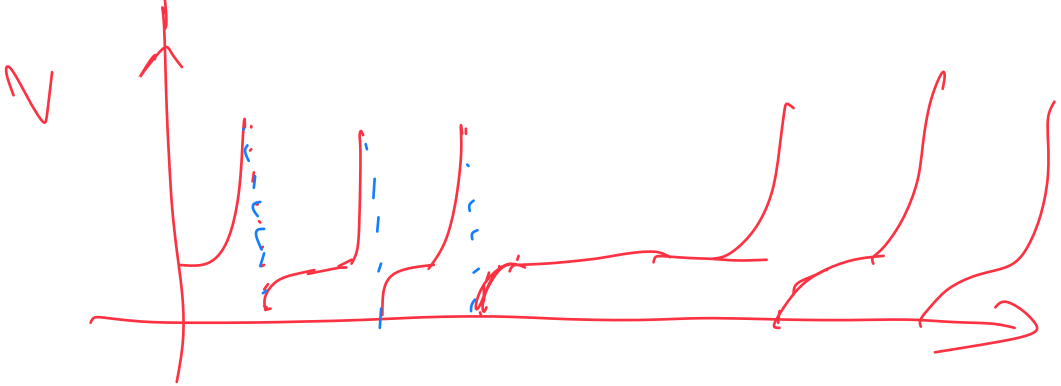
Bursting neuron

$u(0) = 0$

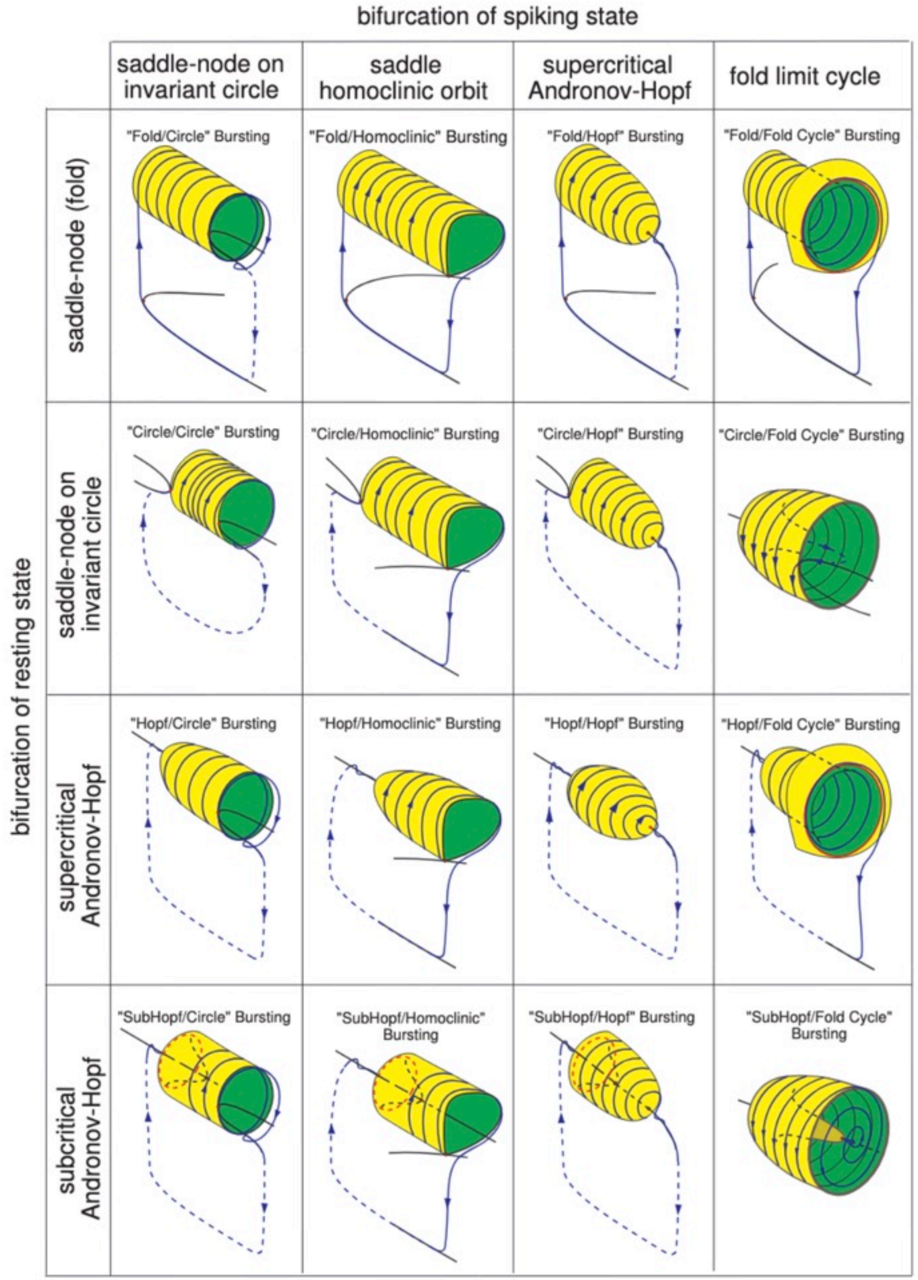
$I > 0$





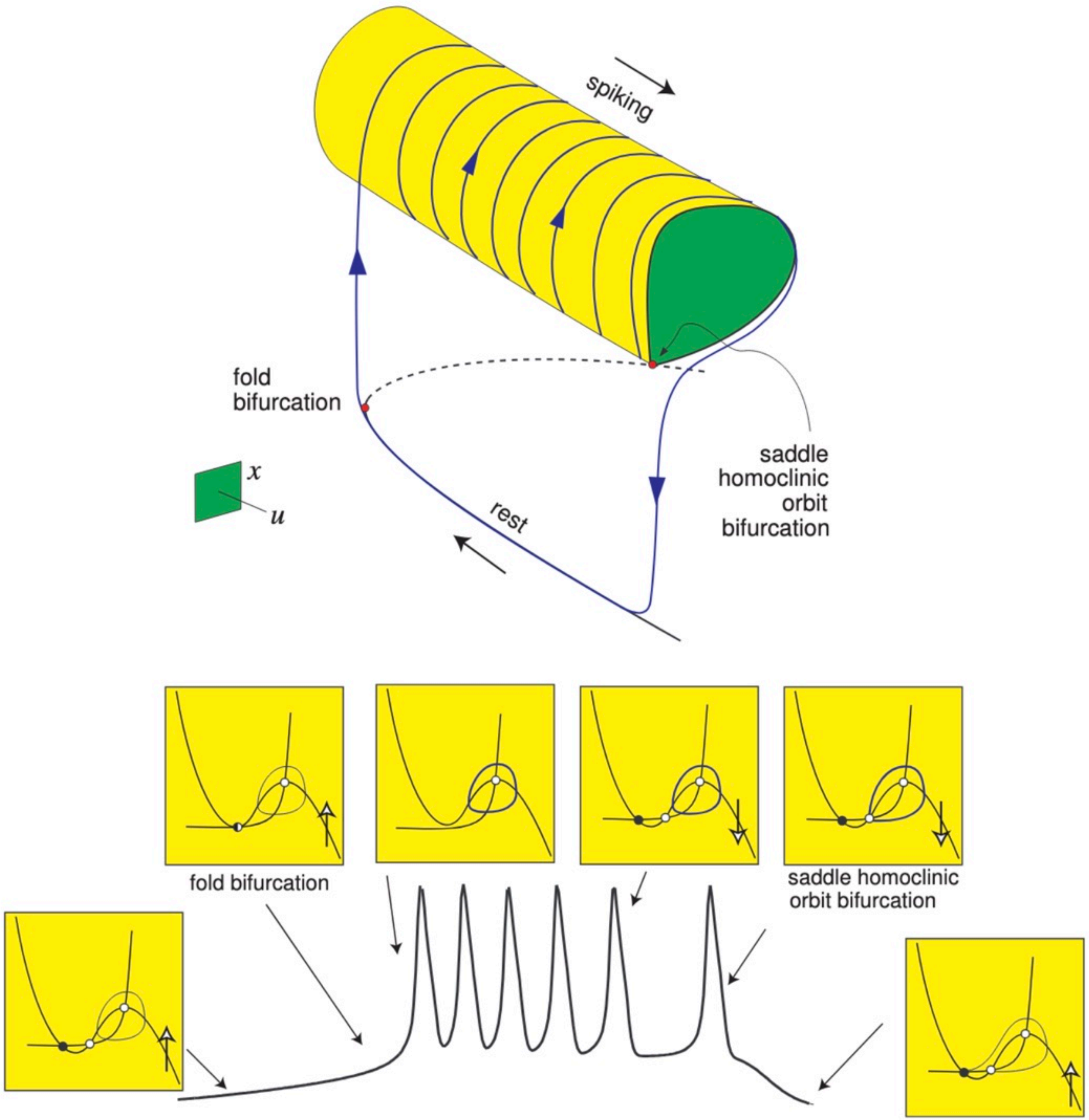


Zoo of bursting neurons

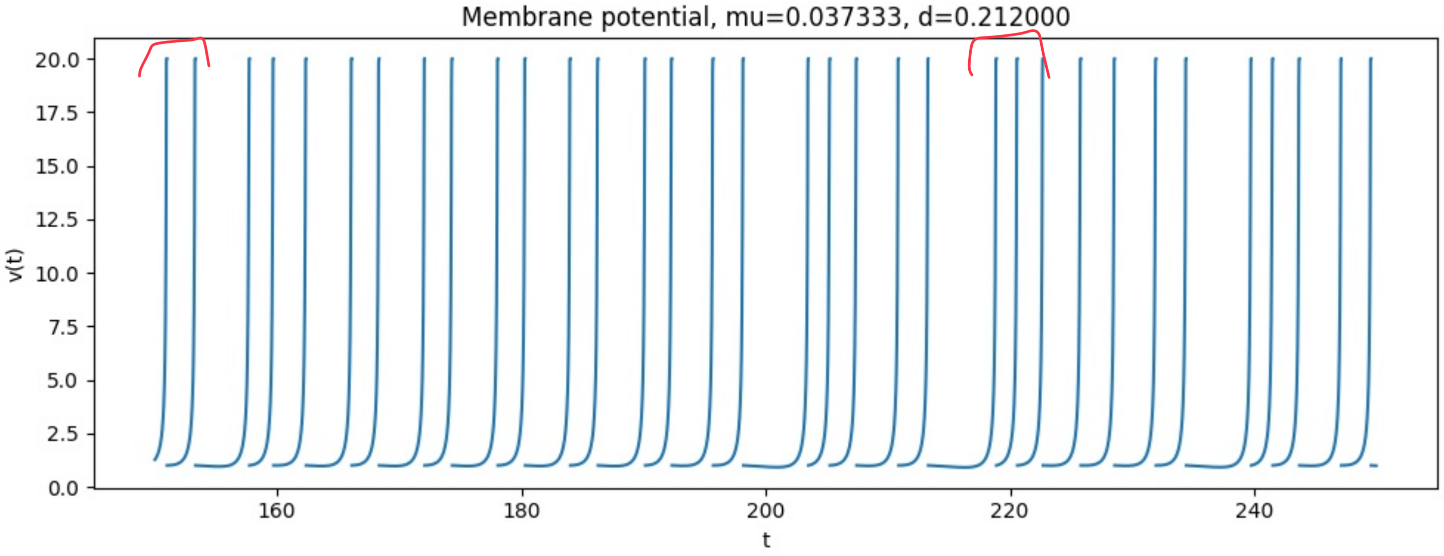


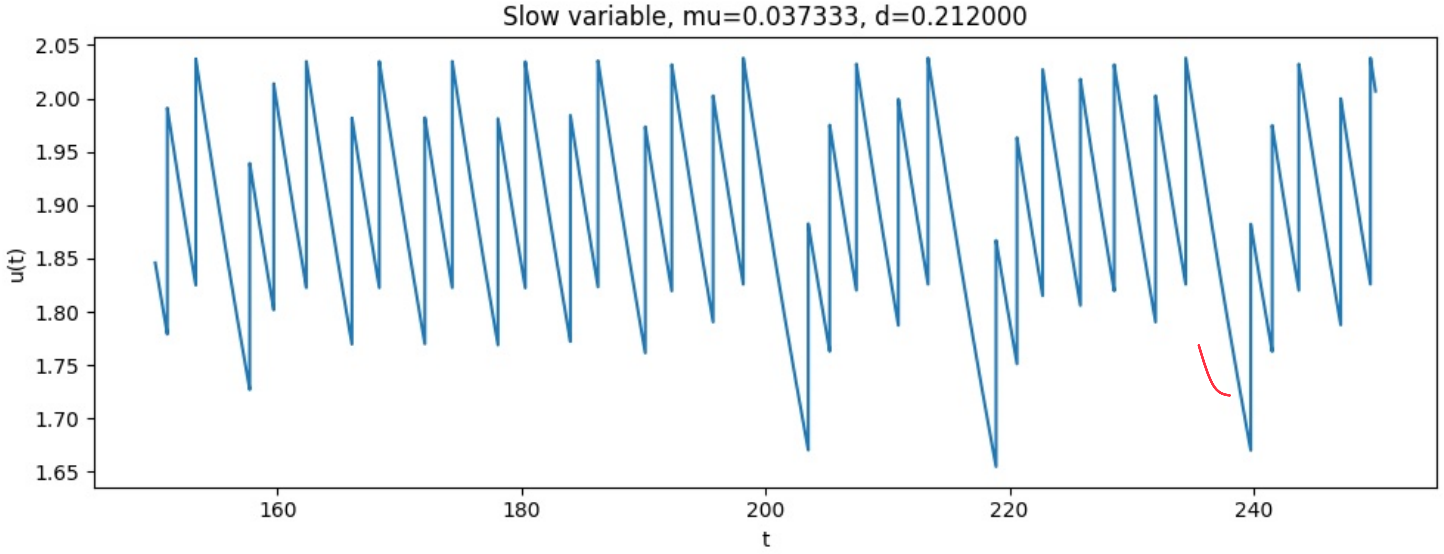
Iz. Fig. 9.24

Fold-homoclinic bursting neuron

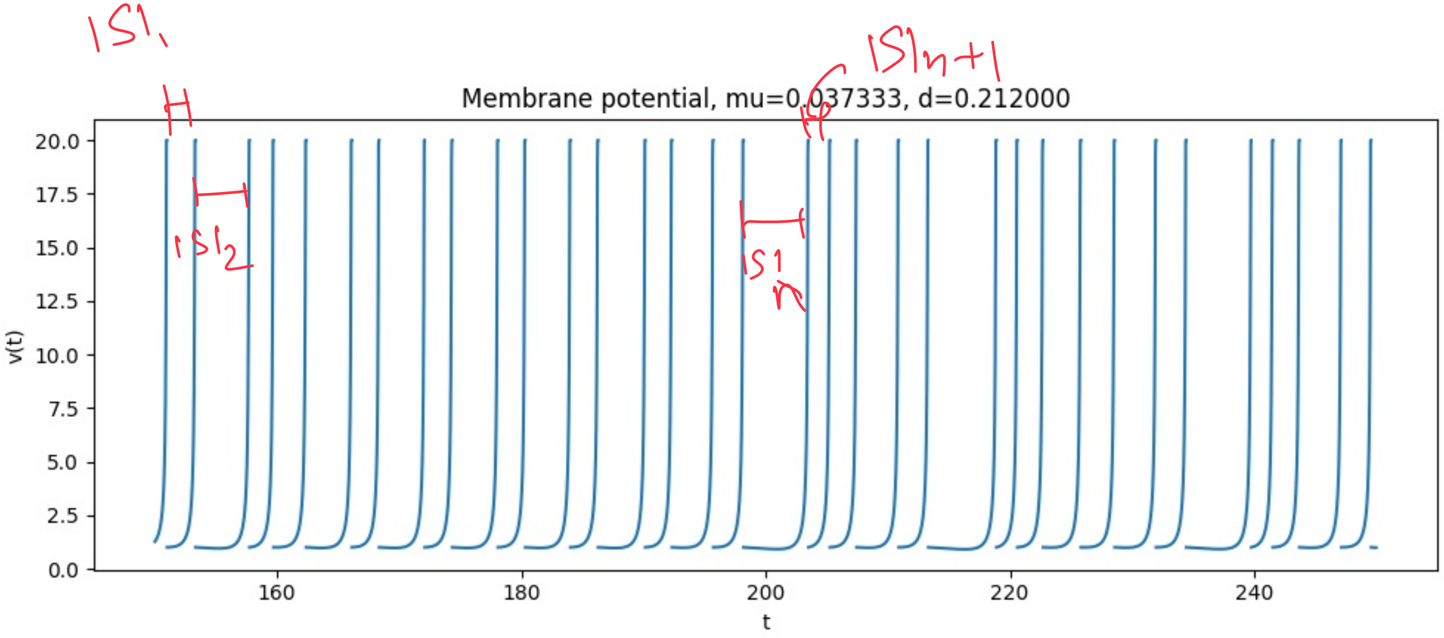


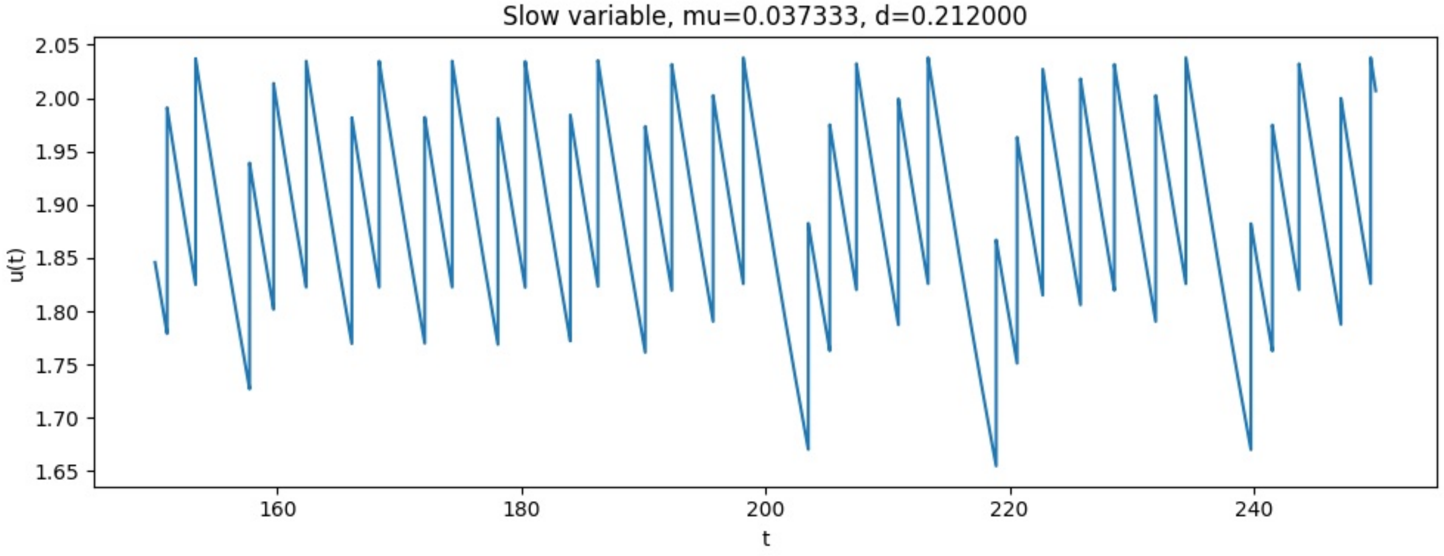
Canonical Fold-homoclinic bursting neuron

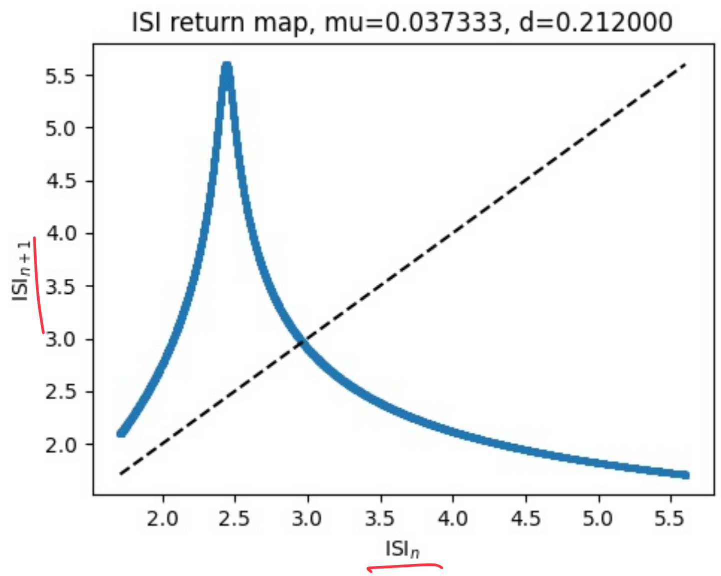




Canonical Fold-homoclinic bursting neuron

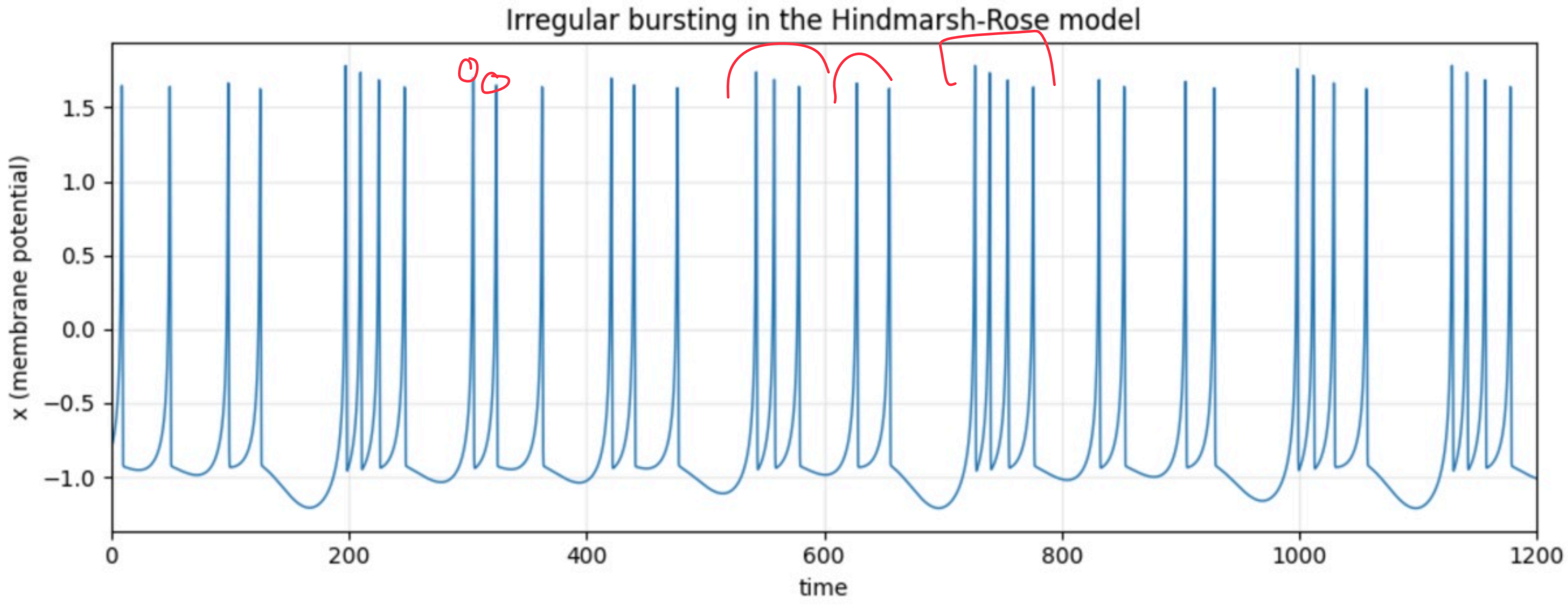


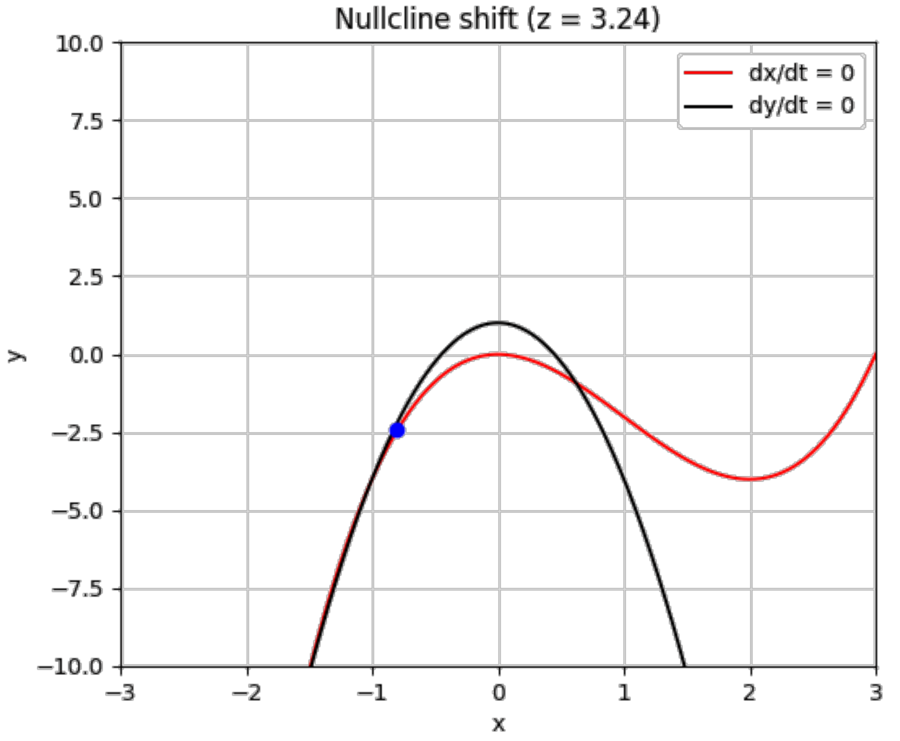




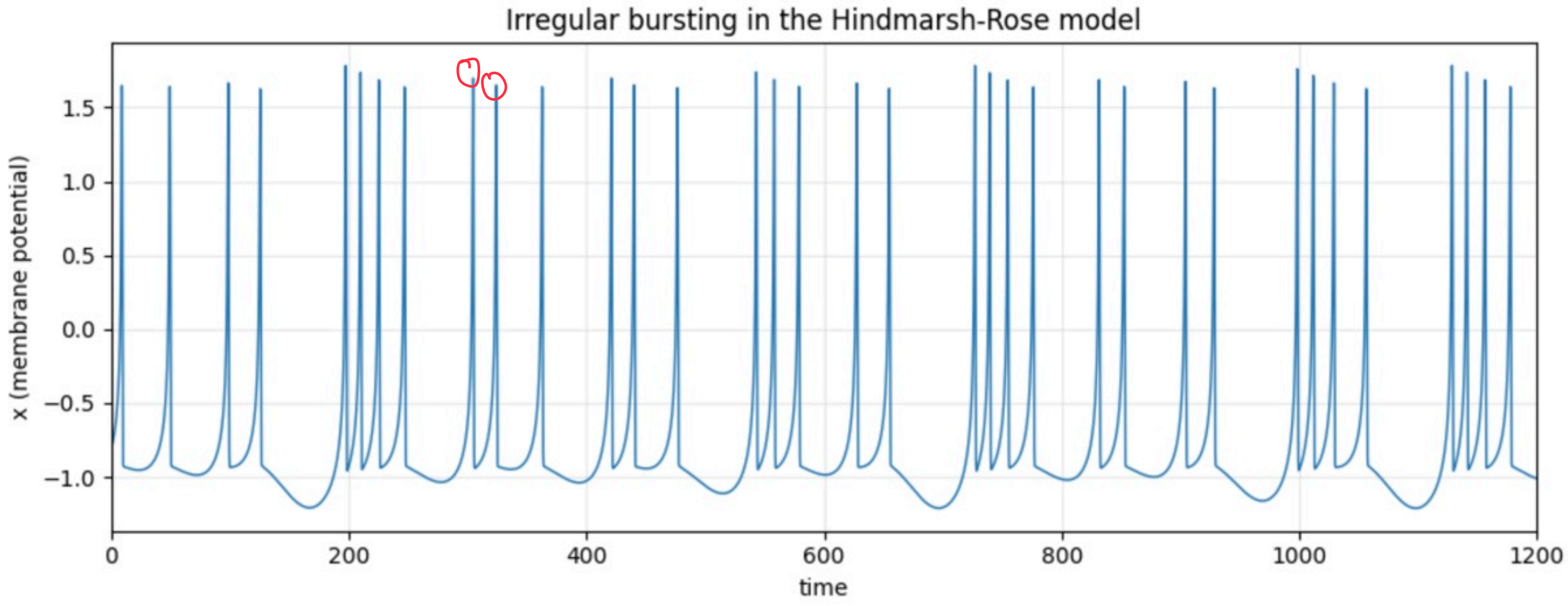
ISI → Inter spike Interval

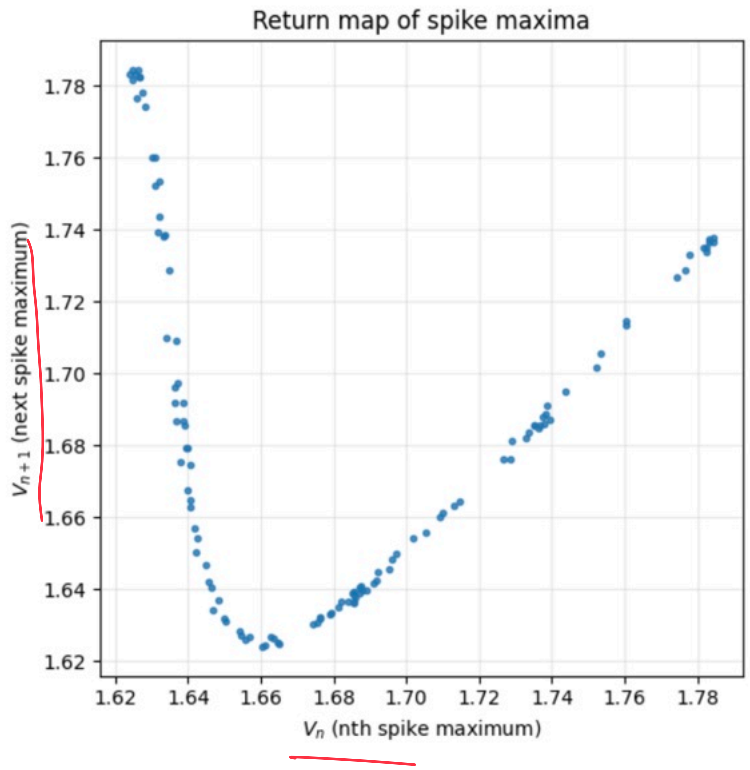
Hindmarsh Rose neuron



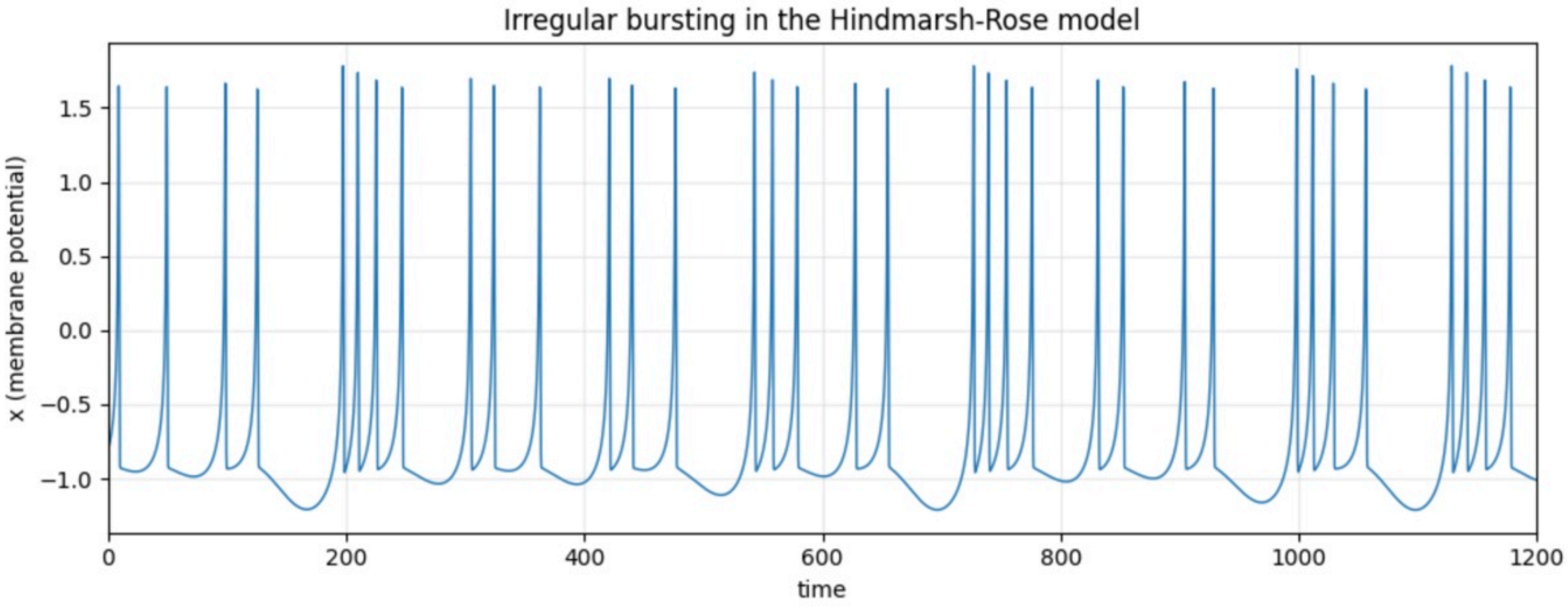


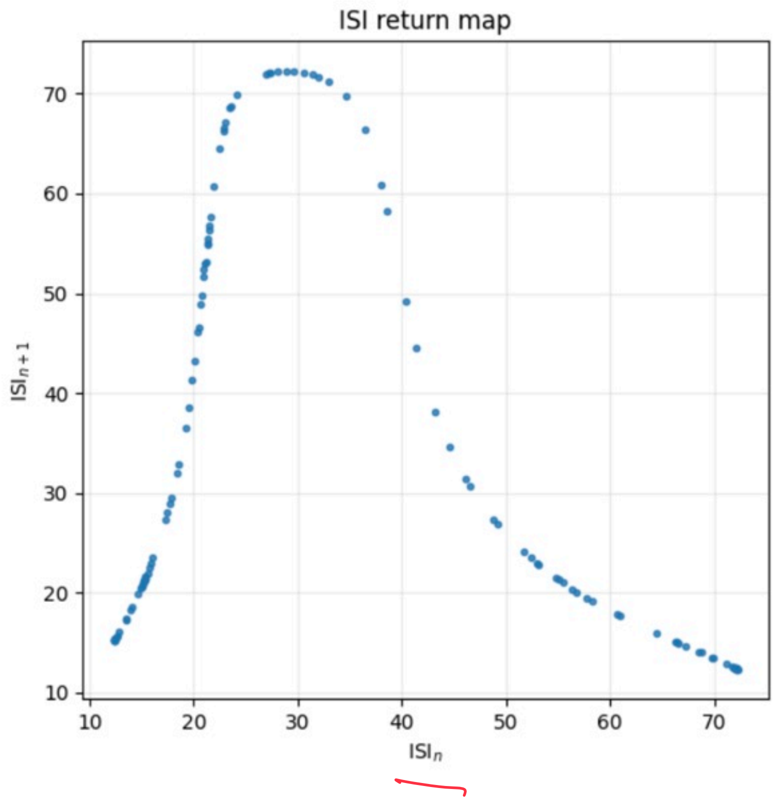
Hindmarsh Rose neuron



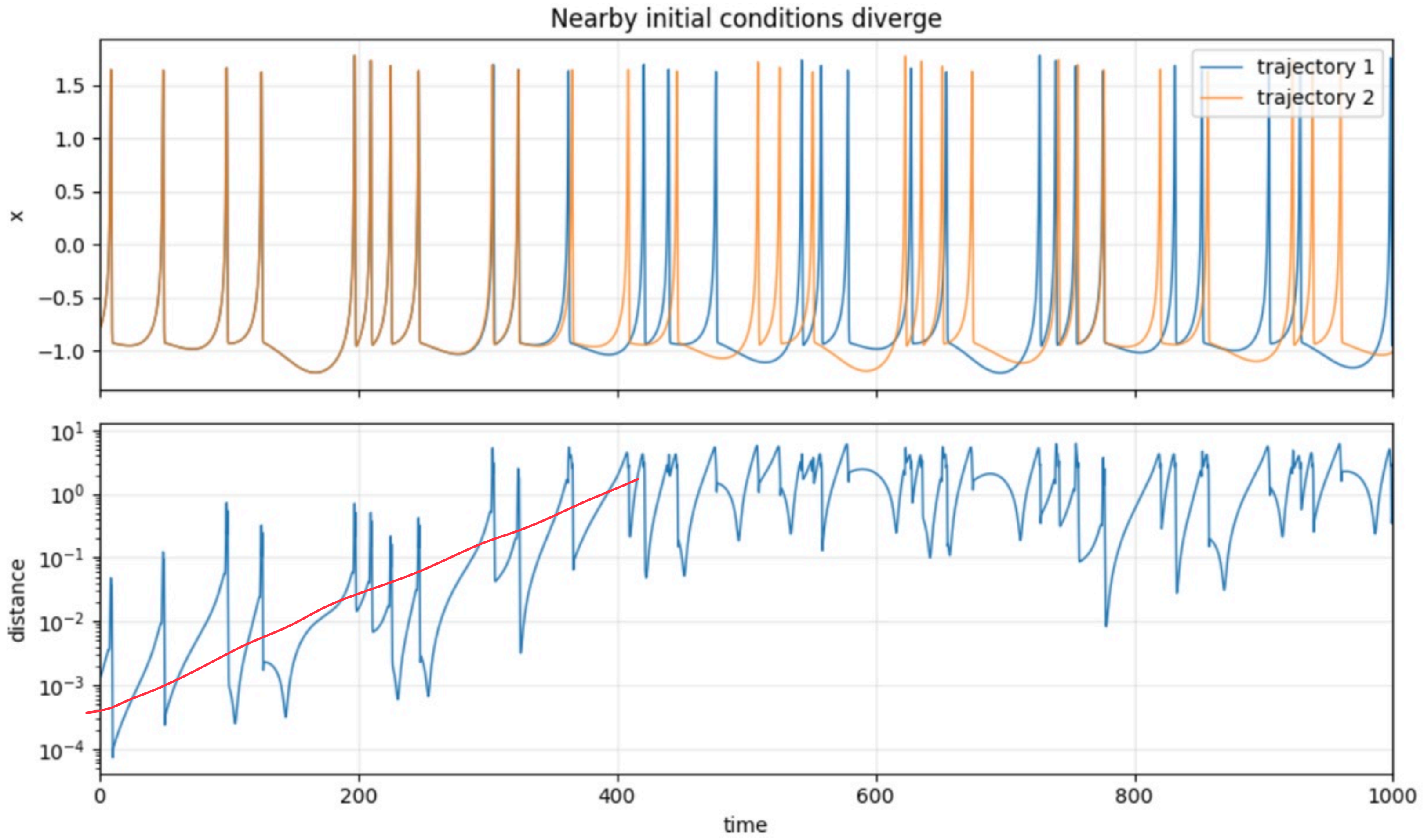


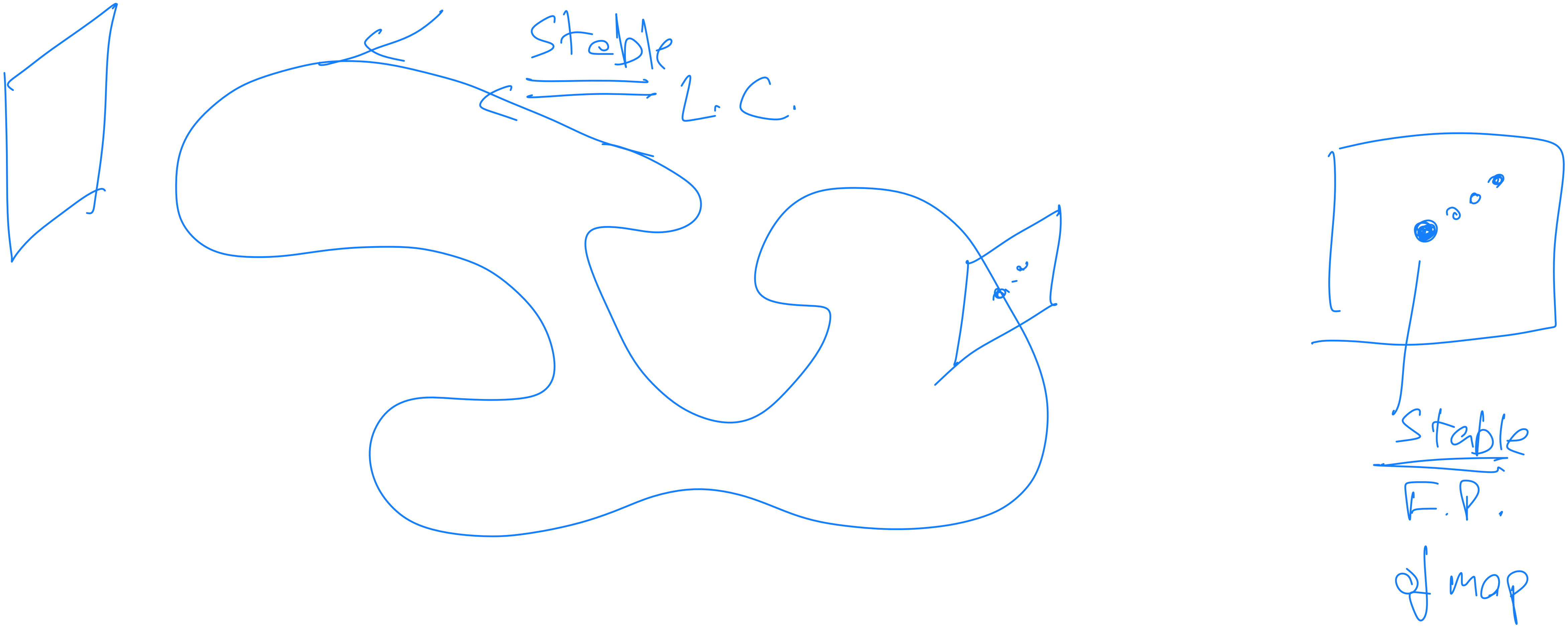
Hindmarsh Rose neuron





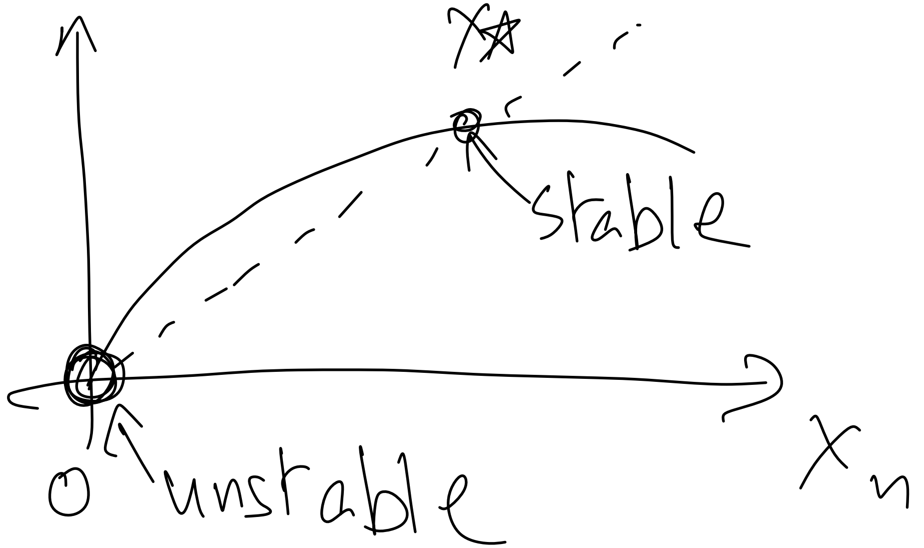
Hindmarsh Rose neuron




Stable L.C.
Stable F.P. of map

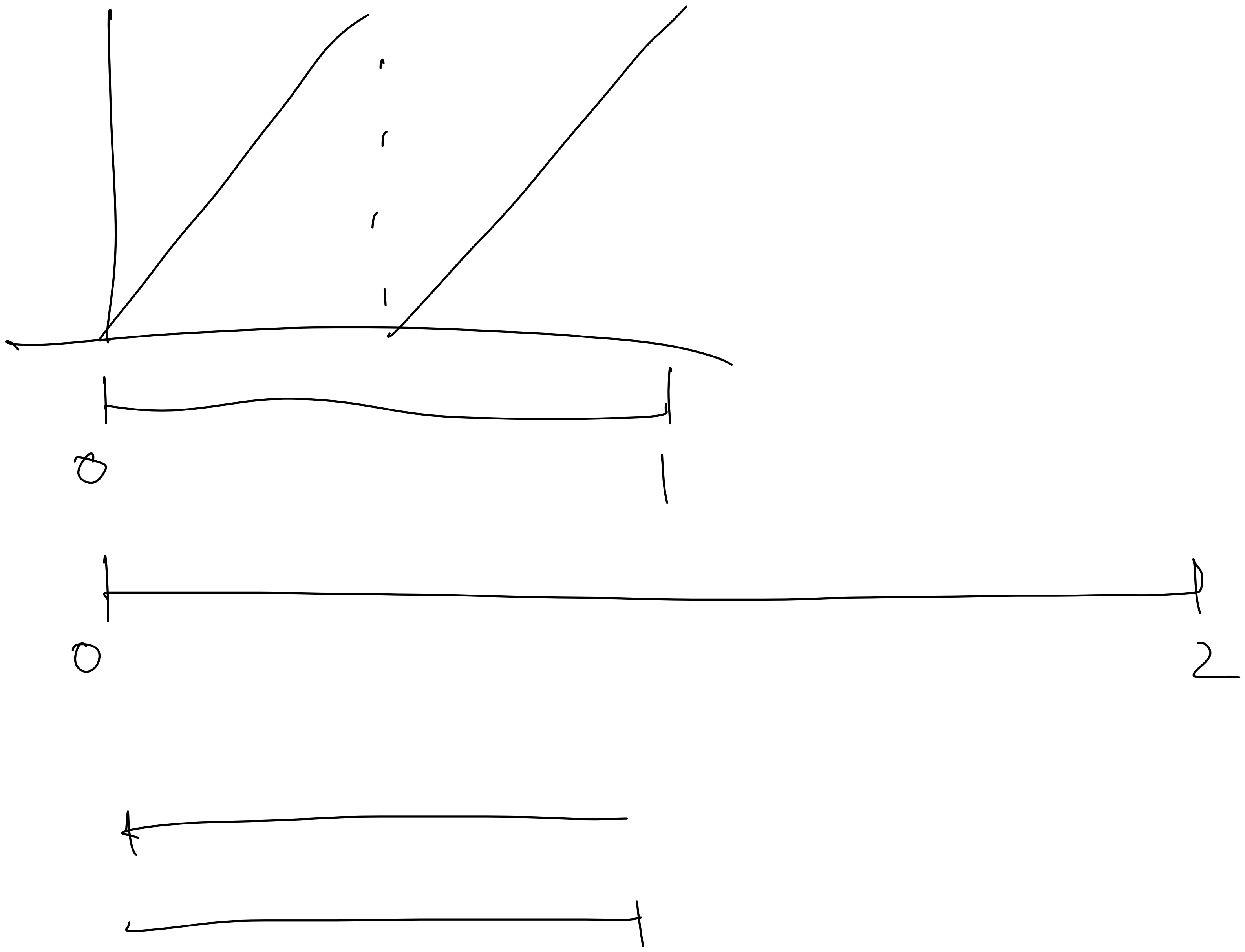
Maps

04-06-2026



$$\lambda = \lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{1}{n} \ln \left| \frac{y_n - x_n}{y_0 - x_0} \right|$$

$$= \left\langle \ln |F'(x)| \right\rangle_n = \ln |F'(x^*)|$$

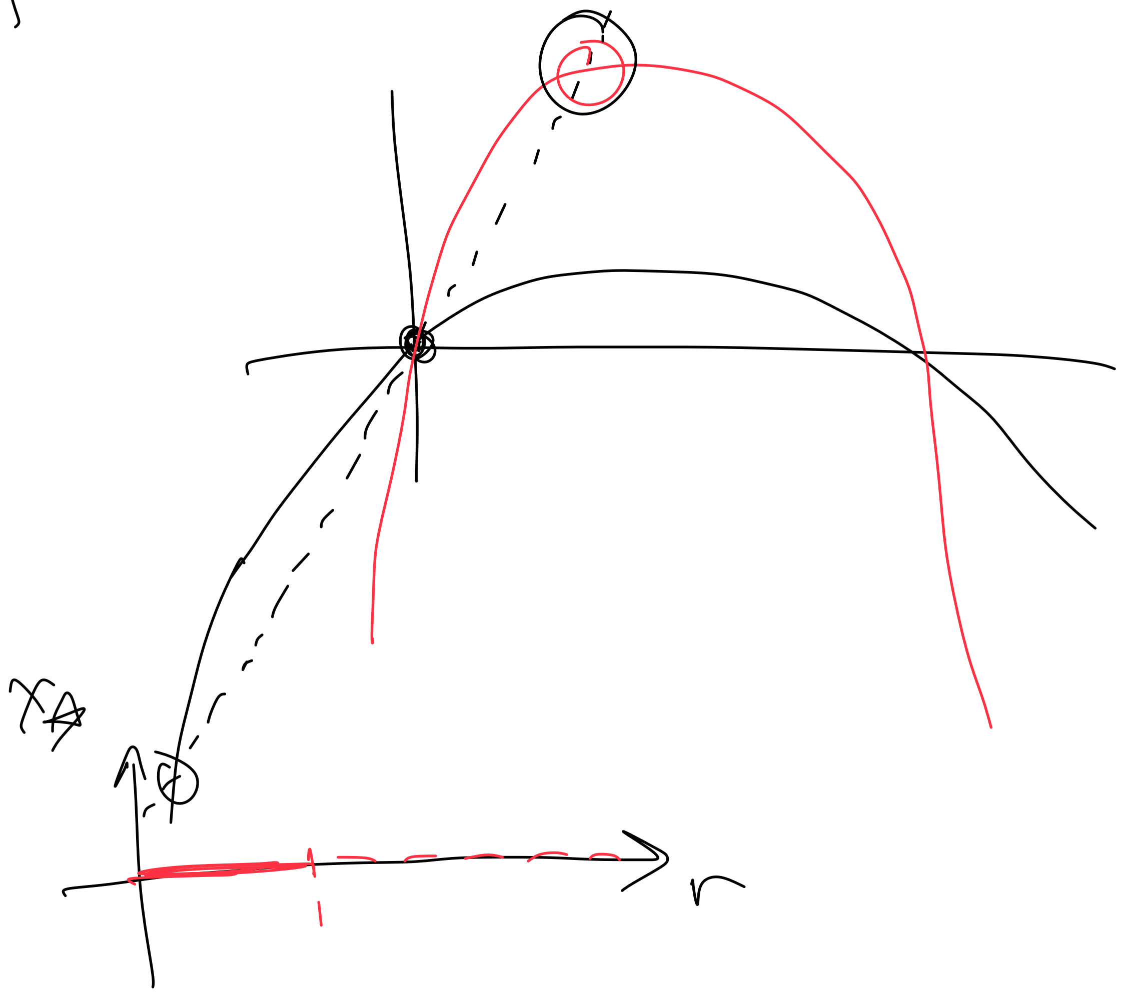

0
1
0
2

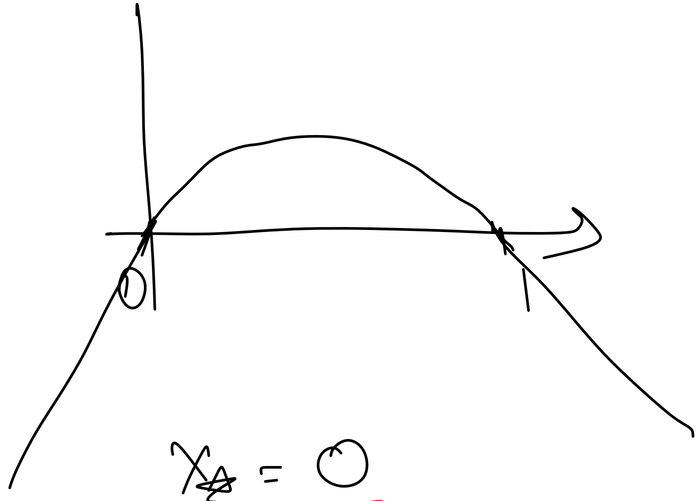
Logistic Map

$x_{n+1} = r x_n (1 - x_n) = F(x_n)$

Strogatz Ch. 10; Ott Ch. 2

if $r < 1$





$x_* = 0$

$F' = r - 2rx$

$F'(x=0) = r$

stable $r < 1$

unstable $r > 1$

$x = rx(1-x)$

if $x \neq 0$

$r(1-x) = 1$

$x = 1 - 1/r$

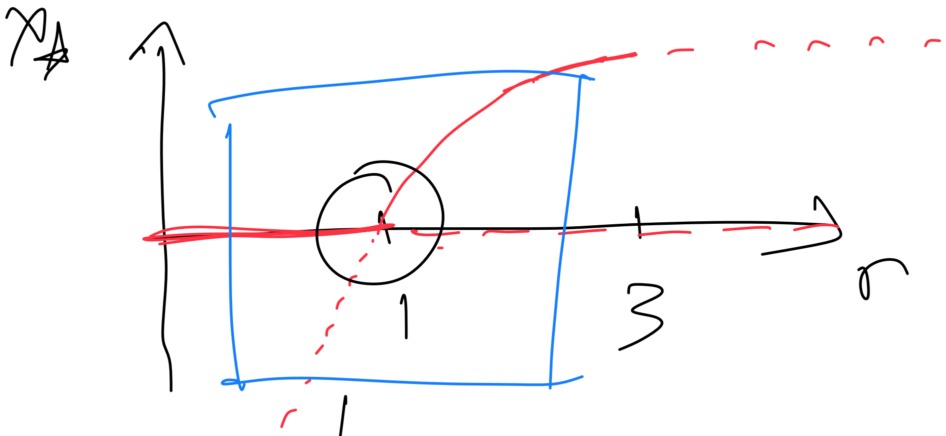
$f' = r - 2rx$

$= r - 2r(1 - 1/r)$

$= r - 2r + 2$

$= 2 - r$ → $x^* = 1 - 1/r$ → stable $1 < r < 3$

unstable $r > 3$



→ Transcritical Bifurcat^n

$$x_{n+1} = F(x_n) = r x_n (1 - x_n)$$

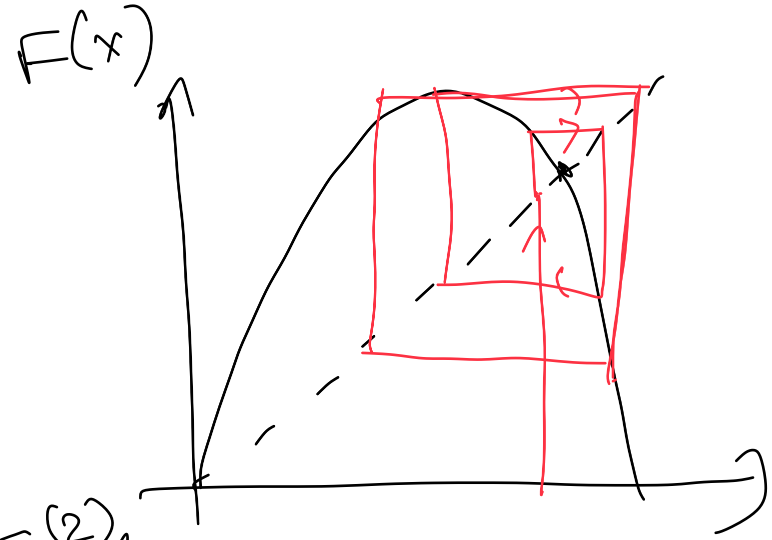
$$x_{\star} = F(F(x_\star))$$

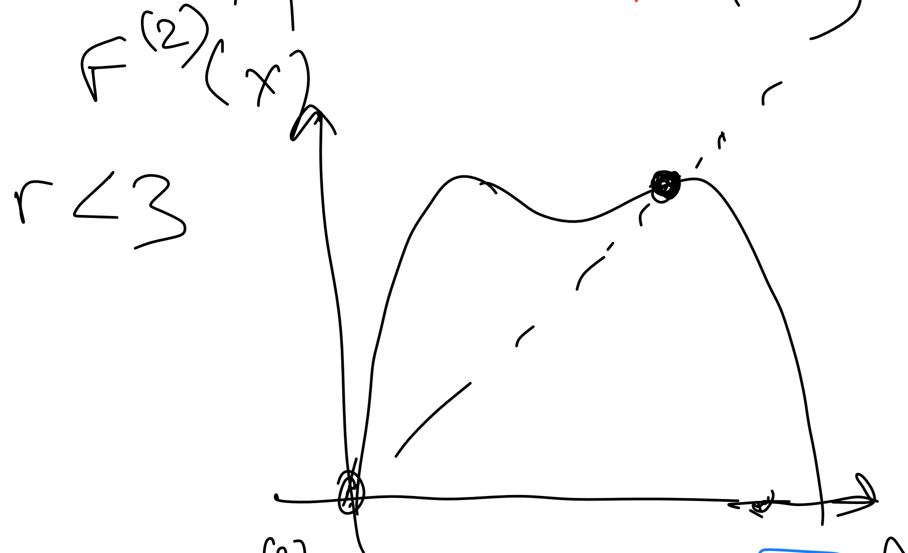
$$\left[x = r[rx(1-x)][1 - rx(1-x)] \right]$$

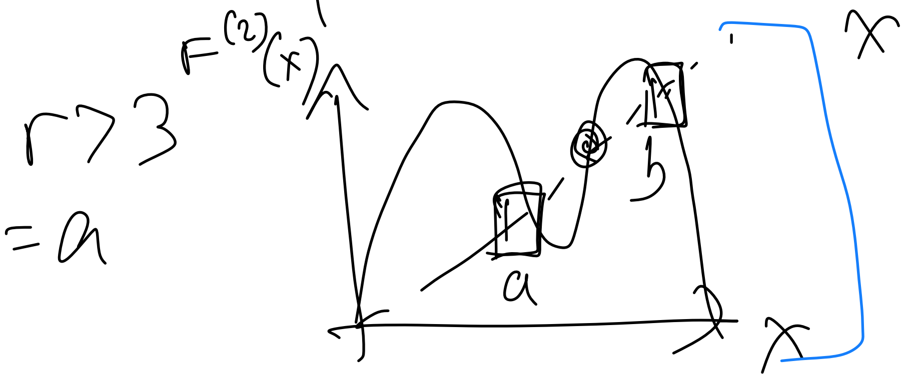
$$x_\star^{1,2} = \frac{r+1 \pm \sqrt{(r-3)(r-1)}}{2r}$$

$r = 3 \rightarrow x_\star^1 = x_\star^2$

$r = 3 + \varepsilon \rightarrow (x_\star^1 - x_\star^2) \sim \sqrt{\varepsilon}$







$F(a) = b$

$F(b) = a$

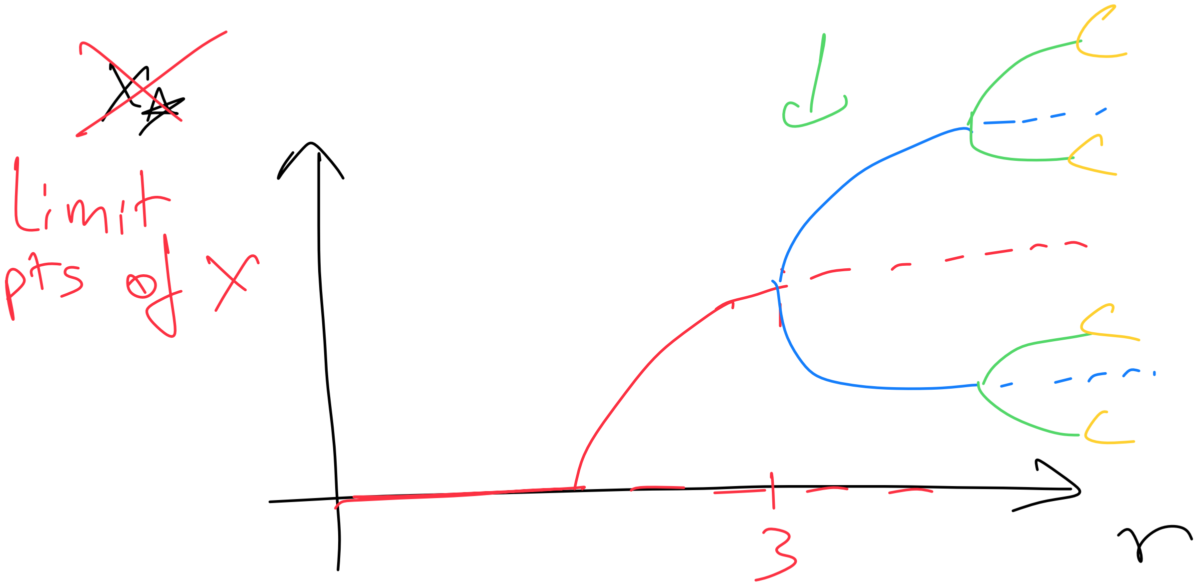
$\Leftarrow F^{(2)}(a) = a$

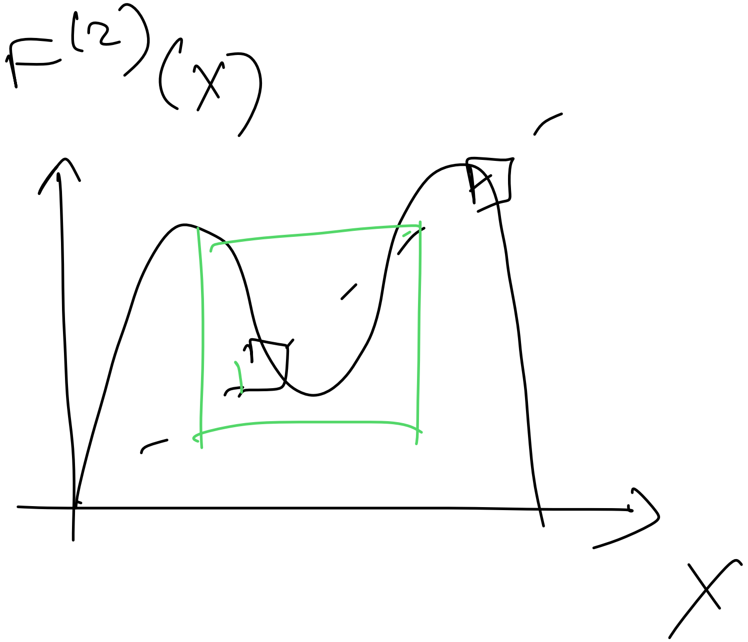
stab. of period 2 orbit

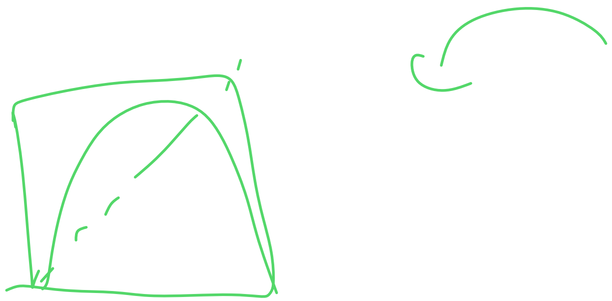
$F'(x^1) \cdot F'(x^2)$

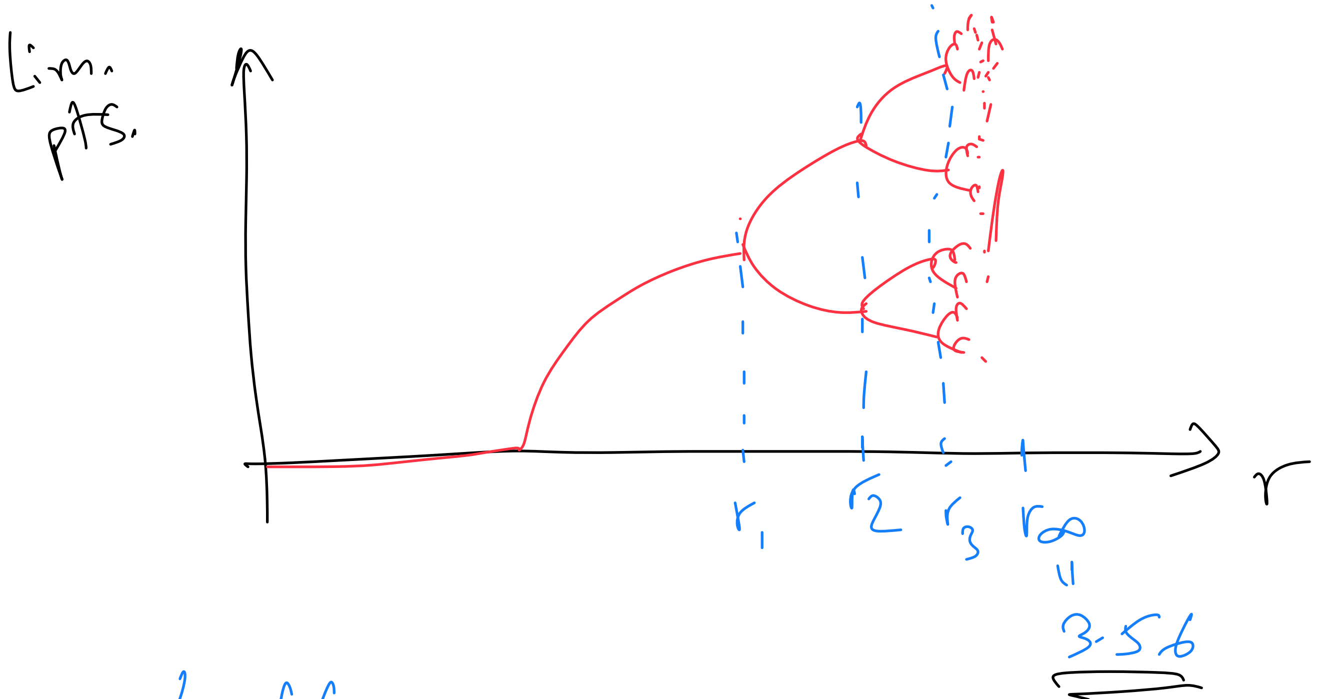
↓

P. 2 orbit stable $3 < r < 1 + \sqrt{6}$









$$\frac{r_m - r_{m-1}}{r_{m+1} - r_m} \sim 4.66.$$

Feigenbaum constant

What of

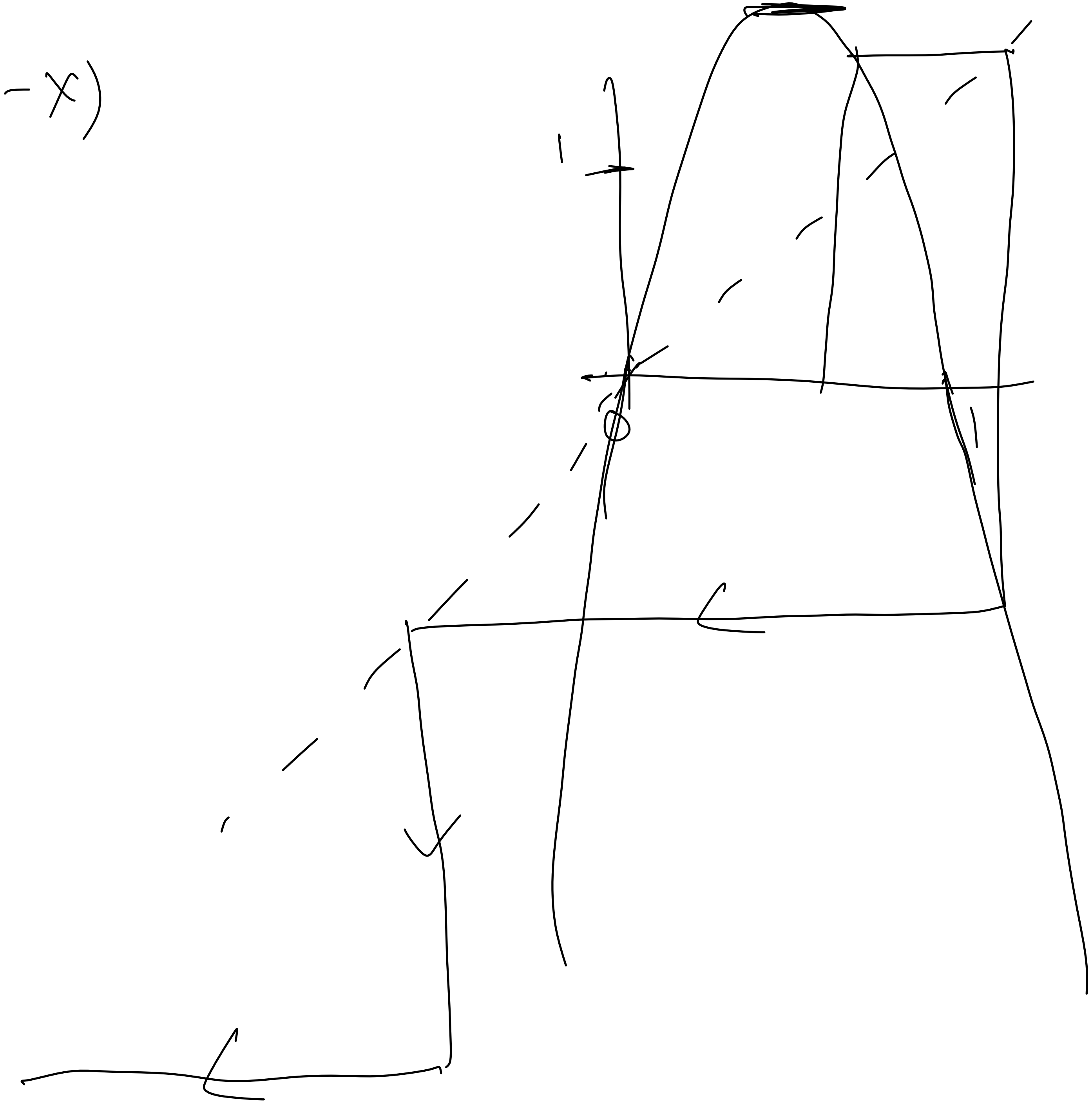
$r_\infty < r < 4$

$r < r_\infty \rightarrow \lambda < 0$

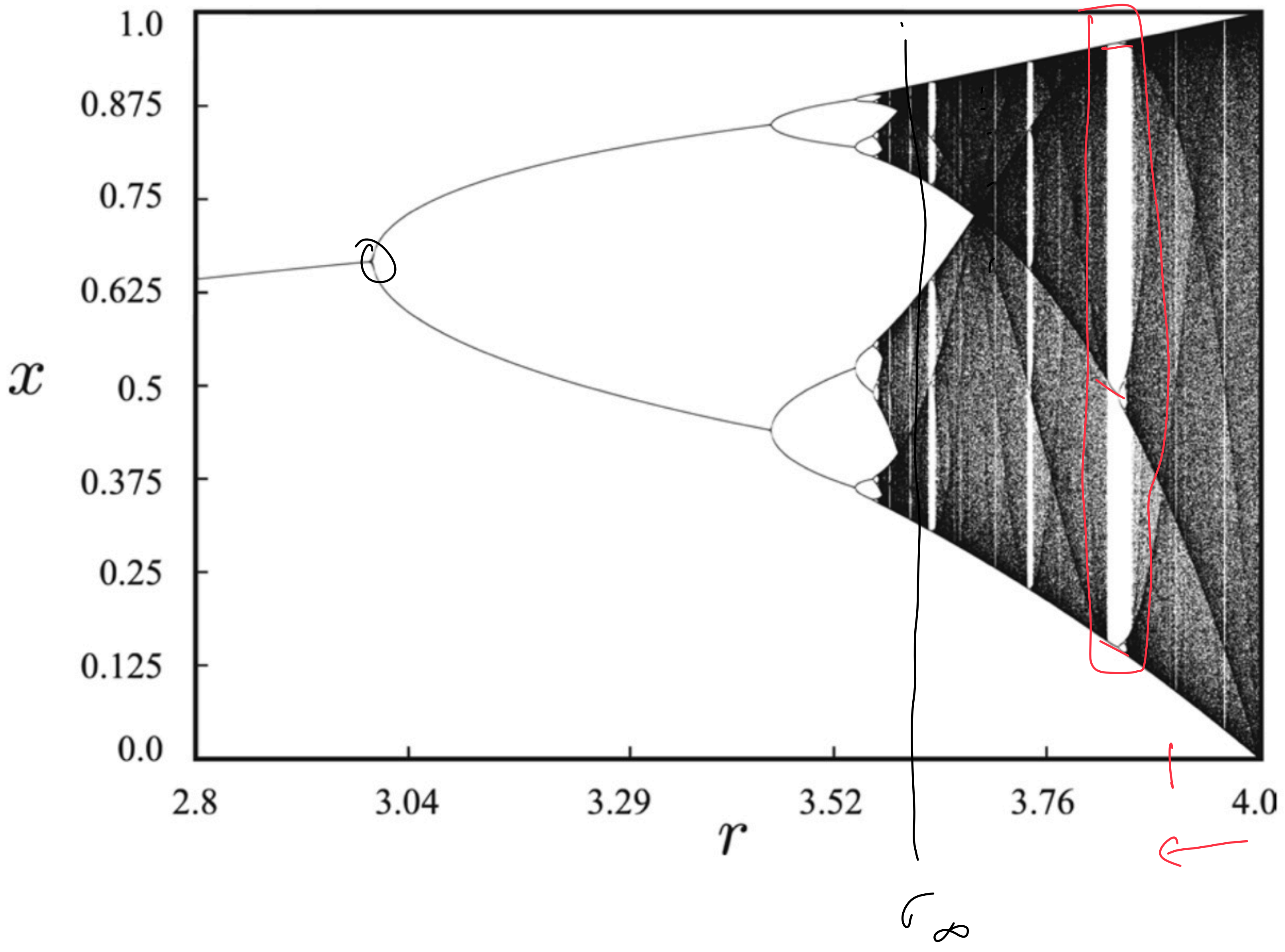
$r > r_\infty \rightarrow$ start of chaos

$$F(x) = rx(1-x)$$

$r < 4$

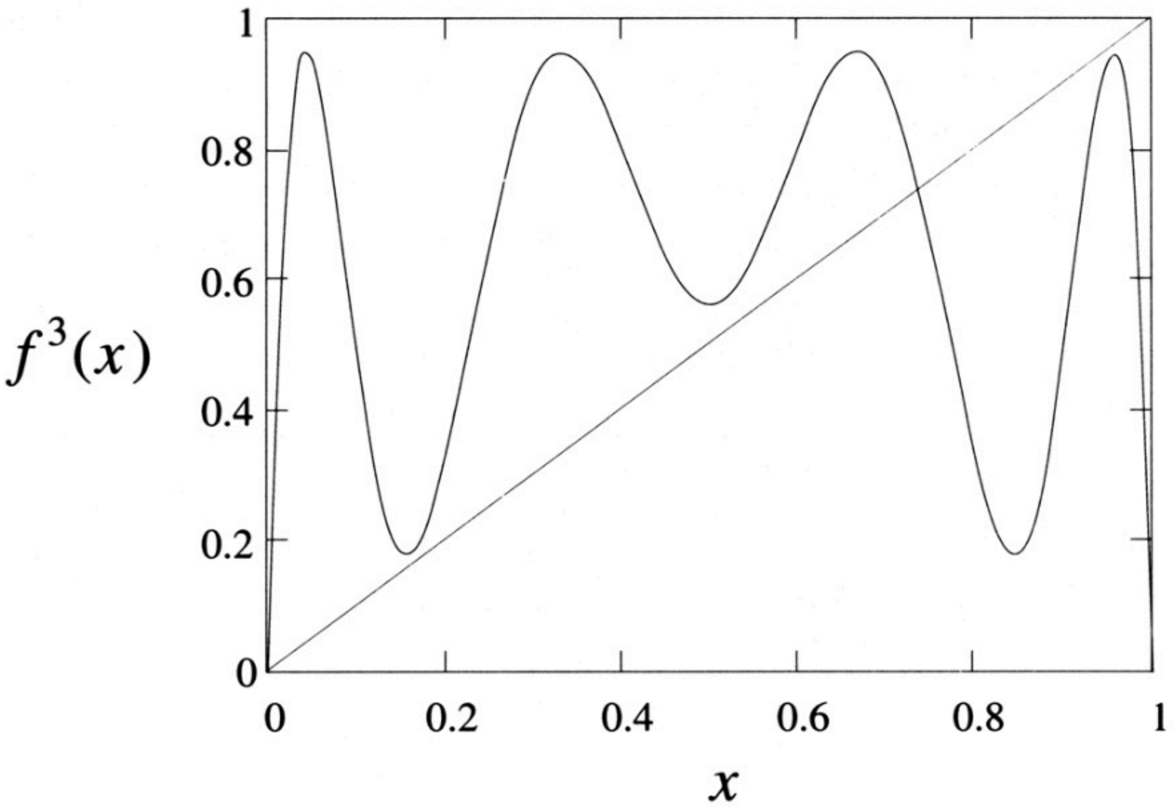


Logistic map (Figs. From Ott Ch 2 + Strogatz Ch. 10)

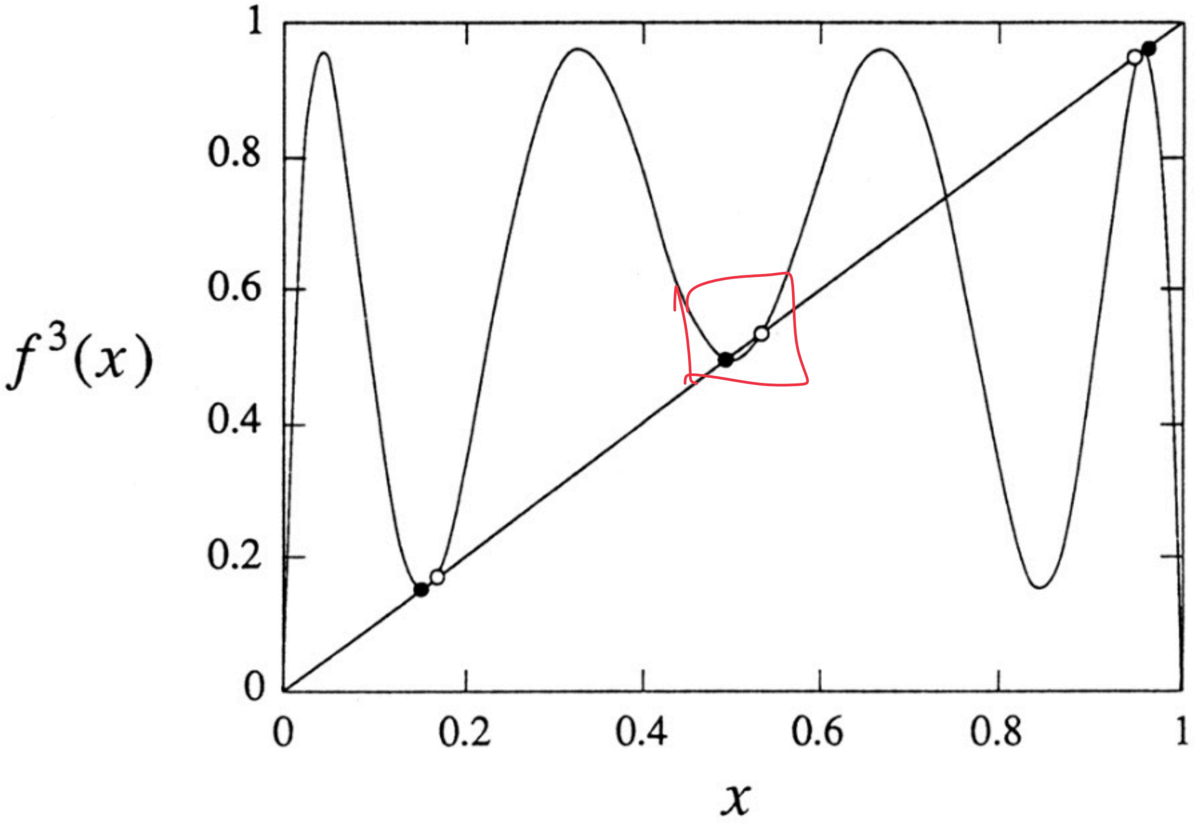


Logistic map

r=3.8

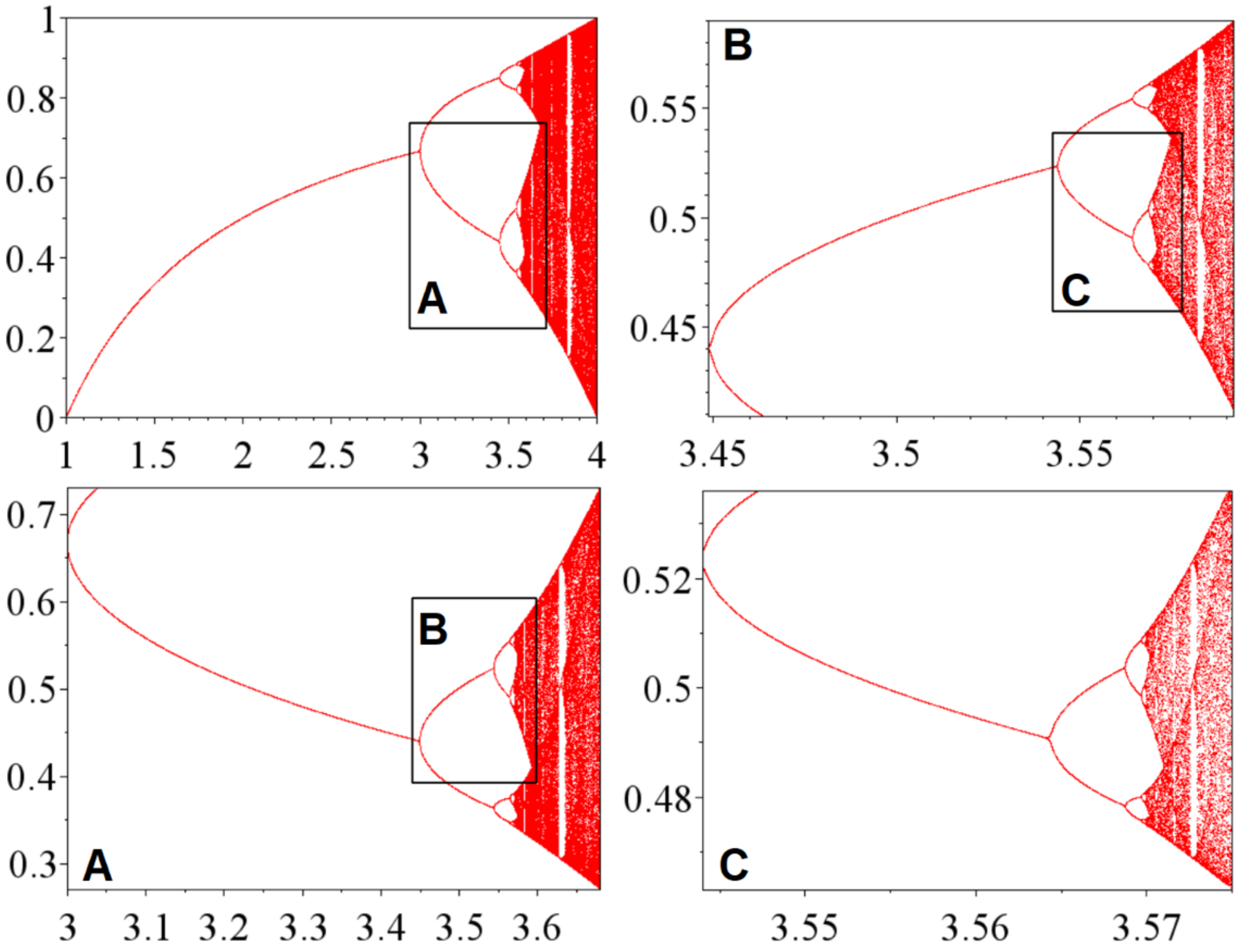


r=3.835



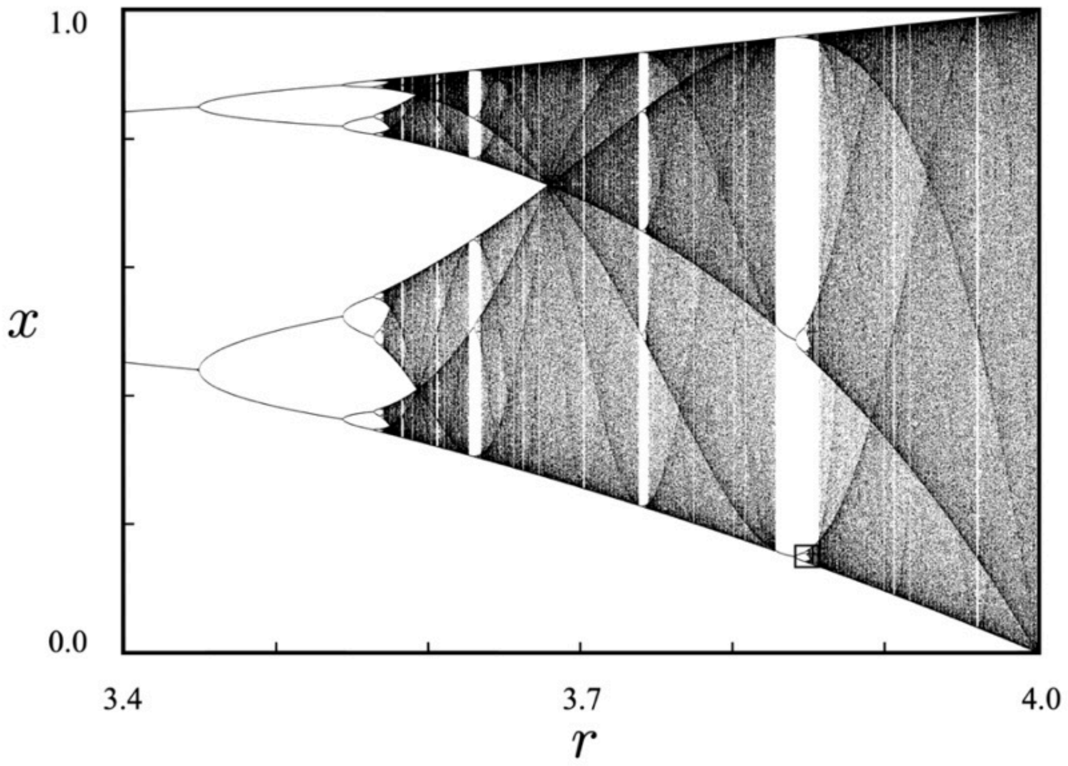
Logistic map

Period doubling route to chaos

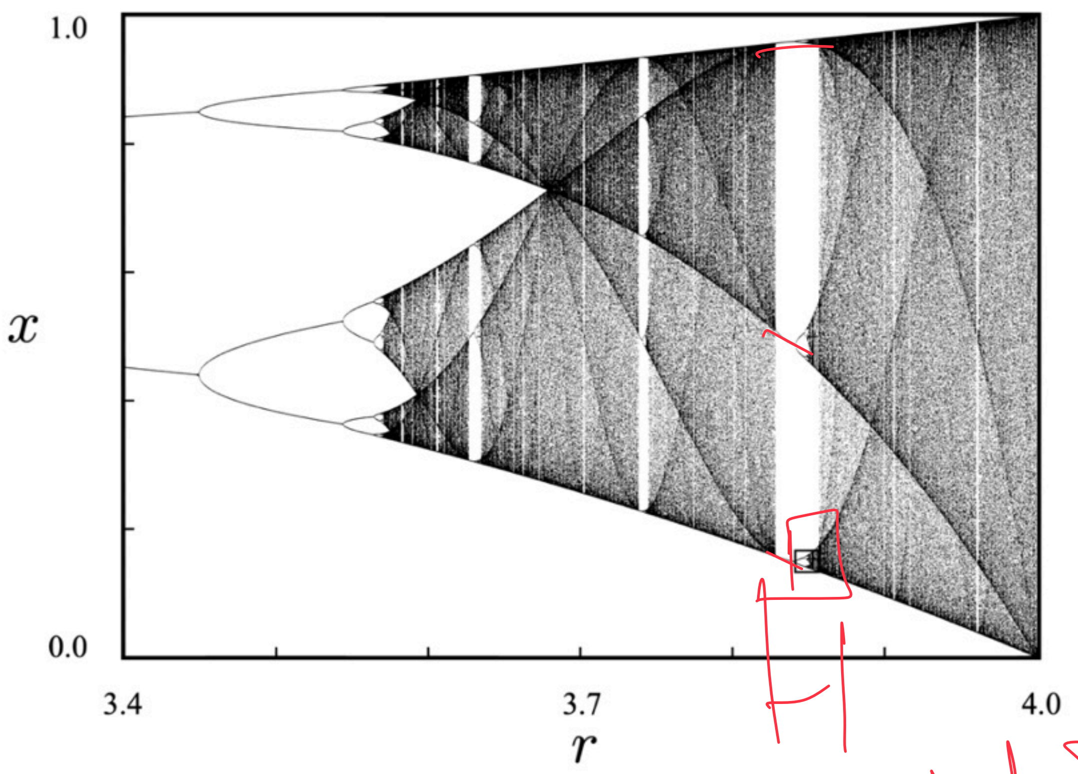


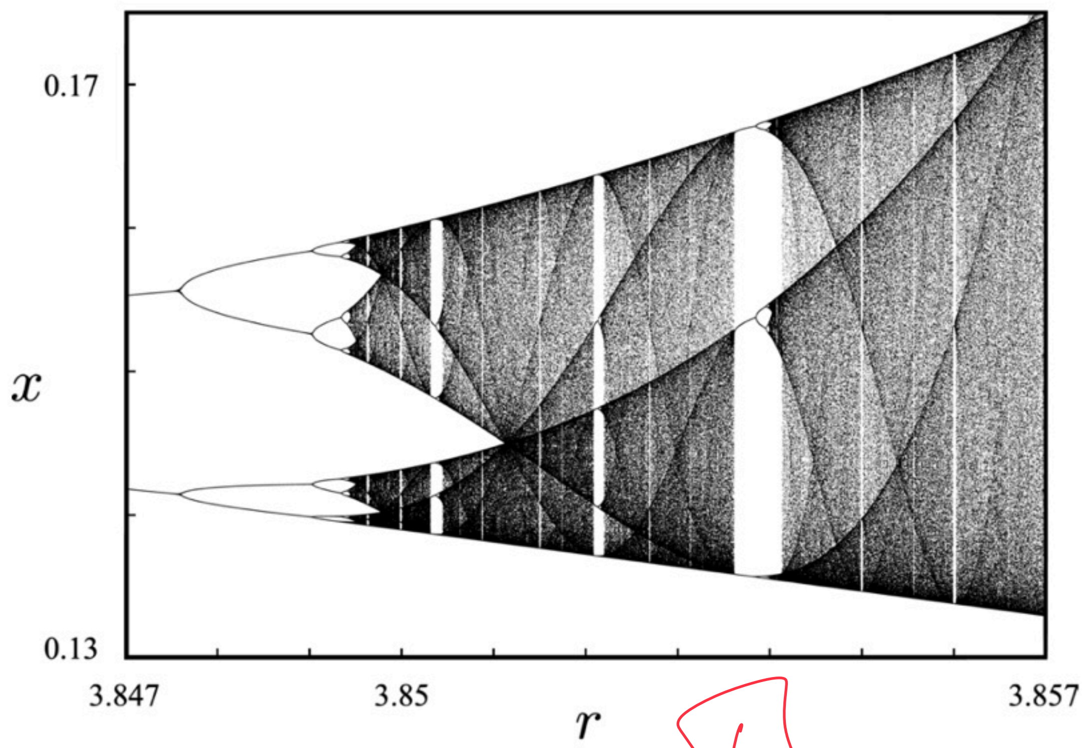
Wikipedia

Logistic map



Logistic map





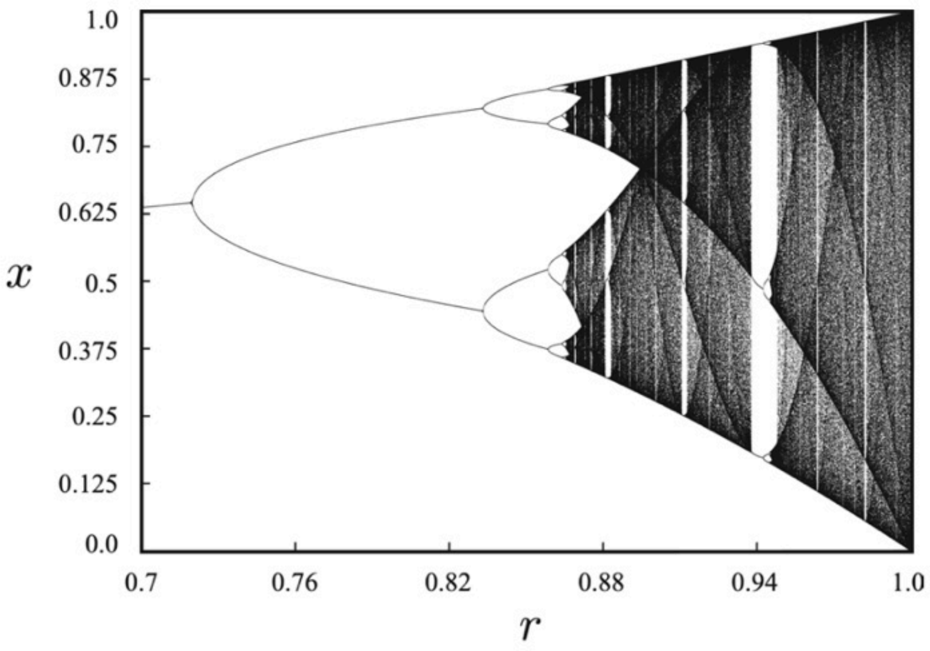
period 3

Period3 implies Chaos

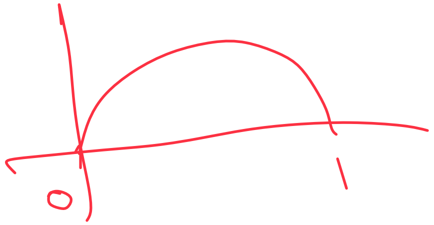
Yorke, Li 1975

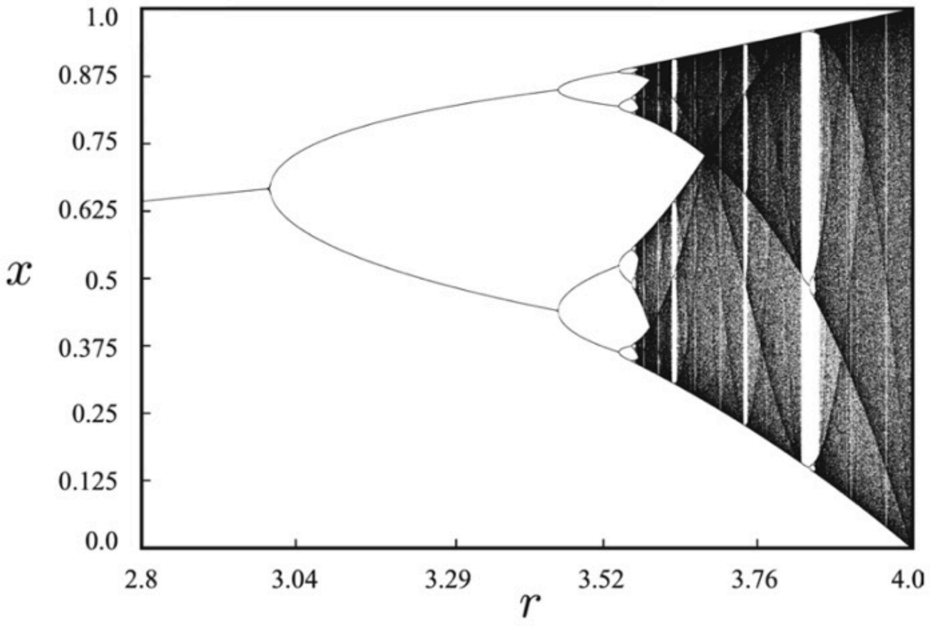
Sine map vs logistic map

$x_{n+1} = r x_n (1 - x_n)$

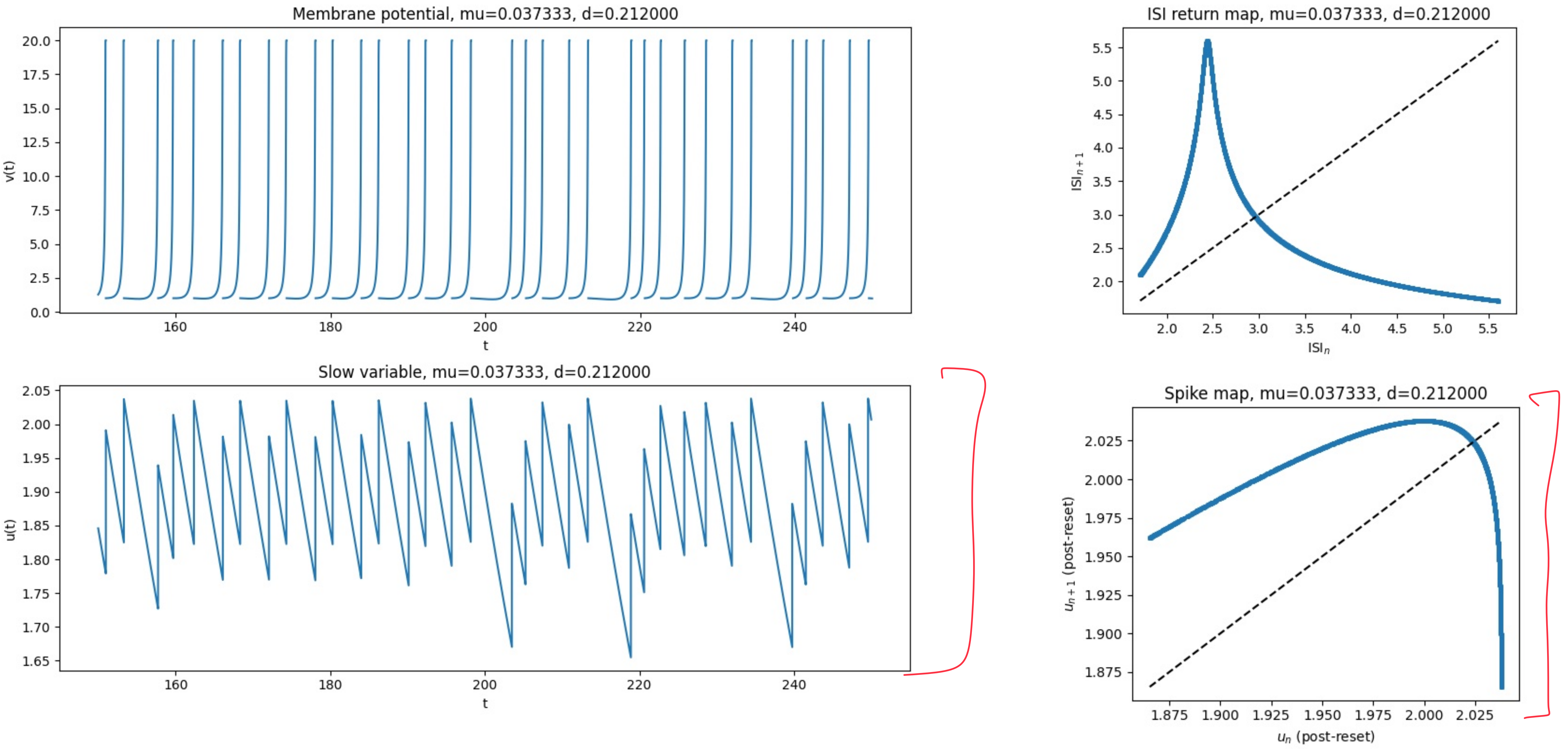


$x_{n+1} = r \sin(\pi x_n)$

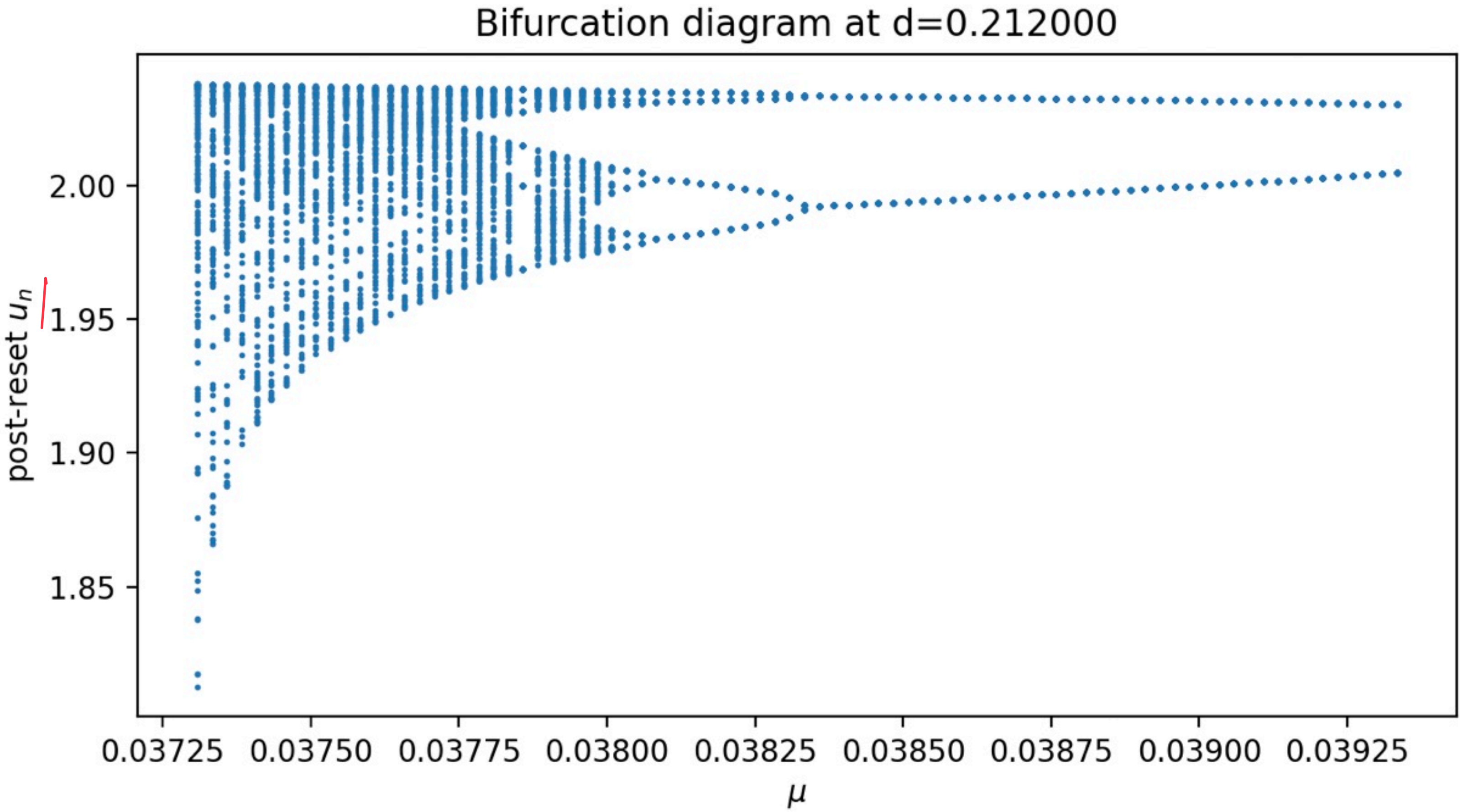




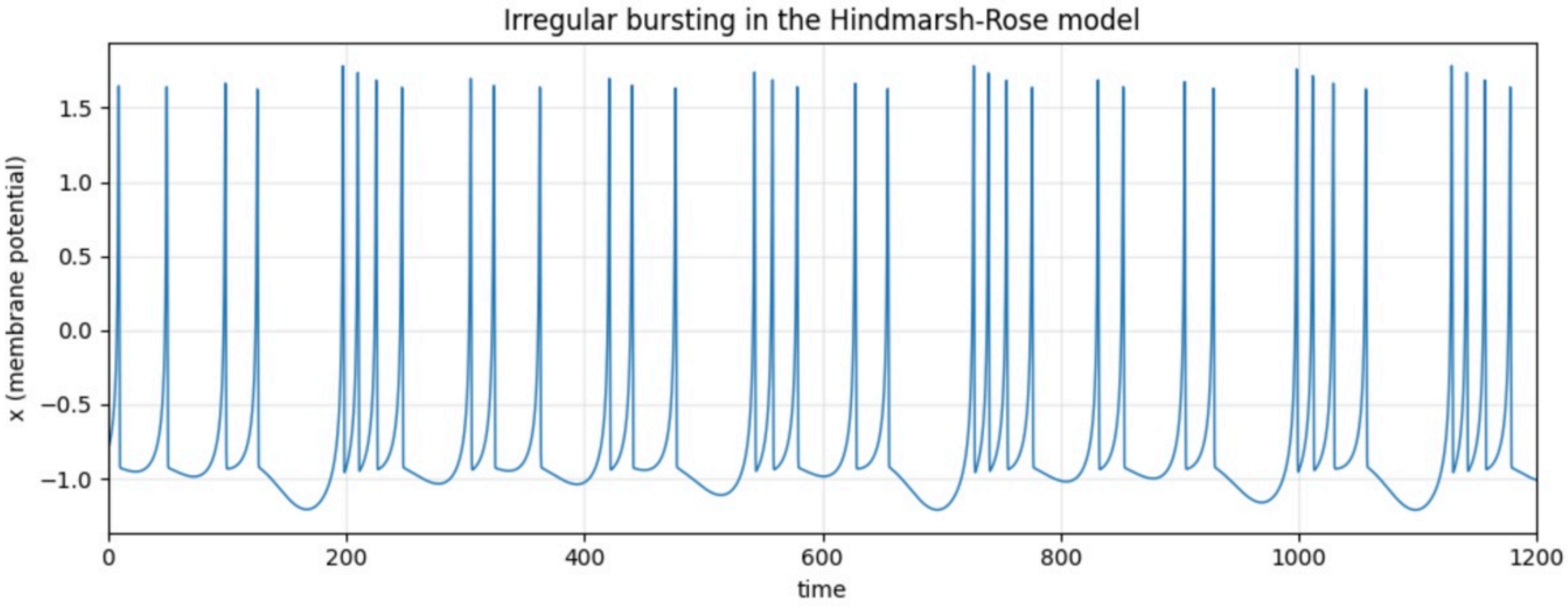
Canonical Fold-homoclinic bursting neuron

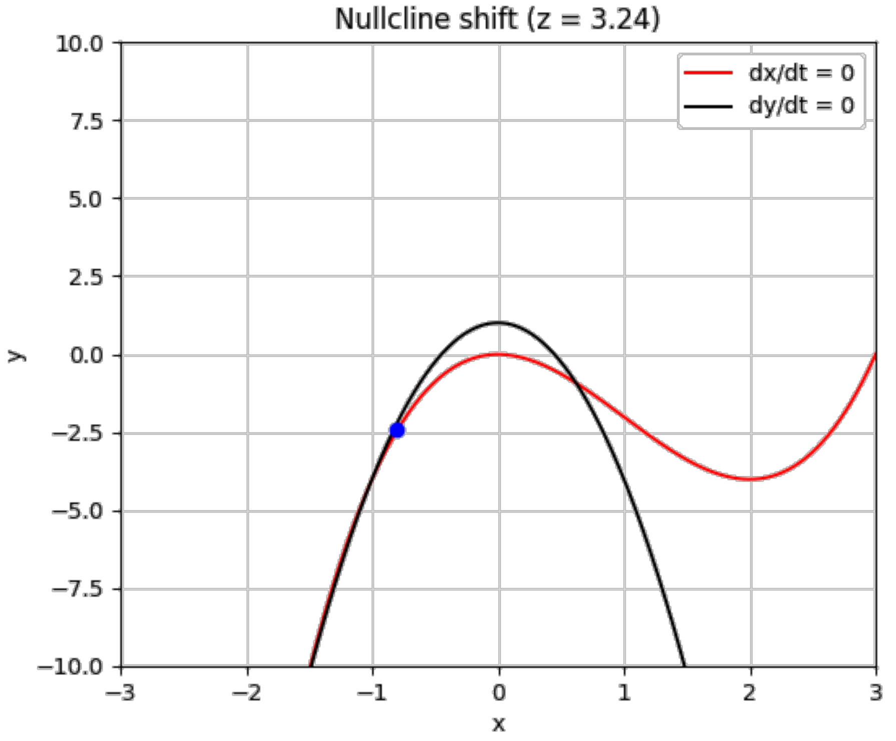


Canonical Fold-homoclinic bursting neuron

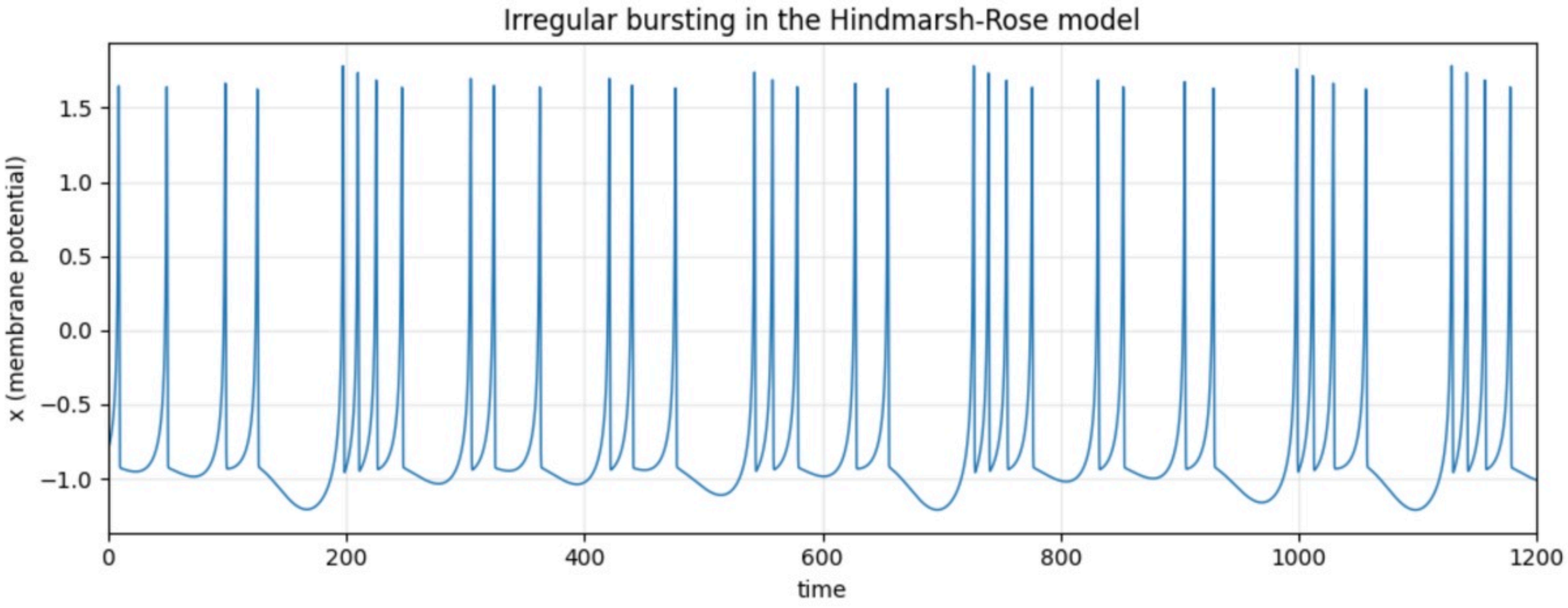


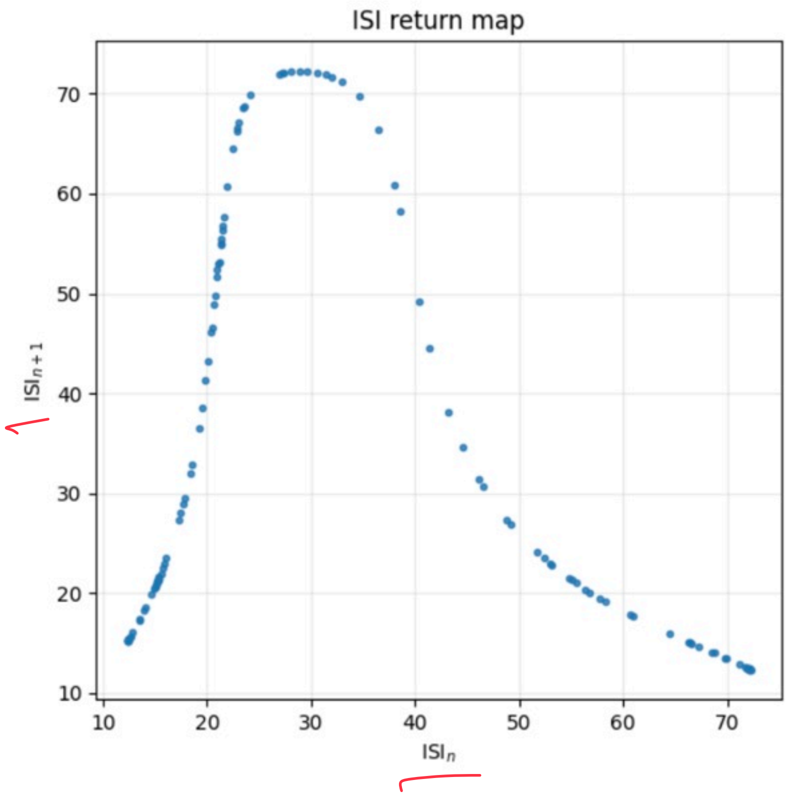
Hindmarsh Rose neuron

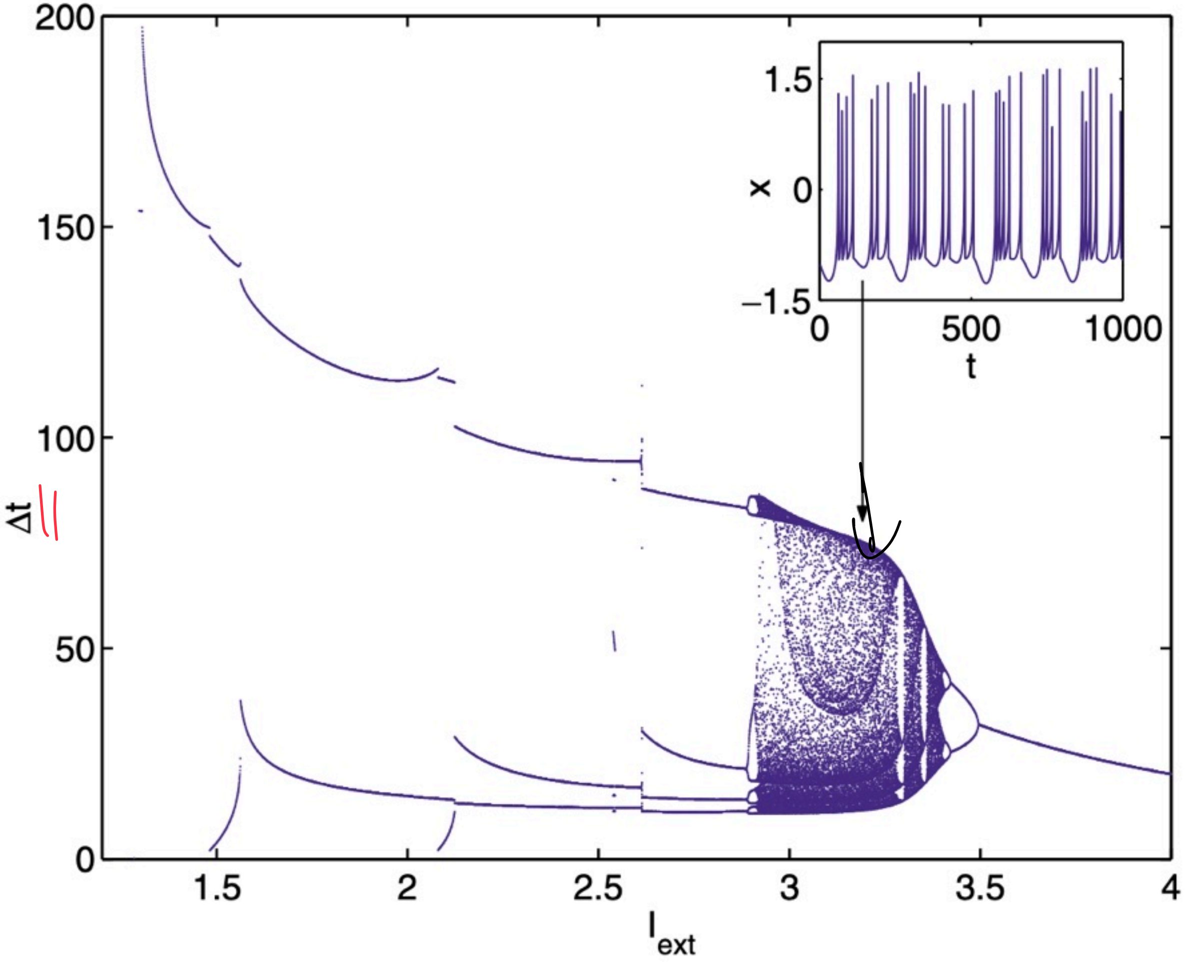




Hindmarsh Rose neuron







VOLUME 92, NUMBER 2 PHYSICAL REVIEW LETTERS week ending 16 JANUARY 2004

Transitions to Synchrony in Coupled Bursting Neurons

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(Received 8 September 2003; revised manuscript received 14 November 2003; published 15 January 2004)

Hyperpolarization-Activated Current Induces Period-Doubling Cascades and Chaos in a Cold Thermoreceptor Model

Kesheng Xu[1], *Jean P. Maidana*[1], *Mauricio Caviedes*[1], *Daniel Quero*[2], *Pablo Aguirre*[2] and *Patricio Orio*[1,3*]

2.1. Mathematical Model

The basis of our model is the Orio et al. (2012) model that reproduces the static firing patterns of cold thermoreceptors. The equation for the membrane voltage V is:

$$C_m \frac{dV}{dt} = -I_{sd} - I_{sr} - I_h - I_d - I_r - I_l , \qquad (1)$$

where C_m is the membrane capacitance; I_d, I_r, I_{sd}, I_{sr} are depolarizing (Na_V), repolarizing (K_{dr}), slow depolarizing (Na_P / Ca_T) and slow repolarizing (K_{Ca}) currents, respectively. I_h stands for hyperpolarization-activated current, and lastly I_l represents the leak current. Currents are defined as:

$$I_i = \rho(T) g_i a_i (V - E_i) \qquad i = d, r, sd, h, l; \qquad (2)$$

$$I_{sr} = \rho(T) g_{sr} \frac{a_{sr}^2}{a_{sr}^2 + 0.4^2} (V - E_{sr}) , \qquad (3)$$

where a_i is an activation term that represents the open probability of the channels ($a_l \equiv 1$), with the exception of a_{sr} that represents intracellular Calcium concentration. Parameter g_i is the maximal conductance density, E_i is the reversal potential and the function $\rho(T)$ is a temperature-dependent scale factor for the current. The activation terms a_r, a_{sd}, and a_h follow the differential equations:

$$\frac{da_i}{dt} = \phi(T) \frac{a_i^{\infty}(V) - a_i}{\tau_i} \qquad i = r, sd, h, \qquad (4)$$

where

$$a_i^{\infty}(V) = \frac{1}{1 + \exp\left(-s_i \left(V - V_i^0\right)\right)}. \qquad (5)$$

On the other hand, a_{sr} follows

$$\frac{da_{sr}}{dt} = \phi(T) \frac{-\eta I_{sd} - \kappa a_{sr}}{\tau_{sr}}. \qquad (6)$$

Finally,

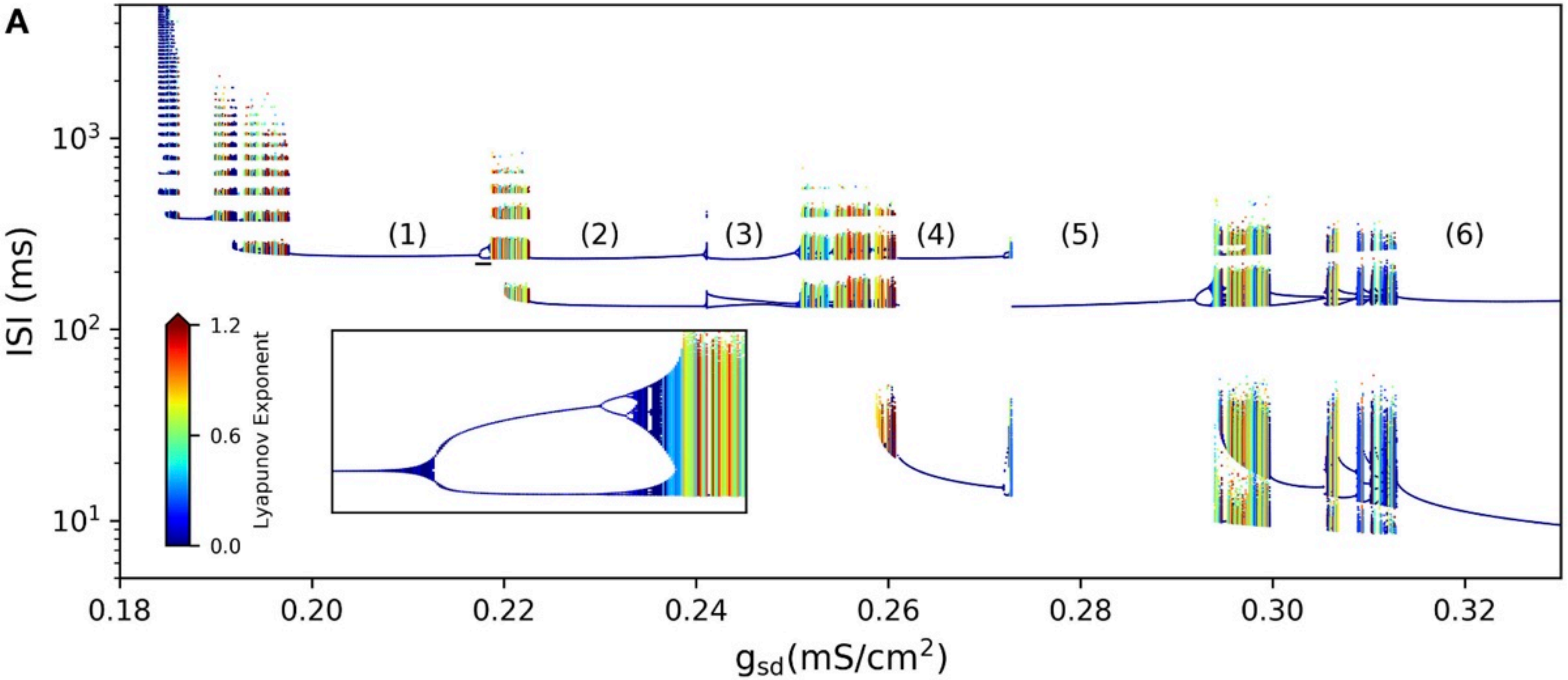
$$a_d = a_d^{\infty} = \frac{1}{1 + \exp\left(-s_d \left(V - V_d^0\right)\right)}. \qquad (7)$$

The function $\phi(T)$ in Equations (4) and (6) is a temperature factor for channel kinetics. The temperature-dependent functions for conductance $\rho(T)$ in Equations (2–3), and for kinetics $\phi(T)$ in Equations (4) and (6) are given, respectively, by:

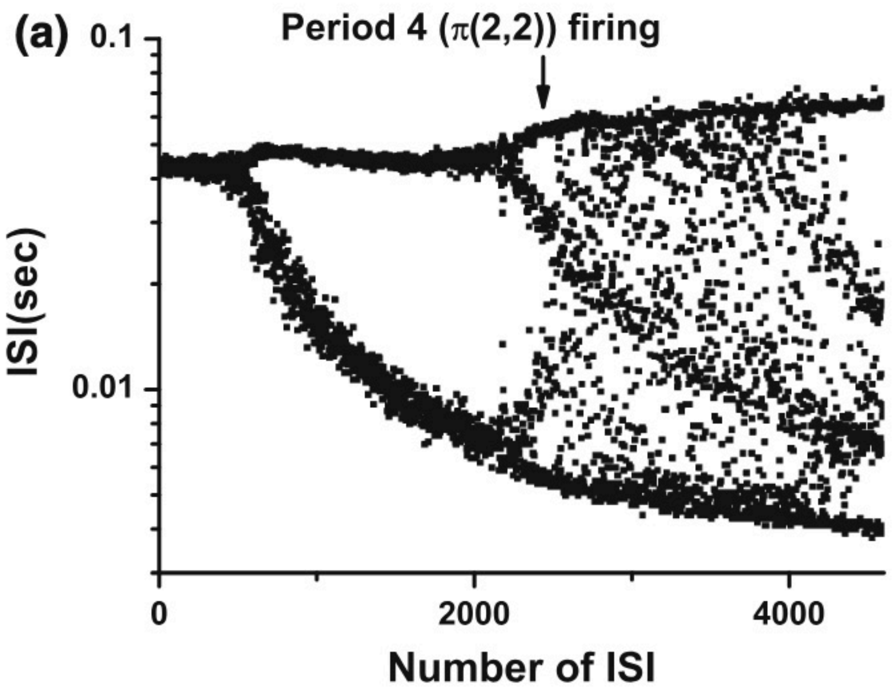
$$\rho(T) = 1.3^{\frac{T-25}{10}} \qquad \phi(T) = 3^{\frac{T-25}{10}}. \qquad (8)$$

TABLE 1 | Parameters of the HB+Ih model.

Parameter	Value	Units
C_m	1.0	$\mu F/cm^2$
g_d	2.5	mS/cm^2
g_r	2.8	
g_{sd}	0.21	
g_{sr}	0.28	
g_l	0.06	
g_h	0.4	
V_d^0	−25	mV
V_r^0	−25	
V_{sd}^0	−40	
V_h^0	−85	
κ	0.18	–
η	0.014	$cm^2/\mu A$
τ_r	2	ms
τ_{sd}	10	
τ_{sr}	35	
τ_h	125	
s_d	0.25	mV^{-1}
s_r	0.25	
s_{sd}	0.11	
s_h	−0.14	
E_d,E_{sd}	50	mV
E_r,E_{sr}	−90	
E_l	−80	
E_h	−30	



A
ISI (ms)
10³
10²
10¹
(1)
(2)
(3)
(4)
(5)
(6)
1.2
0.6
0.0
Lyapunov Exponent
0.18
0.20
0.22
0.24
0.26
0.28
0.30
0.32
g_sd(mS/cm²)



Cogn Neurodyn (2012) 6:89–106
DOI 10.1007/s11571-011-9184-7

RESEARCH ARTICLE

Dynamics of period-doubling bifurcation to chaos in the spontaneous neural firing patterns

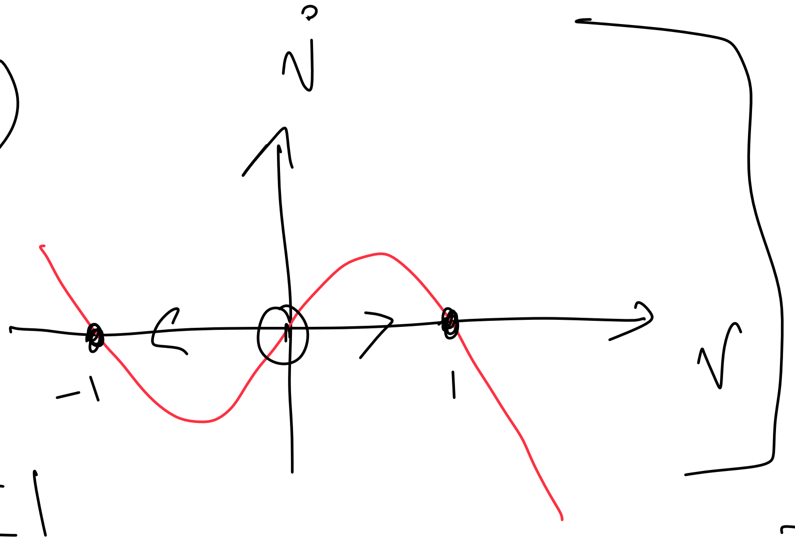
Bing Jia · Huaguang Gu · Li Li · Xiaoyan Zhao

Hopfield Networks

↳ N-neuron model

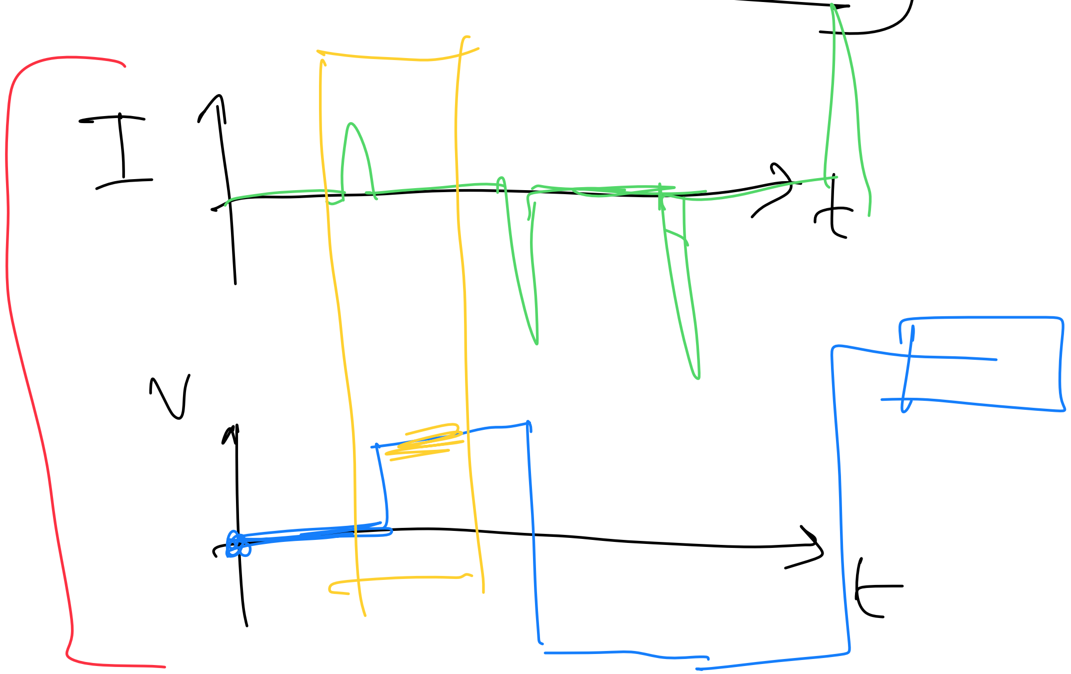
→ Model for associative memory / content-addressable Memory

$$\dot{v} = f(v)$$

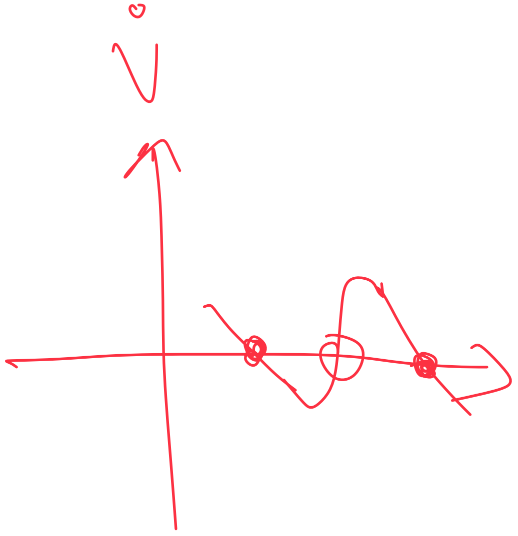


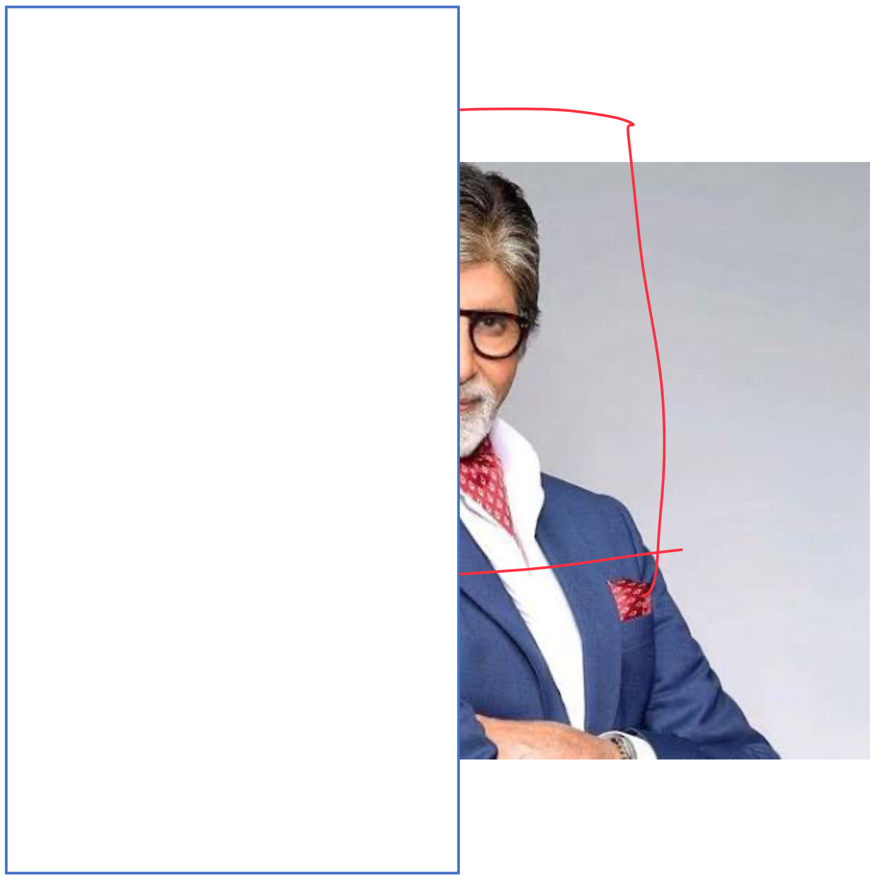
↳ $S_i = \pm 1$

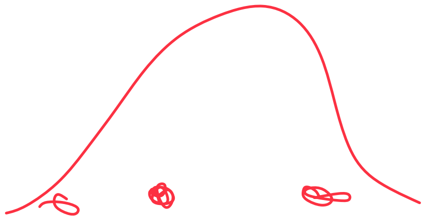
$$s(t) = \text{Sgn}[\text{Inputs}]$$



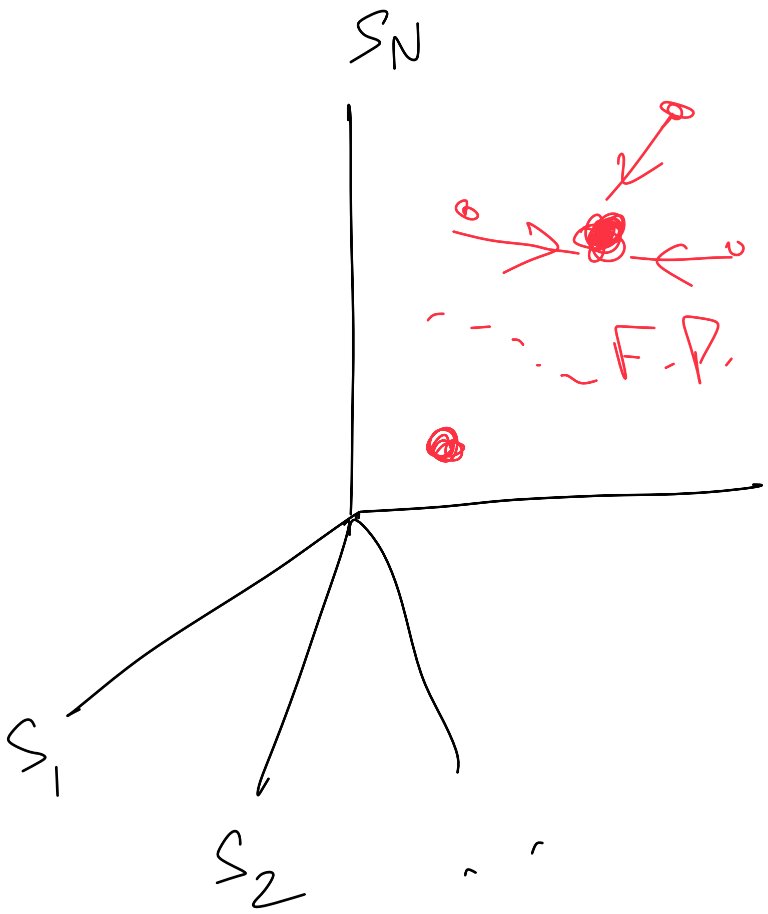
$S_1, S_2 \ldots\ldots S_N$







Exact memory



$$s_i(t) = \text{sgn}[\text{Inputs}]$$

$$= \text{sgn}\left[h_i(t-1) \right] \leftarrow$$

$$\parallel$$

$$S_i(t) = \text{sgn}\left[\sum_j W_{ij} S_j(t-1) \right] \leftarrow$$

$S_i \in \{-1, 1\}$

$\uparrow$ $\downarrow$

$+1$ -1

$$H[\vec{s}] = \sum s_i H_{ij} s_j$$

Zero T, Classical

Hopfield $\longleftrightarrow$ Ising model

$$s_i(t) = \text{sgn}\left[\sum w_{ij}\, s_j(t-1)\right]$$

$$\left[w_{ij} = w_{ji} \;;\; w_{ii} = 0 \right]$$

synchronous updates → $\bar{s}(t) = \text{sgn}\left[W \bar{s}(t-1) \right]$ → Map

Pick i

asynch updates $\quad s_i(t) = \text{sgn}[\quad] \Big\}$ N - times $\Big]$ Map

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad s(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

syn → $s(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $s(2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

↳ periodic orbit

async → $s(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$0 \cdot (-1) + 1 \cdot 1 = 1 \rightarrow s(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ → F.P.

Energy func^n

$$E[\bar{s}] = -\frac{1}{2} \sum_{ij} w_{ij} s_i s_j$$

Goal → Async. updates reduce E ←

$E[\bar{s}]$ → Lower bound ←

Assume updated s_k

$$E = -\frac{1}{2} \sum_{ij} w_{ij} s_i s_j = -\frac{1}{2} \sum_{\substack{i \neq k \\ j \neq k}} w_{ij} s_i s_j - \frac{1}{2} \sum_{i \neq k} w_{ik} s_i s_k$$

$$-\frac{1}{2} \sum_{j \neq k} w_{kj} s_k s_j \quad -\frac{1}{2} w_{kk} s_k^2$$

0

$$E = C - \frac{1}{2}\sum_{i \neq k} W_{ik} S_i S_k \qquad - \frac{1}{2}\sum_{j \neq k} W_{kj} S_k S_j$$

$W_{kj} \downarrow W_{jk}$

$$= C - \sum_{i \neq k} W_{ik} S_i S_k$$

$$= C - S_k \underbrace{\sum_{i \neq k} W_{ik} S_i}$$

Recall
$$S_k(t+1) = \text{sgn}\left[h_k(t)\right]$$

$$E(t+1) - E(t) = - \underbrace{\left(\sum_i W_{ik} S_i\right)}_{h_k} \underbrace{\left(S_k(t+1) - S_k(t)\right)}$$

if $h_k > 0 \Rightarrow [S_k(t+1) - S_k(t)] > 0 \Rightarrow \Delta E < 0$

$h_k < 0 \Rightarrow [\quad] < 0 \Rightarrow \Delta E < 0$

If E changes $\Rightarrow \Delta E < 0$

Either already at F.P.

or E will dec. → eventually go to F.P.

How to choose F.P.s?

↳ W_{ij} ?

Single memory $\bar{\xi}$

↳ $\xi_i = \text{sgn}\left[\sum_j W_{ij} \xi_j \right]$

$\xi_i \in \{-1, 1\}$

$$\xi_i = \text{sgn}\left[\sum w_{ij} \xi_j\right] \leftarrow$$

$$\xi_i \in \{-1, 1\}$$

$$w_{ij} = \xi_i \xi_j \Rightarrow \sum_j w_{ij} \xi_j$$

$$= \sum_j \xi_i \xi_j \xi_j$$

$$= \xi_i \cdot N$$

$$\Rightarrow \text{sgn}[\quad] = \text{sgn}[\xi_i] = \xi_i \Rightarrow \text{F.P.}$$

$$S_i = \operatorname{sgn}\left[\sum w_{ij} S_j\right]$$

$$W = \xi \xi^T$$

$$\bar{S}(0) = \bar{S}$$

$$S(1) = \operatorname{sgn}\left[\xi \underbrace{\xi^T S}\right]$$

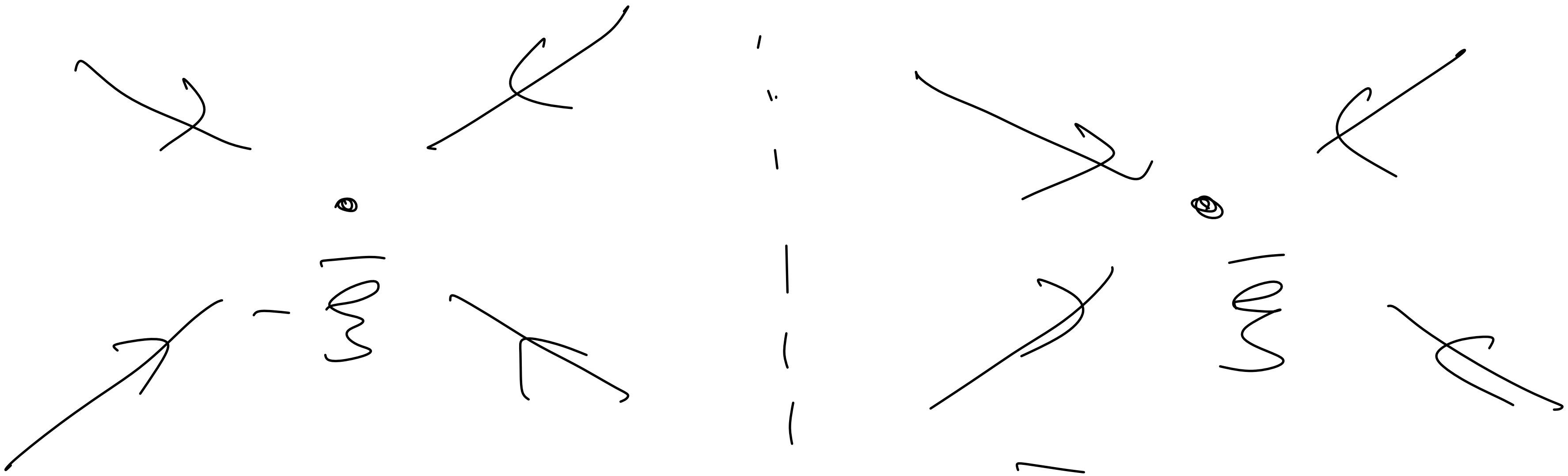
if $\xi^T S > 0$

$$= \operatorname{sgn}[\xi]$$

$$S(1) = \bar{\xi}$$

if $\xi^T S < 0$

$$S(1) = -\bar{\xi}$$



$\underbrace{\xi \cdot \bar{s}} < 0$ | $\underbrace{\bar{\xi} \cdot \bar{s}} > 0$

spurious fixed pts → F.P.s that I didn't store

→ $W_{ij} = \xi_i \xi_j$

Many memories → ξ^μ

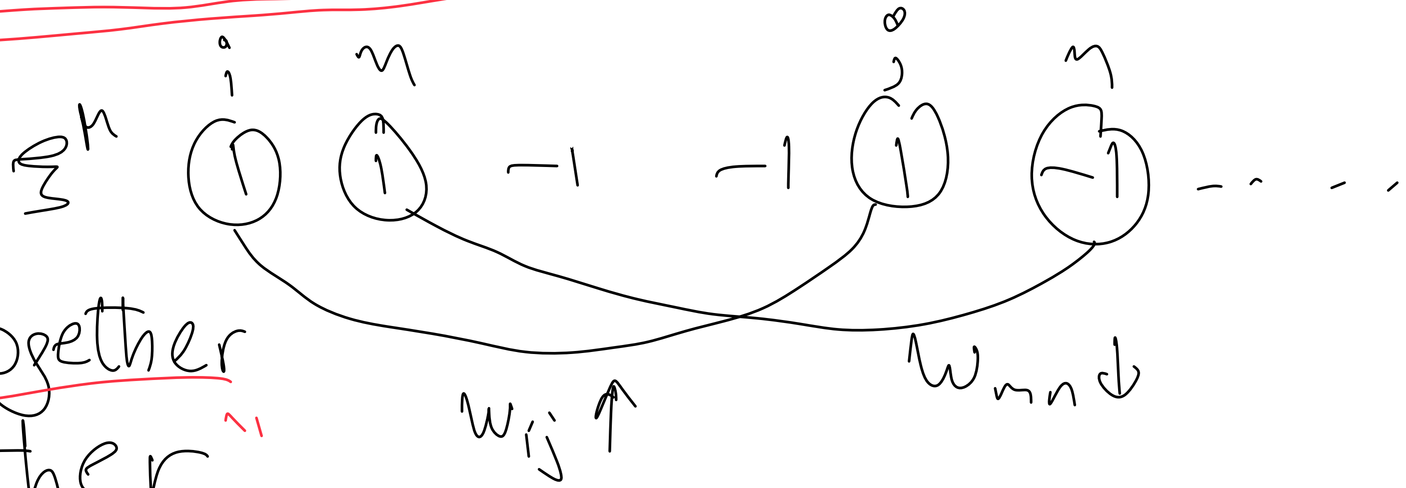
$w_{ij}(0) = 0$

$$w_{ij} = \frac{1}{N}\sum_{\mu} \xi_i^\mu \xi_j^\mu \rightarrow \Delta w_{ij} = \eta\, \xi_i^\mu \xi_j^\mu$$

Hebb rule

↓

"Neurons that fire together wire together"



Examine if ξ^ν is a F.P.

Want $\xi_i^\nu = \text{sgn}\left[\sum_j w_{ij}\xi_j^\nu\right]$

$$w_{ij}\xi_j^\nu = \frac{1}{N}\sum_{\mu,j}\xi_i^\mu \xi_j^\mu \xi_j^\nu$$

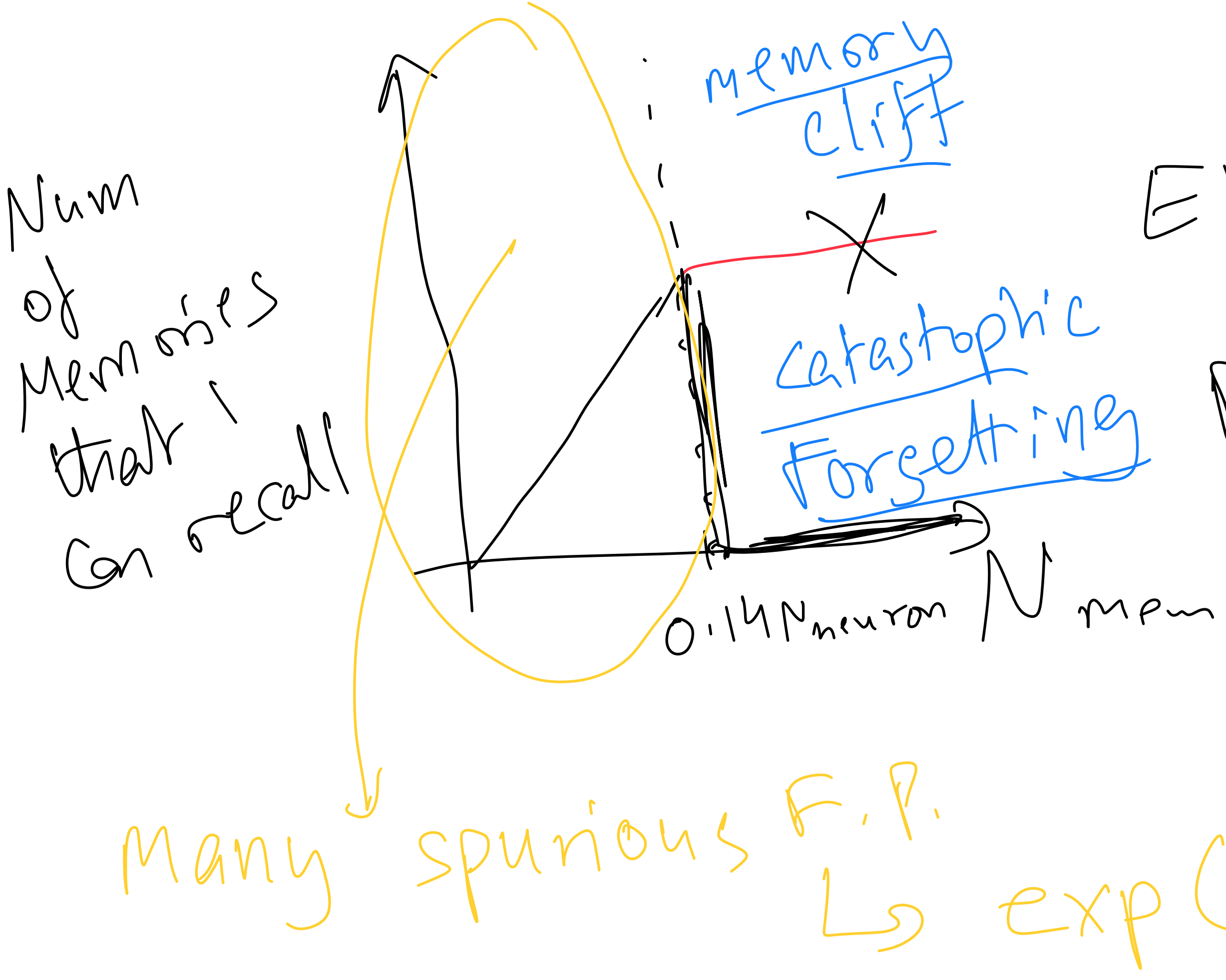
$$= \frac{1}{N}\sum_{\mu\neq\nu}\sum_j \xi_i^\mu \xi_j^\mu \xi_j^\nu + \frac{1}{N}\xi_i^\nu N$$

Want $\xi_i^\nu \approx \xi_i^\nu + \frac{1}{N}\sum \xi_i^\mu (\underline{\xi}^\mu \cdot \underline{\xi}^\nu)$

(interference) (crosstalk)

Upper bound on number of memories

$$N_{mem} \lesssim 0.14 \cdot N_{neurons}$$

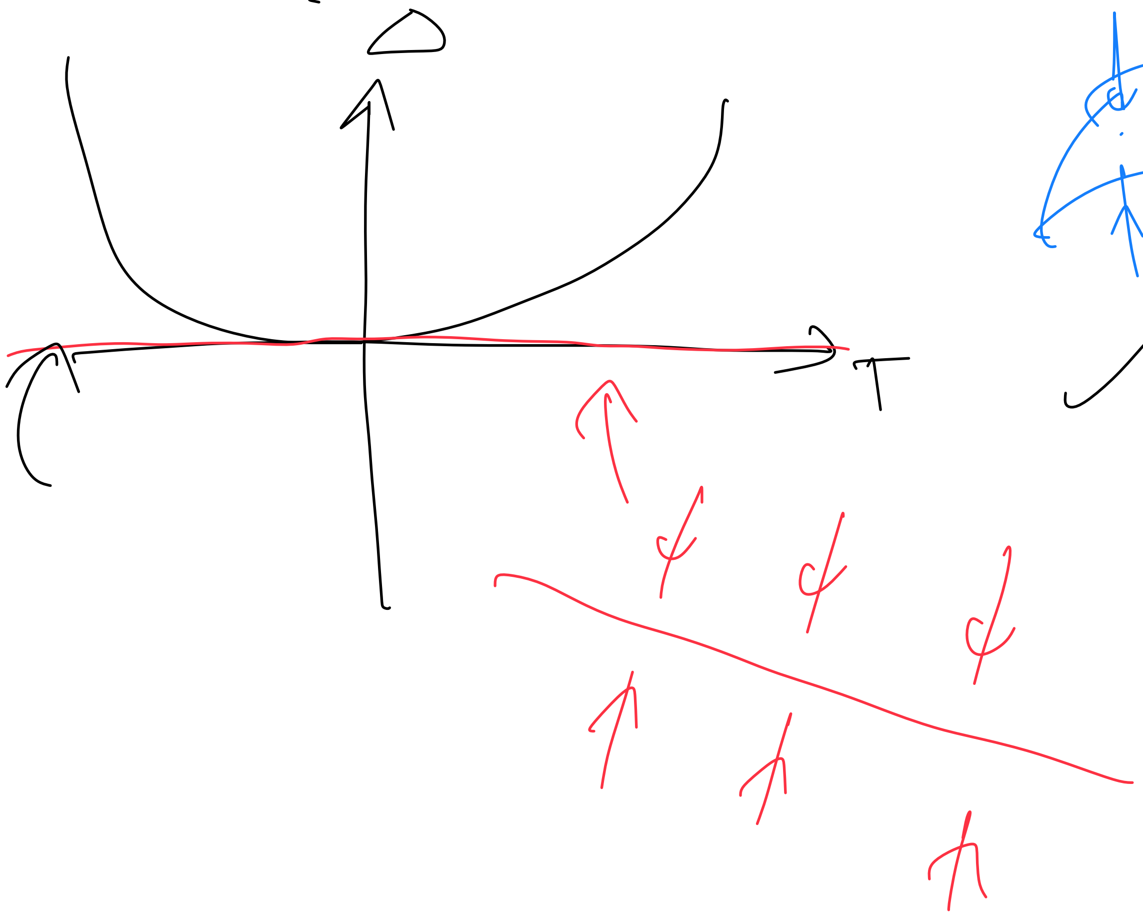


~ 1980s

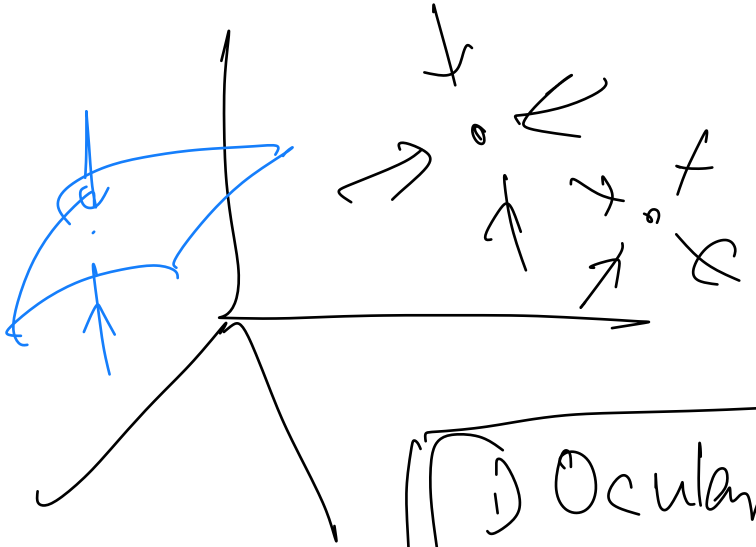
Elizabeth Gardener
Daniel Amit
H. Sompolinsky
Abu Mostafa

2026-06-11

Continuous Attractor



Hopfield



Discrete attractors

1) Ocular Motor Integrator
2) Head Direct^n Cells
3) Grid Cells

Ocular motor integrator

RESEARCH ARTICLE | NEUROSCIENCE |

How the brain keeps the eyes still

H. S. Seung Authors Info & Affiliations

November 12, 1996 | 93 (23) 13339-13344 | https://doi.org/10.1073/pnas.93.23.13339

$$\tau \frac{ds_i}{dt} = -s_i + \phi\left[\sum_j W_{ij} s_j + b\right]$$

$$\tau \dot{s} = -s + Ws + b$$

$$\tau \dot{s} = (W - \mathbb{1}) s + b$$

$$(W - \mathbb{1}) s = -b$$

→ if $(W-\mathbb{1})$ is singular → degenerate sol^{n}s

W has an eigenvalue 1

$Wv = v$

Ocular motor integrator

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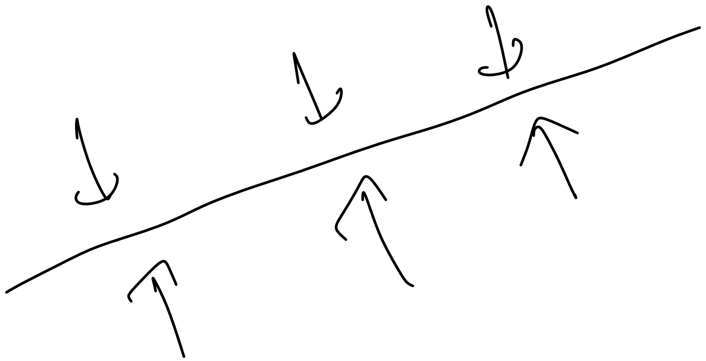
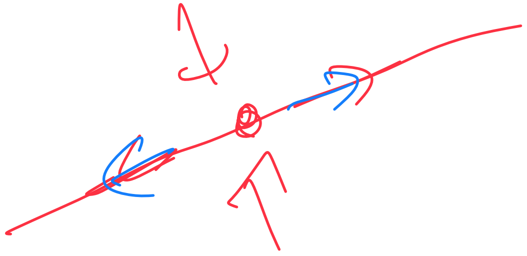
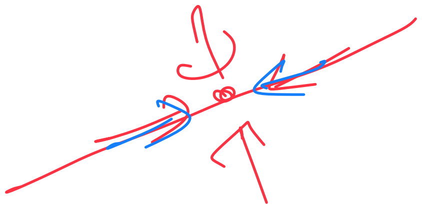


$\lambda = 1$



$\lambda \neq 1$

$\lambda = 1 \pm 10^{-3}$

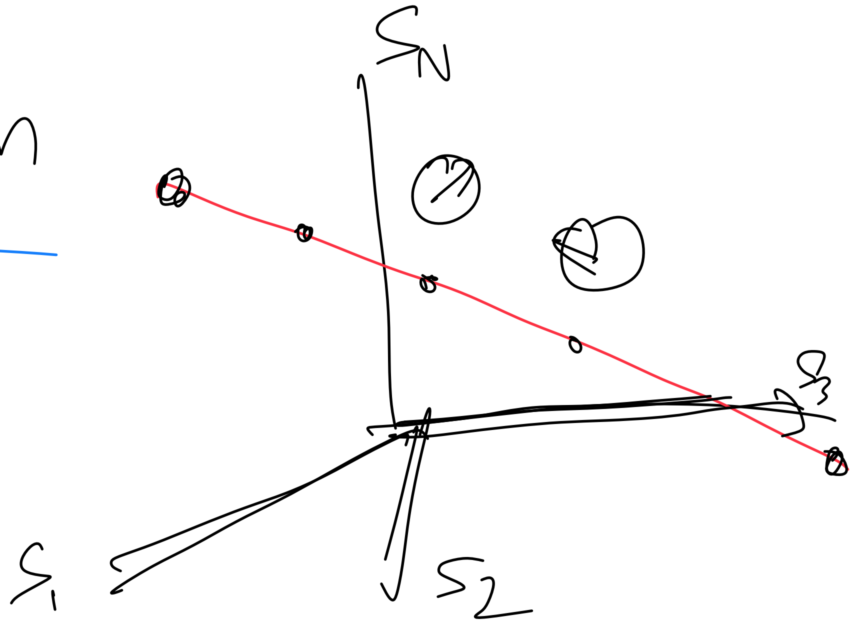


$Wv = v$

$(W - \mathbb{I})\, s = -b$

if s_0 is a soln.

$\Rightarrow \quad s = s_0 + \alpha v$ is also a solution



Ocular motor integrator

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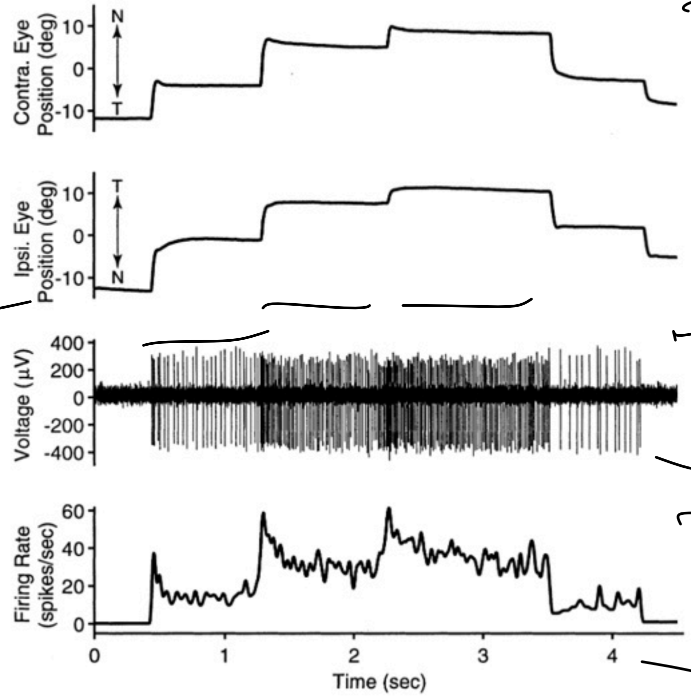
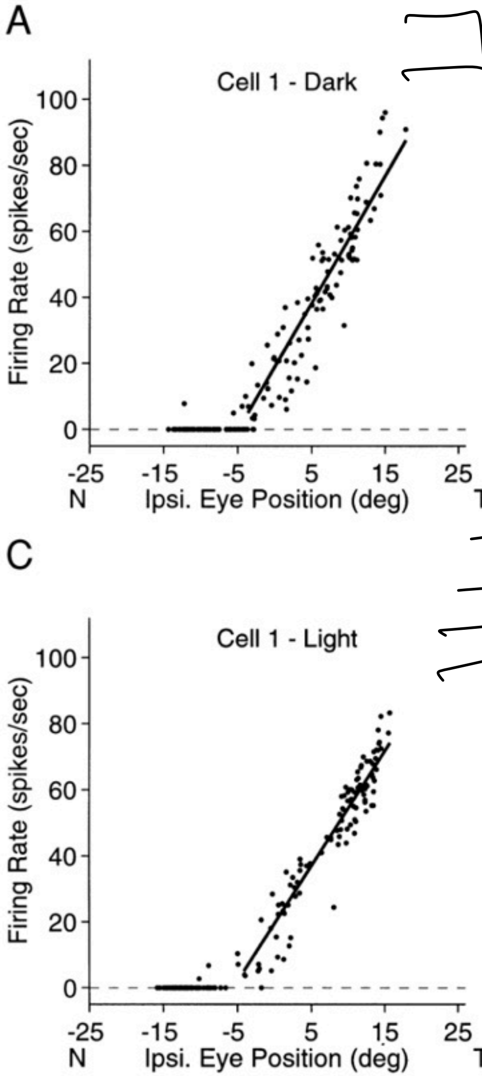
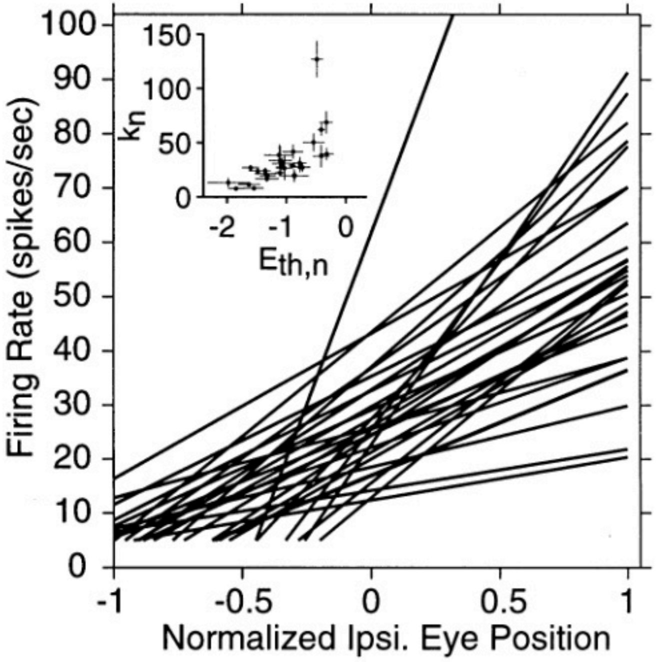


Anatomy and Discharge Properties of Pre-Motor Neurons in the Goldfish Medulla That Have Eye-Position Signals During Fixations

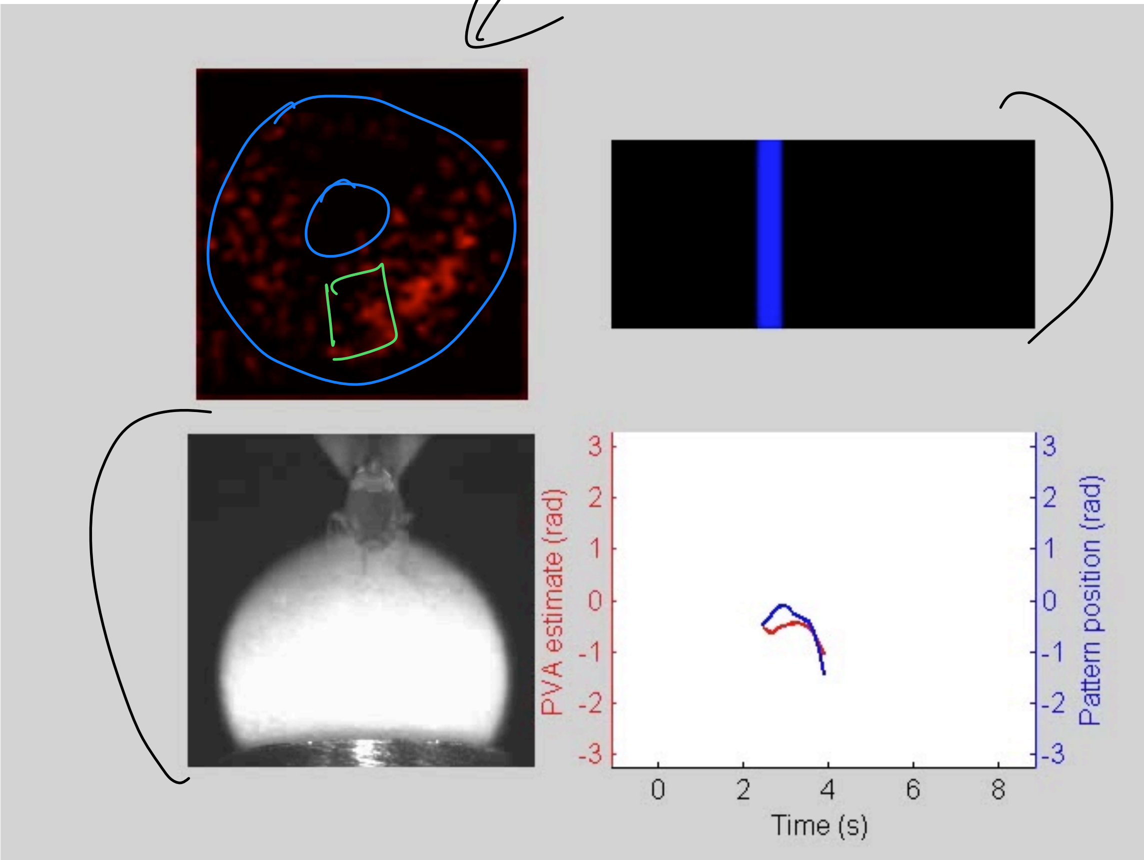
Authors: E. Aksay, R. Baker, H. S. Seung, and D. W. Tank | AUTHORS INFO & AFFILIATIONS

Publication: Journal of Neurophysiology • Volume 84, Issue 2 • https://doi.org/10.1152/jn.2000.84.2.1035



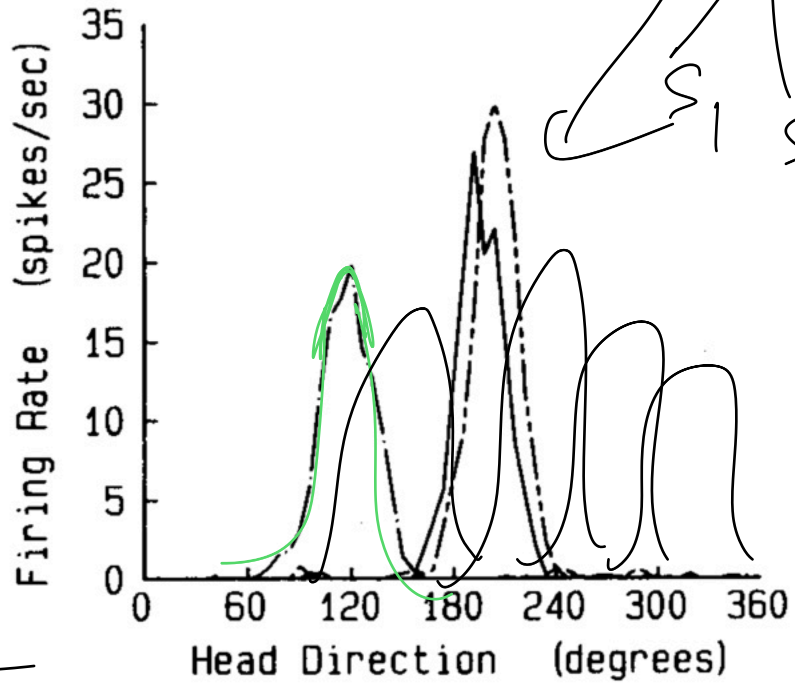


HD cells



Neural dynamics for landmark orientation and angular path integration

Johannes D. Seelig[1] & Vivek Jayaraman[1]



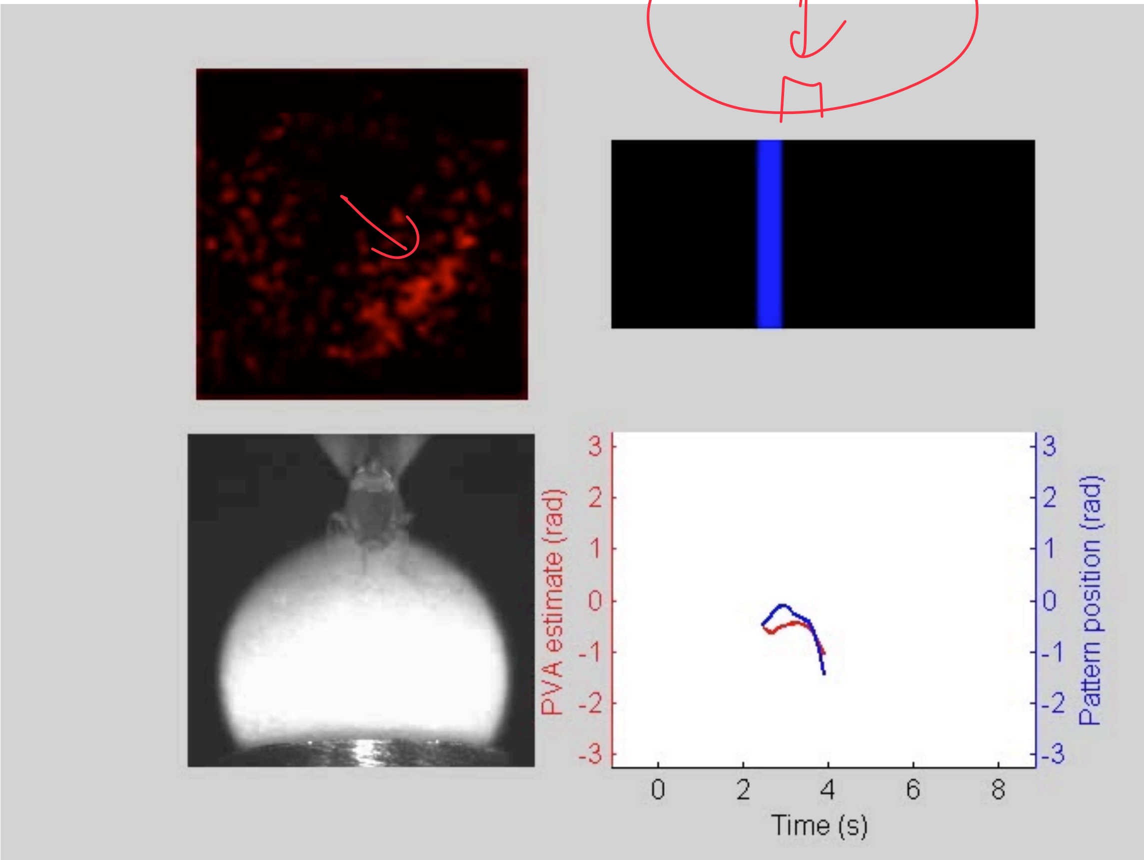
Head-direction cells recorded from the postsubiculum in freely moving rats.

JS Taube, RU Muller, and JB Ranck Jr

Journal of Neuroscience 1 February 1990, 10 (2) 436-447; https://doi.org/10.1523/JNEUROSCI.10-02-00436.1990

HD cells

$\dot{x} = F(x)$



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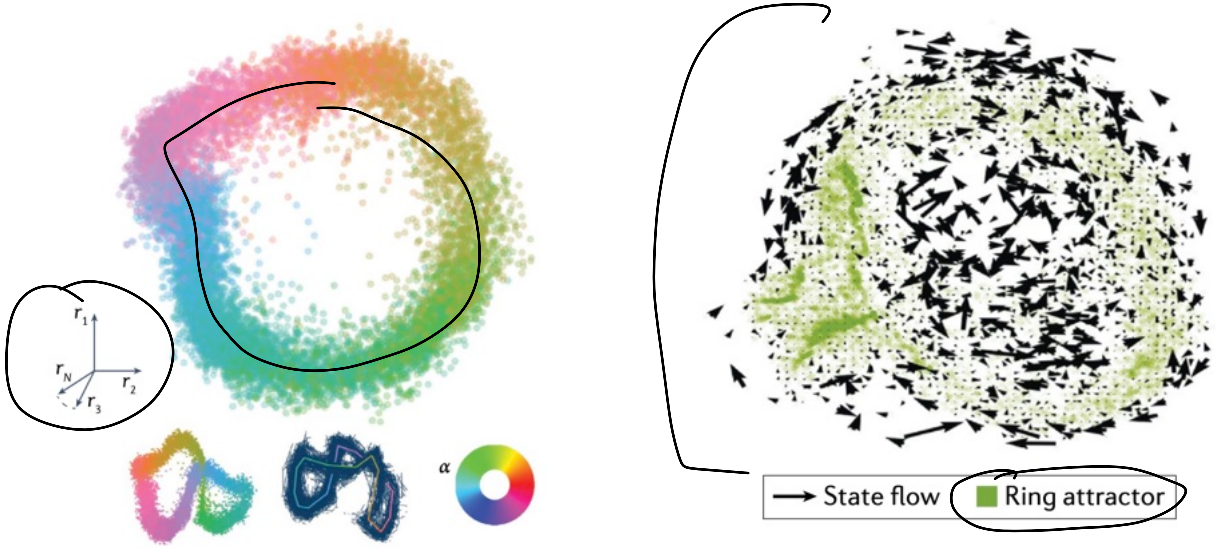


The intrinsic attractor manifold and population dynamics of a canonical cognitive circuit across waking and sleep

Rishidev Chaudhuri, Berk Gerçek, Biraj Pandey, Adrien Peyrache & Ila Fiete

Nature Neuroscience **22**, 1512–1520 (2019) | Cite this article

$$\dot{\bar{s}} = \bar{F}(\bar{s}) \qquad \bar{x} \in \mathbb{R}^n$$

$$S_i$$

Field Model

$s(x)$

$$\tau \frac{\partial s(x,t)}{\partial t} = -s + \phi\left(\int_0^{2\pi} W(x,x')\, s(x')\, dx' + B\right)$$

$$W(x,x') = W(|x'-x'|)$$

↳ F.P.s exist.

MD cells with Finite neurons
Norman et al 2024

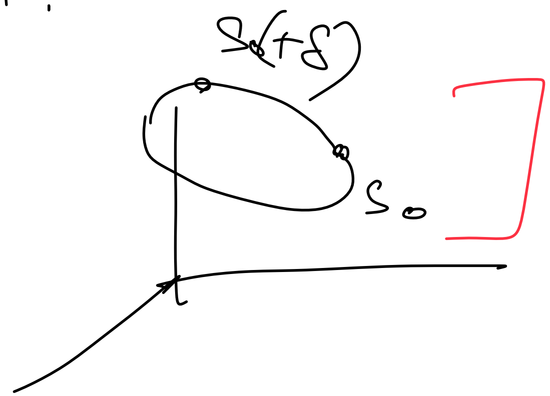
Biswas et al 2026

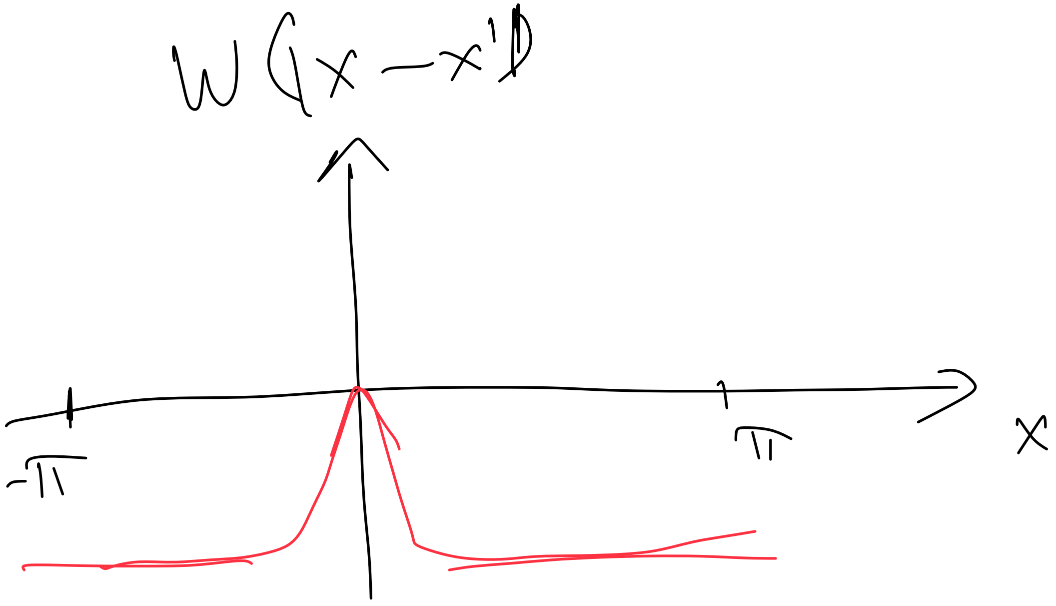
x is on a circle

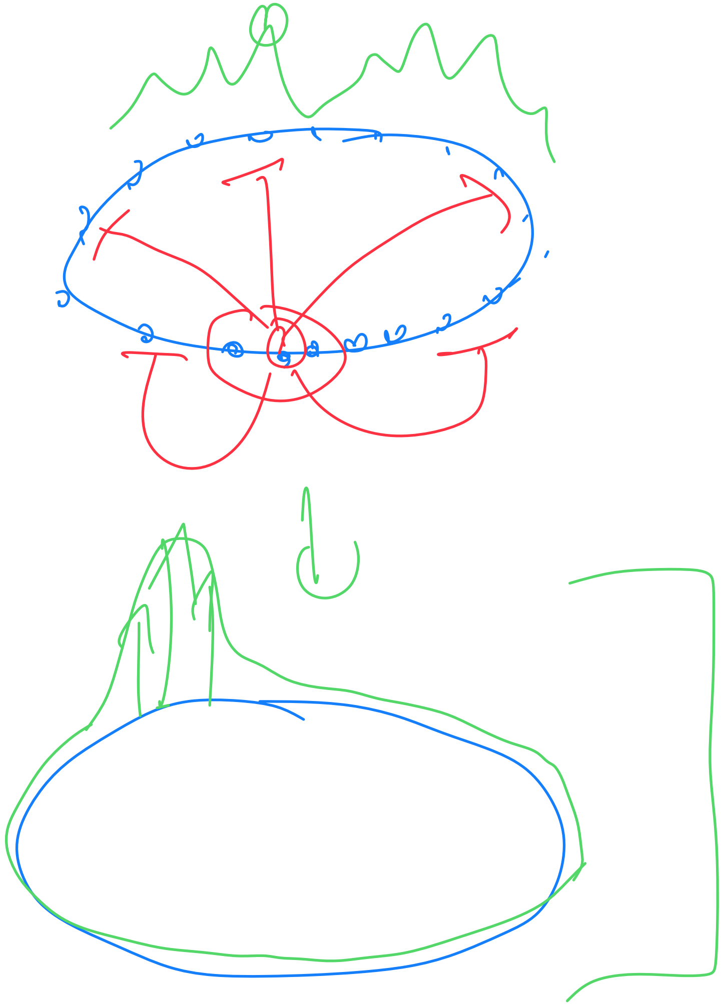
$$s(x+2\pi) = s(x)$$

$S_0(x)$ is a F.P.

$S_0(x+\delta)$ will also be a F.P.







$$S_0(x) + \varepsilon(x,t)$$

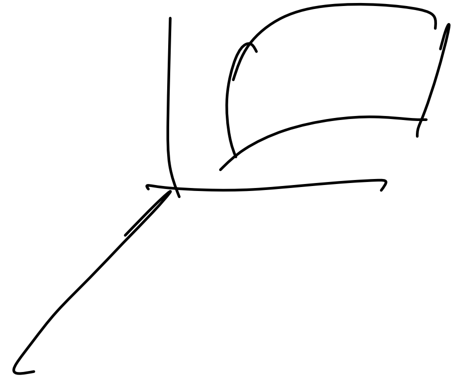
$$\tau \frac{\partial \varepsilon(x,t)}{\partial t} = -\varepsilon + \phi'(\;) \int W \, \varepsilon(x') \, dx'$$

$$\frac{\partial \varepsilon}{\partial t} = \mathcal{L}\left[\varepsilon(x,t)\right] \longrightarrow \mathcal{L}$$ has a zero e.value

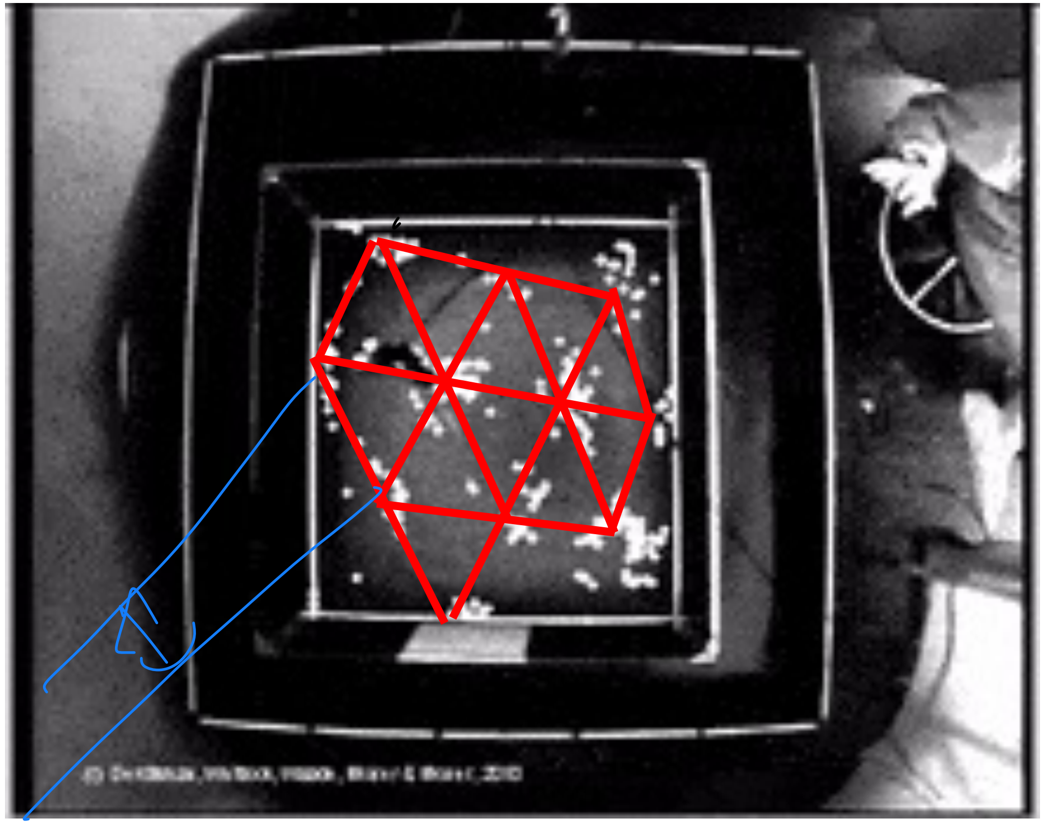
HD cells → represent a pt on a circle

→ circle in neural state space

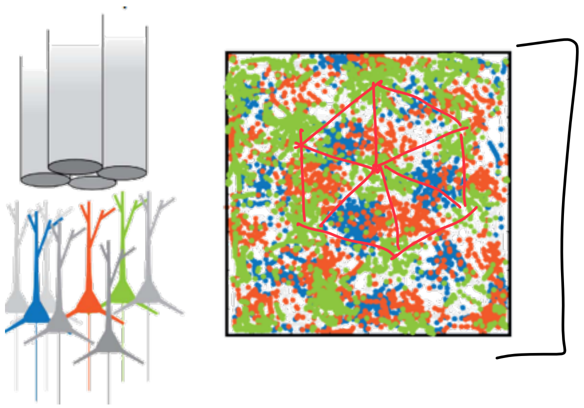
Grid cells → represent a pt. on $\mathbb{R}^2$

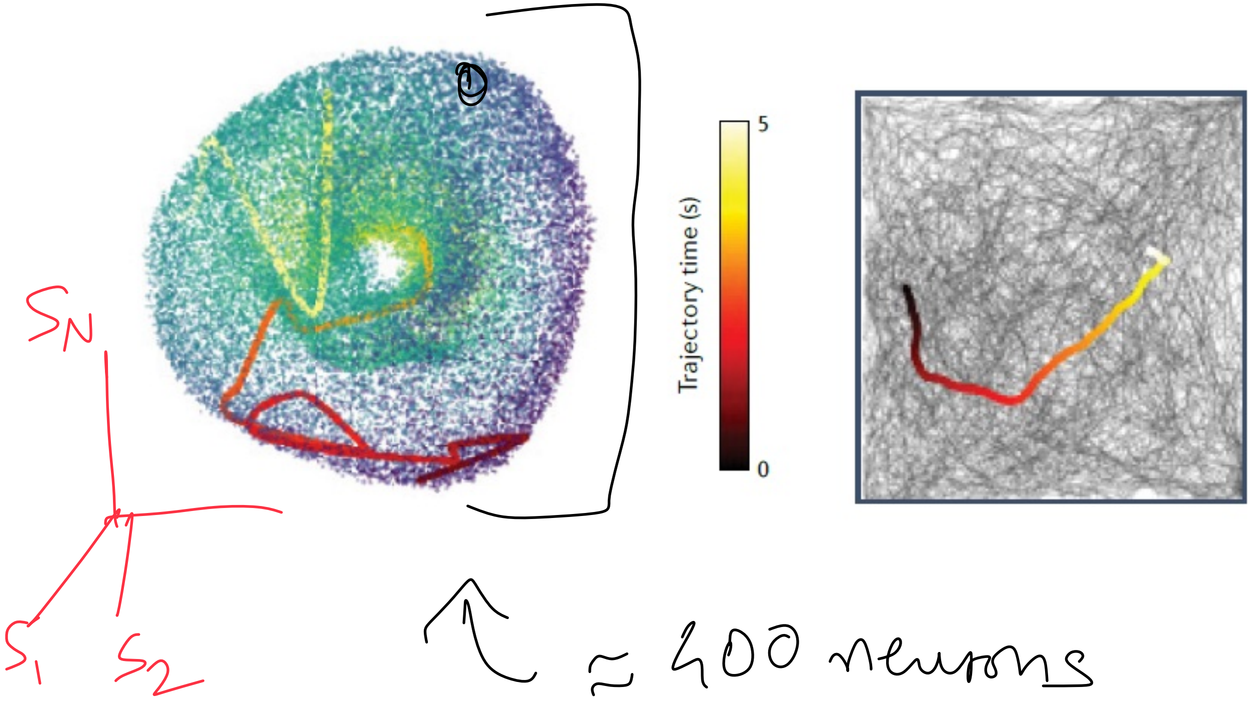


Grid cells



Derdikman et al. 2010

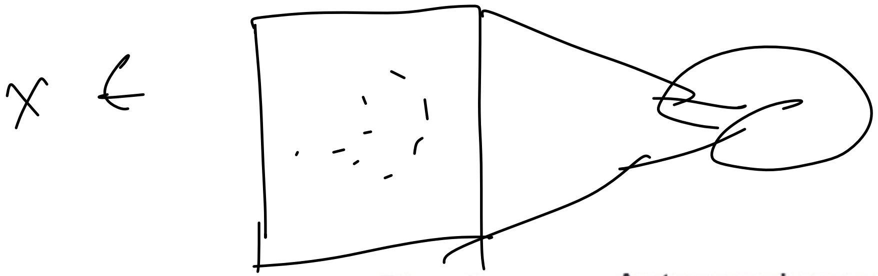


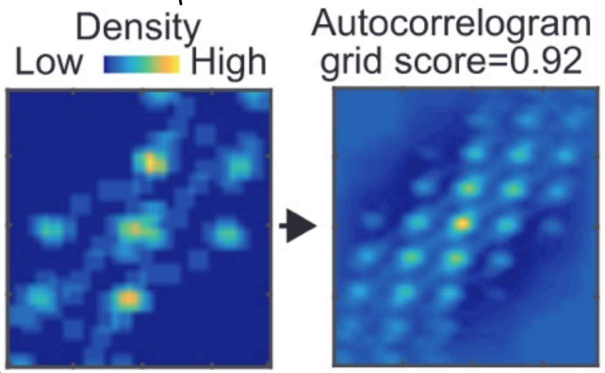


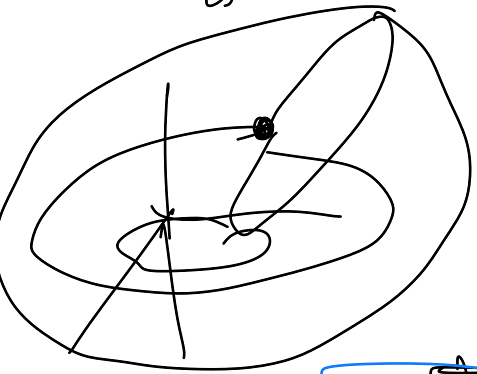
$s(x)$

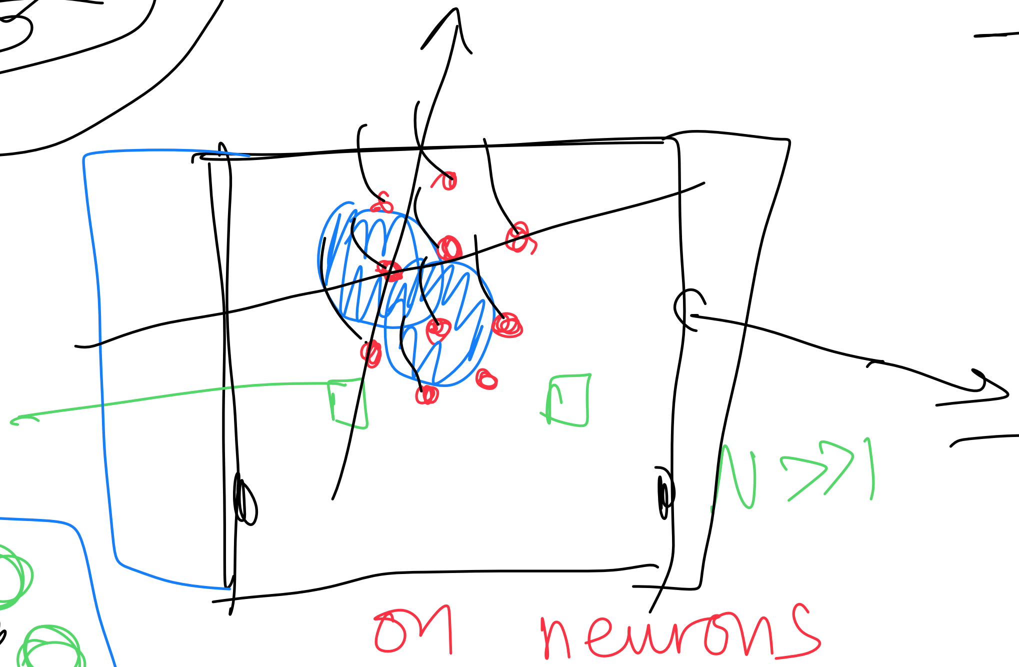
$$\tau \frac{\partial s(x,t)}{\partial t} = -s + \phi\left[\int W(x,x')\, s(x')\, dx' + B\right]$$

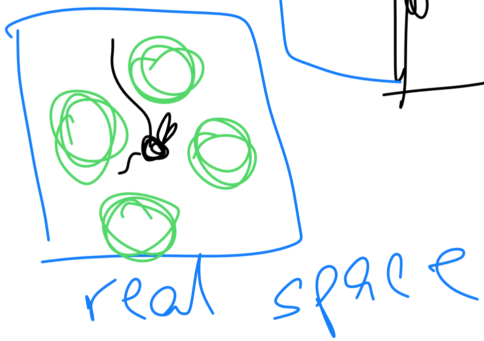
$$W(x,x') = W(|x-x'|)$$

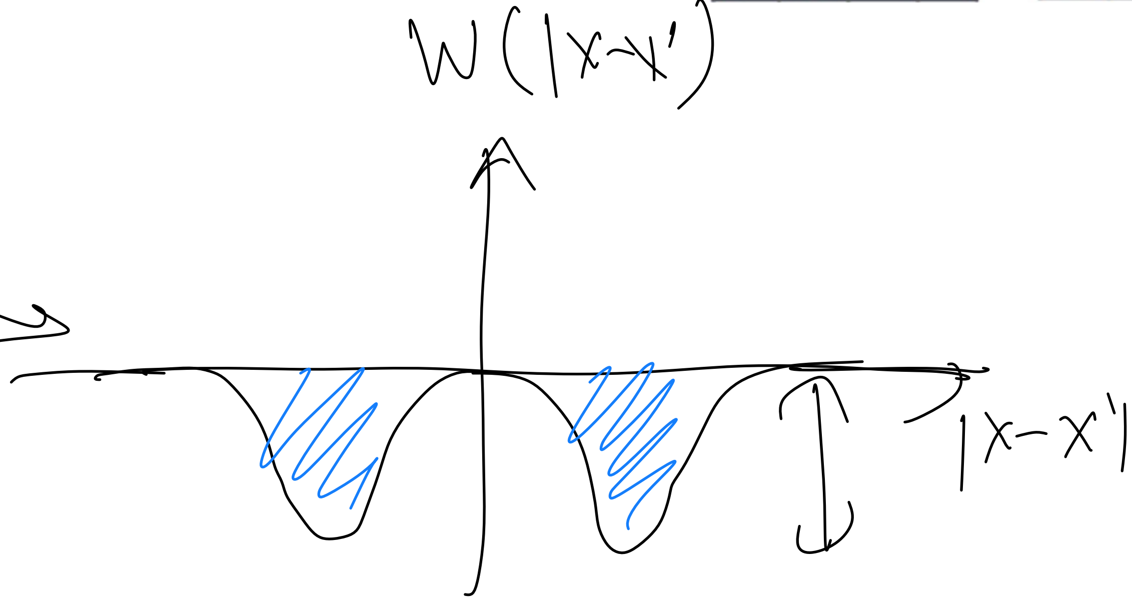












$$\tau \frac{\partial s}{\partial t} = -s + \phi\left(\iint w(|\bar{x}-\bar{x}'|)\, s(\bar{x}')\, d\bar{x}' + B\right)$$

If there is a nontrivial $S_0(\bar{x})$ F.P.

$S_0(\bar{x}+\bar{\xi})$ is also a F.P.

what is the trivial F.P.

$$0 = -S_0 + \phi\left(S_0 \underbrace{\iint w}_{\bar{w}} + B\right)$$

$$S_0 = \frac{B}{1-\bar{w}}$$

$$S(x,t) = S_0 + \varepsilon(x,t)$$

$$\tau \frac{\partial \varepsilon}{\partial t} + \varepsilon = \phi'(\ ) \left[\int W(x-x')\, \varepsilon(x')\, dx' \right]$$

$$\varepsilon(x,t) = \varepsilon \cdot e^{ikx + \alpha(k) t}$$

$$\tau \alpha(k) \varepsilon e^{(ikx)} + \varepsilon e^{(ikx)} = \varepsilon e^{(\ )} \int W(x-x')\, e^{ikx'}\, dx'$$

$$\left[\int W(x-x')\, e^{-ik(x-x')}\, dx' \right.$$

$$= \tilde{W}(k)$$

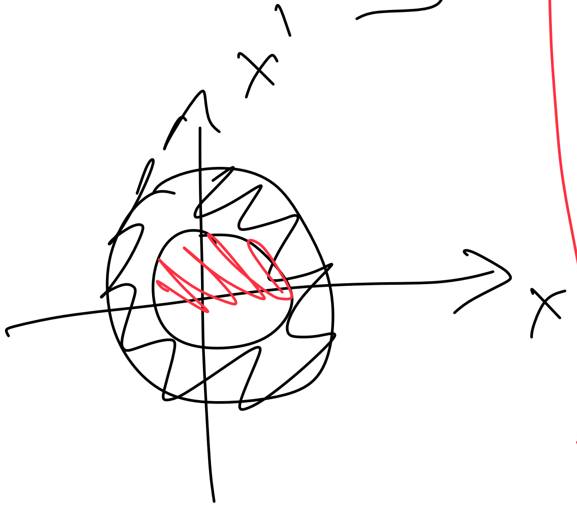
$$\tau \alpha(k) + 1 = \tilde{w}(k)$$

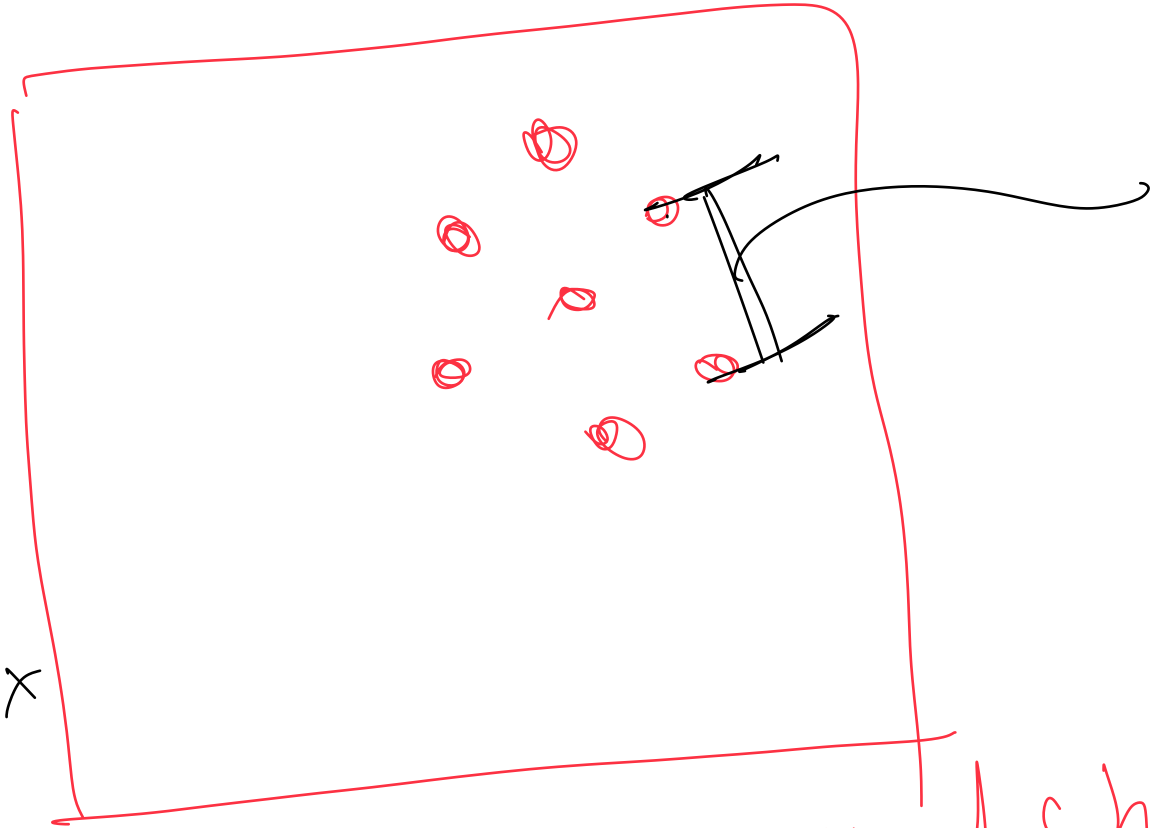
$$\alpha(k) = -\frac{1}{\tau} + \frac{1}{\tau}\tilde{w}(k)$$

→ Pattern format^n at wavenumber

$$k^* = \arg\max\left[\tilde{w}(k)\right]$$

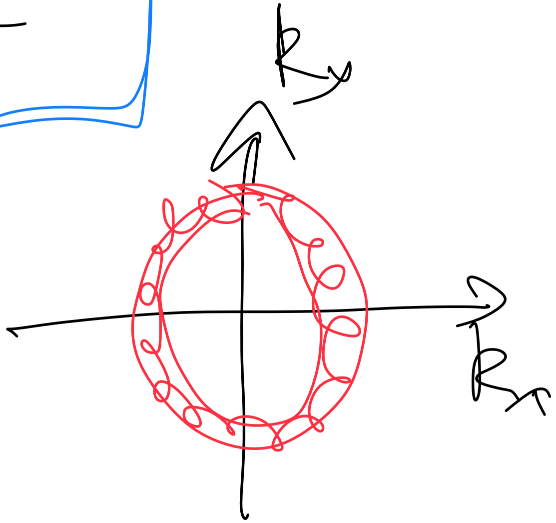
Khona Chandra Fiete (2025)





$$\frac{2\pi}{k^*}$$

neural sheet



Attractors in the Brain

Khona & Fiete (2022)