

COMPLEX STRUCTURES IN PLANAR

BIPARTITE DIMER AND ISING MODELS

VIA T- AND S-SURFACES IN $\mathbb{R}^{2,2}$ AND $\mathbb{R}^{2,1}$

DMITRY CHELKAK, UMICHIGAN, ANN ARBOR

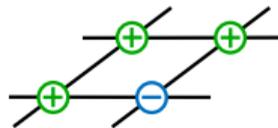
NOTE: BLACKBOARD[!!] TALKS

'GEOMETRIC METHODS IN PERCOLATION AND SPIN MODELS'

ICTS-TIFR, BENGALURU, MARCH 2-6, 2026

Ferromagnetic, w/o external field 2d nearest-neighbor Lenz-Ising model (1920)

Given a piece of the **square grid** and a parameter $x \in (0, 1)$ one assigns random spins $\sigma_u = \pm 1$ to its vertices so that the probability to get a configuration (σ_u) is proportional to $x^{\#\{u \sim u' : \sigma_u \neq \sigma_{u'}\}}$.



Boltzmann-Gibbs:

▷ energy [external field $h=0$]

$$H = - \sum_{u \sim u'} \sigma_u \sigma_{u'} - h \sum \sigma_u$$

▷ probability of a configuration (σ_u) is proportional to

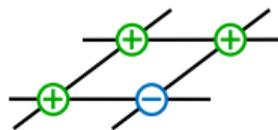
$$\exp(-H[(\sigma_u)]/kT),$$

where T is the temperature

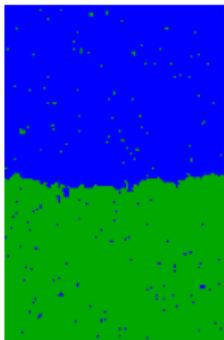
▷ $\sigma_u \sigma_{u'} = \pm 1 \rightsquigarrow x = e^{-2/kT}$.

Ferromagnetic, w/o external field **2d nearest-neighbor Lenz–Ising model (1920)**

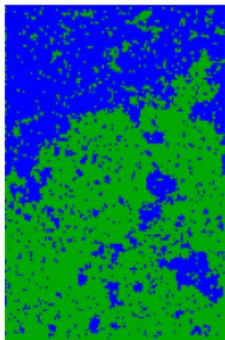
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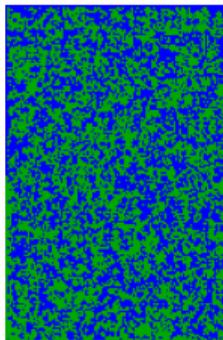
Archetypical example of a phase transition:



$x < x_{\text{crit}}$



$x_{\text{crit}} = \tan \frac{\pi}{8}$



$x > x_{\text{crit}}$

[samples with **+1/-1** (Dobrushin) boundary conditions]

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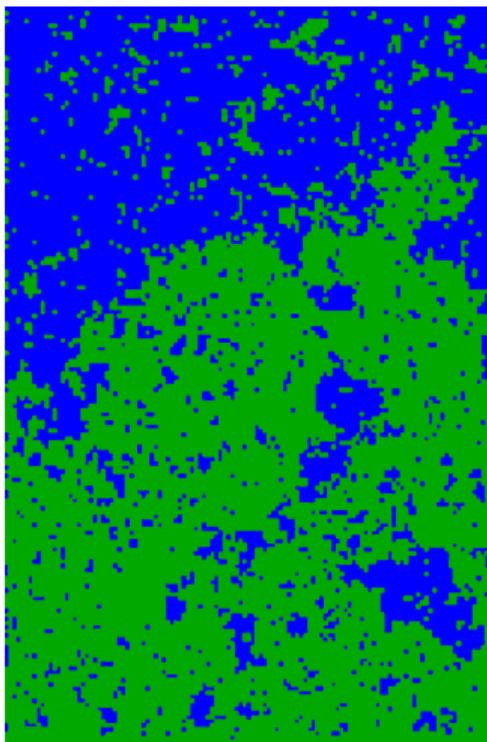
▷ probability of a configuration (σ_u) is proportional to

$$\exp(-H[(\sigma_u)]/kT),$$

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▷ $\sigma_u \sigma_{u'} = \pm 1 \rightsquigarrow x = e^{-2/kT}$.

Critical Ising model is **conformally invariant**



$$x = x_{\text{crit}}$$

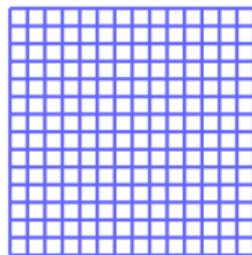
Recap: conformal maps

= holomorphic bijections

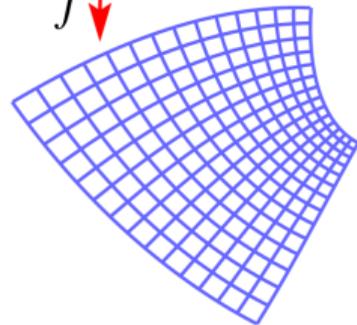
▷ preserve angles

Theorem: [Smirnov'06]

As $\delta \downarrow 0$, critical Ising and FK-Ising interfaces converge to SLE(3) and SLE(16/3), respectively.



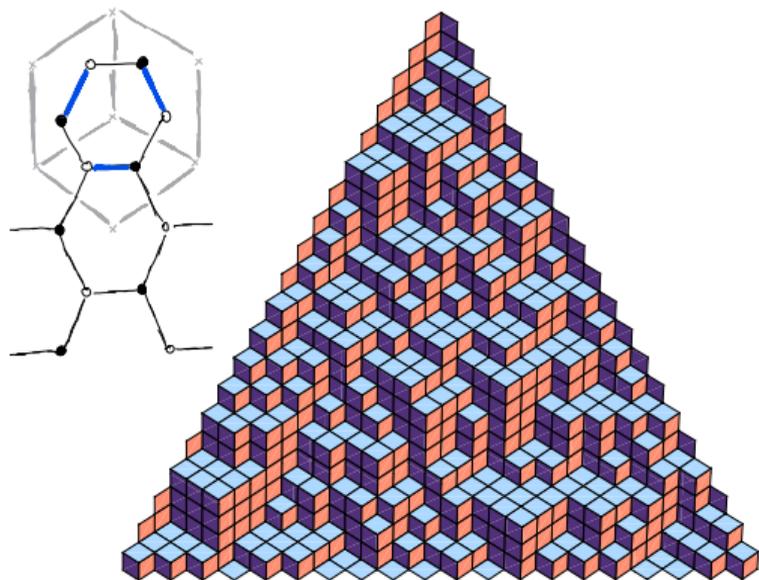
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Question: what to do if the model (i.e., edge weights or the graph itself) is non-homogeneous?

Bipartite dimer model: two setups on the hexagonal grid

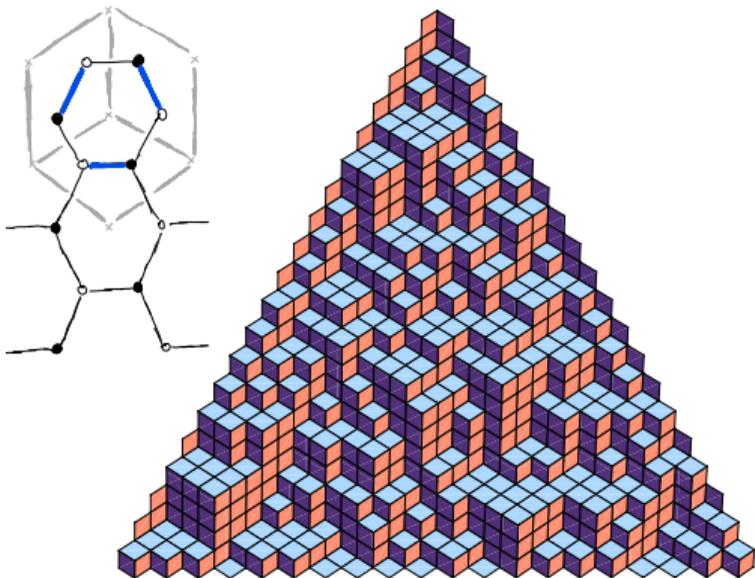
[(c) Kenyon]



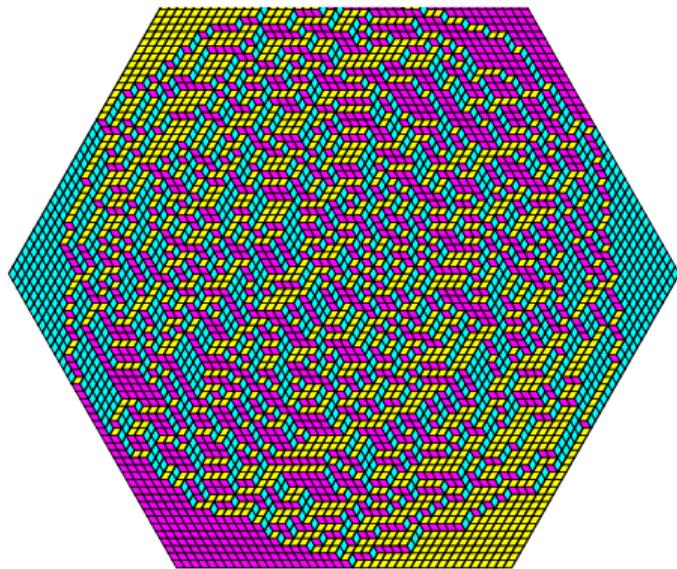
- As $\delta \downarrow 0$, height fluctuations \rightarrow GFF:
 $\text{Cov}(\Gamma(z_1), \Gamma(z_2)) = G_\Omega(z_1, z_2)$, Green's function of the **Euclidean** Laplacian in Ω

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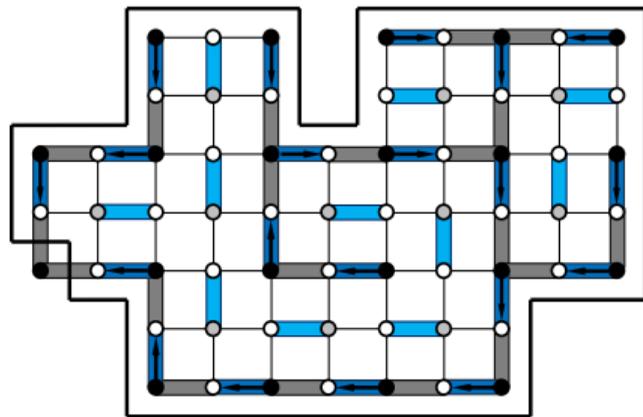
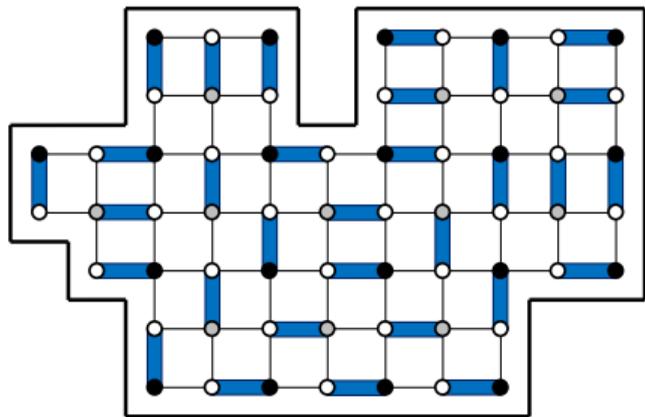


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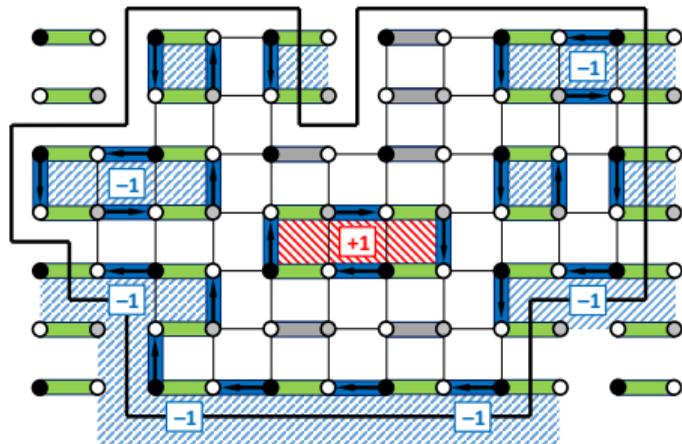


- As $\delta \downarrow 0$, height fluctuations converge to the GFF coming from a **non-trivial complex structure** on the liquid region

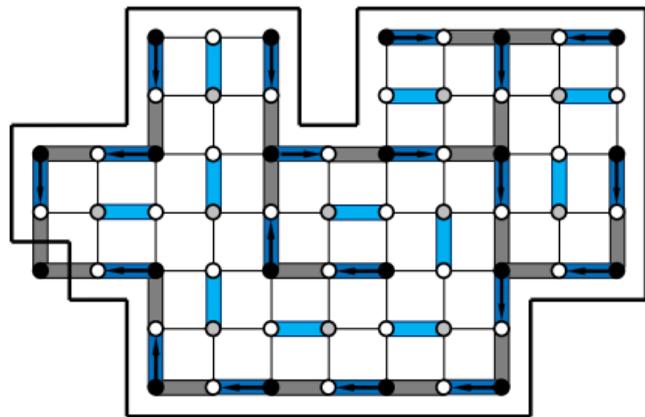
Bipartite dimer model on \mathbb{Z}^2 : very special Temperleyan domains and UST



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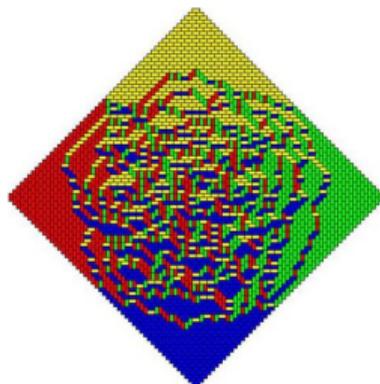
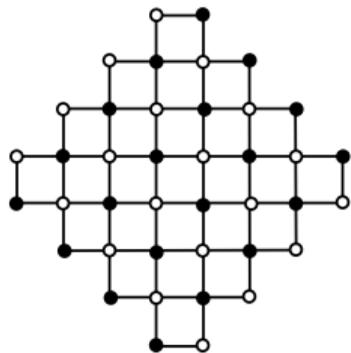


- different **reference cover/flow** \Rightarrow deterministic shift of the height function \Rightarrow **fluctuations** do not[!] change
- independent **random** reference dimer cover \rightsquigarrow **double-dimer model**



Theorem: [KENYON'00] As $\delta \downarrow 0$, fluctuations of the dimer height functions = winding field of the UST/LERW converge to the GFF defined by the **Euclidean** complex structure of Ω

Another example on \mathbb{Z}^2 : Aztec diamonds



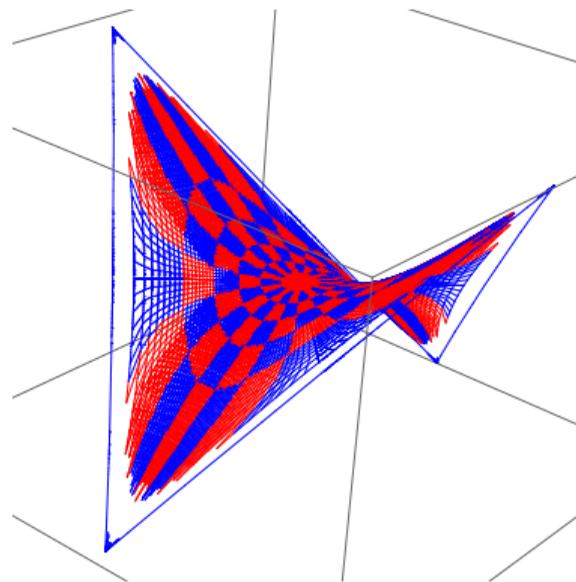
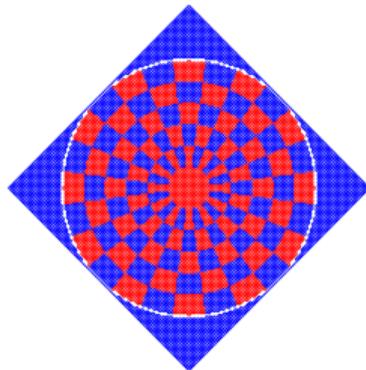
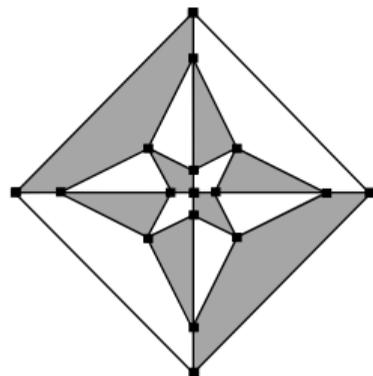
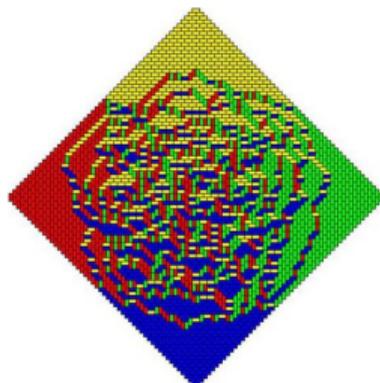
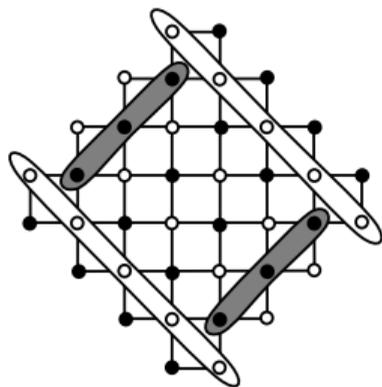
[C.-RAMASSAMY, ARXIV:2002.07540]

- As $\delta \downarrow 0$, fluctuations of the height function converge to the GFF defined by a non-trivial complex structure on the liquid region

Question: can one see this complex structure 'geometrically' by re-drawing/embedding the graph?

Another example on \mathbb{Z}^2 : Aztec diamonds

[C.-RAMASSAMY, ARXIV:2002.07540]



- Height fluctuations converge to the GFF defined by the intrinsic metric of a maximal surface in $\mathbb{R}^{2,2}$