THE FUNDAMENTAL GROUP AND THE WEIGHT LATTICE

1. Root Systems

Definition 1. Let V be a real vector space and $R \subset V$ a finite subset. Then R is a called a (reduced) root system in V, if R satisfies the following conditions:

(1) Vectors in R are non-zero.

(2) If $\alpha \in R, \beta \in R$ are such that β is a multiple of α, then $\beta = \pm\alpha$.

(3) Given $\alpha \in R$, there exists an automorphism s_α of the vector space V such that $s_\alpha(\alpha) = -\alpha$ and for $v \in V$, $s_\alpha(v) - v$ is a real multiple of α.

(4) For every $\alpha \in R$, $s_\alpha(R) \subset R$, and if $\beta \in R$, then $s_\alpha(\beta) - \beta$ is an *integral* multiple of α.

Theorem 1. *Let G be a compact connected semi-simple Lie group and T a maximal torus. Let $\mathfrak{g}_\mathbb{C}$ be the complexified Lie algebra of G and decompose $\mathfrak{g}_\mathbb{C}$ as a direct sum*

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha,$$

where on $\mathfrak{g}_\alpha$ the adjoint action of the group T is by a one dimensional character $\alpha : T \to S^1$ which we write additively as $t \mapsto t^\alpha$. Thus the characters α are viewed in the character group $X^*(T)$ written additively. Let $V = X^*(T) \otimes_\mathbb{Z} \mathbb{R}$.

Then the set $\Phi(G,T)$ is a root system in V.

Proof. Write $R = \Phi(G,T)$. We will check properties (1) through (4).

Let $R(\alpha)$ be the (finite) collection of roots β of (G,T) which are *rational* multiples of α: $\beta = \frac{p}{q}\alpha$. The $\mathbb{Z}$-span of the roots $\beta \in R(\alpha)$ lies in the subgroup $\frac{1}{N}\mathbb{Z}\alpha \simeq \mathbb{Z}$ for some common denominator N of the all the numbers p/q. Therefore, the $\mathbb{Z}$ span of β is of the form $\mathbb{Z}\gamma$ for some character γ (not necessarily a root) of T (each β is an integral power of γ and hence γ is an m-th root of α). Let S be the kernel of the character γ. On the Lie algebra of $Z(S)$, the group $Z(S)/S$ operates and T/S is a one dimensional maximal torus in $Z(S)/S$ with $\gamma : T/S \rightarrow S^1$ an isomorphism. By Lemma **??**, the group $Z(S)/S$ is connected and is either $SO(3)$ or $SU(2)$ and is three dimensional. Thus, the only rational multiples of α in R are $\pm\alpha$. This proves (2) (and part (1) is trivial).

In addition, $\gamma = \pm\alpha$ and we have an isomorphism $\alpha : T/S \rightarrow S^1$. Moreover, all the characters on T/S (written additively) are integral multiples of α.

Since T acts by the non-trivial character α on $\mathfrak{g}_\alpha \subset Lie(Z(S)) \otimes \mathbb{C}$, it follows that $Z(S)/S$ contains an element w which normalises the one dimensional torus T/S and acts by $t \mapsto t^{-1}$ on T/S. Let s_α be an element of $Z(S)$ mapping onto w. Then, s_α acts trivially on S (since it lies in the centraliser of S) and normalises T. Consider the homomorphism $\phi(t) = t \mapsto ts_\alpha t^{-1}s_\alpha^{-1}$ from T into itself. This is trivial on S and hence $\phi : T/S \rightarrow T$. Since the characters of $T/S = S^1$ are just integral multiples of α, it follows that for any character λ of T, there exists an integer m such that $\lambda(ts_\alpha t^{-1}s_\alpha^{-1}) = \alpha(t)^m$. Written additively, this means that for any $\lambda \in X^*(T)$ we have $\lambda - s_\alpha(\lambda) = m\alpha$ for some $m \in \mathbb{Z}$.

In particular, for any $\beta \in R$, $s_\alpha(\beta) - \beta$ is an integral multiple of α. This proves the second part of (4). Since s_α lies in the Weyl group, and the entire Weyl group acts on the set of roots, we have $s_\alpha(R) = R$. This proves (4).

This also proves that for any $\lambda \in X^*(T) \otimes_\mathbb{Z} \mathbb{R}$, the vector $\lambda - s_\alpha(\lambda)$ is a real multiple of α, proving (3) as well. □

Notation 1. Given a root system $R \subset V$, for every $\alpha \in R$ and $\lambda \in V$, the difference $s_\alpha(\lambda) - \lambda$ is a real multiple of α; that is, there exists a linear form H_α on V, such that $s_\alpha(\lambda) - \lambda = \lambda(H_\alpha)\alpha$ for all λ. In the case when $V = X^*(T) \otimes \mathbb{R}$ and R is the set of roots for a compact *semi-simple* connected Lie group G and a maximal torus T, as above, V is the dual of the space $LT = iLie(T)$ and hence $V^* = LT$. The element H_α lies in LT.

(We put a $\mathbb{Q}$-valued metric on $X^*(T) \otimes \mathbb{Q}$ and extend it to a metric on $V = X^*(T) \otimes \mathbb{R}$. We average this metric with respect to the Weyl group W (a finite group) and may assume the metric in W -invariant. This gives an identification of V with its dual V^*. Hence for $\lambda \in V$, the reflection s_α takes the form $s_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$ and thus under the identification of V^* with V, we see that H_α corresponds to $2\frac{\alpha}{(\alpha,\alpha)}$).

We have seen previously that $R = R^+ \coprod R^-$ where the positive roots are defined by $R^+ = \{\alpha \in R : \alpha(H) > 0\}$ for a fixed $H \in V^*$. Given $\alpha \in R^+$ let us call it decomposable if $\alpha = \beta + \gamma$ for some positive roots β, γ. If α is not decomposable, then we call it a simple root. Thus every element of R^+ is a non-negative integral linear combination of simple roots. Let Δ denote the set of simple roots.

Lemma 2. *if $\alpha, \beta \in \Delta$ are distinct simple roots, then $\beta - s_\alpha(\beta)$ is a non-positive multiple of the root α.*

Proof. We first put a $\mathbb{Q}$-valued metric on the character group $V = X^*(T) \otimes \mathbb{Q}$ and average this metric with respect to the action of the Weyl group. We get $\mathbb{Q}$-valued metric on V invariant under all the s_α. If $W = \alpha^\perp \subset V$ is the hyperplane orthogonal to α with respect to this metric, then on W, the action of s_α is trivial, since W is invariant as a subspace under s_α and $(1 - s_\alpha)(W) \subset W \cap \mathbb{Q}\alpha = \{0\}$.

Hence we have for all $v \in V$,

$$v - s_\alpha(v) = \frac{2(\alpha, v)}{(\alpha, \alpha)}\alpha.$$

We have seen that R is a root system and hence for $\beta \in R$, we see that $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = n_{\alpha,\beta}$ is an integer. Similarly, $\frac{2(\alpha,\beta)}{(\beta,\beta)} = n_{\beta,\alpha}$ is also an integer.

The product $n_{\alpha,\beta} n_{\beta,\alpha} = \frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$ is ≤ 4 (by the Cauchy Schwarz inequality) with equality if and only if β, α are proportional. Hence, if $\beta \neq \pm\alpha$, then $n_{\alpha,\beta} n_{\beta,\alpha}$ is a non-negative integer and < 4. Suppose to the contrary, that $n_{\alpha,\beta} > 0$. Then so is $n_{\beta,\alpha}$.

Consequently, one of these two integers, say $n_{\alpha,\beta}$, is one, and hence $s_\alpha(\beta) = \beta - \alpha$ is a root. Then either $\alpha - \beta$ is a positive root and hence $\alpha = (\alpha - \beta) + \beta$ is decomposable, or else $\beta - \alpha$ is a positive root and β is decomposable, a contradiction since α, β are simple. This proves the lemma. □

Lemma 3. *The set Δ of simple roots in Φ^+ is linearly independent. Thus, they form a basis for the $\mathbb{Q}$-span of the roots in Φ. Consequently, for $\alpha \in \Delta$, the elements H_α form a basis of V^*.*

Proof. Suppose a linear combination $\sum_{\alpha \in \Delta} a_\alpha \alpha = 0$ for some rational numbers a_α. Write A for the subset of Δ of α with a_α is non negative, and B for the subset of α with a_α is negative. Then,

$$\lambda = \sum_{\alpha \in A} a_\alpha \alpha = \sum_{\beta \in B} (-a_\beta)\beta.$$

Now, by Lemma 2, if $\alpha \in A$ and $\beta \in B$, then $(\alpha, \beta) \leq 0$. Hence

$$(\lambda, \lambda) = \Big(\sum_{\alpha \in A} n_\alpha \alpha, \sum_{\beta \in B} (-n_\beta)\beta\Big) \leq 0.$$

Since (,) is a positive definite form on V, it follows that $\lambda = 0$. But $\lambda(H) = \sum_{\alpha \in A} a_\alpha \alpha(H)$ is strictly positive unless all the numbers a_α are zero for $\alpha \in A$. Similarly, $a_\beta = 0$ for $\beta \in B$, and hence all the coefficients of the linear combination are zero, and Δ consists of linearly independent vectors.

Since the dual element H_α may be identified with $2\frac{\alpha}{(\alpha,\alpha)}$, it follows that the H_α for $\alpha \in \Delta$, form a basis of $V^* = LT$. □

Definition 2. Let $w \in W$ and $R = \Phi(G, T)$, with a positive system Φ^+. The number of roots β in Φ^+ such that $w(\beta) < 0$ is called the *length* of the element w.

If α is a simple root, the element s_α of the Weyl group is called a *simple reflection.*

Lemma 4. *If $\alpha \in \Delta$ is a simple root, then s_α has length one.*

Proof. Since $s_\alpha(\alpha) = -\alpha$, the length of s_α is at least one.

Since Δ is a base, every root in Φ^+ is a non-negative integral linear combination of elements of Δ. Suppose $\beta \in \Phi^+$ with $\beta \neq \alpha$. Then $\beta = \sum a_\theta \theta$ where θ runs through all the simple roots, and for some simple root $\gamma \neq \alpha$, the coefficient a_γ is non-zero (and strictly positive). Consequently, for some integer m,

$$s_\alpha(\beta) = \beta + (s_\alpha(\beta) - \beta) = \sum a_\theta \theta + m\alpha.$$

This shows that the coefficient of γ (in the expression for $s_\alpha(\beta)$ as a linear combination of simple roots), is the same as that of β, namely a_γ, and is strictly positive. This shows that $s_\alpha(\beta) > 0$. Therefore, s_α takes $\Phi^+ \setminus \{\alpha\}$ into itself and hence the length of s_α is at most one.

The last two paragraphs imply that s_α has length one. □

Lemma 5. *Every Weyl group element is a product of simple reflections.*

Proof. Let $W' \subset W$ be the subgroup generated by simple reflections. If possible, let $w_0 \in W \setminus W'$. Let $w \in W$ be an element with the property that the length of w is minimal among the elements of the left coset $W'w_0$ in the left coset space $W' \backslash W$. Write $\Phi^+ = A \coprod B$ where $w(A) \subset \Phi^+$

and $w(B) \subset \Phi^-$. Then, $\Phi^+ = w(A) \coprod -w(B)$ as well.

Suppose $\alpha \in -w(B)$ and let $\beta \in B$ such that $w(\beta) = -\alpha$. Write $s_\alpha w = w'$. If $\gamma \in A$, then $w(\gamma)$ is not α but is positive and hence $w'(\gamma) = s_\alpha w(\gamma)$ is positive, and $w'(A) \subset \Phi^+$: $w'(A) > 0$.

Suppose $\gamma \in B$. Then $w(\gamma)$ is negative. If $w(\gamma) \neq -\alpha$, then $s_\alpha(-w(\gamma))$ is still positive. Hence $w'(\gamma) = s_\alpha w(\gamma) < 0$. Thus, $w'(B \setminus \{w^{-1}(-\alpha)\})$ consists of negative roots.

If $w(\gamma) = -\alpha$, then $-w(\gamma) = \alpha$ and $s_\alpha(-w(\gamma)) = -\alpha$ and hence $s_\alpha w(\gamma) > 0$. That is $w'(w^{-1}(-\alpha))$ is positive. Therefore, The number of roots in Φ^+ which are taken into their negatives under the element w' is one less than that for w. This is impossible by the minimality of the length of the coset representative w. Therefore, w cannot take any positive root into a negative root. By the Chevalley normaliser Lemma (lemma **??**), $w = 1$ and hence $W' = W$. □

1.1. Integral Forms.

Notation 2. Denote by $L_{sc} = \sum_{\alpha \in \Delta} \mathbb{Z} H_\alpha$, the lattice in LT obtained as the $\mathbb{Z}$ span of the (linearly independent) vectors H_α as α varies in Δ. Denote by Λ_{sc} the lattice of *integral forms*, namely the integral dual of the lattice L_{sc}. It is the $\mathbb{Z}$ span of the basis $(\lambda_\alpha)_{\alpha \in \Delta}$ of $V = X^*(T) \otimes \mathbb{R}$ dual to the basis $(H_\alpha)_{\alpha \in \Delta}$ so that $\lambda_\beta(H_\alpha) = \delta_{\alpha,\beta}$, the Kronecker delta symbol. The integral forms λ_α for $\alpha \in \Delta$ are called *fundamental weights*.

The element $\rho = \frac{1}{2}(\sum_{\alpha \in \Phi^+} \alpha)$ will be seen to be an integral form which is equal to $\sum_{\alpha \in \Delta} \lambda_\alpha$.

A corollary to the preceding lemma is that for w in the Weyl group W, the difference $-w(\rho) + \rho$ is a sum of positive roots α such that $w^{-1}(\alpha) < 0$.

Definition 3. Let $\lambda \in X^*(T)$. We call λ a *dominant integral weight* (with respect to the positive system Φ^+) if $\lambda - s_\alpha(\lambda)$ is a non-negative *integral* linear combination of simple roots.

If V is an irreducible representation with the highest weight λ , then λ is a dominant integral weight. This has been observed before.

1.2. **Stiefel Diagram.** Consider G a connected semi-simple group of adjoint type, and let LT denote the set of real elements of the complexified Lie algebra $\mathbf{t}_{\mathbb{C}}$ on which the roots take real values. We denote the exponential map by $X \mapsto exp(2\pi i X) \in T$ from LT onto T.

The kernel L_{ad} of this map is precisely the set of vectors $X \in LT$ with the property: for each root α, $\alpha(X) \in \mathbb{Z}$. The roots are viewed as elements of the dual of LT and the simple roots $\Delta = \{\alpha : \alpha \quad simple\}$ form a basis of LT. Let $\{v_\alpha : \alpha \in \Delta\}$ denote the *dual* basis in LT to the basis Δ of simple roots. Then the kernel L_{ad} is the integral linear span of $\{v_\alpha : \alpha \in \Delta\}$. That is, $L_{ad} = \sum_{\alpha \in \Delta} \mathbb{Z} v_\alpha$. This is called the *adjoint lattice*. Since it is the kernel to the exponential map, L_{ad} is stable under the action of the Weyl group: $\sum_{\alpha\in\Delta} \mathbb{Z}\alpha$ is W stable.

Denote by L_{sc} the $\mathbb{Z}$-span of the vectors $v_\alpha - s_\alpha(v_\alpha)$ as α varies over Δ. Then $L_{sc} \subset L_{ad}$. It follows from definitions that $H_\alpha = v_\alpha - s_\alpha(v_\alpha)$.

The inverse image in LT under the exponential map $LT \to T$ of the set T_{reg} of regular elements of the torus T, will be referred to as strongly regular elements LT_{sreg}. This is precisely the set of elements $X \in LT$ where no $\alpha(X)$ can be integral for any root α. For an integer k and a root α, denote by $W_{\alpha,k}$ the set of elements $X \in LT$ such that $\alpha(X) = k$. This is a hyperplane in LT (if α is simple, then the hyperplane $W_{\alpha,k}$ passes through the point kv_α; thus $W_{\alpha,k} = kv_\alpha + ker(\alpha)$). Denote by W_α the union over all integers k of the $W_{\alpha,k}$. Then for a simple root α, $W_\alpha \supset \mathbb{Z}v_\alpha + Ker(\alpha) \supset \mathbb{Z}v_a + \sum_{\beta\in\Delta,\beta\neq\alpha} \mathbb{Z}v_\beta = L_{ad}$. Since every root α is of the form $w(\beta)$ for simple root β, it follows that $W_\alpha = w(W_\beta) \supset w(L_{ad}) = L_{ad}$. Hence for each root α, the union of hyperplanes W_α is stable under translation by elements of L_{ad}. Consequently, the union over all W_α as α varies over all roots, is also W stable and L_{ad} stable. Therefore, the strongly regular elements LT_{sreg} of LT are stable under the action of $L_{ad} \rtimes W$.

The set of strongly regular elements of LT_{sreg} is called the *Stiefel diagram* of G_{ad}. The set LT_{sreg} has countably many connected components and each connected component is an *alcove* of the form $\gamma + P$ where γ is an element of the integral lattice L_{sc}, and P is the "fundamental alcove" consisting of elements p of the form $p = \sum t_\alpha v_\alpha$ with $0 < t_\alpha < 1$ and $\alpha(p) < 1$ for all positive roots α (this is equivalent to saying that $0 < \beta(p)$ for simple roots β and $\alpha\ (p) < 1$ where α is the *highest root*). Moreover, the Weyl group W also acts on LT_{sreg} since $w \in W$ takes each $W_{\alpha,k}$ into $W_{w(\alpha),k}$.

We can describe the reflection $s_{\alpha,k}$ about the translated hyperplane $W_{\alpha,k}$ as the map $v \mapsto v - (\alpha(v) - k)H_\alpha$. Then a computation shows that $s_{\alpha,1}s_{\alpha,0}$ is the translation by the element H_α on LT. The group generated by the reflections $s_{\alpha,k}$ for $\alpha \in R$ and $k \in \mathbb{Z}$ is the semi-direct product of the lattice $L_{sc} = \sum \mathbb{Z}H_\alpha$ and the Weyl group W.

The extended Weyl group $W_e = L_{sc} \rtimes W$ acts on all of $LT_{reg} = LT - \bigcup_{\alpha\in R,k\in Z} W_{\alpha,k}$ and permutes the connected components. Therefore, if $\sigma \in W_e$ and $\sigma(P) \cap P$ is non-empty, then $\sigma(P) = P$.

We first consider the seemingly smaller group W' generated by the reflections s_α where α is a simple root and by the reflection $s_{\alpha\ ,1}$ about the wall $\alpha\ (x) = 1$. We will refer to these reflections as the simple reflections of the extended Weyl group W_e. Thus, $W' \subset W$.

Lemma 6. *Every alcove may be translated into the fundamental alcove by an element of the smaller group W'.*

Proof. The action of W_e and therefore of W' is properly discontinuous on the Lie algebra LT. Furthermore, we have an inner product $(,)$ on LT invariant under the Weyl group by averaging any inner product under the action of the finite group W. Given vectors $x, y \in LT$, denote by $d(x,y) = \sqrt{(x-y, x-y)} =\mid x-y \mid$ the distance of y from x. This defines a metric which is invariant under translations by elements of LT and under the action of W; therefore, it is invariant under W_e. Given an element x in the fundamental alcove P and $z \in LT_{sreg}$ a strongly regular element, consider the function on LT_{sreg} defined by

$$f(z) = inf_{\gamma\in W'} d(\gamma z, x).$$

That is, take all possible translates of z under the smaller group W', and take the infimum of the distances of these translates of z from x. The proper discontinuity of the action of W' ensures that this minimum is attained by $y = \gamma_0 z$ for some $\gamma_0 \in W'$. We then get $d(y,x) \le d(\gamma y, x)$ for all $\gamma \in W'$.

We will show that y must necessarily lie in the fundamental alcove P. This will prove the lemma since z being an element of LT_{sreg} may be translated into $y \in P$ by the element γ of W'; but any strongly regular element z of LT lies in an alcove Q and $y = \gamma z \in \gamma(Q) \cap P$. But if two alcoves intersect nontrivially, then they coincide since they are connected components. Hence $\gamma Q = P$.

Let α be a simple root. Suppose $\alpha(y) < 0$. Consider a point $q \in LT$ on the line joining x and y; it is of the form $q = ty + (1-t)x$. Then $t = \frac{\alpha(x)-\alpha(q)}{\alpha(x)-\alpha(y)}$. If we choose $t = \frac{\alpha(x)}{\alpha(x)-\alpha(y)}$ then its denominator is greater than $\alpha(x) > 0$ since $\alpha(x) > 0$ and $\alpha(y) < 0$; hence $0 < t < 1$ and the formula for t shows that $\alpha(q) = 0$. Hence $s_\alpha(q) = q$. Then by the triangle inequality,

$$d(s_\alpha(y), x) < d(s_\alpha y, q) + d(q, x) =$$

$$= d(s_\alpha y, s_\alpha q) + d(q, x) = d(y, q) + d(q, x) = d(y, x),$$

and the last equality holds because q is on the line joining x and y between x and y. This contradicts the choice of y and hence $\alpha(y) < 0$ cannot hold: $a(y) > 0$ for all simple roots α.

Suppose $\alpha\ (y) > 1$. By using the element $s_{\alpha,1}$ instead of s_α in the preceding paragraph, we get a contradiction as in the preceding paragraph. We therefore see that $\alpha\ (y) < 1$. Hence the element y lies in the fundamental alcove P. This proves the lemma. □

Lemma 7. *The extended Weyl group W_e is the group W' generated by the simple reflections.*

Proof. The group W_e is generated by the reflections $s_{\alpha,k}$ for all positive roots α and all integers k. The reflection $s_{\alpha,k}$ leaves the wall $W_{\alpha,k}$ pointwise stable and the wall $W_{\alpha,k}$ is a boundary of some alcove Q. By the preceding lemma, there exists an element γ of W' which moves Q into the fundamental alcove P; hence it moves $W_{\alpha,k}$ into some wall of P. But the only walls which meet the boundary of P are of the form $W_{\alpha\ ,1}$ or W_β for some simple root β. Hence $s_{\alpha,k} = \gamma^{-1} r \gamma$ where r is the simple reflection about a wall E of the fundamental alcove P, and hence r lies in the "smaller" group W'. Therefore, $s_{\alpha,k} = \gamma^{-1} r \gamma$ also lies in W', since γ and r both are in W'. □

Given $\sigma \in W$, we may write $\sigma = s_1 s_2 \cdots s_k$ where s_i are simple reflections as in the lemma. The smallest such k is called the *length* of the element σ. For example, the length of a simple reflection is 1.

We will say that two alcoves Q and Q' are on opposite sides of a wall E, if E is set $E = \{x \in LT : \lambda(x) = k\}$ for some linear form λ and some number k, and for *all* $x \in Q$ and *all* $x' \in Q'$, either $\lambda(x) < k < \lambda(x')$ or the other way around: $\lambda(x) > k > \lambda(x')$. We will also say that the wall E separates the alcoves Q and Q'. If s is a simple reflection about a wall E of the fundamental alcove P, then the alcoves P and $s(P)$ are on opposite sides of the wall E. Moreover, if E' is any wall such that P and $s(P)$ are on opposite sides of E', then $E' = E$.

Lemma 8. *Let $\sigma = s_1 s_2 \cdots s_k$ be an element of W_e of length k and with s_i simple reflections. Suppose s_1 is a reflection about the wall E. Then P and $\sigma(P)$ are on opposite sides of the wall E.*

Proof. We prove this by induction on the length of σ. If $k = 1$, then σ is a simple reflection s_1 about a wall E, and we have already observed that P and $s_1(P)$ are on opposite sides of E.

Suppose that the lemma is false for a smallest such k (then $k \geq 2$) and for some σ of length k, the alcoves $\sigma(P)$ and P are on the *same* side of the wall E. Let $u = s_1 s_2 \cdots s_{k-1}$. Then u has length $k-1$ and by induction assumption, $u(P)$ and P are on opposite sides of the wall E. Hence $u(P)$ and $\sigma(P)$ are on the opposite sides of E.

Now, compare the alcoves $\sigma(P) = us_k(P)$ and $u(P)$; the alcoves P and $s_k(P)$ are on opposite sides of the wall E_k (and s_k is the reflection about the wall E_k)). Moreover, E_k is the only wall separating P and $s_k(P)$. Consequently, the alcoves $u(P)$ and $us_k(P) = \sigma(P)$ are on opposite sides of the wall $u(E_k)$ and $u(E_k)$ is the only wall separating $u(P)$ and $\sigma(P)$. By the preceding paragraph, $E = u(E_k)$, and hence the reflection $s_1 = us_k u^{-1}$. Since $\sigma = us_k$, we get : $\sigma = s_1 u = s_2 \cdots s_{k-1}$ has length $k-2$, contradicting our assumption that σ has length k.

This proves the lemma. □

Proposition 9. *The extended Weyl group acts simply transitively on the set of alcoves.*

Proof. We have already seen that W' acts transitively on the set of alcoves. We need only show that if $\sigma(P) = P$ then $\sigma = 1$. Suppose $\sigma \in W_e$ stabilises P and has length k. Write $\sigma = s_1 s_2 \cdots s_k$. Suppose $k \geq 1$ and let s_1 be a reflection about the wall E. By the lemma, $\sigma(P)$ and P are on the opposite sides of E, but $\sigma(P) = P$. This contradiction proves that $k = 0$ and $\sigma = 1$. □

Corollary 1. *The quotient of the strongly singular elements in LT by the action of the extended Weyl group is simply connected.*

Proof. Since the group W_e acts transitively on the connected components of LT_{sreg}, it follows that $LT_{sreg}/W_e = P/stab(P)$; by the proposition, the action on the set of alcoves has no isotropy and hence $LT_{sreg} \to LT_{sreg}/W_e = P$ is a (disconnected) covering, and the quotient is the alcove P which is simply connected. □

The Weyl group W is a quotient of the extended Weyl group $W_e = (\sum_{\alpha\in\Delta} \mathbb{Z}H_\alpha) \rtimes W = L_{sc} \rtimes W$. The group W acts by right multiplication on the quotient G/T where G is a compact connected LIe group of adjoint type and by conjugation on T_{reg}. Hence it acts diagonally on the product $G/T \times T_{reg}$ and the conjugation map $(gT, t) \to gtg^{-1}$ is an isomorphism from $G/T \times T_{reg}/diag(W)$ onto G_{reg}. Since $T_{reg} = LT_{sreg}/L_{ad}$, we see that $G/T \times LT_{sreg}/L_{sc}$ is a covering of $G/T \times T_{reg}$. Going modulo the diagonal action of the Weyl group W on both, we get a covering from $G/T \times P$ onto G_{reg}, with deck transformation group isomorphic to $L_{ad} \rtimes W/L_{sc} \rtimes W$ and the latter is isomorphic to L_{ad}/L_{sc}.

We recall that $G/T \times P$ is simply connected. Therefore, we arrive at the conclusion that the fundamental group of G_{reg} is (isomorphic to) the finite quotient L_{ad}/L_{sc}.

Its dual group $(L_{ad}/L_{sc})^*$ may be identified with the quotient of the "weight lattice" (the lattice $Hom(\sum \mathbb{Z}H_\alpha, \mathbb{Z})$ of integral forms which is also the integral span of the fundamental weights) of the Lie algebra $\mathfrak{g}$ modulo the root lattice:

$$(L_{ad}/L_{sc})^* = Hom\Big(\sum \mathbb{Z}H_\alpha, \mathbb{Z}\Big)/Hom\Big(\sum \mathbb{Z}v_\alpha, \mathbb{Z}\Big) = (\sum \mathbb{Z}\lambda_\alpha)/(\sum \mathbb{Z}\alpha).$$

If $\lambda \in X^*(T_{sc})$ is a dominant integral weight of the inverse image T_{sc} of T in G_{sc} (T_{sc} is then a maximal torus), then λ is the highest weight of an irreducible representation $V(\lambda)$ of G_{sc}. Suppose now that λ is a dominant integral weight of the Lie algebra $\mathfrak{g}$ of G.

Theorem 10. *(Existence Theorem) Conversely, given a dominant integral weight of the Lie algebra* $\mathfrak{g}$*, there exists an irreducible representation with highest weight* λ*.*

Proof. Let G be a compact semi-simple group of adjoint type. Consider the open set $G_{reg} = (G/T \times T_{reg})/Diag(W)$. Let $V = LT$ be i times the Lie algebra $\mathfrak{t}$ of T.We note that the character group of T is the root lattice and hence the torus $T = LT/\sum_{\alpha\in\Delta} \mathbb{Z}v_\alpha$, where v_α is the basis of LT dual to the basis consisting of the simple roots α. Thus, $\beta(v_\alpha) = \delta_{\alpha,\beta}$ where δ is the Kronecker delta symbol. Consequently, for any root α, its evaluation on the $\mathbb{Z}$ span L_{ad} of all the v_β is an integer, and H_α lies in L_{ad}. The $\mathbb{Z}$ span L_{sc} of H_α also lies in L_{ad} (that is, $L_{sc} = \sum_{\alpha\in R} \mathbb{Z}H_\alpha \subset L_{ad}$). Therefore, we have a covering $LT/L_{sc} \to LT/L_{ad}$. Thus the order of the Deck transformation group of the *universal* covering of G_{reg} is the order of L_{ad}/L_{sc}.

The inclusion $G_{reg} \to G$ induces an isomorphism of fundamental groups. Hence the universal cover of G has the same number of sheets over G as that of G_{reg}. In other words, the fundamental group of G has order equal to the order of the quotient L_{ad}/L_{sc} by the preceding paragraph. But, if G_{sc} denotes the simply connected cover of G, then the Deck transformation group of $G_{sc} \to G$ may be identified with the centre of G_{sc}, and the dual of the centre may be identified $\Lambda/\sum_\alpha \mathbb{Z}\alpha$, where $\Lambda = X^*(T_{sc})$ is the weight lattice of T_{sc}, a maximal torus of G_{sc} mapping onto T under the covering map $G_{sc} \to G$ (by weights of a torus, we mean the group of characters on the torus). By the preceding paragraph, Λ must be the lattice of integral forms.

As a consequence, given a dominant integral weight λ of the Lie algebra $\mathfrak{g}$ (i.e.a dominant integral form), there exists an irreducible representation $V(\lambda)$ of the *simply connected group* G_{sc} with highest weight λ (by differentiating, we get an irreducible (finite dimensional) representation of the Lie algebra $\mathfrak{g}$ as well).This proves the theorem.

□

We have therefore classified all the irreducible representations of a semi-simple Lie algebra $\mathfrak{g}$, in terms of a dominant integral weight. We can also classify all the compact (simply connected) semi-simple Lie groups G in terms of the root system. The root systems may in turn be classified in terms of the Dynkin diagram.

References

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