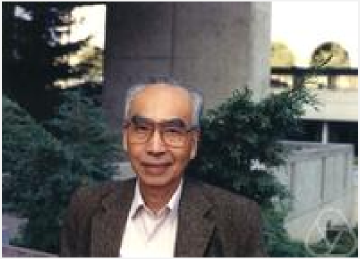
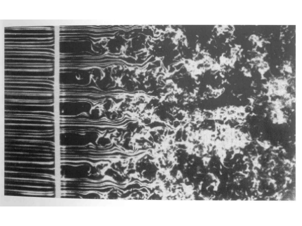
From the d'Alembert paradox to the 1984 Kato criteria via the 1941 1/3 Kolmogorov law and the 1949 Onsager conjecture

Claude Bardos
Retired .
https://www.ljll.math.upmc.fr/˜bardos

Turbulence: Problems at the Interface of Mathematics and Physics.





Several of my recent contributions , mostly with Edriss Titi and more recently with E. Wiedemann , A. and P. Gwiadza, were motivated by the following issues:
The role of boundary effects in mathematical theory of fluid mechanic and the similarity , in presence of these effects, of the weak convergence in the zero viscosity limit with the anomalous energy dissipation in the statistical theory of turbulence.
As consequences.

- I will recall the Kolmogorov $1/3$ law and the Onsager conjecture and compare them to the issue of anomalous energy dissipation.
- Give extensions : For general systems with an extra conservation laws about local and global conservation.
- Give several forms of a basic theorem of Kato in the presence of a Lipschitz solution of the Euler equations. Insisting that in such case the absence of anomalous energy dissipation is **equivalent** to the persistence of regularity in the zero viscosity limit.

Energy Balance and Kolmogorov 1/3 law

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|u_\nu|^2dx + \nu\int_\Omega |\nabla_x u_\nu|^2dx = 0$$

compare the two points correlation and the rate of energy dissipation:

$$\frac{\delta u_\nu(x,l,t)}{l^\beta} = \frac{u_\nu(x+l,t) - u_\nu(x,t)}{l^\beta}$$

$$\text{And}\quad \epsilon(t) = -\frac{d}{dt}\int_\Omega \frac{1}{2}|u_\nu|^2dx = \nu\int_\Omega |\nabla_x u_\nu|^2dx$$

Assuming **Isotropy** and **Homogeneity** by dimensional analysis :

$$\text{Kolmogorov Law}\quad \langle \frac{u_\nu(x+l,t) - u_\nu(x,t)}{|l|^{\frac{1}{3}}}\rangle \simeq \langle \nu\int_\Omega |\nabla_x u_\nu|^2dx\rangle^{\frac{1}{3}}\,.$$

Assuming only **Homogeneity**, Karman-Howard-Monin (cf. Frisch page 77) derived the 4/5 law:

$$\langle (u_\nu(x+l,t) - u(x,t))\cdot \frac{l}{|l|})^3\rangle = -\frac{4}{5}\epsilon|l|$$

The Onsager Conjecture. Deterministic version of Komogorov law

Any weak solution which belongs to the space $C^{0,\alpha}$ with $\alpha > \frac{1}{3}$ conserves the energy.

1. Complete mathematical proofs were given in (1994) by Eyink then Constantin , E, Titi. in 1994 and several other extensions A Cheskidov, Constantin, Friedlander, and Shvydkoy. (2008).
2. Recent papers (2017) Buckmaster, Isett , De Lellis, Szkelyhidi and Vlad Vicol have shown, for $\alpha < \frac{1}{3}$, the existence of wild solutions in $C^{0,\alpha}((0,T)\times\mathbb{T}^3)$. Hence $\alpha > \frac{1}{3}$, is **a necessary and sufficient condition for all solutions of the $3d$ Euler equation to conserve energy.**

More Remarks

1. Simple examples (shear flows for instance) which are not regular but nevertheless at the Euler level conserve the energy and at the zero viscosity limit no anomalous energy dissipation(C.B, E. Titi).
2. The condition $\alpha > \frac{1}{3}$ generalized to the class of systems of conservation laws is also a sufficient condition for the conservation of extra (entropy) conservation law. However there are examples for this which condition can be relaxed . Cf. For many examples CB. ... arXiv:1806.02483 and for Vlasov Maxwell Eyink arXiv:1803.03691 .
3. Very useful to analyze first the local validity of a companion law in a sub domain following Duchon and Robert and then to extend this result to a domain with boundary, controlling the effect of the boundary.

With $Q \subset \mathbb{R}^{d+1}$ a system of conservation laws, $x \mapsto u(x)$ is defined in Q with values in an open convex set $\mathcal{M} \subset\subset \mathbb{R}^k$.
$A_i\,, i = 0, 1, \cdots, d \in C^2(\mathcal{M}; \mathbb{R}^l)\,,\ A_i^j$, for $j = 1, \cdots, l$, denotes the *j-th* component of A_i .

$$\sum_{0\le i\le d} \partial_{x_i} A_i(u) = 0\,, \tag{1}$$

Theorem 1 Let
$u \mapsto B(u) \in \mathcal{C}^1(M; L(\mathbb{R}^l; \mathbb{R})) : B(u) = (b_1(u), b_2(u), \cdots, b_l(u))$, be a *generalized entropy*: for $i = 0, 1, \cdots, d$

$$\begin{aligned} &B(u)\cdot\nabla_u A_i(u) = \nabla_u q_i(u) \Leftrightarrow \\ &\sum_{ij} b_k(u)\partial_{u_j} A_i^k \partial_{x_i} u_j = \sum_j \partial_{u_j} q_i(u)\partial_{x_i} u_j\,. \end{aligned} \tag{2}$$

Suppose that u is a weak solution of (1). Suppose also that for every $\mathcal{K} \subset\subset Q$ $u \in C^{0,\alpha(\mathcal{K})}(\mathcal{K})$ with $\alpha(\mathcal{K}) > \frac{1}{3}$ then in $\mathcal{D}'(Q)$

$$\sum_{0\le i\le d} \partial_{x_i} q_i(u) = 0\,. \tag{3}$$

The Euler equation in the above Formalism

$$\partial_t v + \nabla\cdot(v\otimes v) + \nabla p = 0\,,\quad \nabla\cdot v = 0\,.$$

$$u = \begin{pmatrix} p \\ v \end{pmatrix}\,,\qquad B(u) = (0\,,\ v),$$

$$x_0 = t\ ,\quad A_0(u) = \begin{pmatrix} 0 \\ v \end{pmatrix}\,,$$

$$A_i^0(u) = v_i\,,\quad A_i^j(u) = \left(v_i v_j + p\delta_{ij}\right)\quad \text{for}\quad i,j = 1,2,3\,.$$

$$q_0(u) = \eta(u) = \frac{|v|^2}{2}\,, q_i(u) = ((\frac{|v|^2}{2} + p)v_i)\,.$$

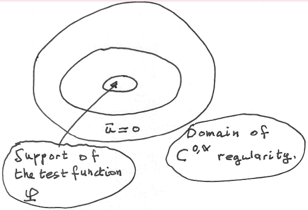
For given test function $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$ with support compact in Q using extension, mollifiers and Fubini theorem.

$$\sum_{0\le i\le d} \langle \partial_{x_i}\phi, q_i(u)\rangle = 0\,,\ T \mapsto \overline{T}\,,\ \rho_\epsilon(x) = \frac{1}{\epsilon^{d+1}}\rho(\frac{|x|}{\epsilon})\overline{T}^\epsilon = \overline{T}\star\rho_\epsilon\,.$$

$$\langle \sum_{0\le i\le d} \partial_{x_i} A_i(u), \psi\rangle = 0 \Rightarrow \langle \sum_{0\le i\le d} A_i(u), \partial_{x_i}\psi\rangle = 0\,. \qquad (4)$$

Test function $\quad \Psi^\epsilon = \phi B((\overline{u})^\epsilon) \in C^2_c(Q_1; \mathcal{L}(\mathbb{R}^l, \mathbb{R}))$:

$$\langle \sum_{0\le i\le d} \overline{A_i(u)}, \partial_{x_i}\psi_\epsilon\rangle = 0 \Rightarrow \langle \sum_{0\le i\le d} A_i(\overline{u}), \partial_{x_i}\psi_\epsilon\rangle = 0\,.$$



$$
\begin{aligned}
\text{Fubini}\quad 0 &= \langle \Psi^\epsilon, \sum_{0\le i\le d} \partial_{x_i} A_i(\overline{u})\rangle = -\sum_{0\le i\le d} \langle (\partial_{x_i}\Psi)^\epsilon, A_i(\overline{u})\rangle \\
&= -\sum_{0\le i\le d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i}\Psi^\epsilon(x)\cdot (A_i(\overline{u}))^\epsilon(x)\, dx \\
&= -\sum_{0\le i\le d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i}\Psi^\epsilon(x)\cdot A_i((\overline{u})^\epsilon)(x)\, dx \\
&- \sum_{0\le i\le d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i}\Psi^\epsilon(x)\cdot \Big((A_i(\overline{u}))^\epsilon(x) - A_i((\overline{u})^\epsilon)(x)\Big)\, dx\,.
\end{aligned}
\tag{5}
$$

The right-hand side of (5) is the sum of two terms:

$$
\begin{aligned}
J_\epsilon &= -\sum_{0\le i\le d}\int_{\mathbb{R}_x^{d+1}} \partial_{x_i}(\phi B((\overline{u})^\epsilon)(x)\cdot A_i((\overline{u})^\epsilon))(x)\,dx\,,\\
K_\epsilon &= \sum_{0\le i\le d}\int_{\mathbb{R}_x^{d+1}} \partial_{x_i}(\phi B((\overline{u})^\epsilon)(x)\cdot\Big(A_i((\overline{u})^\epsilon)(x)-(A_i(\overline{u}))^\epsilon(x)\Big)\,dx\,.
\end{aligned}
\tag{6}
$$

For the first term $(\overline{u})^\epsilon$ being smooth one uses the chain rule and then the Lebesgue theorem:

$$J_\epsilon = -\sum_{0\leq i\leq d}\int_{\mathbb{R}_x^{d+1}} \partial_{x_i}(\phi B((\overline{u})^\epsilon))(x)\cdot A_i((\overline{u})^\epsilon)(x)\,dx$$

$$= \sum_{0\leq i\leq d}\int_{\mathbb{R}_x^{d+1}} (\phi B((\overline{u})^\epsilon))(x)\cdot \partial_{x_i} A_i((\overline{u})^\epsilon)(x)\,dx =$$

$$\sum_{0\leq i\leq d}\int_{\mathbb{R}^d} \phi(x)\partial_{x_i} q_i((\overline{u})^\epsilon(x))dx = -\sum_{0\leq i\leq d}\int_{\mathbb{R}^d} \partial_{x_i}\phi(x) q_i((\overline{u})^\epsilon(x))dx\,.$$

$$\to -\sum_{0\leq i\leq d}\int_Q \partial_{x_i}\phi(x) q_i(u)dx = \sum_{0\leq i\leq d}\langle \phi, \partial_{x_i} q_i(u)\rangle\,.$$

To complete the proof it remains to show that for $\alpha > \frac{1}{3}$ the term

$$K_\epsilon = \sum_{0\le i\le d} \int_{\mathbb{R}_x^{d+1}} \partial_{x_i}(\phi B((\overline{u})^\epsilon)(x) \cdot \Big(A_i((\overline{u})^\epsilon)(x) - (A_i(\overline{u}))^\epsilon(x) \Big)\, dx\,.$$

converges to 0, as $\epsilon \to 0$.

1. $\|\partial_{x_i}(\phi B((\overline{u})^\epsilon))\|_{L^\infty(Q_1)} \le C\|\phi\|_{C^1(Q_1)}\|B\|_{C^2(\mathcal{M})}\|u\|_{C^{0,\alpha(Q_2)}}\epsilon^{\alpha(Q_2)-1}$.
2. $\|A_i((\overline{u})^\epsilon)(x) - (A_i(\overline{u}))^\epsilon(x)\|_{L^\infty} \le \epsilon^{2\alpha}\|A\|_{C^2(\mathcal{M}))}\|u\|^2_{C^{0,\alpha(\mathcal{K})}(\mathcal{K})}$.

F affine map $\Rightarrow (F(v))^\epsilon - F(v^\epsilon) = 0$. Taylor formula $\Rightarrow$

$$|(F(v))^\epsilon(x) - F(v^\epsilon(x))| = |\Big(\int_{\mathbb{R}_y^d} F(v(x-y))\rho_\epsilon(y)dy \Big) - F(v^\epsilon(x))|$$

$$= | \int_{\mathbb{R}_y^d} \rho_\epsilon(y)dy \Big(\int_0^1 \Big(\nabla_v^2 F(sv(x-y) + (1-s)v^\epsilon(x))\Big)(1-s)ds \Big)$$

$$\le \|F\|_{C^2(M))} \int_{\mathbb{R}_y^d} | \int_{\mathbb{R}_z^d} \Big(v(x-y) - v(x-z)\Big)\rho_\epsilon(z)dz|^2 \rho_\epsilon(y)dy$$

$$\le C\epsilon^{2\alpha}. \qquad \square$$

Consider in $Q=(T_1,\,T_2)\times\Omega$ (Ω an open set with Lipschitz boundary) solutions $u\in L^\infty(Q)$ of the following:

$$\partial_t A_0(u)+\sum_{1\le i\le d}\partial_{x_i}(A_i(u))=0\,, \tag{7}$$

Assume that this equation has an extra conservation law $u\mapsto\eta(u)$ (or entropy as often called) with corresponding fluxes $q_j(u)$, for $j=1,2,\dots d$, satisfying

$$\begin{aligned}&\nabla_u\eta(u)\cdot\nabla_u A_0(u)=\nabla_u\eta(u)\\ \text{and }&\nabla_u\eta(u)\cdot\nabla_u A_j(u)=\nabla_u q_j(u)\quad\text{for } j=1,2,\dots d\,.\end{aligned} \tag{8}$$

which gives formally the extra conservation law:

$$\partial_t\eta(u)+\sum_{1\le i\le d}\partial_{x_i}q_i(u)=0 \tag{9}$$

With, $x_0 = t$ and $(B(u) = \nabla_u \eta(u))$ one has the following:
Theorem 2 Assume that u is a weak solution of (7) with the following properties:
1. For any $\tilde{Q} \subset Q$ one has $u_{|\tilde{Q}} \in C^{0,\alpha}(\tilde{Q})$ with $\alpha > \frac{1}{3}$.
2. With $\delta \in (0, \frac{\delta_0}{2})$, small enough $\hat{x}$ being defined by $\hat{x} = d(x, \partial\Omega)$, $Q_\delta = (T_1, T_2) \times \Omega_\delta$ assume that:

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{[T_1 T_2] \times \{\frac{\delta}{4} < d(x) < \frac{\delta}{2} < \frac{\delta_0}{2}\}} |\sum_{1 \leq i \leq d} q_i(u(t,x)) \vec{n}_i(\hat{x})| dt = 0\,; \qquad (10)$$

Then in the sense of $\mathcal{D}'(T_1, T_2)$ the solution u conserves the total entropy

$$\frac{d}{dt} \int_\Omega \eta(u(t,x)) dx = 0\,. \qquad (11)$$

Test functions $\phi(t,x) = \theta(t) \times \chi\left(\frac{d(x,\partial\Omega)}{\delta}\right)$, with $\delta \in (0,\delta_0)$, and χ a C^∞ cutoff function $s \mapsto \chi(s)$ equal to 0 for $s < \frac{1}{4}$ and equal to 1 for $s > \frac{1}{2}$:

$$\langle \phi(t,x), (\partial_t \eta(u) + \sum_{1\le i\le d} \partial_{x_i} q_i(u)) \rangle$$
$$= -\int_Q \eta(u(t,x))\chi\left(\frac{d(x,\partial\Omega)}{\delta}\right)\frac{d}{dt}\theta(t)dxdt$$
$$-\int_{Q_\delta} \theta(t)\Big(\sum_{1\le i\le d} q_i(u(t,x))\vec{n}_i(\hat{x})\frac{1}{\delta}\chi'((\frac{d(x,\partial\Omega)}{\delta}))\Big)dxdt\,. \tag{12}$$

Then let $\delta \to 0$ use the hypothesis (10) and the Lebesgue Dominant Convergence theorem .

Theorem 3 Navier-Stokes Leray-Hopf in $(0,T)\times\Omega$ with $u_\nu=0$ on $\partial\Omega$:

$$\partial_t u_\nu + (u_\nu\cdot\nabla_x)u_\nu - \nu\Delta u_\nu + \nabla p_\nu = 0\,,\; u_\nu(0,\cdot)=u_0$$

1 There exists an open subset $V_{\eta_0}=\{x\in\Omega\,,d(x)<\eta_0\}$, and $\beta<\infty$ (both being independent of ν):

$$\text{No Holder hypothesis on } p \text{ but } \sup_\nu \|p_\nu\|_{L^{3/2}((0,T);H^{-\beta}(V_{\eta_0}))}<\infty; \tag{13}$$

2 For any $\Omega\subset\subset\Omega$ there exists $\alpha=\alpha(\Omega)>\frac{1}{3}$ and a constant $M(\Omega)$ such that for any $\nu>0$ one has:

$$\text{Local hypothesis:}\quad \|u_\nu\|_{L^3((0,T);C^{0,\alpha}(\overline{\Omega}))}\le M(\Omega); \tag{14}$$

3 Boundary Flux control

$$\lim_{\eta\to0}\lim_{\nu\to0}\int_0^T\frac{1}{\eta}\int_{\{x\in\Omega:\ \frac{\eta}{4}<d(x)<\frac{\eta}{2}<\frac{\eta_0}{2}\}}\left(\frac{|u_\nu|^2}{2}+p_\nu\right)u_\nu(t,x)\cdot\vec{n}(\sigma(x))\,dt\le0\,. \tag{15}$$

Modulo subsequence u_ν converges weak$-*$ in $L^\infty((0,T);L^2(\Omega))$ to a a weak solution of the Euler equations $\overline{u_\nu}\in C_{\mathrm{weak}}([0,T);L^2(\Omega))$ with the same initial data $u_0(\cdot)$ and which also satisfies the hypotheses of Theorem II .
Moreover, $\overline{u_\nu}$ belongs to $C([0,T);L^2(\Omega))$ and conserves the energy.
Eventually , there is no anomalous energy dissipation in the vanishing viscosity limit, i.e., for every $T^*\in(0,T)$ one has:

$$\lim_{\nu\to 0}\nu\int_0^{T^*}\int_\Omega |\nabla_x u_\nu(t,x)|^2 dxdt = 0. \tag{16}$$

(13) is used to prove on a very weak assumption the local $C^{0,\alpha}$ regularity of the pressure with (14) this gives the convergence to a weak solution of the Euler equations which satisfies the hypothesis of the theorem I .
Then with (15) which describes the vanishing of the Bernouilli pressure on the boundary one obtains the global conservation of energy.
Eventually the absence of anomalous energy dissipation follows from the Leray-Hopf energy inequality and from the relation:

$$\|\overline{u_\nu}\|^2_{L^2(\Omega)} \leq \lim_{\nu\to 0} \|u_\nu\|^2_{L^2(\Omega)}$$

- The uniform no flux hypothesis concerns the **normal** component of the velocity. This hypothesis is not in contradiction with a Prandlt type boundary layer.
- This hypothesis is not redundant. In the presence of boundary there exists dissipative wild solutions that do not conserve the energy and that are not characterized by their value at $t = 0$ B. Szkelyhidi and Wiedemann.
- It is only in the presence of a smooth (Lispchitz) solution of the Euler equation that weak convergence to such solution turns out to be equivalent to the absence of anomalous energy dissipation. This is an avatar (in the sense of weak convergence) of the Kolmogorov 1/3 law and the object of the Kato theorem.

Relative estimate with $\nabla u \in L^\infty((0,T)\times\Omega)$

$$
\begin{aligned}
&\partial_t(u_\nu - u) + u_\nu\cdot\nabla u_\nu - u\cdot\nabla u - \nu\Delta u_\nu + \nabla p_\nu - \nabla p = 0\\
&(u_\nu\cdot\nabla u_\nu - u\cdot\nabla u, u_\nu - u) = (u_\nu - u, S(u)(u_\nu - u))\,;\\
&\qquad S(u) = \frac{\nabla u + (\nabla u)^t}{2} \in L^\infty((0,T)\times\Omega)\,;\\
&\frac{d}{dt}\frac{1}{2}|u_\nu - u|^2_{L^2(\Omega)} + \nu\int_\Omega |\nabla u_\nu|^2 dx \le |(u_\nu - u, S(u)(u_\nu - u))|\\
&\qquad + \nu\int_\Omega (\nabla u_\nu\cdot\nabla u)dx - \nu\int_{\partial\Omega}(\partial_{\vec n} u_\nu)_\tau u d\sigma\,.\ \text{The bad term!}\,.
\end{aligned}
$$

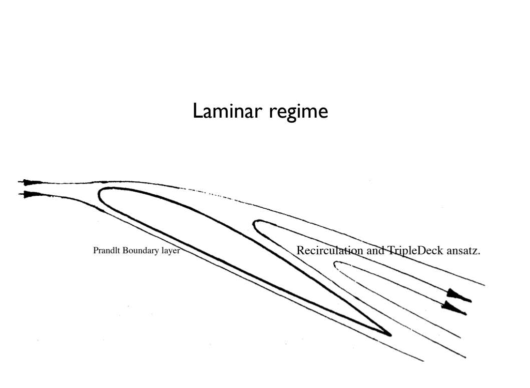
Without physical boundary u_ν converges to u in $C((0,T);L^2(\Omega))$ and $\lim_{\nu\to 0}\int_0^T \|\nabla u_\nu\|^2_{L^2(\Omega)}dt = 0$. Otherwise the situation is much more subtle!!!

It is enough to have a moderate recirculation:

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} \left((\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau u_\tau(\sigma,t) \right)_{-} d\sigma dt = 0$$

or a moderate backward vorticity using:

$$(u_\nu = 0, u\cdot\vec{n} = 0) \Rightarrow \nu(\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau u_\tau = \nu((\nabla \wedge u_\nu)\wedge \vec{n})\cdot u$$



Theorem 4 Are equivalent:

$$\forall w(x,t) \in L^\infty((0,T)\times\partial\Omega) \quad \text{with} \quad w\cdot\vec{n}=0\,,$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\partial\Omega}(\frac{\partial u_\nu}{\partial\vec{n}}(\sigma,t))_\tau w(\sigma,t)d\sigma dt=0 \tag{17}$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\partial\Omega}((\frac{\partial u_\nu}{\partial\vec{n}}(\sigma,t))_\tau u_\tau(\sigma,t))_- d\sigma dt=0 \tag{18}$$

$$u_\nu(t)\to u(t) \text{ in } L^2(\Omega) \text{ uniformly in } t\in[0,T]\,, \tag{19}$$

$$u_\nu(t)\to u(t) \text{ weakly in } L^2(\Omega) \text{ for each } t\in[0,T]\,, \tag{20}$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_\Omega|\nabla u_\nu(x,t)|^2dxdt=0\,, \tag{21}$$

$$\lim_{\nu\to 0}\nu\int_0^T\int_{\Omega\cap\{0<d(x,\partial\Omega)<\frac{\nu}{2}\}}|\nabla u_\nu(x,t)|^2dxdt=0\,. \tag{22}$$

$$\lim_{\nu\to 0}\frac{1}{\nu}\int_0^T\int_{\Omega\cap\{\frac{\nu}{4}<d(x,\partial\Omega)<\frac{\nu}{2}\}}|u_\nu(x,t)|^2dxdt=0\,. \tag{23}$$

The fact (17) implies (18) is trivial then that it implies (19) has already been observed. It implies (20) which gives (21) with the energy inequality

$$\|u_\nu(t)\|^2_{L^2(\Omega)} + 2\nu \int_0^t \int_\Omega |\nabla u_\nu(s)|^2 ds \leq \|u_0\|^2_{L^2(\Omega)} \tag{24}$$

and of course (22).
With Poincaré inequality (22) implies (23).

$$\forall\, w \in L^{lip}(\partial\Omega \times (0,T))\,, \quad w\cdot\vec n = 0\,; \quad s \mapsto \Theta(s) = 1 - \chi(s)\,,$$

$$\hat w_\nu(x,t) = \nabla \wedge ((\vec n(\sigma(x)) \wedge w(\sigma(x),t) d(x,\partial\Omega)\Theta(\frac{d(x,\partial\Omega)}{\nu}))$$

$$\Rightarrow \text{On } \partial\Omega \quad \hat w_\nu(x,t) = w(x,t) \text{ and } \;, \text{ in } \Omega\,,\; \nabla\cdot \hat w_\nu = 0\,.$$

Multiplication of the Navier-Stokes equation by $\hat w_\nu$ integration, use of the Kato estimates on $\hat w_\nu$ shows that (23) implies (17):

$$\nu \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial \vec n}(\sigma,t))_\tau w(\sigma,t) d\sigma =$$

$$\nu(\nabla u_\nu, \nabla \hat w_\nu)_{L^2(\Omega)} - (u_\nu \otimes u_\nu, \nabla \hat w_\nu)_{L^2(\Omega)} + (\partial_t u_\nu, \hat w_\nu)_{L^2(\Omega)}$$

$$|\nabla_x^p \hat w_\nu|_\infty \le C\nu^{-p} and \quad = 0 \quad \text{for} \quad \{x \notin \frac{\nu}{4} < d(x,\partial\Omega) < \frac{\nu}{2}\}$$

$$(23)\; \lim_{\nu\to 0} \frac{1}{\nu} \int_0^T \int_{\Omega\cap\{\frac{\nu}{4} < d(x,\partial\Omega) < \frac{\nu}{2}\}} |u_\nu(x,t)|^2 dx dt = 0$$

$$\Rightarrow \int_0^T (u_\nu \otimes u_\nu, \nabla \hat w_\nu)_{L^2(\Omega)} dt = 0 \text{ and } \int_0^T |\nu(u_\nu, \Delta \hat w_\nu)_{L^2(\Omega)}| dt \to 0\,.$$

- If the notion of vorticity on the boundary which appears in (17) is not (experimentally) well defined, the condition (19) means that moderate recirculation ($o(\nu^{-1})$) does not create turbulence or anomalous energy dissipation.
- (20) means that in presence of a smooth solution of the Euler equations the absence of anomalous energy dissipation in the vanishing vsicosity limit is equivalent to the persistence of regularity in this limit.
- (23) involve the behaviour of the fluid in a subregion $\Omega \cap \{\frac{\nu}{4} < d(x, \partial\Omega) < \frac{\nu}{2}\}$ cf. Drivas and Nguyen SIAM J. Math. Anal. **50**(5): 4785–4811, 2018

- On one hand the Kato criteria is consistent with situation which can be described by the (1904) Prandlt ansatz.
- On the other hand cases where it is not valid agrees with experimental and numerical experiment (Farge , Klein, Nguyen, Schneider) about Energy dissipating structures produced by walls in two-dimensional flows at vanishing viscosity. Phys. Rev. Lett. 106.18 (2011) where it is observed that turbulent vorticity is generated in such boundary layer .
- With the introduction of a sub layer $\{x \in \Omega/0 \leq d(x, \partial\Omega \leq \frac{\nu}{4U^\star}\}$ this is also in agreement with the wall law proposed in (1930) by Prandlt and Von Karman under the name of turbulent layer cf. Landau and Lifshitz Fluid Mechanics 42 .
- This raises the issue of a very localised asymptotic singularity of the Navier-Stokes equations (cf. for Prandlt E and Enquist).

- The above derivation underline the importance of no slip boundary condition in the generation of boundary turbulence. For instance if the no slip boundary condition is relaxed in the following boundary condition

$$u_\nu \cdot \vec{n} = 0\,, \quad \text{and} \quad \nu \partial_{\vec{n}} u_\nu + \lambda_\nu u_\nu = 0 \qquad \text{With} \quad 0 \le \lambda_\nu < \infty \tag{25}$$

the Kato criteria is satisfied and the limit is the smooth solution of the Euler equation.
This indicates that situations where this criteria is not satisfied,(keeping in mind the d'Alembert paradox for potential solutions of the Euler equation) would be the common case.

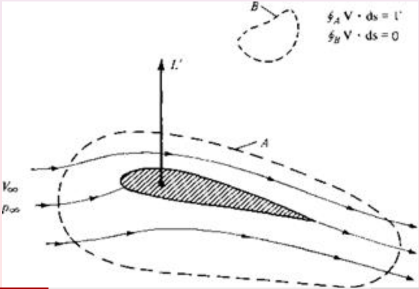
- The no slip boundary condition is justified by the fact that it generates observed turbulence
- It May contribute also to the Kutta Joukowski condition.

Computation of lift and drag via the Kutta Joukowski formulation introduces a solution with line or plane of discontinuity Γ which is called the Treffz plane.

1 The flow is potential outside Γ but not every where. A Kelvin Helmoltz problem which is an approximation of the reality as explained in Landau−Lifshitz.

2 Such solution is not the regular one and hence may be generated by a zero viscosity limit in the presence of no slip boundary condition.

Is it related to one of the wild solutions constructed by Francisco Mengual and Laslo Szkelyhidi???



THANKS FOR INVITATION AND ATTENTION.