Congruent Numbers II

Recall the definition of a congruent number.

Definition(**Congruent number**): A positive rational number n is called a *congruent number* if there is a rational right triangle with area n: there are rational $a, b, c > 0$ such that $a^2 + b^2 = c^2$ and $(1/2)ab = n$.

The congruent number problem makes its earliest appearance in an Arab manuscript traced to the tenth century and around 1000 years old.

1 Questions

Question 1.1 *Show that a rational right triangle with area n produces a rational solution to the equation $E_N : y^2 = x^3 - n^2 x$.*

(Hint! Question 2.2 from last time, a number n is congruent if and only if there exists a rational number a such that $a^2 + n$ and $a^2 - n$ are both squares of rational numbers.

Question 1.2 *Let $x_0, y_0 \in \mathbb{Q}$, so that*

$$y_0^2 = x_0^3 - n^2 x_0$$

Assume that x_0 satisfies:

(1) x_0 is the square of a rational number

(2) x_0 has even denominator

(3) the numerator of x_0 is relatively prime to n

There exists a right triangle with rational sides and area n which corresponds to x_0.

Question 1.3 *Define sets A and B by*

$$A = \{(X, Y, Z) \in \mathbb{Q}^3 : \frac{1}{2} XY = n, X^2 + Y^2 = Z^2\}$$

$$B = \{(x, y) \in \mathbb{Q}^2 : y^2 = x^3 - n^2 x, y \neq 0\}$$

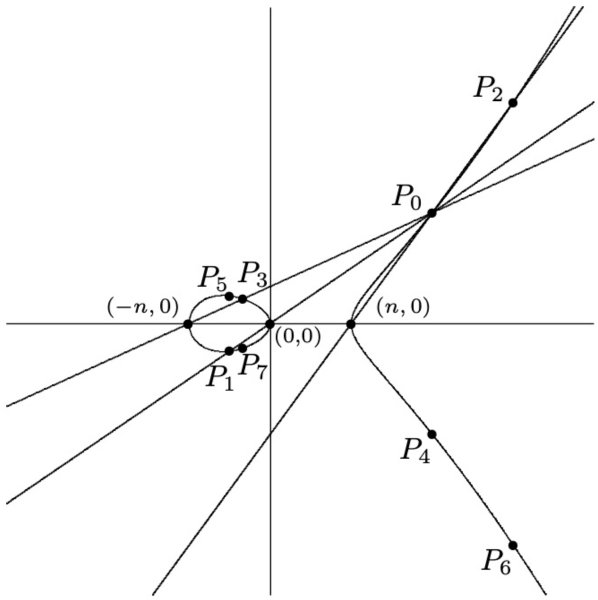
prove that there is a bijection between A and B given by maps

$$f(X, Y, Z) = \left(\frac{nY}{X + Z}, \frac{2n^2}{X + Z} \right)$$

and

$$g(x, y) = \left(\frac{N^2 - x^2}{y}, -\frac{2xn}{y}, \frac{n^2 + x^2}{y} \right)$$

Graph of $E_N : y^2 = x^3 - n^2 x$ intersecting with lines through P_0 and $(0,0)$, $(n,0)$, $(-n,0)$ and reflected points.



This geometric viewpoint is connected to a million-dollar problem (namely), Birch and Swinnerton–Dyer (BSD) conjecture. Assuming the conjecture, there is a tentative solution to the congruent number problem.

Theorem 1.4 *(Tunell) Let n be a squarefree positive integer. Define*

$$f(n) = \#\{(x,y,z) \in \mathbb{Z}^3 : x^2 + 2y^2 + 8z^2 = n\}$$
$$g(n) = \#\{(x,y,z) \in \mathbb{Z}^3 : x^2 + 2y^2 + 32z^2 = n\}$$
$$h(n) = \#\{(x,y,z) \in \mathbb{Z}^3 : x^2 + 4y^2 + 8z^2 = n/2\}$$
$$k(n) = \#\{(x,y,z) \in \mathbb{Z}^3 : x^2 + 4y^2 + 32z^2 = n/2\}$$

For odd n, if n is congruent then $f(n) = 2g(n)$. For even n, if n is congruent then $h(n) = 2k(n)$. Moreover, if BSD conjecture is true, then the converse of both implications are true: $f(n) = 2g(n)$ implies n is congruent when n is odd and $h(n) = 2k(n)$ implies n is congruent when n is even.

Remark 1.5 *Tunnell's theorem provides an unconditional method of proving a squarefree positive integer n is not congruent (show $f(n) \neq 2g(n)$ or $h(n) \neq 2k(n)$, depending on the parity of n), and a conditional (BSD conjecture) method of proving n is congruent.*

Example 1.6 *As $f(1) = g(1) = 2$ and $f(3) = g(3) = 4$, we have $f(n) \neq 2g(n)$ for $n = 1$ and 3, so Tunell's criterion shows that 1 and 3 are not congruent.*

Question 1.7 *What about 2, 5, and 7?*

Question 1.8 *Conditionally (on BSD conjecture) show that any squarefree positive integer n satisfying $n \equiv 5, 6, 7 \pmod 8$ is a congruent number.*

Question 1.9 *Can you determine whether 260124 is congruent? (Hint! $260124 = 2^2.3.53.409$)*