Dynamical Systems in Neuroscience

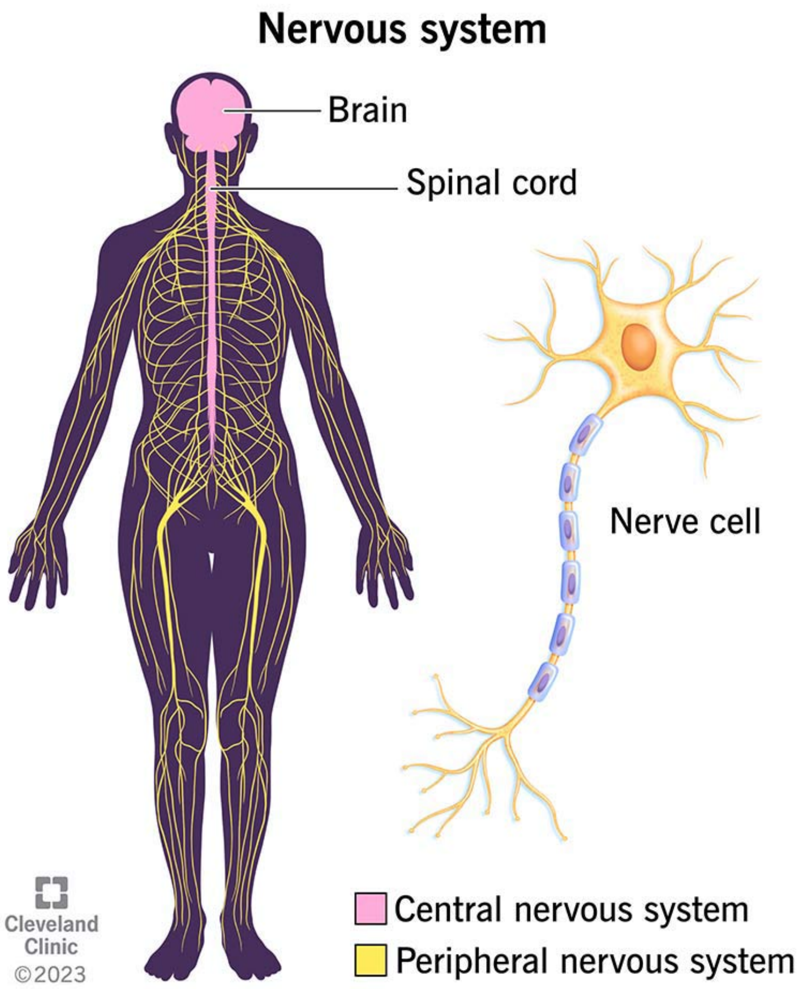
2026-05-21

Textbooks

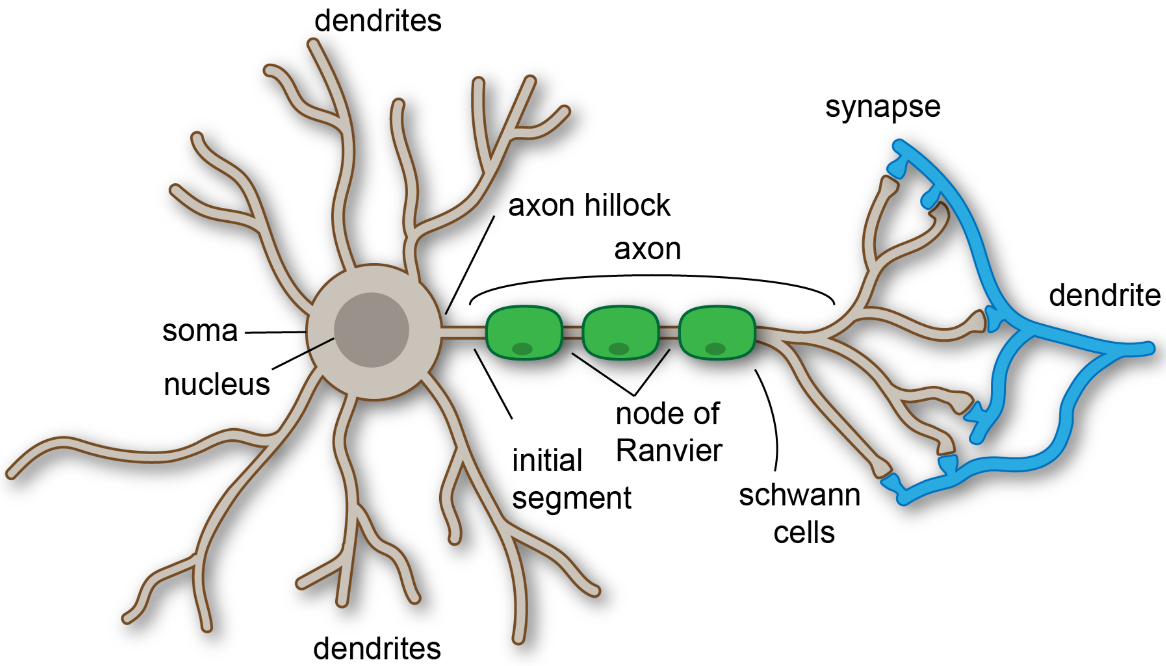
- Nonlinear Dynamics and Chaos - Steven Strogatz
- Chaos in Dynamical Systems - Edward Ott
- Dynamical Systems in Neuroscience - Eugene M. Izhikevich
- Mathematical Foundations of Neuroscience - Bard Ermentrout and David Terman
- Neuronal Dynamics - Gerstner, Kistler, Naud and Paninski

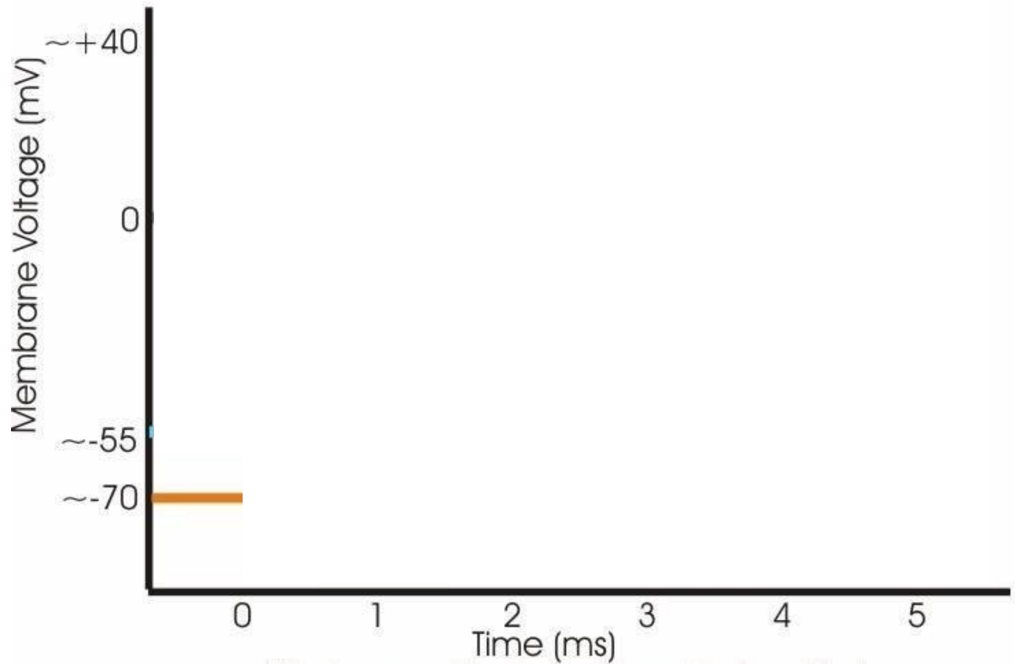
What is neuroscience?

Study of the nervous system



What is a neuron?





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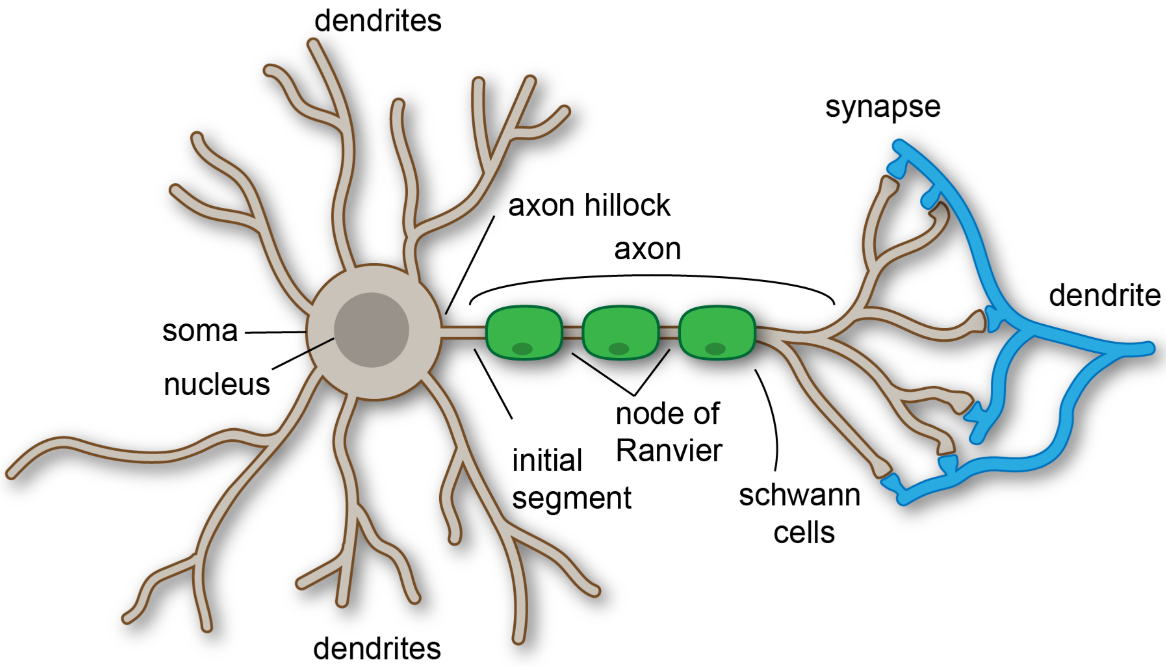

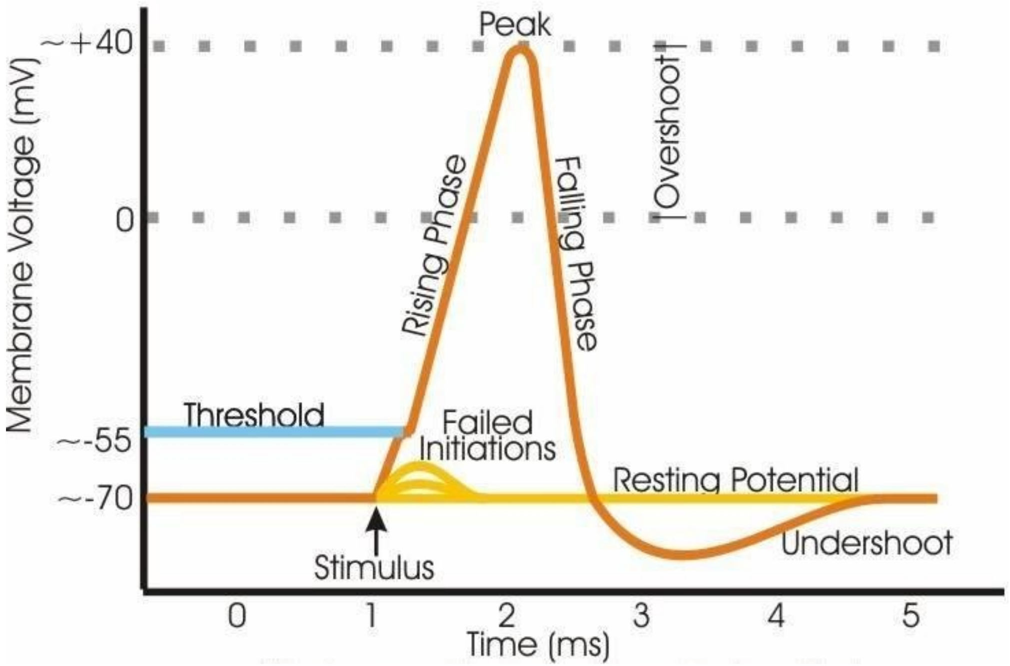

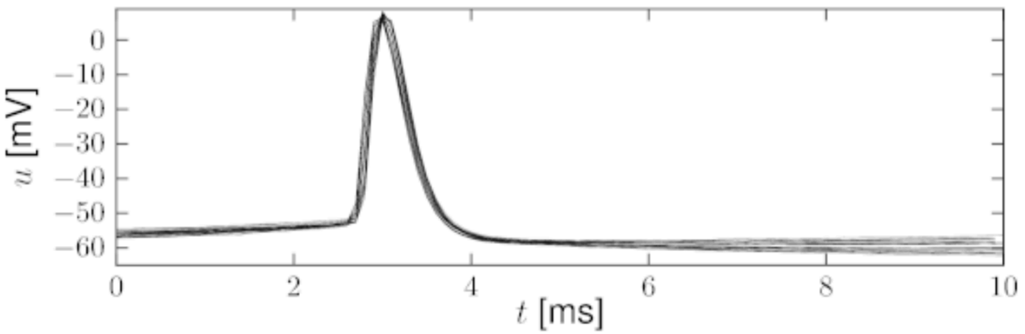

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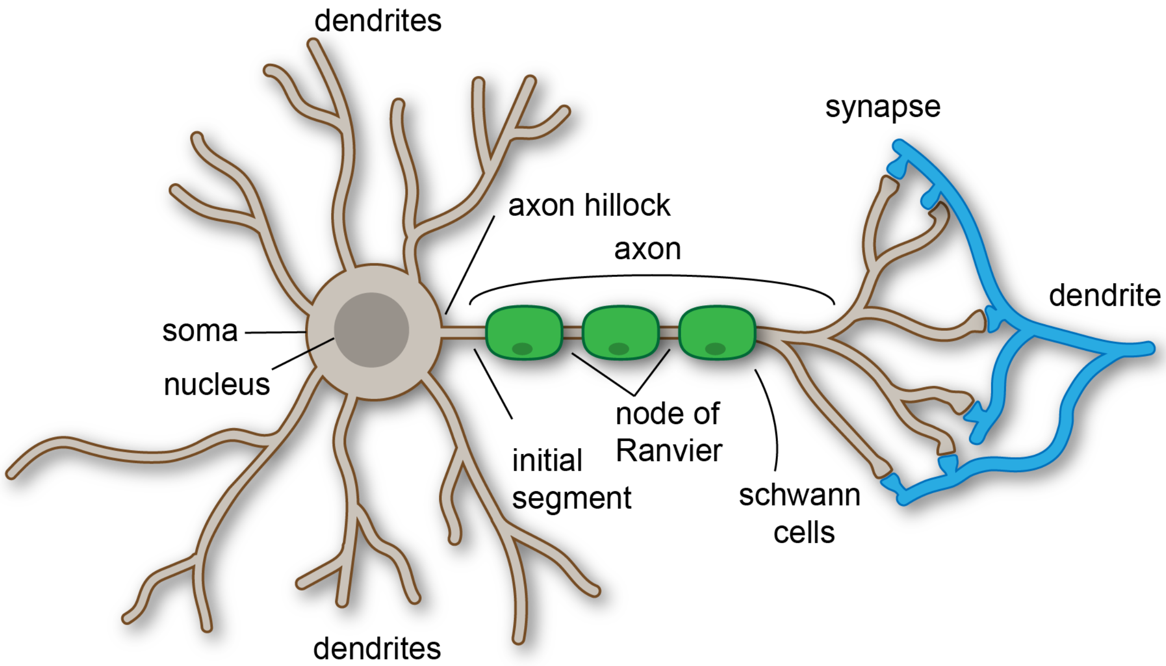
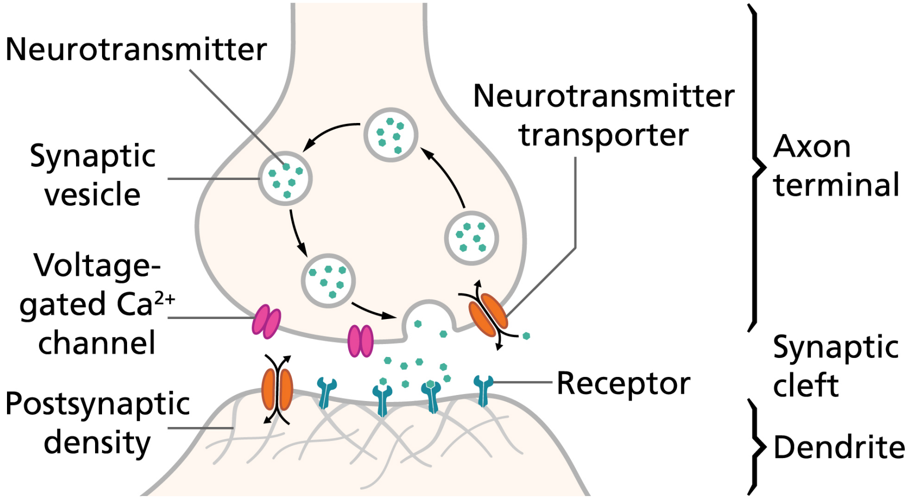
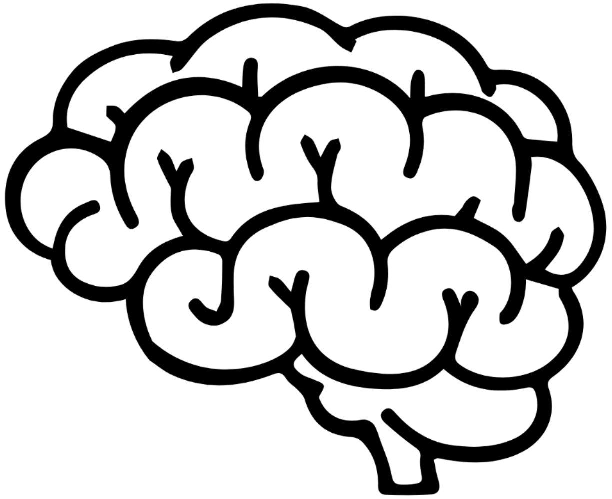
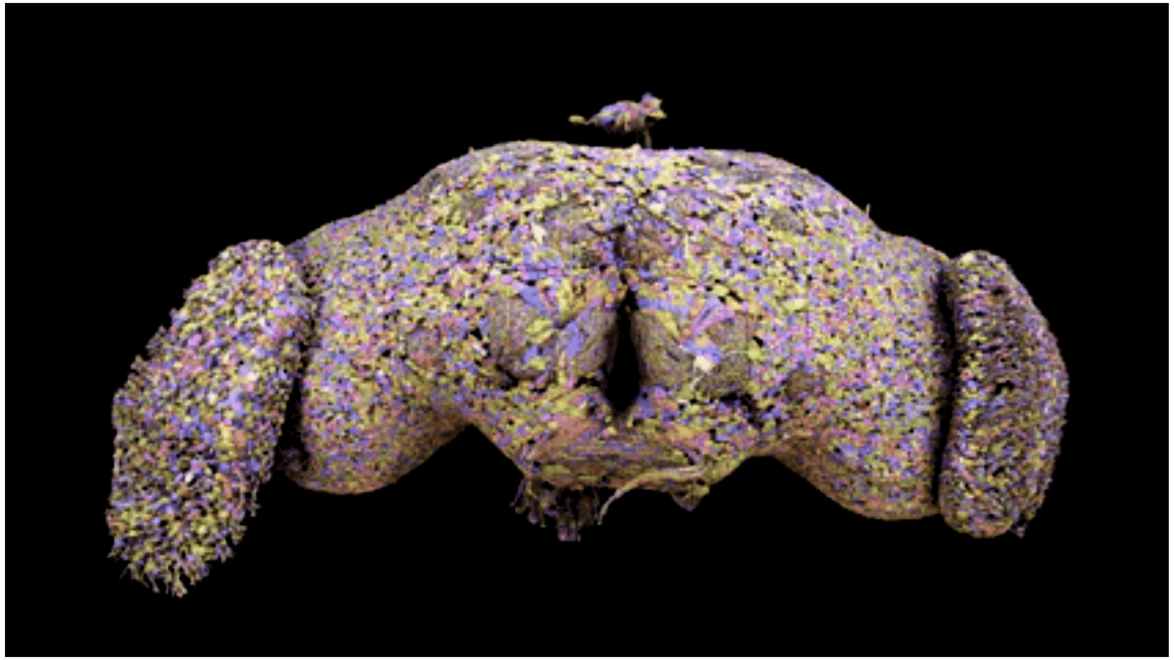




Neurons to neural circuits

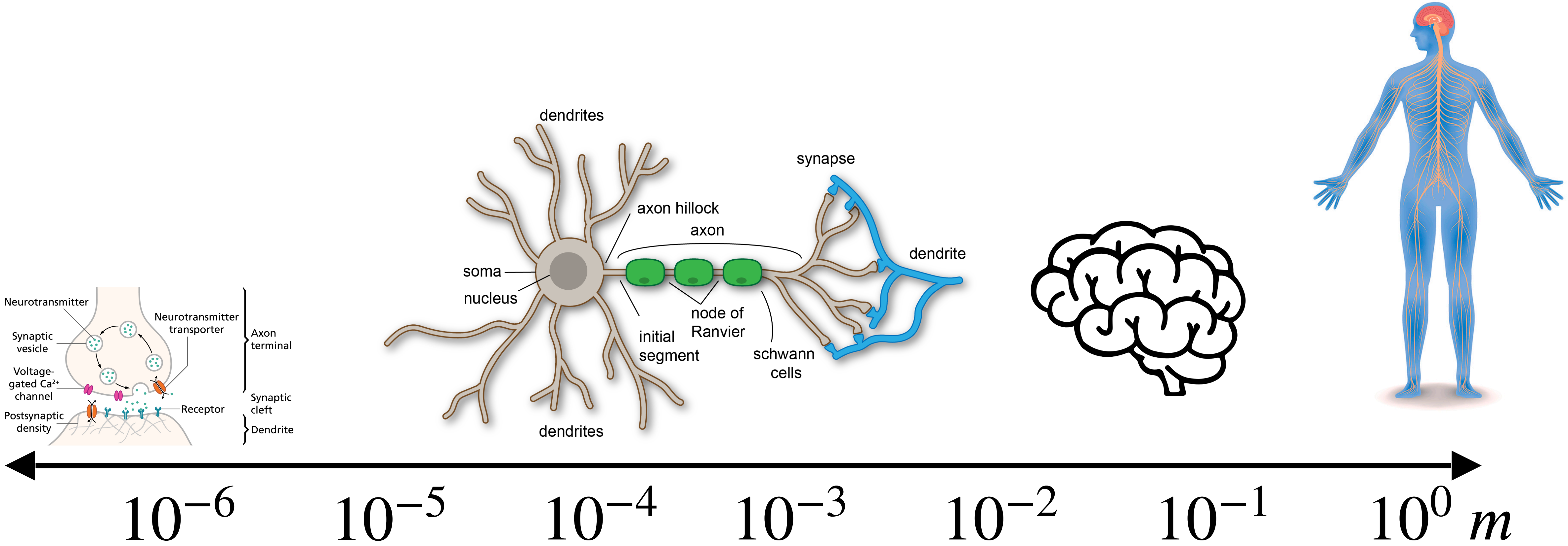




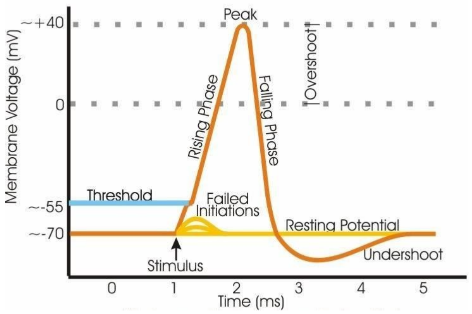
10^{11} neurons

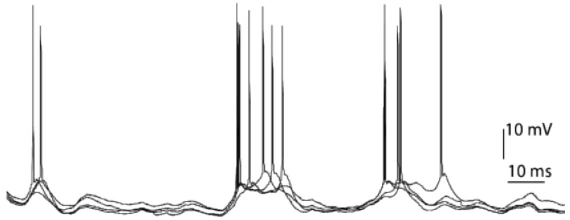
10^{14} synapses

Relevant lengthscales



Relevant timescales





Synaptic plasticity

Sensory perception

Planning, reasoning

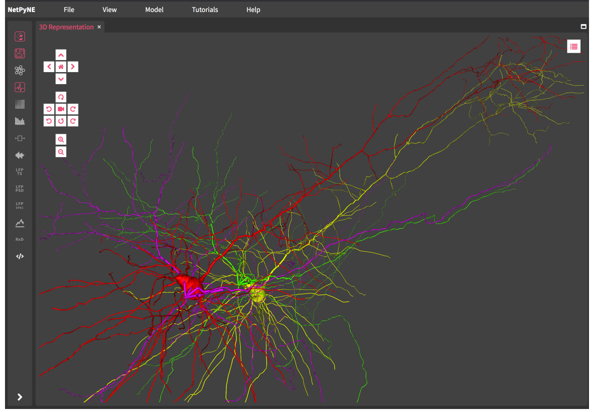
Long-term memory, learning

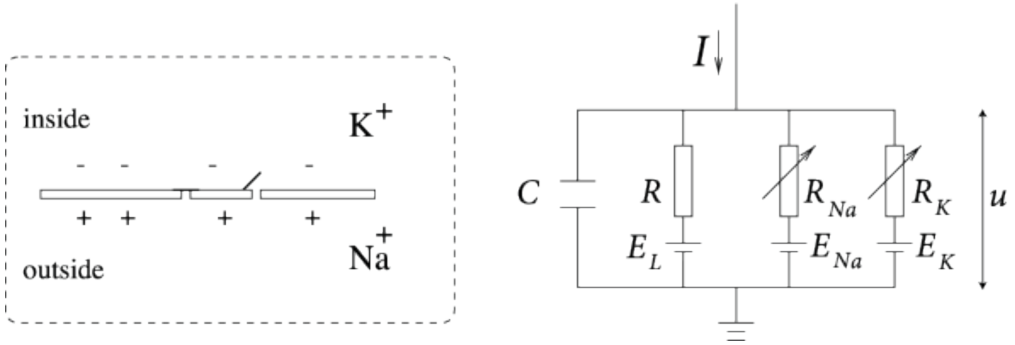
Short-term memory

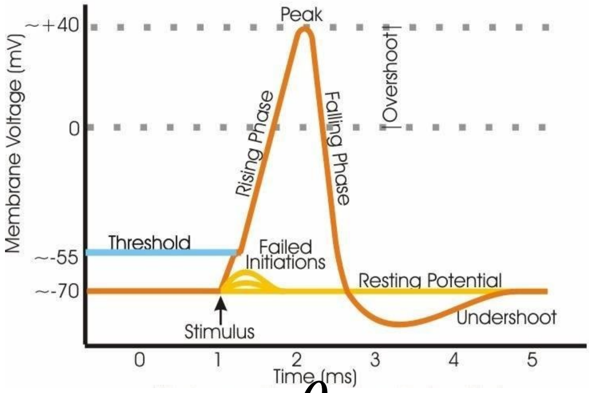
10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} $10^{5}s$

Different levels of abstraction

- Model detailed biochemistry and biophysics
- Model individual spikes







θ

$$I = C_m \frac{\mathrm{d}V_m}{\mathrm{d}t} + \bar{g}_{\mathrm{K}} n^4 (V_m - V_K) + \bar{g}_{\mathrm{Na}} m^3 h (V_m - V_{Na}) + \bar{g}_l (V_m - V_l),$$

$$\frac{dn}{dt} = \alpha_n(V_m)(1-n) - \beta_n(V_m)n$$

$$\frac{dm}{dt} = \alpha_m(V_m)(1-m) - \beta_m(V_m)m$$

$$\frac{dh}{dt} = \alpha_h(V_m)(1-h) - \beta_h(V_m)h$$

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)I(t),$$

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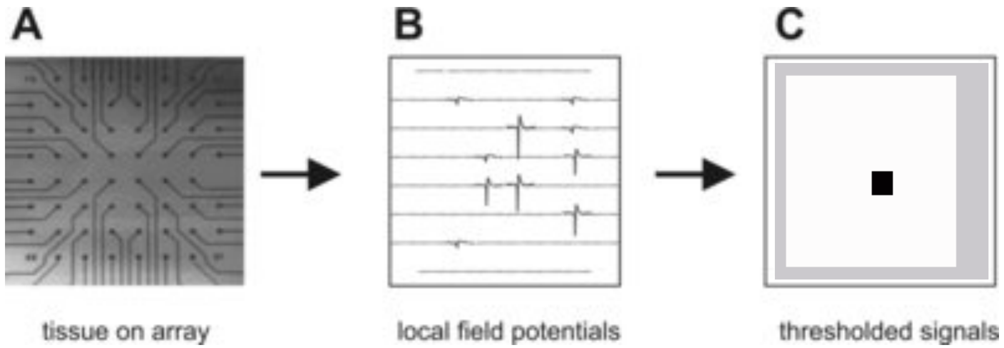
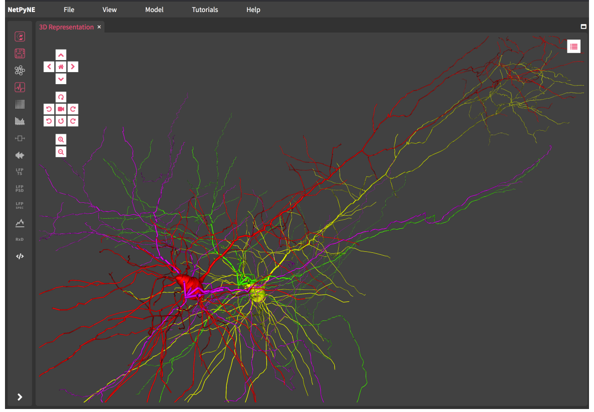
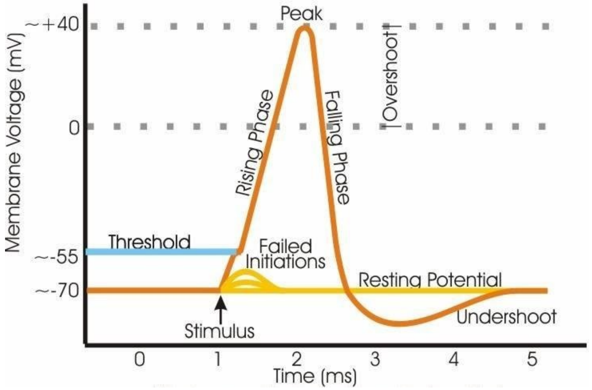
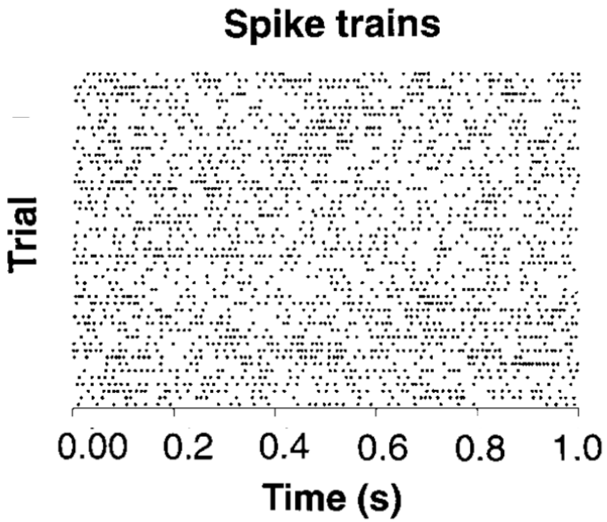
- Model binary neurons



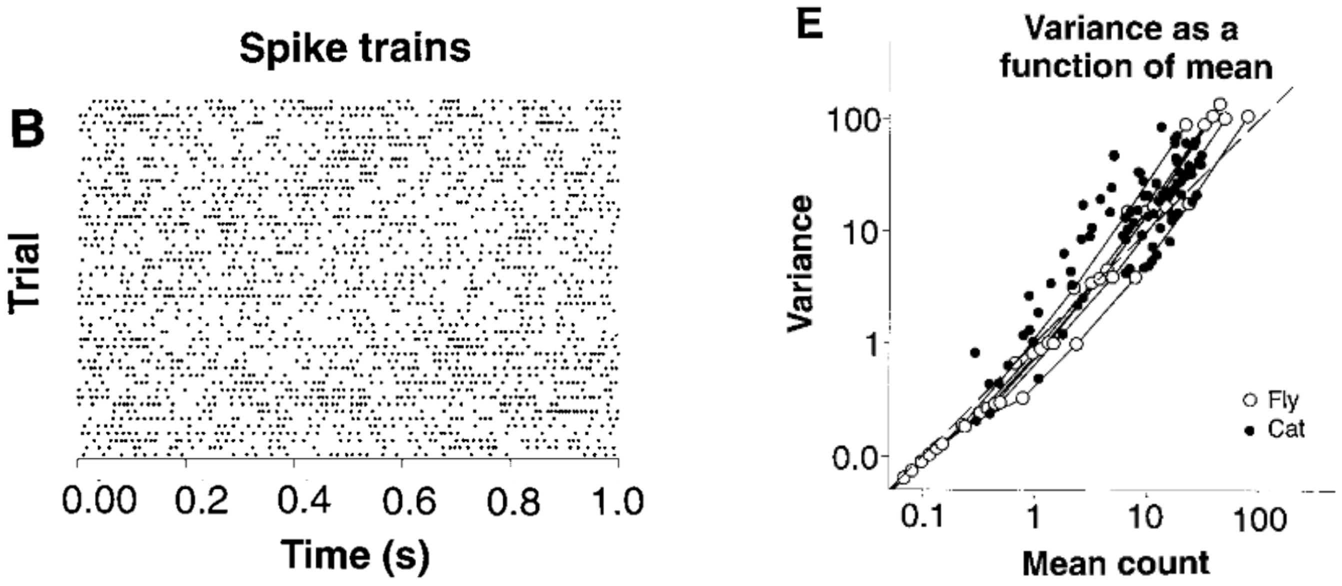






Different levels of abstraction

- Neuron spiking can seem stochastic
- Poisson spiking model: neuron action potential spike distribution



Ruyter van Steveninck et al 1997

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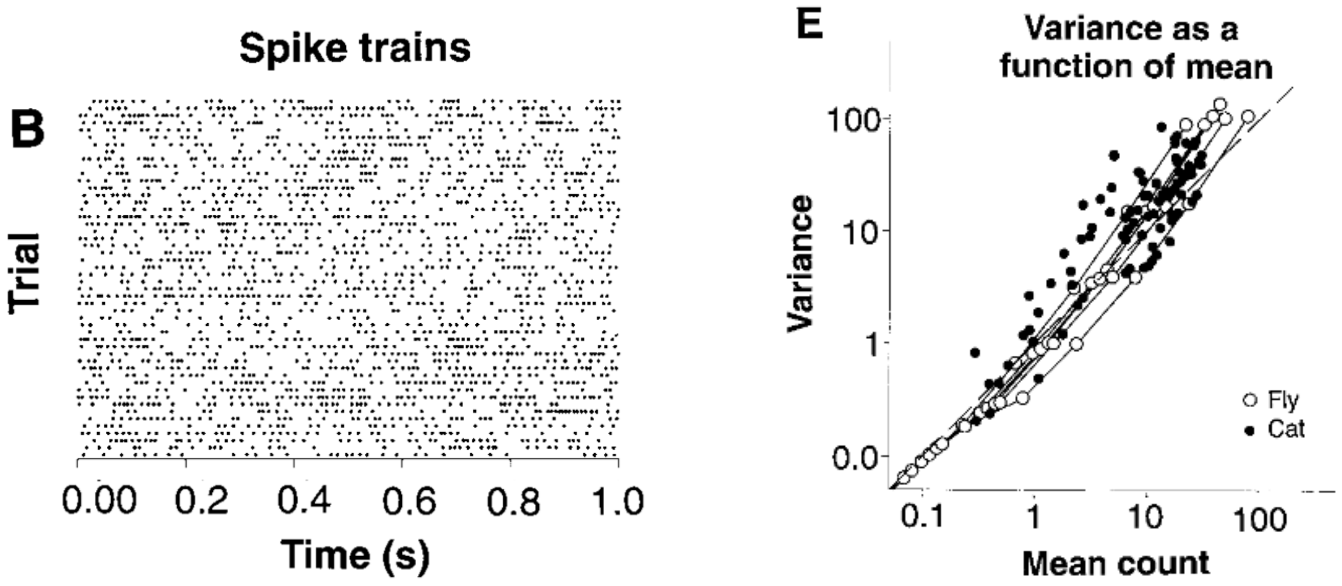

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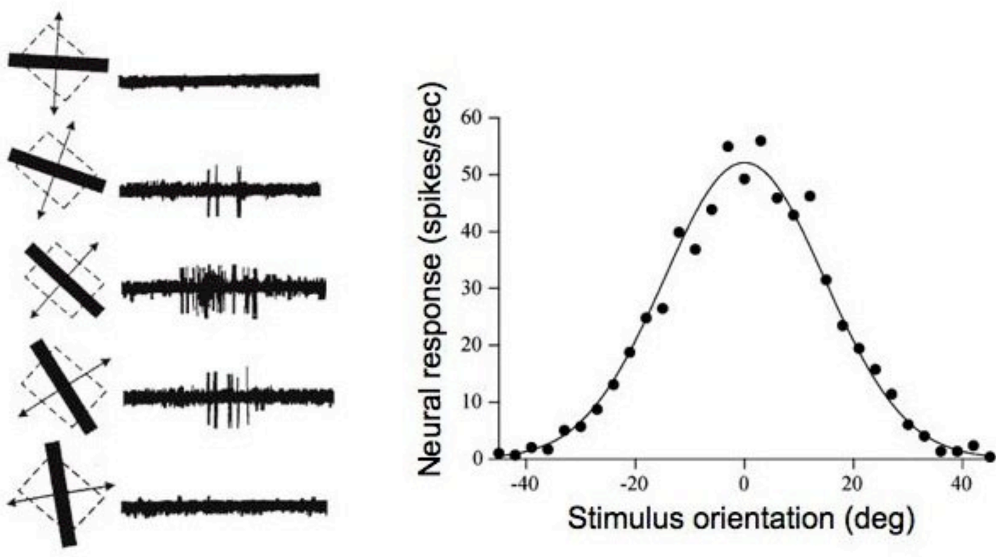

Hubel & Wiesel 1968

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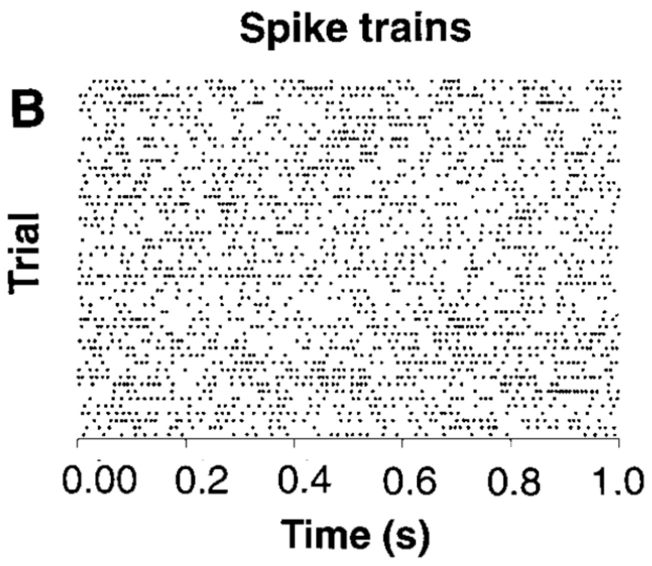

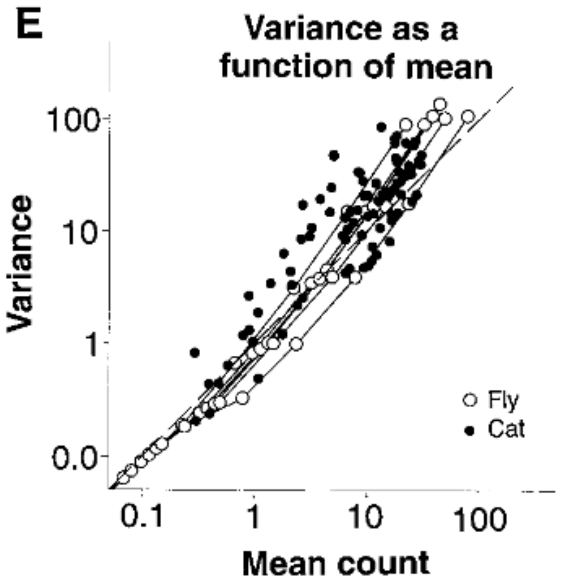

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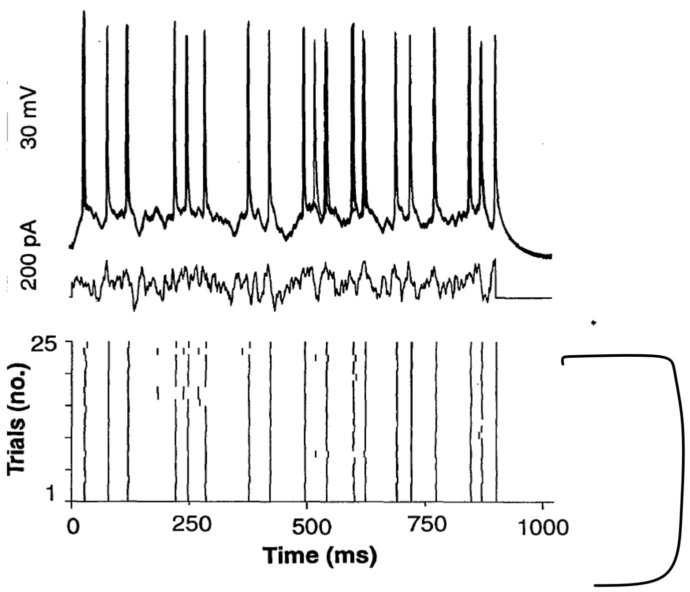

Mainen and Sejnowski 1995

Different levels of abstraction

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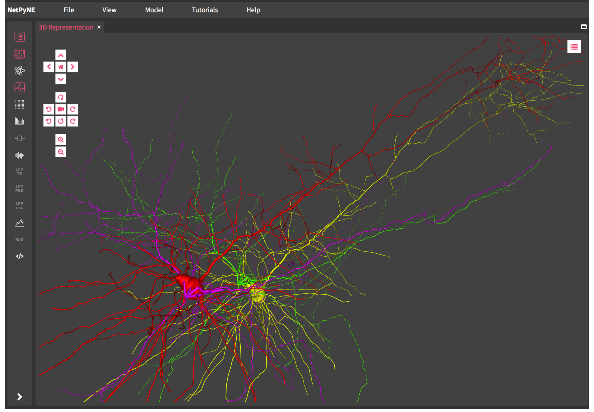
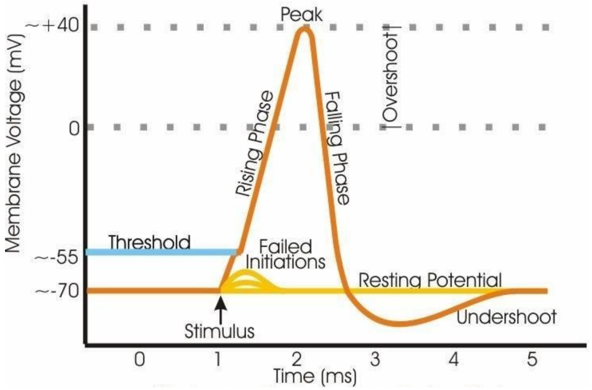
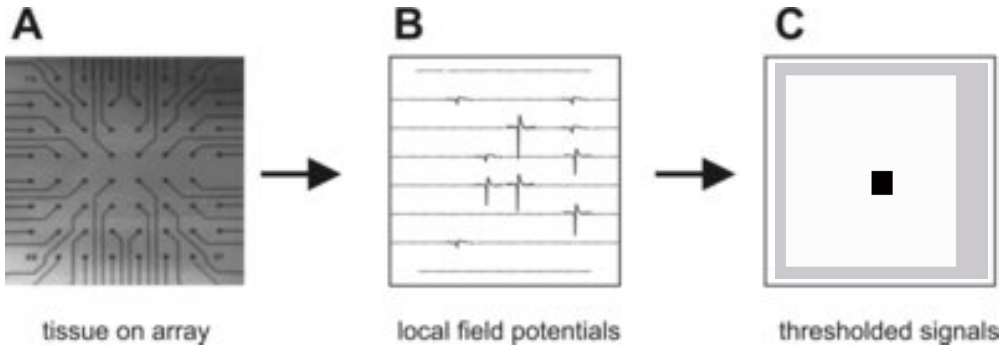
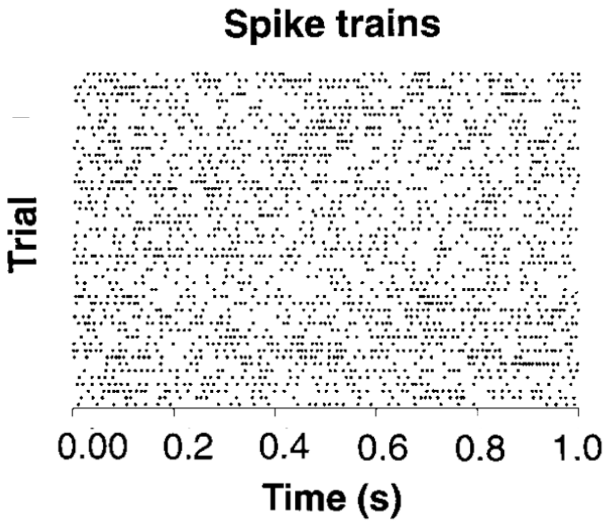
- Model individual spikes
- Model binary neurons
- Model firing rate dynamics





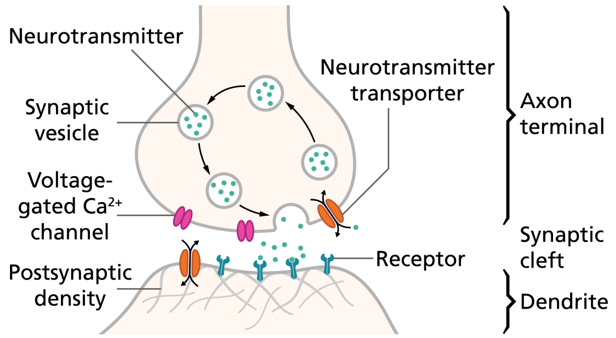




$$\frac{\partial s_i}{\partial t} + \frac{s_i}{\tau} = \phi\left[\sum_j W_{ij} s_j + B_i\right]$$

Dynamics of the circuits themselves

Neural plasticity modeled at various levels of abstraction

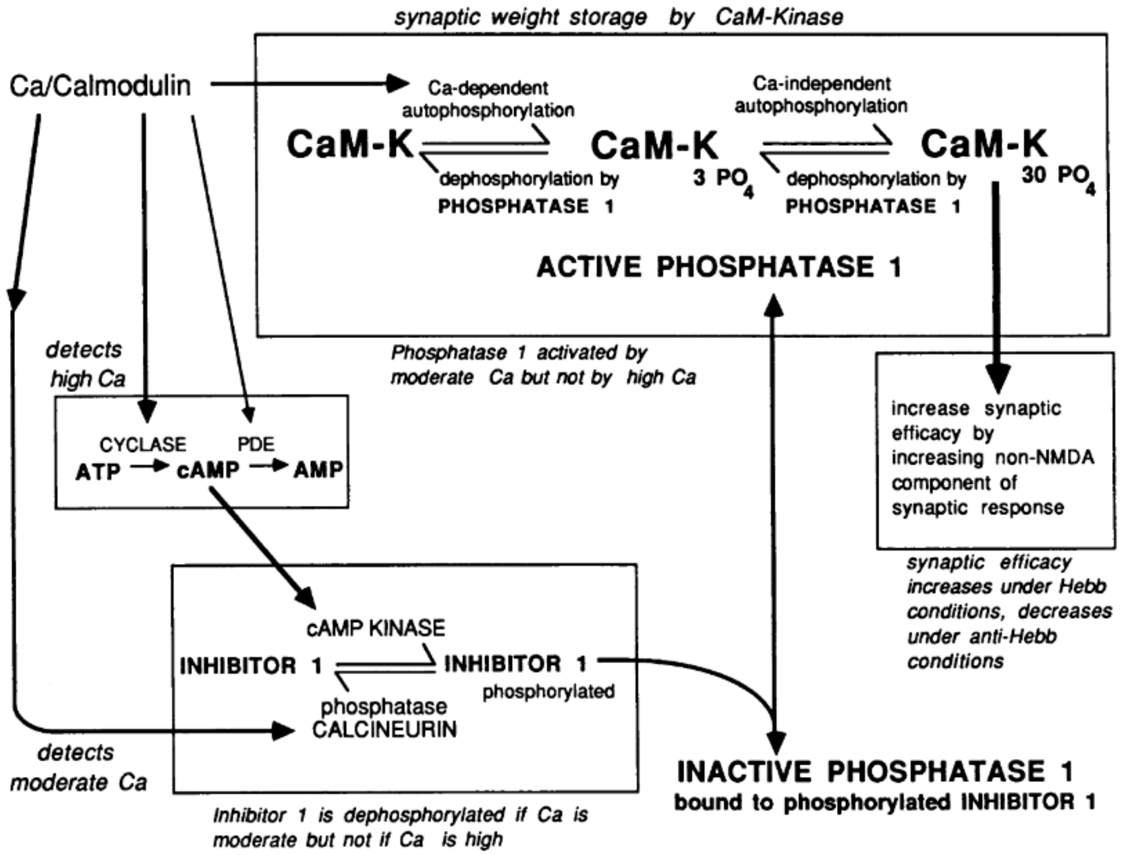


Proc. Natl. Acad. Sci. USA
Vol. 86, pp. 9574–9578, December 1989
Neurobiology

A mechanism for the Hebb and the anti-Hebb processes underlying learning and memory

(calmodulin/calcium/calmodulin-dependent protein kinase II/protein phosphatase 1/adenylate cyclase/calcineurin)

John Lisman
Department of Biology, Brandeis University, Waltham, MA 02254



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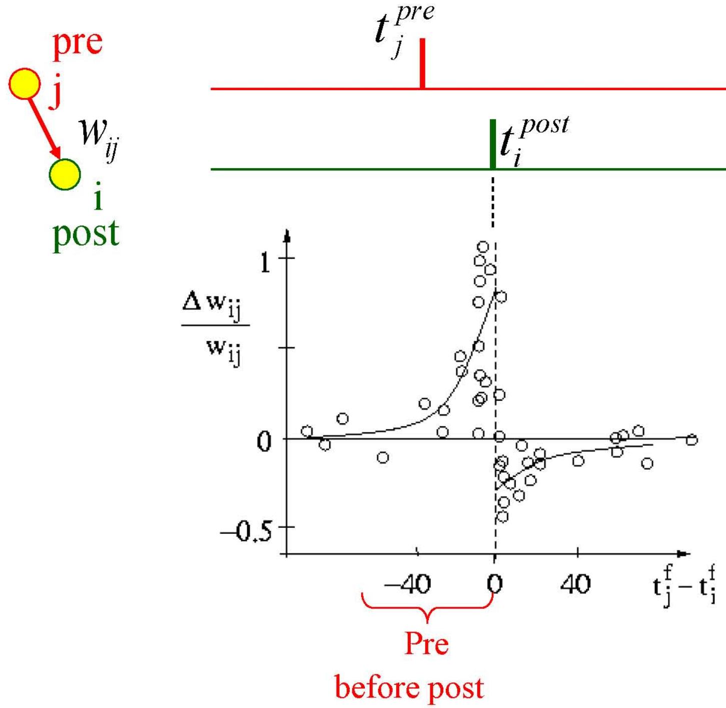

Bi & Poo (1998)

The Spike-Timing Dependence of Plasticity

Daniel E. Feldman[1,*]
[1]Department of Molecular and Cell Biology, and Helen Wills Neuroscience Institute, University of California, Berkeley, Berkeley, CA 94720-3200, USA
*Correspondence: dfeldman@berkeley.edu
http://dx.doi.org/10.1016/j.neuron.2012.08.001

Timescales of plasticity

- STDP ~10 milliseconds
- BTSP ~1-10 seconds
- LTP, LTD ~ 1-1000 minutes
- Neurodevelopment ~ hours to months
- Evolutionary?

Nonlinear dynamical systems

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  Understand and analyze nonlinear dynamics **without** solving the entire system

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$\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$

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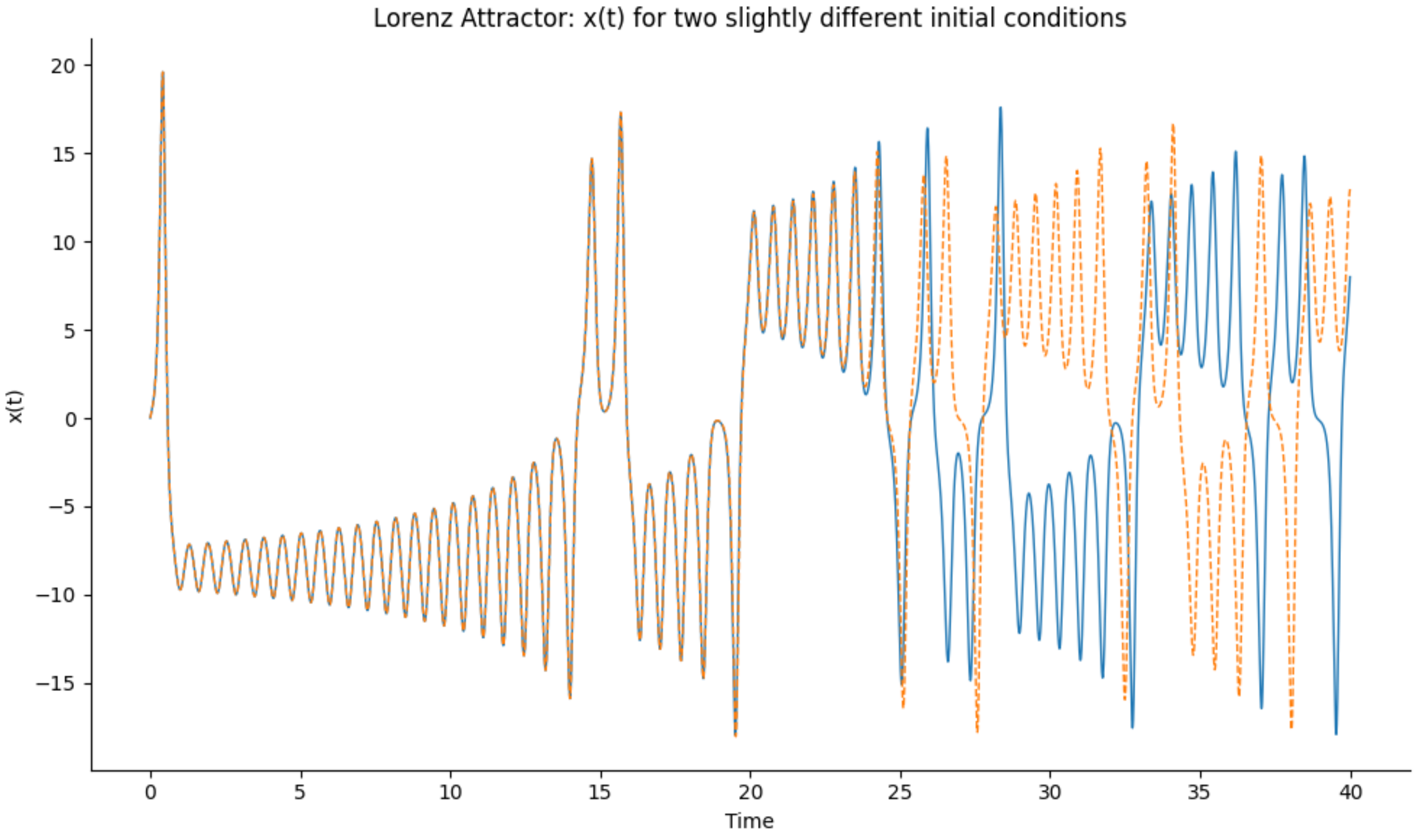

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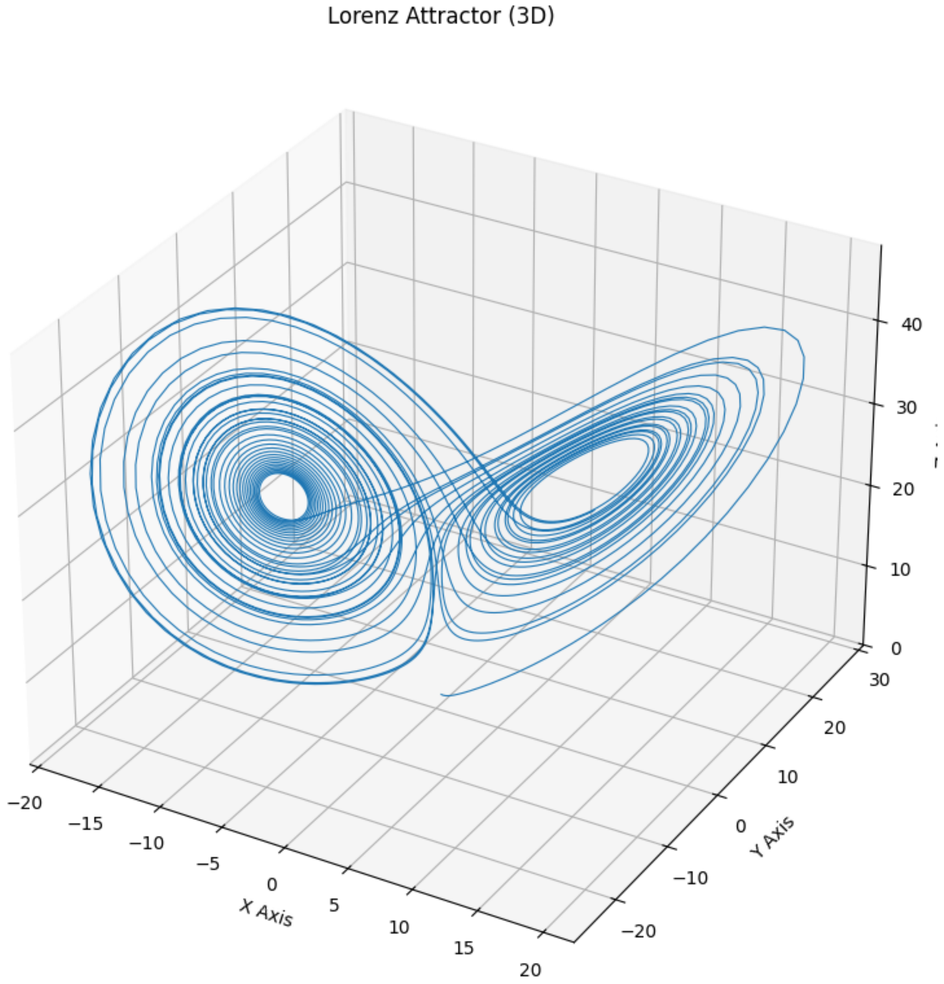

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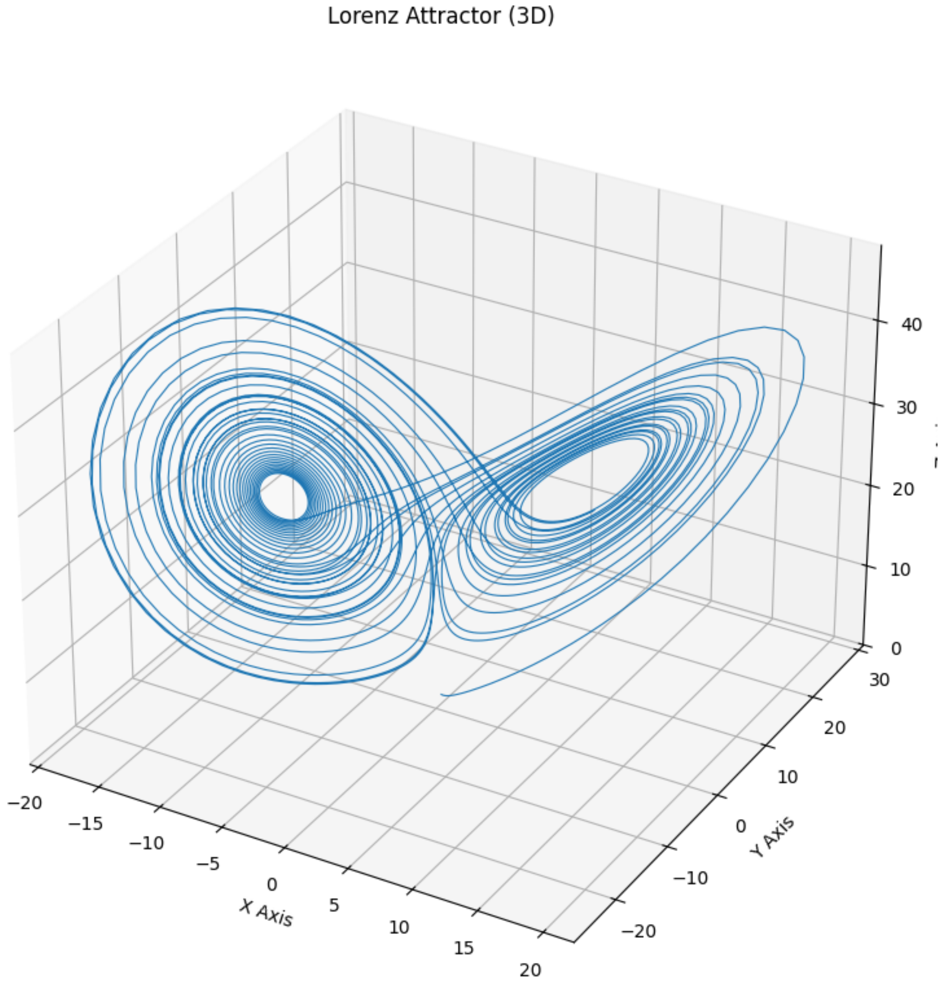
Edward Lorenz



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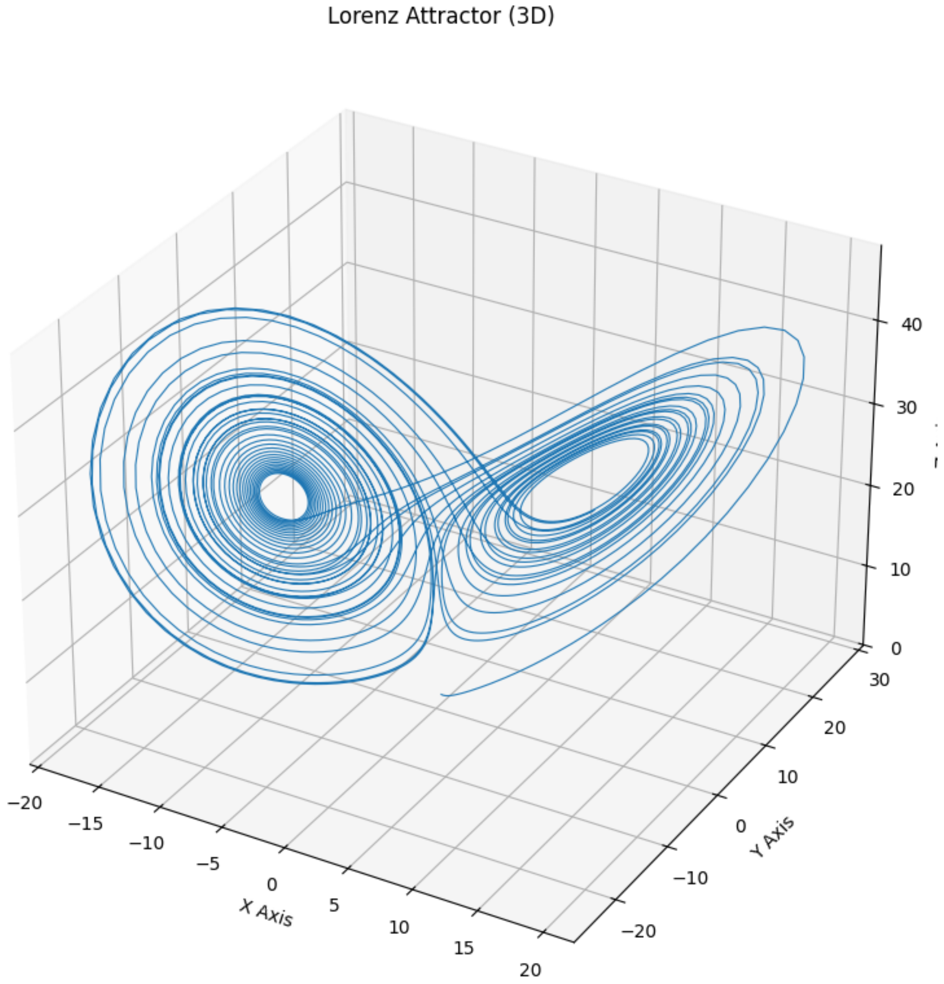
Edward Lorenz

Ellen Fetter

Margaret Hamilton



What is a Dynamical System?

$$\dot{\bar{x}} = F(\bar{x})$$ Autonomous D.S.

$$\dot{\bar{x}} = A\bar{x}$$

$$\dot{x}_1 = F_1(x_1, \dots, x_n)$$
$$\dot{x}_2 = F_2(\quad)$$
$$\vdots$$
$$\dot{x}_n = F_n(x_1, \dots, x_n)$$

1) Time explicit ←
2) Higher order Deriv. ←
3) Implicit Dyn ←
4) Neural Fields
5) Discrete systems

Non. Aut. D.S. → $\dot{\bar{x}} = F(\bar{x}, t)$

$$\bar{y} = \begin{pmatrix} \bar{x} \\ t \end{pmatrix} \qquad \dot{\bar{y}} = \begin{pmatrix} F(\bar{x}) \\ 1 \end{pmatrix}$$

$$\dot{t} = 1$$

$$\ddot{x} = -x$$
$$y = \dot{x}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$\dot{x} = F(x) \leftarrow$

$\dot{x} = F(x,t)$

F is cts

DF or F' is cts

Implicit

$F(\dot{x}, x, t) = 0$

$\dot{\bar{x}} = A\bar{x}$ (non-normal dynamics ← v.v. interesting)

$\bar{x}(t) = c_1 e^{\lambda_1 t}\hat{e}_1 + c_2 e^{\lambda_2 t}\hat{e}_2 + \dots$

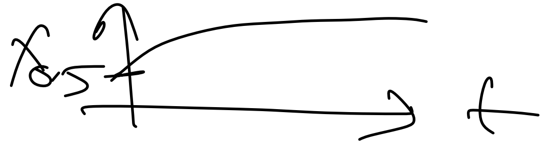
1D Flows

$x(0) = 0.5$

$\dot{x} = F(x) = -x(x-1)(x-2)$

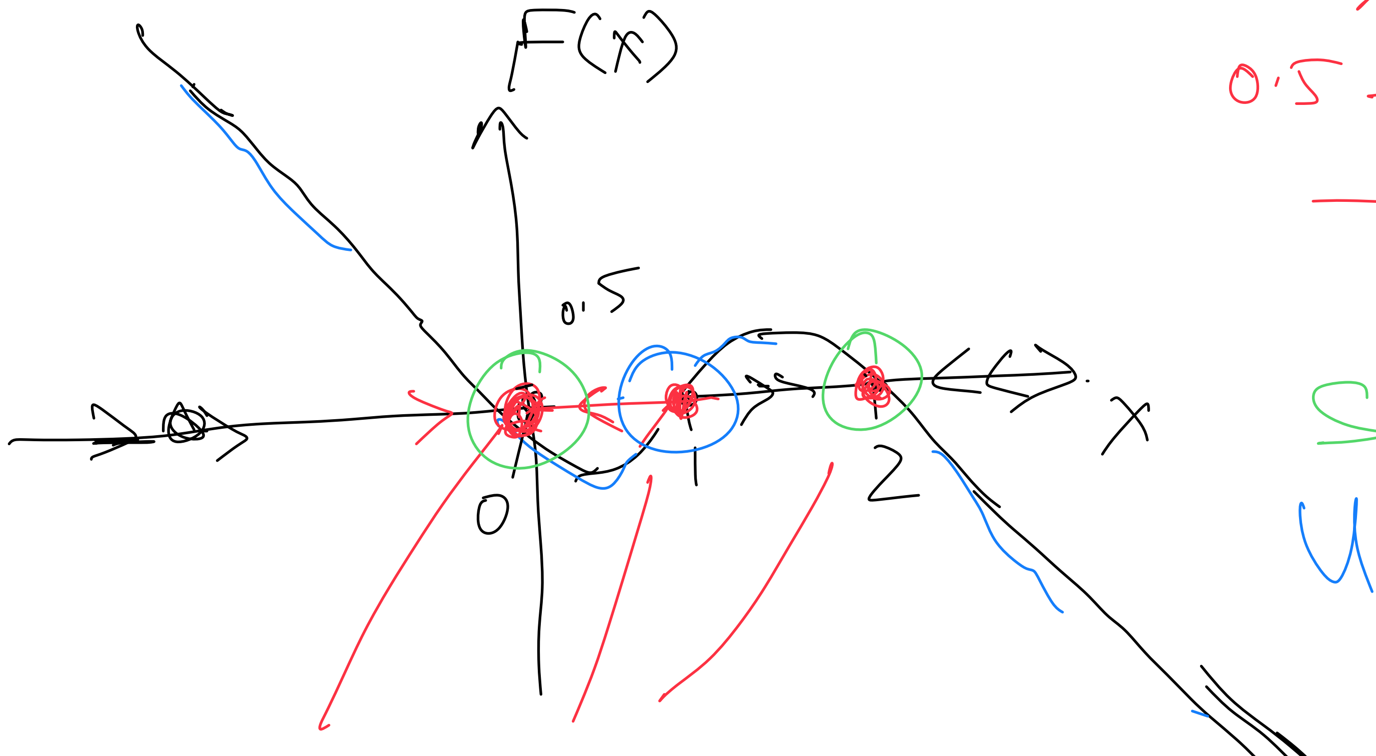
$$-\frac{dx}{x(x-1)(x-2)} = dt \longrightarrow t = -\frac{1}{2}\ln x + \ln(x-1) - \frac{1}{2}\ln(x-2) + C$$

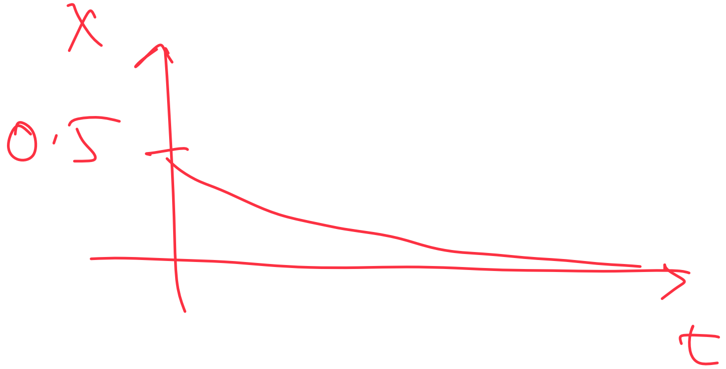
$x =$ ____



$$\dot{x} = -x(x-1)(x-2) + \varepsilon \sin x$$

$$= F(x)$$





Stable → $F' < 0$

Unstable → $F' > 0$

Fixed Pt

→ $F(x_*) = 0$

$x = x_* + \varepsilon$

$\dot{x} = F(x)$

$\dot{x}_* + \dot{\varepsilon} = F(x_* + \varepsilon)$ (with $\dot{x}_* = 0$)

$\dot{\varepsilon} = \cancel{F(x_*)} + F'(x_*) \cdot \varepsilon$

$\dot{\varepsilon} = F'(x_*)\, \varepsilon$

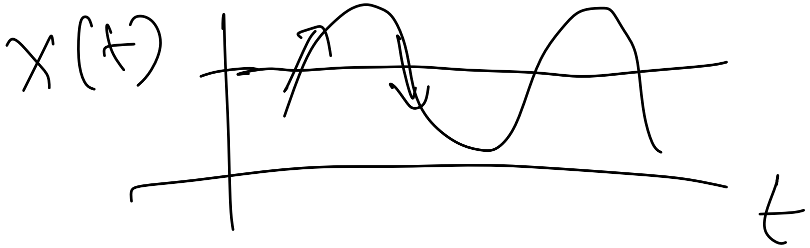
$$\dot{x} = F(x)$$
$$x(0) = x_0$$

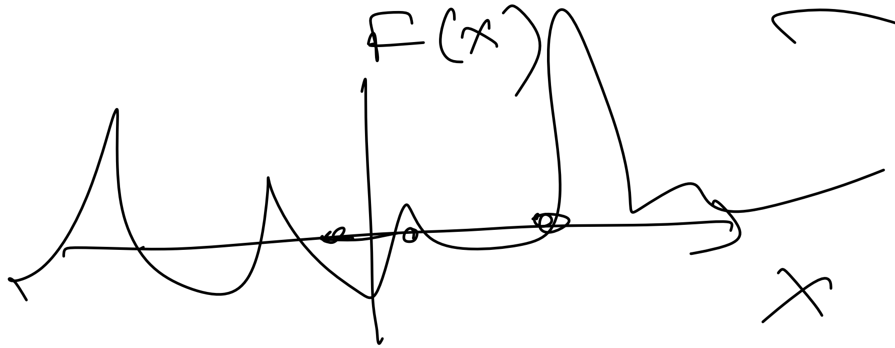
What happens ~~at~~ as $t \to \infty$

1) $x(t) \to x^*$ (goes to a F.P.) ← Can be multiple F.P. depending on init. cond.

2) $x(t) \to \infty$ or $x(t) \to -\infty$

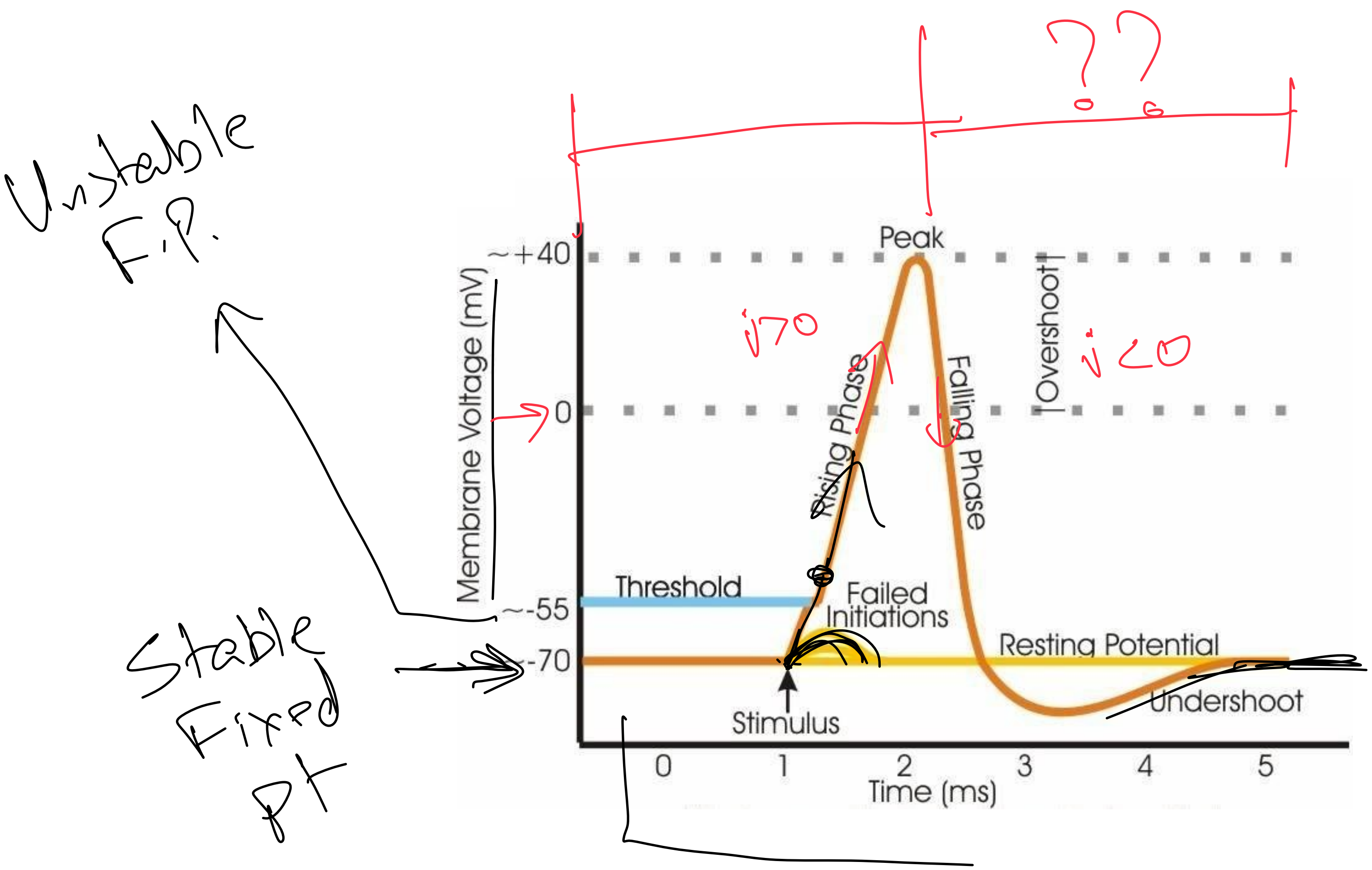
~~Periodic $x(t+T) = x(t)$~~ ✗





1 D Flow

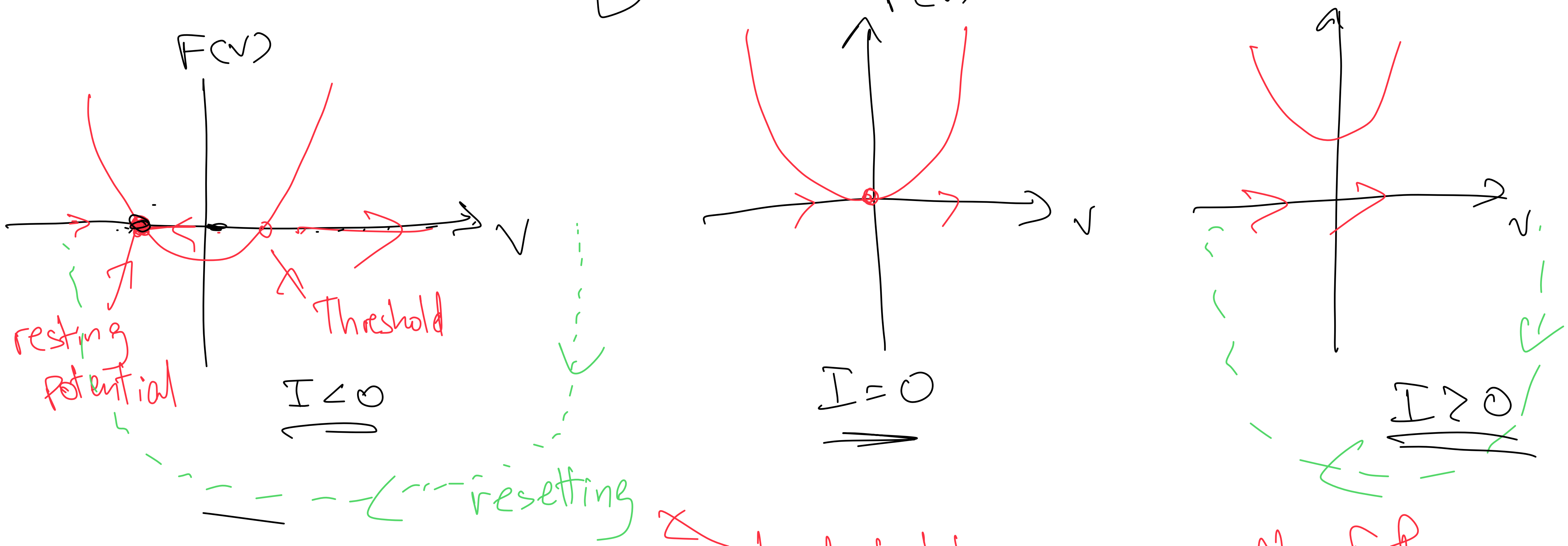
↳ cannot have periodicity



Unstable F.P.
Stable Fixed pt
??
Peak
~+40
0
~-55
-70
Membrane Voltage (mV)
Rising Phase
Falling Phase
Overshoot
Threshold
Failed Initiations
Resting Potential
Undershoot
Stimulus
0
1
2
3
4
5
Time (ms)

Quadratic Integrate and Fire Neuron

$$\dot{V} = F(V) = V^2 + I$$ → Inputs (I)



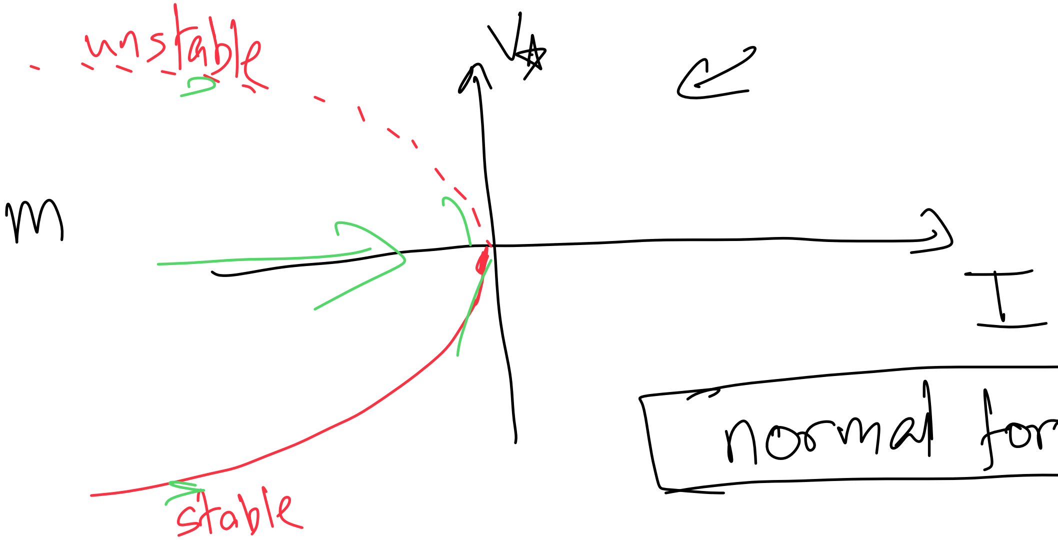
F.P. → $V_{\pm} = \pm\sqrt{-I}$

kind of like an extra dim

No. F.P

QIF

Bifurcat^n Diagram



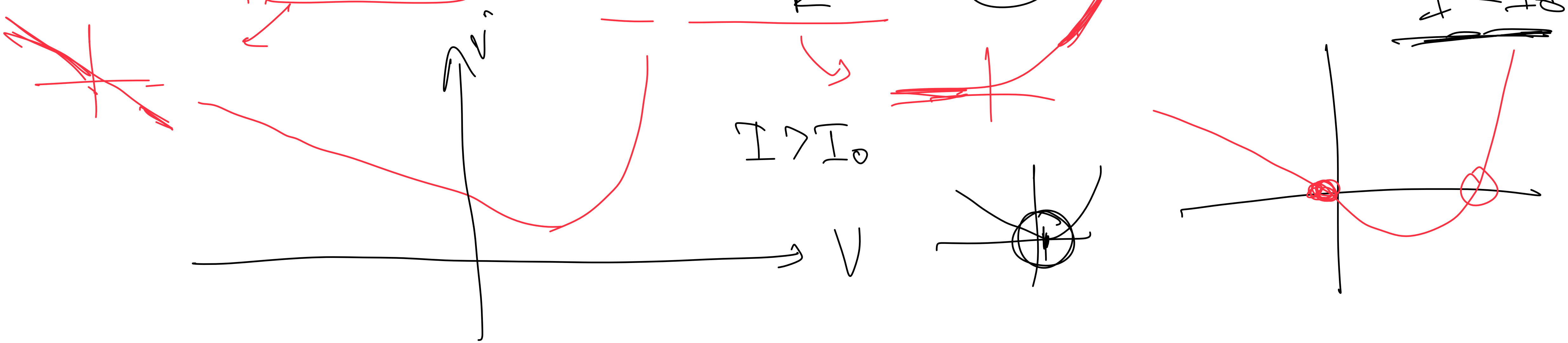
$$\dot v = v^2 + I$$

"Saddle Node Bifurcat^n"

normal form of an S.N. Bif.

Exponential Int. & Fire Neuron (EIF)

$$\dot V = -(V - V_0) + k \exp\left(\frac{V - V_\theta}{k}\right) + I$$



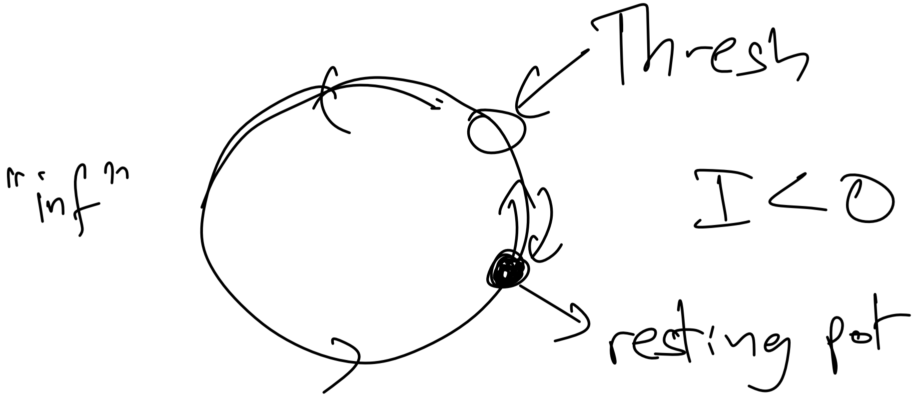
~~$\dot{x}$~~ $\dot{\theta} = F(\theta)$

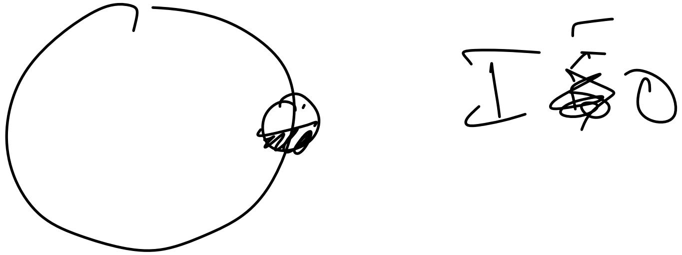
Theta Neuron Model

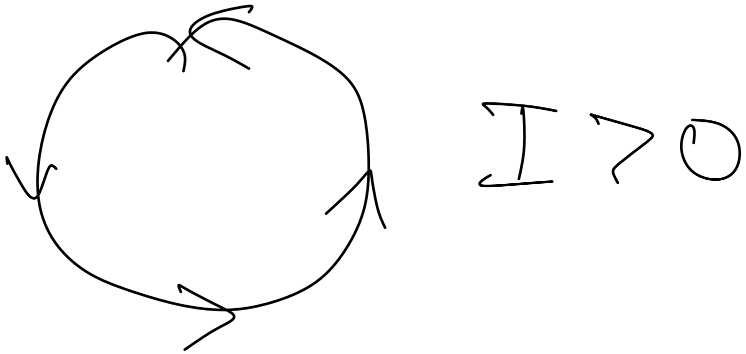
$$\dot{\theta} = 1 - \cos\theta + (1 + \cos\theta) I$$

Iz. §3.38

Saddle Node Bif on an Invariant Cycle (SNIC)

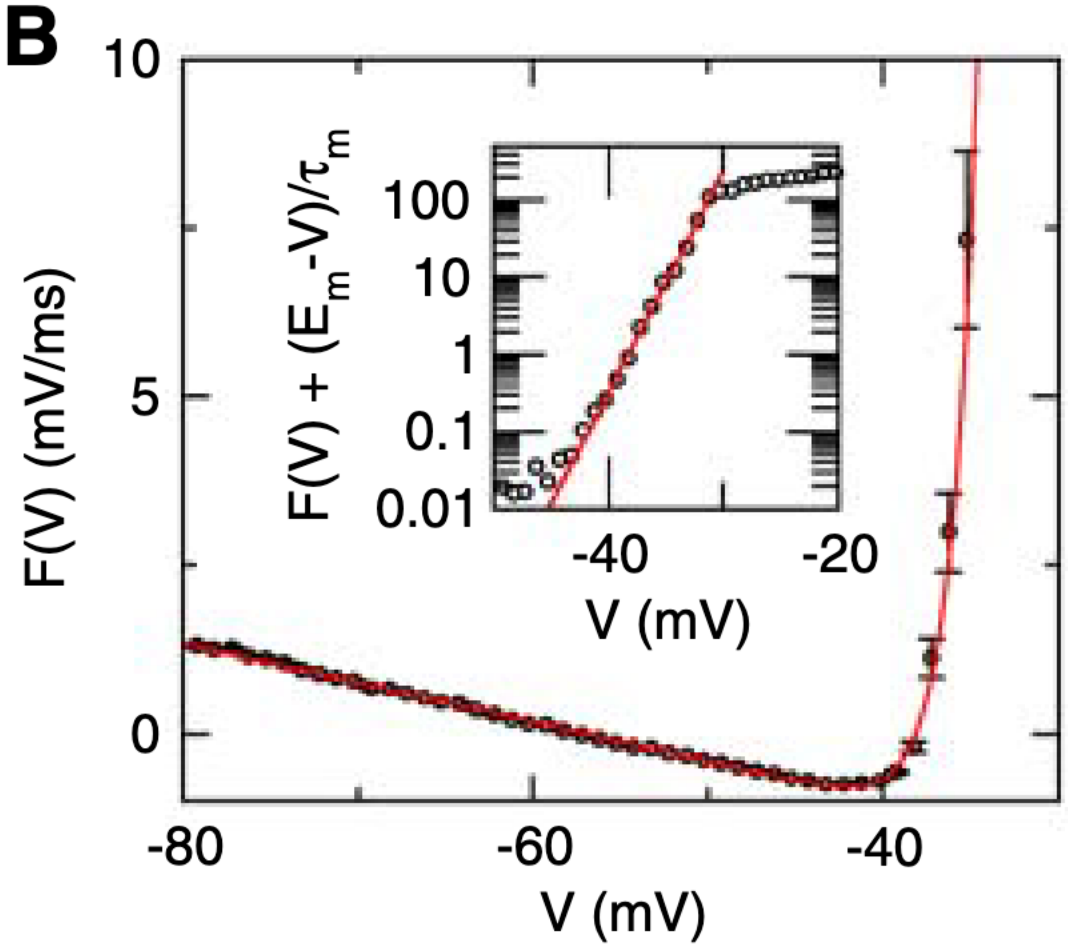




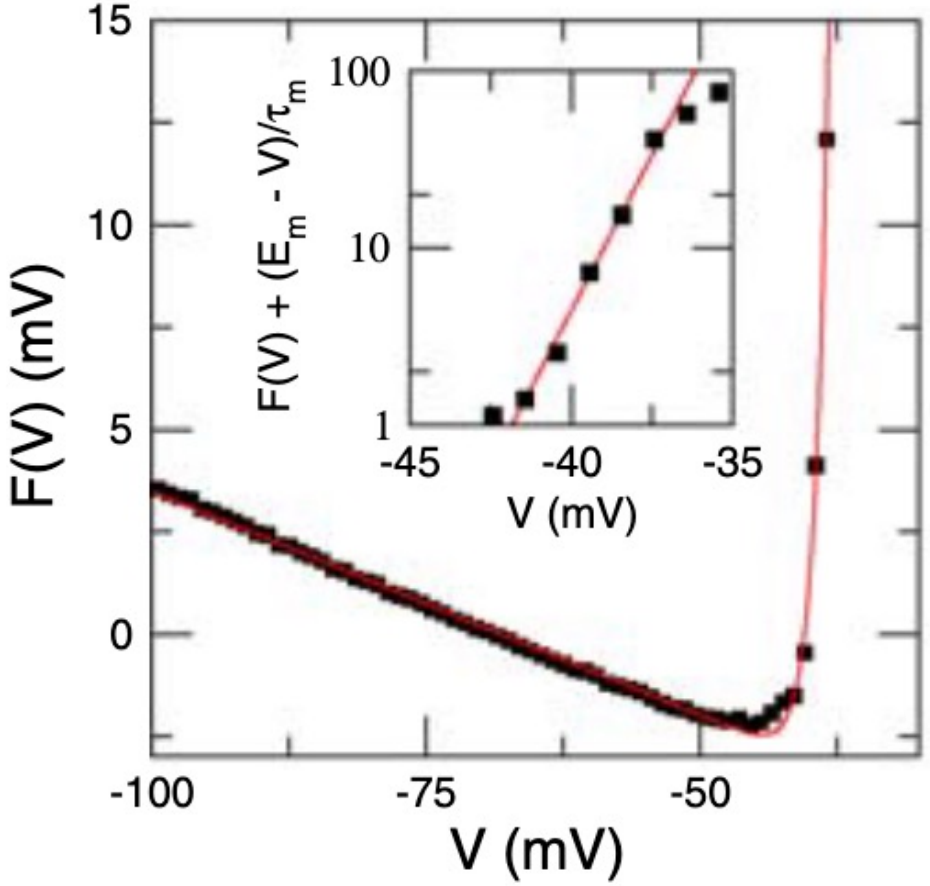


Expoential Integrate and Fire neuron

Badel et al. Biol Cybern (2008)



Pyramidal cell



Interneuron

Transcritical Bifurcation

Avalanche Dyn

$a \rightarrow$ fraction of active neurons

$a \in [0,1] \longrightarrow a = 0$ is an absorbing boundary condition

$\uparrow$

$a = 0$ is a F.P $\leftarrow$ unstable

$\dot{a} = -a + \lambda a(1-a) \longrightarrow$ growth $\propto$ active frac $\propto$ available frac

$\frac{da}{dt} = a(\lambda - 1) - \lambda a^2$

$\tau = \lambda t$

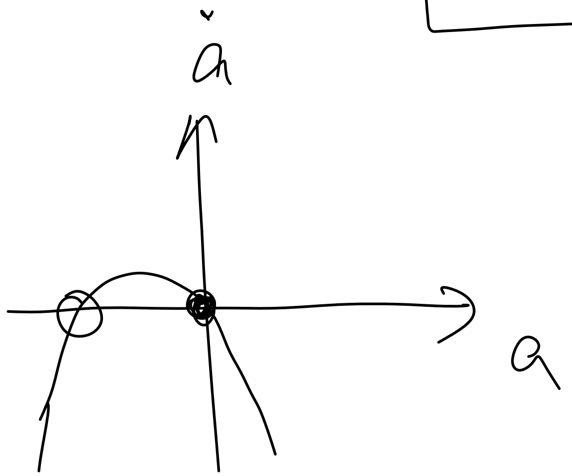
$\frac{da}{\lambda dt} = a\frac{(\lambda - 1)}{\lambda} - a^2$

$\dot{a} = a\left(\frac{\lambda - 1}{\lambda}\right) - a^2$

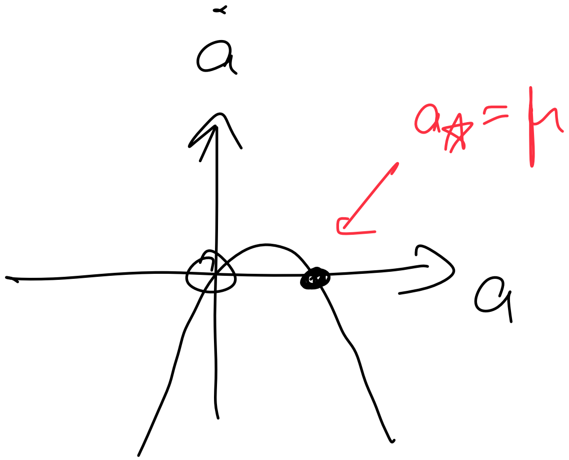
$$\dot a = a\underbrace{\left(\frac{\lambda-1}{\lambda}\right)}_{\mu} - a^2$$

$$\boxed{\dot a = \mu a - a^2}$$

$\lambda \approx 1+\mu$

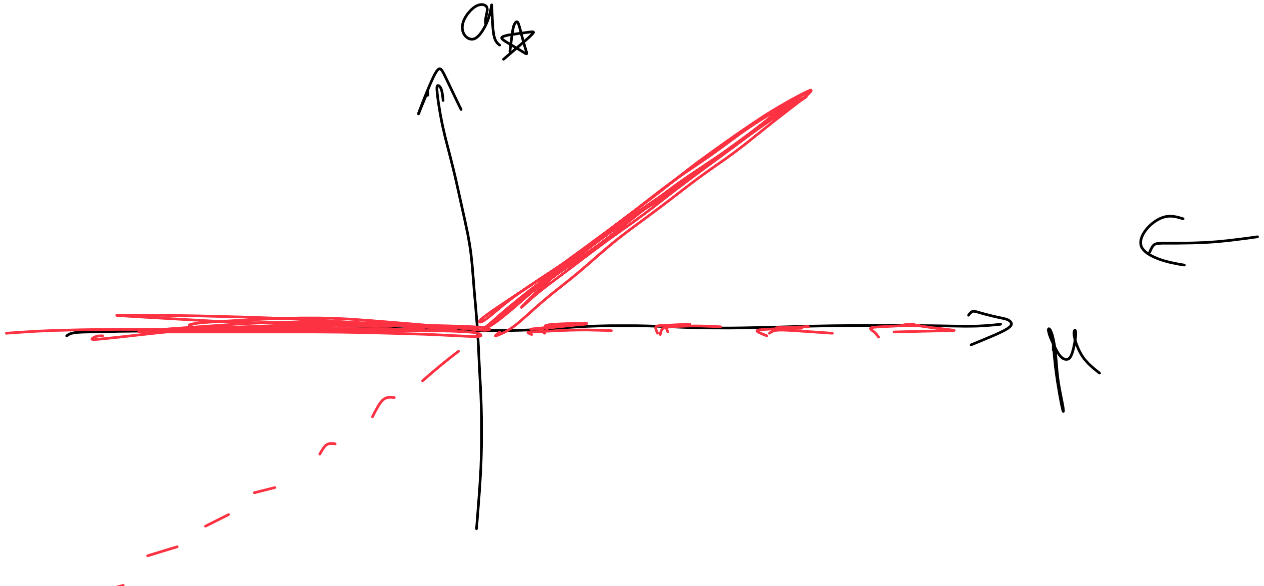


$\mu < 0$

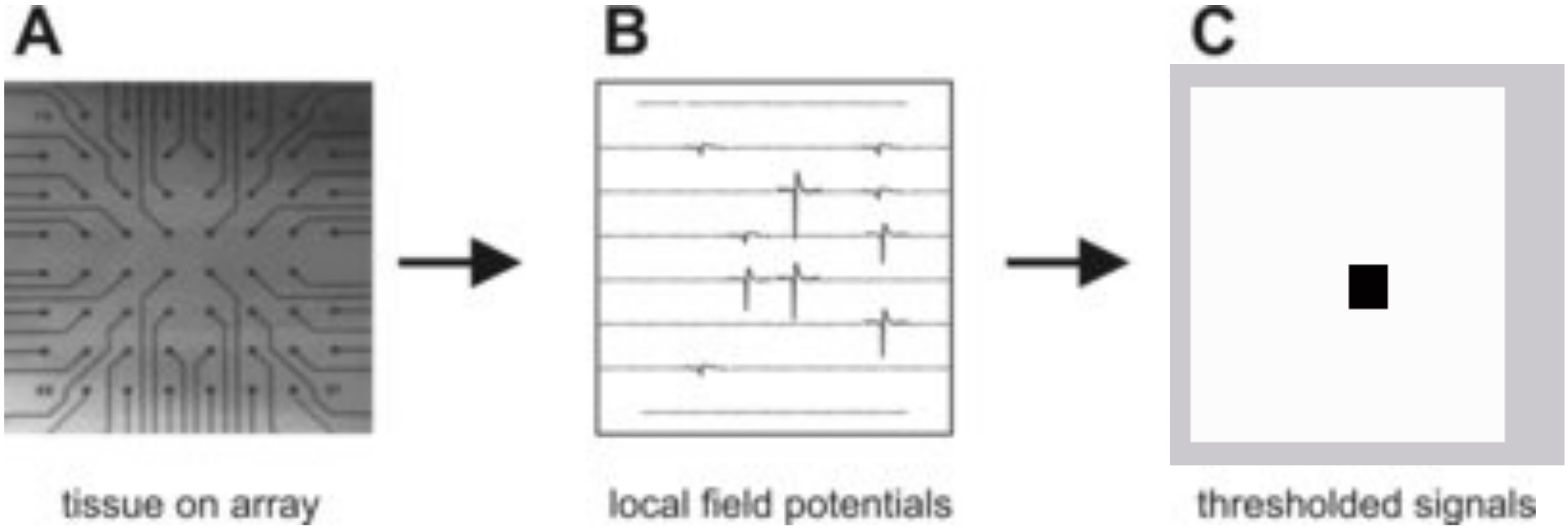


$\mu > 0$

Transcritical Bifurcatⁿ



Avalanche Dynamics



J. Beggs (2007)

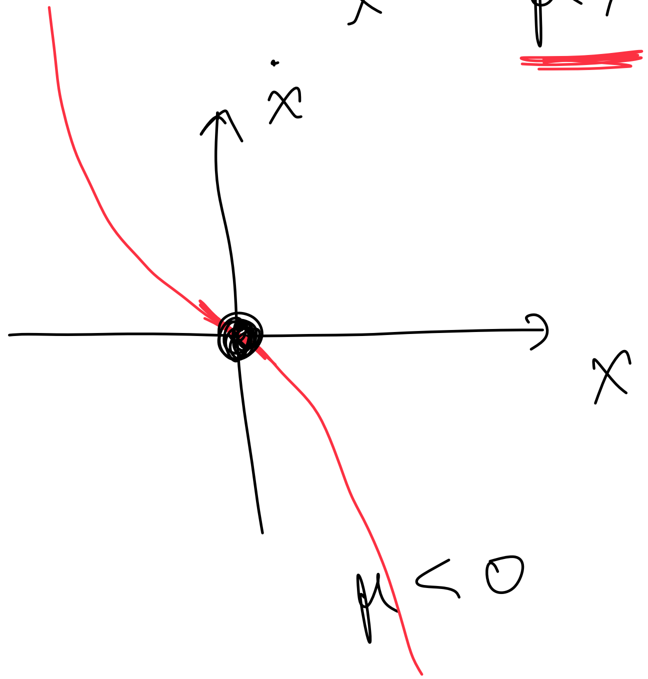
Pitchfork Bifurcation

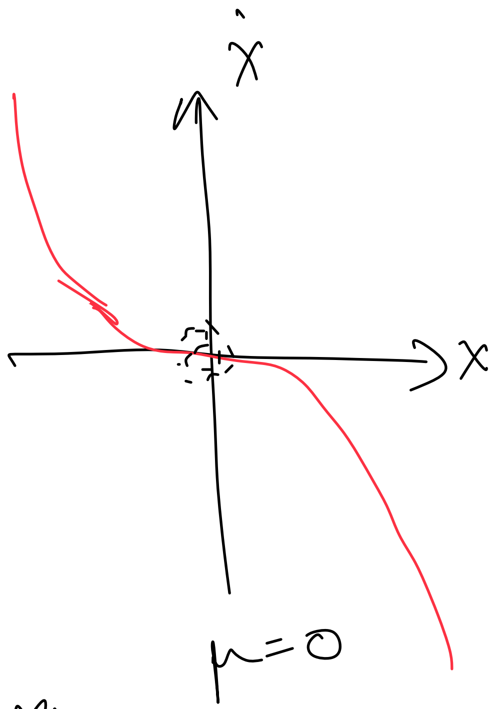
$$\dot{x} = \mu x - x^3$$

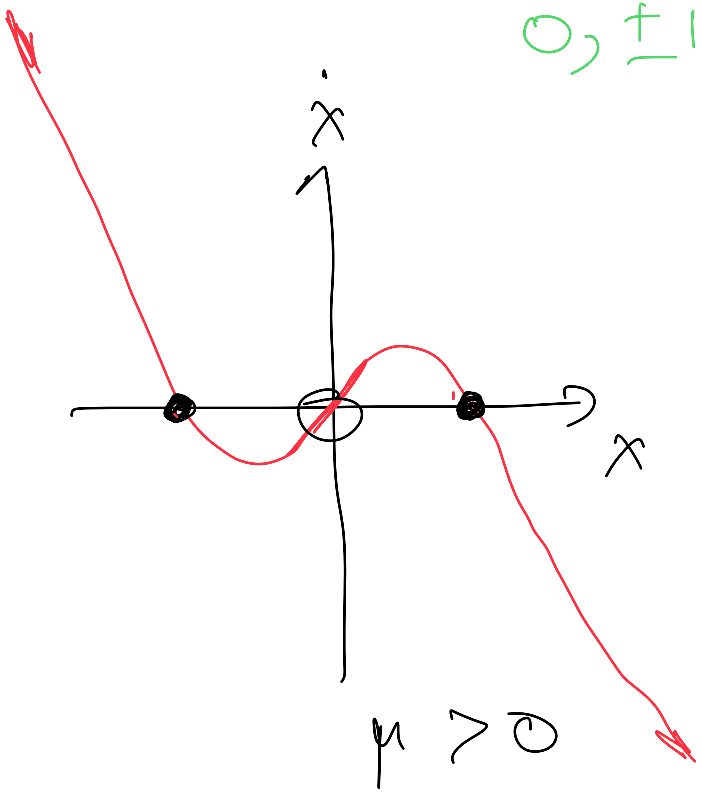
$\mu x - x^5$

$+x - x^3$

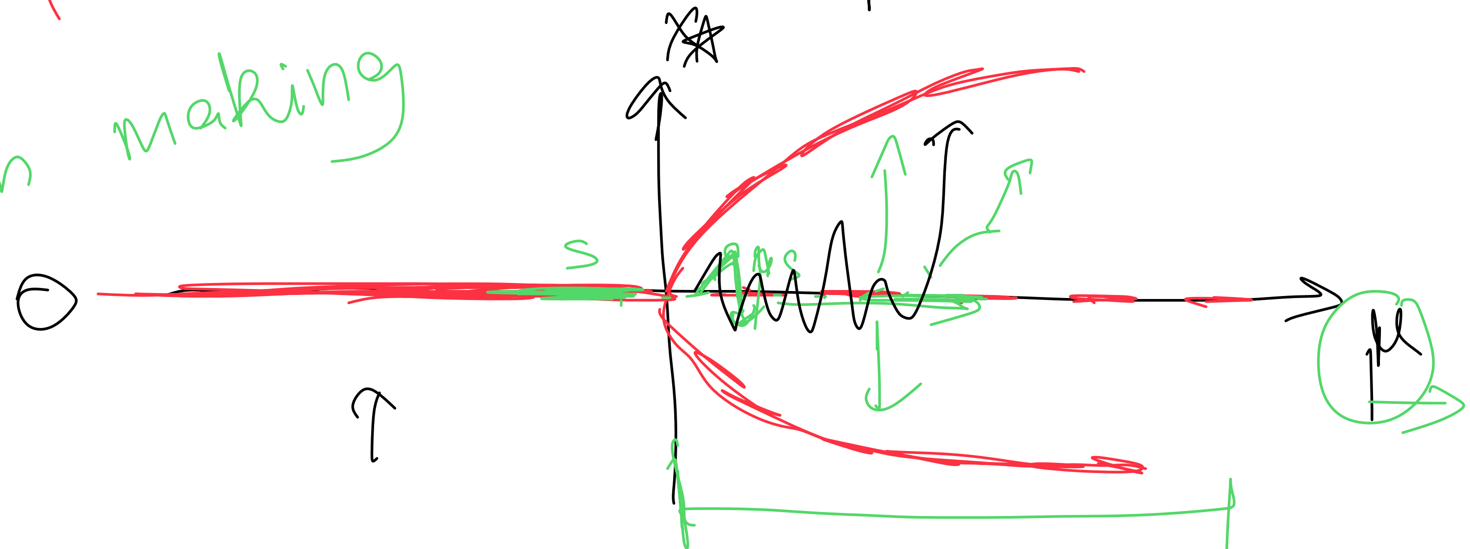
$0, \pm 1$



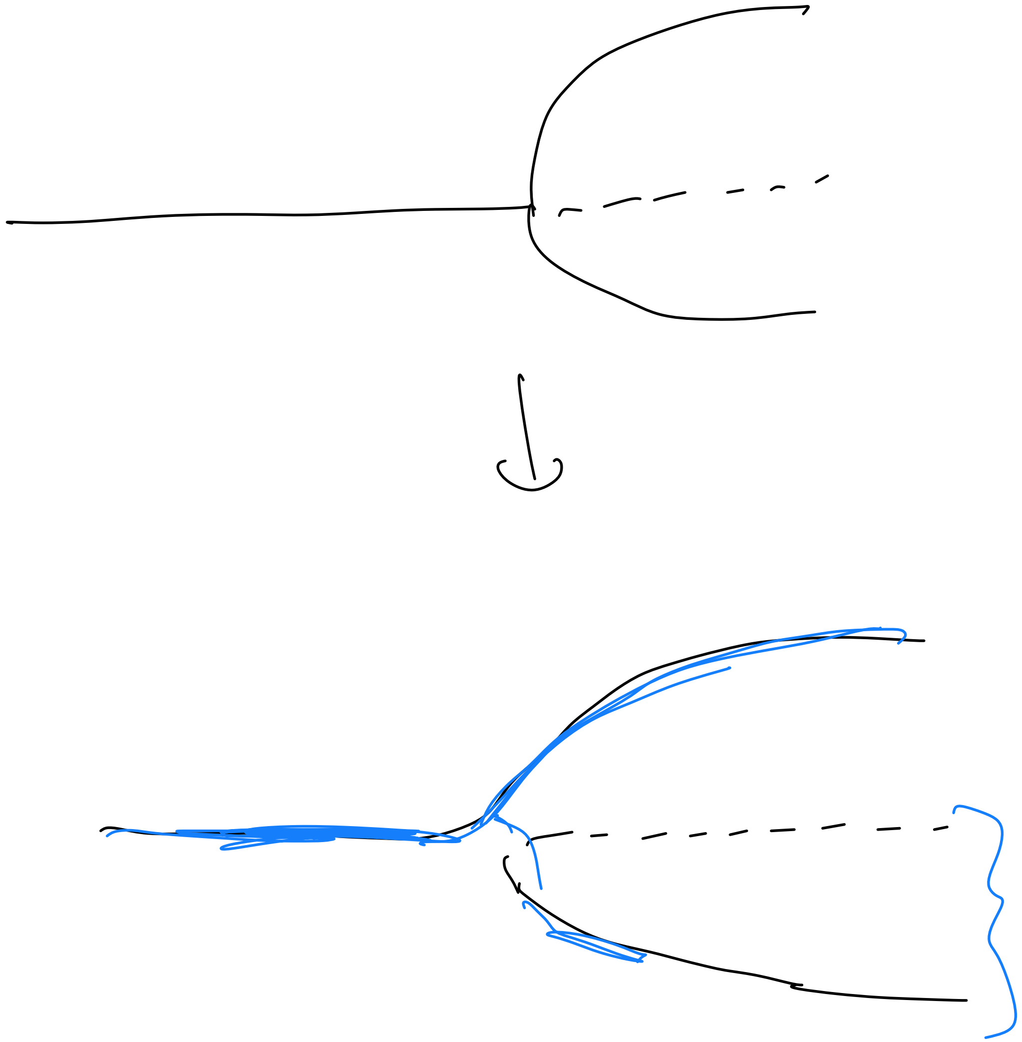




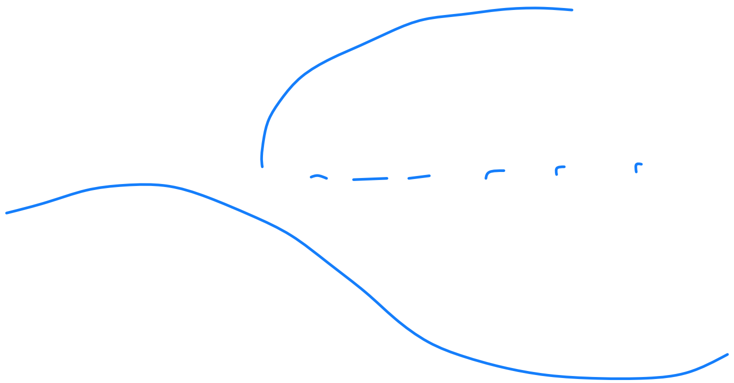
2-choice decision making



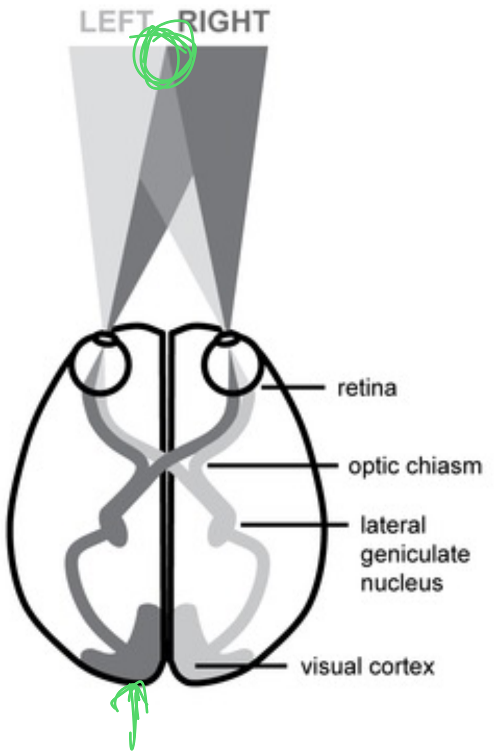
$$\dot{x} = \mu x - x^3 + f(x)$$



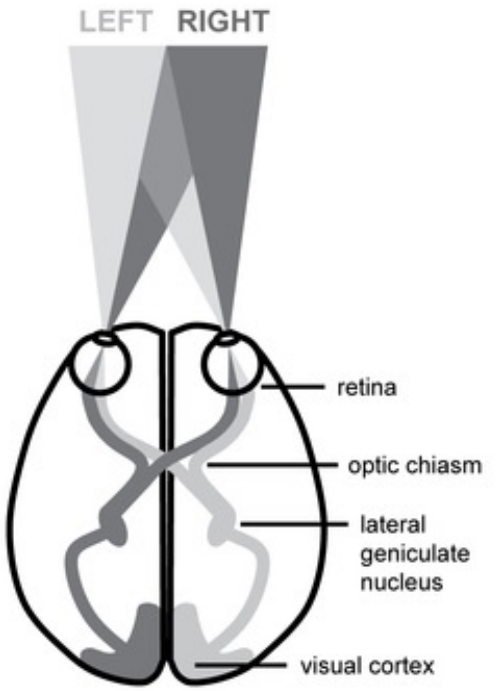
S.N. Bif.

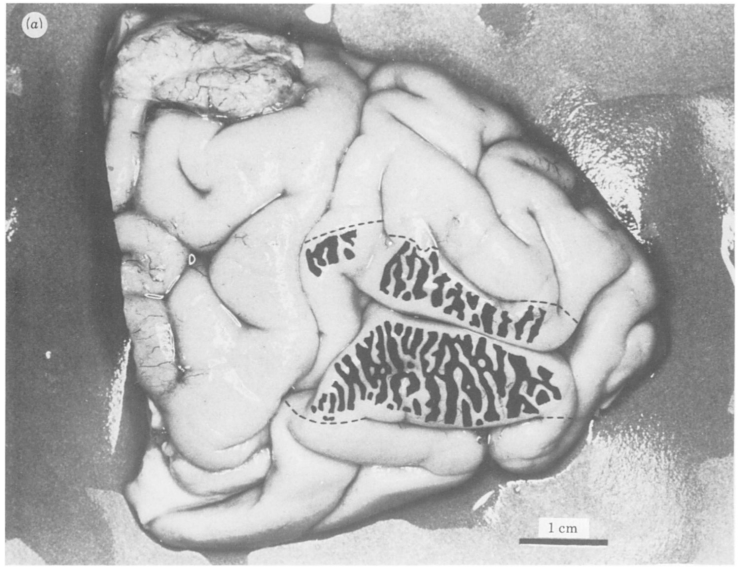


Ocular dominance columns

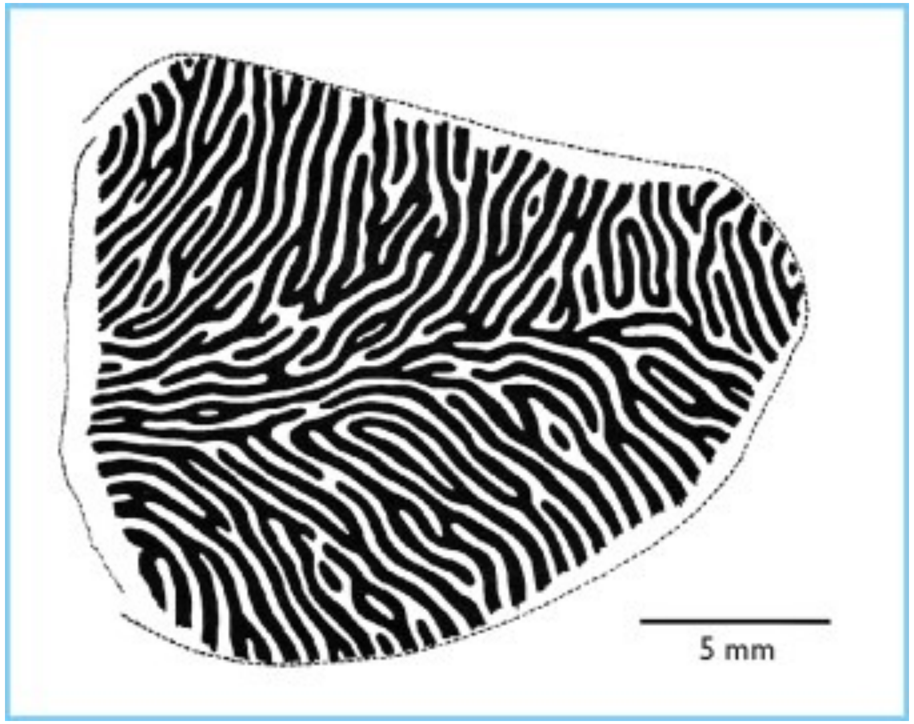


Ocular dominance columns



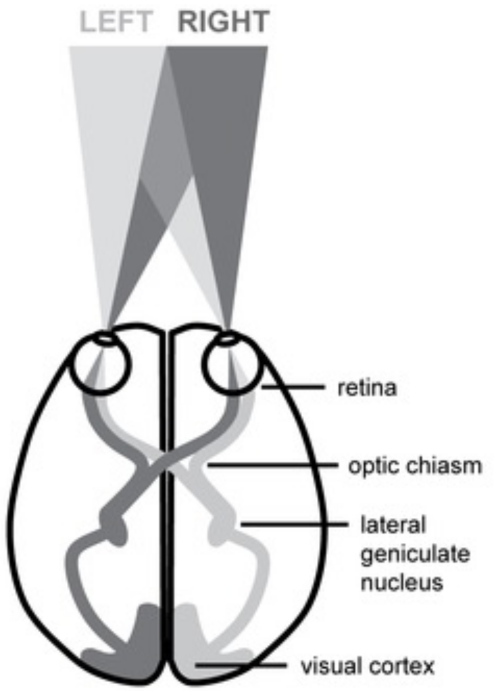


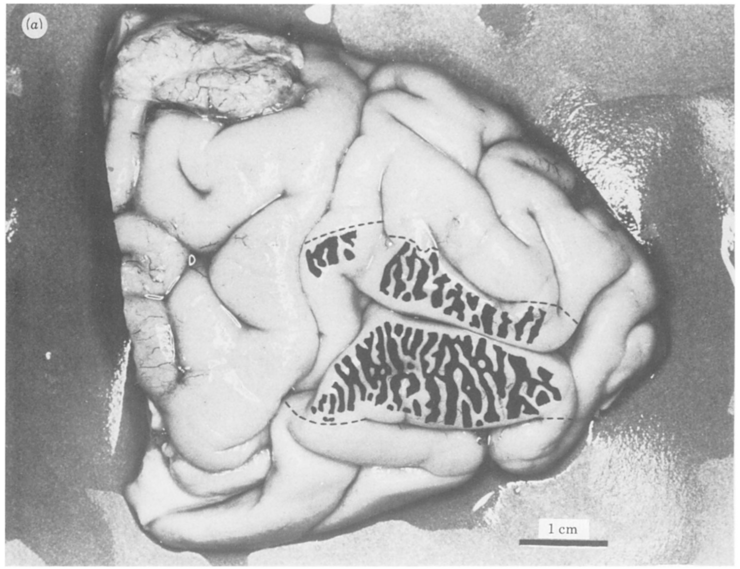
Horton & Hedley-Whyte (1984)



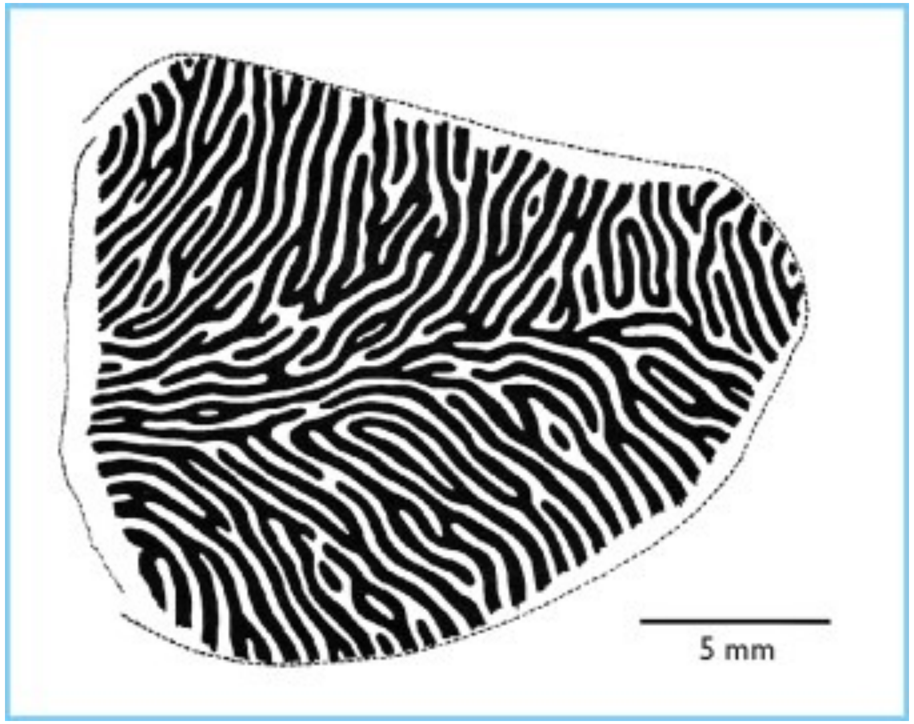
Hubel & Wiesel 1977

Ocular dominance columns





Horton & Hedley-Whyte (1984)



Hubel & Wiesel 1977

Proc. Natl. Acad. Sci. USA
Vol. 94, pp. 9944–9949, September 1997
Neurobiology

Check for updates

A model of ocular dominance column development by competition for trophic factor

Anthony E. Harris*†‡§, G. Bard Ermentrout†¶, and Steven L. Small*†‡

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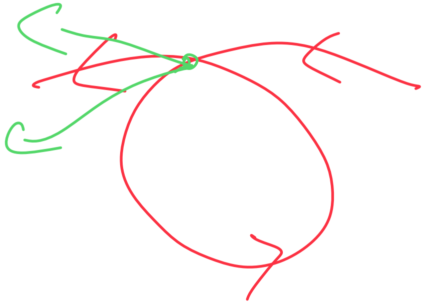
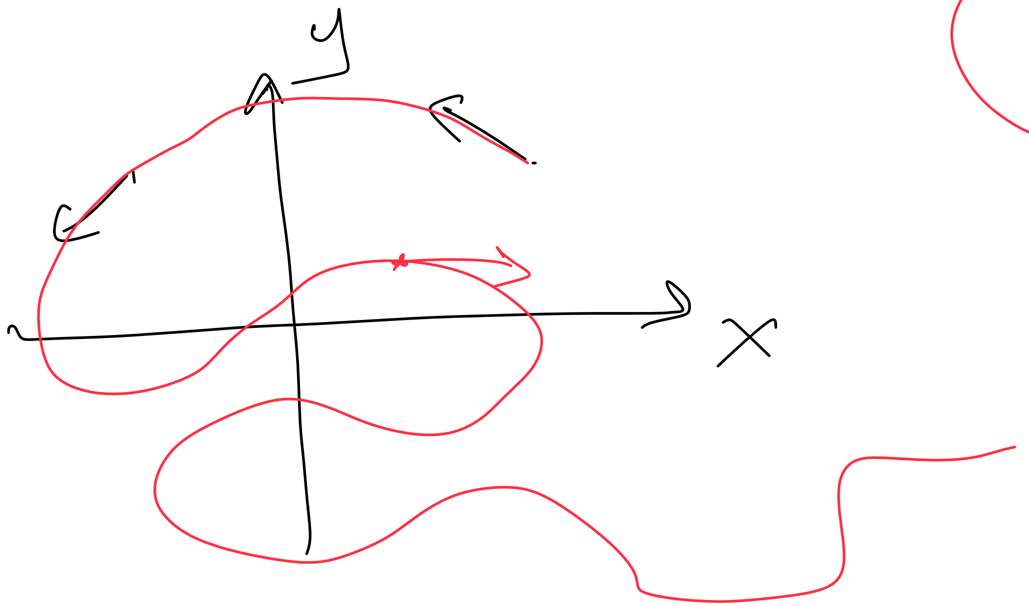
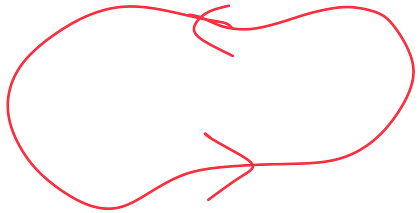
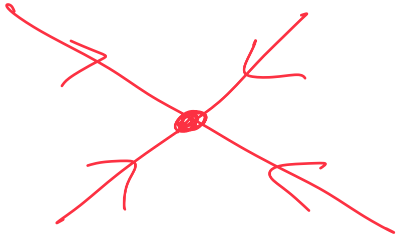
2D Flows

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} \qquad \dot{\bar{x}} = \bar{F}(\bar{x})$$









1) Go to infinity
2) Go to a F.P.
3) Go to a periodic Orbit

→ Poincare-Bendixson Theorem ←

$$\dot{\bar{x}} = F(\bar{x})$$

$$\dot{\bar{x}} = 0 \Rightarrow F(\bar{x}_*) = 0$$

Jecobien

$$\bar{x} = \bar{x}_* + \bar{\varepsilon}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigen values $\lambda_1, \lambda_2 \rightarrow$ stable. F.P $\Longleftrightarrow$ $Re[\lambda_1] < 0$, $Re[\lambda_2] < 0$

unstable $\rightarrow$ all other cases

$$F(x_*) = 0$$

$F' > 0$ unstable

$F' < 0$ stable.

$$x = x_* + \varepsilon$$

$$\dot{\varepsilon} = F' \cdot \varepsilon$$

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$Tr[J] = T$$

$$Det[J] = \Delta$$

$$\det[\lambda \mathbb{1} - J) = 0$$

$$\det\begin{bmatrix} \lambda - a & b \\ c & \lambda - d \end{bmatrix} = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$\lambda^2 - Tr[J]\lambda + Det[J] = 0$$

$$\lambda^2 - T\lambda + \Delta = 0$$

$$\hookrightarrow \boxed{\lambda = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}}$$