Congruent Numbers II

Recall the definition of a Magic square.

Definition (**Magic square**): A Magic square is a $n \times n$ grid of numbers such that the sum of each row, the sum of each column and the sum of the diagonal are the same.

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

There has been some recent interest in whether there can exist a three-by-three magic square whose nine entries are all perfect squares. The answer is of course YES, for example,

$$\begin{bmatrix} 5^2 & 1^2 & 7^2 \\ 7^2 & 5^2 & 1^2 \\ 1^2 & 7^2 & 5^2 \end{bmatrix}$$

which is a particular case of the parametrized square

$$\begin{bmatrix} (m^2+n^2)^2 & (m^2-2mn-n^2)^2 & (m^2+2mn-n^2)^2 \\ (m^2+2mn-n^2)^2 & (m^2+n^2)^2 & (m^2-2mn-n^2)^2 \\ (m^2-2mn-n^2)^2 & (m^2+2mn-n^2)^2 & (m^2+n^2)^2 \end{bmatrix}.$$

Martin Gardner has offered \$100 for an example of a three-by-three magic squares of squares in which the nine entries are distinct or for a proof of the non-existence of such a square.

One of the major open problems is:

Conjecture 1. *Prove or disprove the existence of a three-by-three magic square whose nine entries are distinct perfect square.*

1 Questions

Question 1.1 *Any three-by-three magic square of integers is of the form*

$$\begin{bmatrix} a-b & a+b+c & a-c \\ a+b-c & a & a-b+c \\ a+c & a-b-c & a+b \end{bmatrix}$$

with $a, b, c \in \mathbb{Z}$. The square has repeated entries precisely when

$$bc(b^2-c^2)(b^2-4c^2)(4b^2-c^2) = 0. \tag{1}$$

Question 1.2 *Show that Conjecture 1 is equivalent to proving or disproving the existence of three arithmetic progressions of length 3 having the same common difference, each terms are perfect squares such that their middle terms themselves form an arithmetical progression.*

Question 1.3 *Two arithmetical progressions of squares of length 3 have the same common difference of terms if and only if the corresponding Pythagorean right-angled triangles have the same area.*

Question 1.4 *Show that Question 1.2 is equivalent to proving or disproving that there are three right-angled triangles with the same area such that squares of the hypotenuses are in arithmetical progression i.e. that there exists one congruent number n such that n is the area of three different right angled triangles whose sides X_i, Y_i, Z_i for $i = 1, 2, 3$ and the squares of the hypotenuses are in arithmetical progression.*

We are interested in a special type of graph of the curve $E_N : y^2 = x^3 - n^2x$ where n is an congruent number.

Question 1.5 *Prove that Question 1.4 is equivalent to proving or disproving the existence of three integral points satisfying E_N such that their x-coordinates are in arithmetical progression.*