COMPACT CONNECTED LIE GROUPS

1. The Lefschetz Fixed Point Formula

Suppose M is a Hausdorff manifold and $f : M \to M$ a smooth map with *isolated* fixed points. Given a fixed point p of f, and a coordinate neighbourhood of p in M, define the *local Lefschetz Number* $L(f)_p$ of f at p to be the sign of the determinant $det(df_{|p} - I)$ where I is the identity. Denote by $H^k(M, \mathbb{Q})$ the k-th cohomology group of M with rational coefficients and by $H^k(f)$ the linear transformation induced by the map f on $H^k(M, \mathbb{Q})$. The *total Lefschetz number* of f is the alternating sum of traces $\sum_{k=0}^{dimM} (-1)^k trace(H^k(f))$.

Theorem 1. *(Lefschetz Fixed Point Formula) Let f be a smooth self map of a manifold M. If the total Lefschetz number is nonzero, then f has a fixed point.*

If f has only isolated fixed points, we have the identity

$$\sum_{k=0}^{dimM} (-1)^k trace(H^k(f)) = \sum_{p:f(p)=p} L(f)_p$$

That is, the total Lefschetz number of f is the sum of all the local Lefschetz numbers of f over all the (by assumption, isolated) fixed points of the map f.

2. Compact Connected Lie Groups

2.1. The Weyl Group.

G is a compact connected Lie group and T is a maximal torus. Denote by $N(T)$ the normaliser of T in G. The quotient group $N(T)/T$ is the *Weyl Group* of T.

Lemma 2. *The group $N(T)/T$ is finite.*

Proof. The group $N(T)/Z(T)$ is compact, but, on the other hand, is a closed subgroup of the discrete group $Aut(T) = Aut(\mathbb{Z}^l) \simeq GL_l(\mathbb{Z})$ where $T = (S^1)^l$ for some integer $l \geq 1$. Therefore, the group $N(T)/Z(T)$ is finite.

Since $Z(T)$ is a compact *Lie group*, its connected component of identity $Z(T)^0$ is open (and closed) and hence has finite index in $Z(T)$. Fix $X \in z(T)$ the Lie algebra of $Z(T)$. Then the closed subgroup generated by T and the one-parameter group $\{exp(tX) : t \in \mathbb{R}\}$ is a compact *connected abelian* group and is therefore a torus. By the maximality of T, this means that $X \in \mathfrak{t}$, the Lie algebra of T. Therefore, $z(T) = \mathfrak{t}$ and $Z(T)^0 = T$. Consequently $N(T)/T$ is finite. □

Consider the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G. Consider the complexification $\mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$. Since, by the proof of Lemma 2, the set of fixed points of T in $\mathfrak{g}$ is the Lie algebra $\mathfrak{t}$ of T, it follows that we have a decomposition $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{t} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where Φ is a finite set of non-trivial characters of T and for each $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ is the subspace of $\mathfrak{g} \otimes \mathbb{C}$ of vectors X with $Ad(t)(X) = \alpha(t)X$. Note that since $\mathfrak{g} \otimes \mathbb{C}$ is the complexification of the T-representation $\mathfrak{g}$, if $\alpha \in \Phi$ then $\alpha^{-1} \in \Phi$, and is different from α. We may write $\Phi = \Phi^+ \cup (\Phi^+)^{-1}$ for some subset $\Phi^+ \subset \Phi$.

Let T_{reg} denote the subset of elements $t \in T$ such that $\alpha(t) \neq 1$ for any $\alpha \in \Phi$; elements of T_{reg} are referred to as *regular elements* of the maximal torus T. Clearly, T_{reg} is a dense open subset of T.

Lemma 3. *The determinant of $Ad(t) - 1$ on the quotient space $\mathfrak{g}/\mathfrak{t}$ is strictly positive.*

Proof. The determinant is the same as the determinant of $Ad(t) - 1$ on the complexification $(\mathfrak{g} \otimes \mathbb{C})/\mathfrak{t} \otimes \mathbb{C}$. The latter is clearly

$$\prod_{\alpha \in \Phi} (\alpha(t) - 1) = \prod_{\alpha \in \Phi^+} (\alpha(t) - 1)(\alpha(t)^{-1} - 1) = \prod_{\alpha \in \Phi^+} \mid (\alpha(t) - 1) \mid^2,$$

and is therefore strictly positive. □

2.2. **Maximal Tori.**

Theorem 4. (*Conjugacy of Maximal Tori) Every element of G may be conjugated into a maximal torus.*

All maximal tori in G are conjugate in G. Hence the dimension of a maximal torus is an invariant of the group G, called the (absolute) rank of the group G.

Proof. We use the Lefschetz Fixed Point Formula. Consider the action by left translation, of an element g in G, on the quotient manifold G/T. Since G is connected, this translation is homotopic to the translation by identity namely, the identity transformation on G/T. Therefore, the Lefschetz number of this transformation g is the same as the Euler Characteristic of the manifold G/T.

The Lefschetz number of $t \in T_{reg}$ is positive, since the local Lefschetz number at each fixed point wT (for $w \in W$) i.e. the determinant of $Ad_{\mathfrak{g}/\mathfrak{t}}(wtw^{-1}) - 1$, is positive (the second part of Theorem 1 and Lemma 3). Since the Lefschetz number of any left translation is the same, this implies that the Euler characteristic of G/T is positive and therefore, the Lefschetz number of the left translation L_g by g is positive.

By the Lefschetz fixed point formula, the transformation g does have a fixed point in G/T. That is, there exists $x \in G$ such that $x^{-1}gx$ lies in T. Thus every element of G may be conjugated into T.

Take now a maximal torus T' and fix a generating element $t' \in T'$. Then, t' may be conjugated into T by the foregoing paragraph, and since t' generates T' it follows that T' is conjugate to T, proving the theorem. □

Lemma 5. *Let G be a compact connected Lie group and $S \subset G$ a torus. Then the centraliser of S in G is connected.*

Proof. Let $z \in G$ centralise S. Consider $H = Z(z)^0$ the identity component of the centraliser of the element z. Since S centralises z, it follows that $S \subset H$.

By the theorem, z lies in a maximal torus T, and hence $T \subset H$ as well, and T being a maximal torus in G, is a maximal torus in H. Now S has an element s which generates S (by the Kronecker density theorem of the previous chapter), which, by the theorem, can be conjugated into T by an element h of H. Hence S can be conjugated into T by h. Now, z lies in hTh^{-1} since $z \in T$ and h commutes with z. Hence both z and S lie in $hTh^{-1} \subset Z(S)^0$ which shows that $z \in Z(S)^0$, the identity component of $Z(S)$. That is, $Z(S) = Z(S)^0$. □

Define T_{reg} as the subset of elements $t \in T$ on which no nontrivial character α of T acting on the complexified Lie algebra $\mathfrak{g}$ is trivial. Let G_{reg} be the set of elements of G which may be conjugated into T_{reg}.

The map $G/T \times T_{reg} \longrightarrow G_{reg}$ given by $(gT, t) \mapsto gtg^{-1}$ is a surjection whose fibers are in one one correspondence with elements of the Weyl Group W.

2.3. **The Set of Roots.** Let T be a maximal torus of a compact connected Lie group G. On the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ of G, the group T operates by adjoint action and we may decompose $\mathfrak{g}_{\mathbb{C}}$ as a sum of $\mathfrak{t}_{\mathbb{C}}$ and of subspaces $\mathfrak{g}_\alpha$ where on each $\mathfrak{g}_\alpha$, T acts by the character $\alpha : T \longrightarrow \mathbb{C}^*$. We may write this character in the form $\alpha(exp(X)) = e^{\alpha(X)}$ where, for $X \in \mathfrak{t} = Lie(T)$, $exp(X) \in T$, and by an abuse of notation we denote by $\alpha(X)$ the associated linear form on the Lie algebra $\mathfrak{t}$. This takes imaginary values (i.e. values in $i\mathbb{R}$). The collection $\Phi = \Phi(G, T)$ of these characters (or linear forms) are called the *roots* of $\mathfrak{g}$ (with respect to T; since all maximal tori are conjugate, we obtain that these roots are essentially the same, up to G conjugacy). Moreover, if α is a root, then so is its inverse (in Lie algebra terms, its negative).

If $t_0 \in T$ is a regular element, then $\alpha(t) \neq 1$ for any α. Write $t_0 = exp(iH)$ with $H \in i\mathfrak{t}$. Then $\alpha(H)$ is a nonzero real number. We get a decomposition of the set Φ of roots as a disjoint union: $\Phi = \Phi^+ \coprod -(\Phi^+)$, where Φ^+ is the set of roots on which H is positive. If $\alpha \in \Phi^+$ we also write $\alpha > 0$. *We may even choose $t_0 = exp(iH_0)$ so that all the values $\alpha(t_0)$ are all distinct for distinct $\alpha \in \Phi^+$. That is the positive real numbers $\alpha(H_0)$ are all distinct.*

Fix any regular element $t = exp(iH)$ in T_{reg}. If t has a fixed point xT in the quotient space G/T, then $x^{-1}tx \in T$ i.e. $t \in xTx^{-1}$ and the connected group xTx^{-1} is abelian. Hence the Lie algebra $xLie(T)x^{-1}$ is in the trivial eigenspace of $Ad(t)$; this is simply $Lie(T)$. This proves that $xTx^{-1} = T$ and hence that $x \in N(T)$, and $xT = wT$ for some $w \in W$. The only fixed points of t in G/T are the translates wT by the Weyl group elements of the trivial coset T.

Theorem 6. (*The Weyl Integral Formula*) *The Haar measure dg of G decomposes as*

$$dg = \frac{1}{\mid W \mid} dt \mid D(t) \mid^2 dg^*,$$

where dt is the Haar measure on T, dg^* is the Haar measure on G/T, $D(t) = \prod_{\alpha \in \Phi^+} (\sqrt{\alpha}(t) - \frac{1}{\sqrt{\alpha(t)}})$ and W is the order of the Weyl group.

As a corollary, we see that

$$\int_T dt \mid D(t) \mid^2 = \mid W \mid .$$

Proof. We first compute the Jacobian of the map $\psi : G/T \times T \to G$ given by $(gT, t) \mapsto gtg^{-1} = x$. We assume (Theorem **??**) that $G \subset GL_n(\mathbb{C})$ is linear . Since $GL_n(\mathbb{C})$ is an open subset of the vector space $M_n(\mathbb{C})$, the tangent space to the element $x \in G$ may be viewed as the subspace $x\mathfrak{g}$ where, for $X \in \mathfrak{g} \subset M_n(\mathbb{C})$, xX denotes the multiplication of the matrix x with X, and $x\mathfrak{g}$ denotes the real vector subspace of $M_n(\mathbb{C})$ consisting of vectors xX with $X \in \mathfrak{g}$. Suppose $X \in T_e(G/T) = \mathfrak{g}/\mathfrak{t}$.

Then for $s \in \mathbb{R}$ the curve (write ${}^g(y) = gyg^{-1}$)

$$s \mapsto \psi(gexp(sX)T, t) = gexp(sX)texp(-sX)g^{-1} =$$

$$= gtg^{-1}g(t^{-1}exp(sX)texp(-sX))g^{-1} = x \quad ({}^g(t^{-1}exp(sX)texp(-sX))),$$

has the derivative

$$x({}^g(t^{-1}Xt - X)) = xAd(g)(Ad(t^{-1}) - id)(X) \in x\mathfrak{g} = T_x(\mathfrak{g}),$$

at $s = 0$.

Similarly, if $Y \in \mathfrak{t}$ then for $s \in \mathbb{R}$, the curve $s \mapsto \psi(gtexp(sY)g^{-1}) = xgexp(sY)g^{-1}$ has the derivative $xAd(g)(Y)$ at $s = 0$. Consequently the derivative $d\psi$ at (gT, t) of of the map ψ is given by

$$(g(X), Y) \mapsto x(Ad(g)((Ad(t^{-1}) - id)(X), Y)).$$

Since $Ad(g)$ has determinant 1 the determinant of this derivative $d\psi$ becomes $det(Ad(t^{-1}) - id)_{\mathfrak{g}/\mathfrak{t}}$. We can replace $\mathfrak{g}/\mathfrak{t}$ by the complexification without changing the determinant. But $\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}} = \oplus_{\alpha \in \phi^+}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Therefore, the determinant of $Ad(t^{-1}) - id$ on $\mathfrak{g}/\mathfrak{t}$ is the product

$$\prod_{\alpha \in \Phi^+} (\alpha(t) - 1)(\frac{1}{\alpha(t)} - 1) = D(t)\overline{D(t)}.$$

The integral formula then follows since the pull back $\psi^*(\omega)$, under the map ψ of the top exterior form ω (obtained by wedging left invariant differential 1 forms on G and which gives volume 1 on G), is simply the Jacobian of ψ times the top exterior form ω' on $G/T \times T$. Since $G/T \times T \longrightarrow G$ is a $\mid W \mid$ fold covering on the open subset G_{reg} of regular elements (and G_{reg} has total measure 1), it follows that $\psi^*(\omega)$ has total measure $\mid W \mid$. □

2.4. **Consequences of the Weyl Integral Formula.**

Lemma 7. *If the Weyl group is trivial, then the connected group G is a torus.*

Proof. From the Weyl Integral formula, it follows that

$$\int_G dg = 1 = \frac{1}{\mid W \mid} \int_T dt \quad D(t)\overline{D(t)} = \int_T dt \quad D(t)\overline{D(t)},$$

where $D(t) = \prod_{\alpha \in \Phi^+}(\alpha - 1) = \sum_\chi m(\chi)\chi$ is an integral linear combination of distinct characters $\chi$of T . Here each χ is a product of positive roots α. Since each $\alpha(H) > 0$ it follows that $m(1) = \pm 1$ and that $\pi = \prod_{\alpha \in \Phi^+} \alpha$ is different from the trivial character *if* the set Φ^+ is non-empty. Moreover, π is different from every other character χ with $m(\chi) \neq 0$, because such a χ is a partial product of the characters α with $\alpha > 0$, and viewed as linear forms on $iLie(T)$, $\pi(H) > \chi(H)$ for any other χ.

By orthogonality of characters on T, it follows that

$$1 = \int_T dt \mid \Delta(t) \mid^2 = \sum_\chi m(\chi)^2,$$

and the trivial character 1 certainly occurs. Hence we get a contradictory inequality : $1 \geq m(1)^2 + m(\pi)^2$, unless Φ^+ is empty. Thus the set of roots is empty and $\mathfrak{g} = \mathfrak{t}$. That is, $G = T$. □

Lemma 8. *(A) If the Weyl group has order two, then* $dimT = 1+dim(Z)$ *where* Z *is the centre of* G*, and* $dim(G/Z) = 3$*. More precisely,* $G/Z = SO(3)$ *or* $SU(2)$*.*
(B) Suppose G *is a connected compact Lie group of semi-simple rank one, i.e. if* Z *is the connected component of identity of the centre of* G *and* T *is a maximal torus, suppose* $dim(T/Z) = 1$*. Then* G/Z *is either* $SO(3)$ *or* $SU(2)$*.*

Proof. Let $D(t) = \prod_{\alpha\in\Phi^+}(\alpha - 1)$ as before. From the Weyl integral formula, it follows that

$$2 = \mid W \mid = 2\int_G dg = \int_T dt \quad D(t)\overline{D(t)}.$$

Again, $\Delta(t) = \sum_\chi m(\chi)\chi$ is an integral linear combination of characters χ of the torus T, and $m(1) = \pm 1$, and $m(\pi) = 1$. Therefore, $2 \geq m(1)^2 + m(\pi)^2$ and therefore, there are no other characters χ. Recall that $t_0 = exp(iH_0)$ was chosen so that the non-zero real numbers $\beta(H_0)$ are all distinct for distinct roots $\beta \in \Phi$. Let us order the distinct numbers $\beta(H_0)$ for $\beta \in \Phi^+$ in increasing order:

$$0 < \beta_1(H) < \beta_2(H) < \cdots < \beta_m(H).$$

The equation $\Delta(t) = \prod_{\alpha\in\Phi^+}(\alpha - 1)$ shows that the signed multiplicity $m(\beta_1)$ is nonzero. But, by the previous paragraph, $m(\pi) = 1$ and hence $\pi = \beta_1$ and there are no other characters. Moreover, the multiplicity $m(\beta_1) = 1$. That is $\Phi^+ = \{\beta_1 = \beta\}$ is a singleton and $dim(\mathfrak{g}_\beta) = 1$. Further, clearly, $Z = ker(\beta)$ and therefore has co-dimension one in T. Then,

$$dim_{\mathbb{R}}(G/Z) = dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{z}) = dim(\mathfrak{g}_\beta) \oplus dim(\mathfrak{g}_{-\beta}) \oplus dim(\mathfrak{t}/\mathfrak{z}) = 1+1+1 = 3.$$

Now, the adjoint representation Ad of the semi-simple group G/Z has finite kernel and preserves the killing form which is a negative definite quadratic form in three variables; hence $Ad(G/Z) \subset SO(3)$. Since the dimensions of G/Z and $SO(3)$ are both 3, it follows that $Ad(G/Z) = SO(3)$. Hence G/Z is $SU(2)$ or $SO(3)$. This proves part

(A).

Since $T \neq Z$, G cannot be a torus; then G/Z cannot be a torus either, since in that case, $T/Z = G/Z$. Therefore, the Weyl group W has order at least two. Since W acts non-trivially on $T/Z = S^1$, the group W can only act by $t \mapsto t^{-1}$ on T/Z. Hence W has order exactly two and by part (A), G/Z is $SO(3)$ or $SU(2)$. This proves part (B).

□

Lemma 9. *If $w \in N(T)$ and $w(\Phi^+) = \Phi^+$ then $w \in T$.*

Proof. (This is essentially the Chevalley normaliser theorem and the proof is essentially Chevalley's proof). We argue by induction on the semi-simple rank (dimension T/Z) of G.

Let m be the dimension of $\mathfrak{n} = \oplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$ and let d be the dimension of $\mathfrak{n}$. Consider the d-th exterior power of $\mathfrak{g}$. Then $\wedge^d(\mathfrak{n})$ is a line in the G/Z representation $\wedge^d\mathfrak{g}$ fixed by w and by the torus T/Z. Let v be a non-zero vector in $\wedge^d\mathfrak{n}$. Then the group T/Z and by the assumption on w, the element w, both take the vector v into a multiple of itself.

Hence the commutator $wtw^{-1}t^{-1}$ acts trivially on v for $t \in T/Z$. If T/Z is the commutator, then T/Z fixes the vector v. But the character by which T/Z acts on v is the character $\pi = \prod_{\alpha>0}\alpha$ (which, viewed additively on the complex Lie algebra $\mathfrak{t}_\mathbb{C}$ is strictly positive on the element H and hence cannot be trivial on T/Z). Therefore, the commutator map $t \mapsto wtw^{-1}t^{-1}$ is a character on T/Z with positive dimensional kernel with non-trivial identity component S/Z, say for a torus $S \subset T$ containing Z. Then S is strictly bigger than Z and by definition, w centralises S.

Hence $w \in Z(S)$ and $Z(S)$ is connected (by Lemma 5). The centre of $Z(S)$ contains the torus $S \neq Z$. Hence the semi-simple rank of $Z(S)$ is $dim(T/S) < dim(T/Z)$. By induction, $w \in T$. □

Consequently, given $w \in W$ with $w \neq 1$, the set of positive roots which get taken into negative roots is non-empty. Let $A \subset \Phi^+$ be the subset with $w(A) \subset \Phi^+$ and $B \subset \Phi^+$ with $w(B) \subset -\Phi^+ = (\Phi^+)^{-1}$, so that $A \coprod B = \Phi^+$. Then $w(A) \coprod w(B)^{-1} = \Phi^+$ as well. We compute the effect of w on the function $D(t) = \prod_{\alpha>0}(\alpha-1) = \prod_{\alpha\in w(A)}(\alpha-1)\prod_{\alpha\in -w(B)}(\alpha-1)$. Then

$$w(D(t)) = \prod_{\alpha\in A}(w(\alpha)-1)\prod_{\alpha\in B}(w(\alpha)-1) = D(t)(-1)^{CardB}\prod_{\alpha\in B}w(\alpha).$$

We now introduce a character ρ not necessarily on T but on a covering T^* of T such that $\rho^2 = \pi = \prod_{\alpha>0} \alpha$. Then we see that $w(\rho)/\rho = \prod_{a\in B} w(\alpha)$. Consequently,

$$w(D(t)/\rho) = (-1)^{CardB} D(t)/\rho = sgn(w)D(t)/\rho,$$

where the latter equation defines $sgn(w)$. Since $D(t)/\rho$ is the product $\prod_{\alpha>0}(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}})$, it follows that the latter product is alternating.

Lemma 10. *If $t \in T_{reg}$ and $w \in W$ such that $wtw^{-1} = 1$ then $w = 1$.*

Proof. Write $t = exp(iH)$ for $H \in iLie(T)$. For any root α, then $\alpha(H)$ is either positive or negative, and hence we get $\Phi = \Phi^+ \coprod -\Phi^+$. The element w takes Φ^+ into itself since it takes H into itself. By lemma 9, $w = 1$.

□

2.5. **The alternating sum.** Given a character λ of T such that , define the alternating sum $A(\lambda) = \sum_{w\in W} sgn(w)t^{w(\lambda)}$. Let us say that two characters λ and μ are equivalent if there exists an element w of W with $w(\lambda) = \mu$. If λ, μ are inequivalent, then, by the orthogonality of distinct characters on the torus T, the integral

$$\int_T dt(A(\lambda)\overline{A(\mu)}) = 0.$$

The same observation holds true if λ, μ are two characters on a fixed finite covering T^* of T, such that $\lambda - w(\mu)$ is a (non-trivial) character on T (not only on T^*) for all elements w of the Weyl group W: since, in that case, the sum $A(\lambda)\overline{A(\mu)}$ is still a function (a finite linear combination of characters on T) on T although $A(\mu)$ and $A(\lambda)$ are functions on T^*.

Compute $\frac{1}{|W|}\int_T dt \quad (A(\rho)\overline{A(\rho)})$: since $w(\Phi^+) = \Phi^+$ if and only if $w = 1$, it follows that $w(\rho) = \rho$ if and only if $w = 1$. Therefore, $A(\rho) = \sum_{w\in W} sgn(w)t^{w(\rho)}$ is a linear combination of *distinct* characters $w(\rho)$ as w varies in W. By the orthogonality relations for T, we then get

$$\frac{1}{\mid W \mid}\int_T dt \quad A(\rho)\overline{A(\rho)} = 1.$$

The function $D(t)$ is alternating and is a sum (with coefficients ±1) of characters of the form ρ/χ where χ is a product of positive roots α. Consequently, $D(t)$ is a sum - with integral coefficients $m(\mu)$- of the alternating sums $A(\mu)$ where μ runs through a set of inequivalent (regular) characters of T^*. The coefficient of $A(\rho)$ is clearly positive. However, we have, by the last formula of the preceding subsection and the orthogonality of the $A(\mu)$, that $1 = \sum_{\mu \not\equiv \rho} m(\mu)^2 \frac{1}{|W|} \int_T dt \quad A(\mu)\overline{A(\mu)} + m(\rho)$. Therefore, we have proved that $m(\mu) = 0$ if $\mu \not\equiv \rho$ and that $m(\rho) = 1$. That is

$$\prod_{\alpha>0}(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}) = D(t)/\rho = A(\rho) = \sum_{w \in W} sgn(w) t^{w(\rho)}.$$

2.6. **The Weyl Character Formula.** Suppose V is an irreducible representation of the compact connected group G. We have, by orthogonality of characters and the Weyl integral formula, that

$$1 = \int_G dg \mid \chi_V \mid^2 = \frac{1}{\mid W \mid} \int_T dt \mid \chi_V(t) D(t) \mid^2 .$$

The function $\chi_V(t)D(t)$ is alternating and is a sum of characters on T^* with integral coefficients. Therefore, there exist finitely many inequivalent characters μ of T^* and integers $m(\mu)$ corresponding to them such that $\chi_V(t)D(t)$ is a sum of the basic alternating sums $A(\mu)$: $\chi_V(t)D(t) = \sum m(\mu)A(\mu)$.

Using now the orthogonality of $A(\mu)$ we see that $1 = \sum_\mu m(\mu)^2$ which shows that only one of the $m(\mu)^2$ is 1, and the rest are zero. Consequently,

$$\chi_V(t)D(t) = \pm A(\mu) = \pm \sum_{w \in W} sgn(w) t^{w(\mu)}.$$

Let us now introduce a partial order on the characters λ, ν on T by writing $\lambda > \nu$ if the character $\lambda\nu^{-1}$ is a product of positive roots. The trace function $\chi_V(t)$ is a sum of characters of T, and among these pick out one -call it λ- which is maximal with respect to this order ("a highest weight" of V). Suppose it occurs with multiplicity $e \geq 1$. Then $\chi_V(t)D(t)$ is alternating and is a sum of $eA(\lambda+\rho)$ and other sums $A(\nu)$; but the Weyl integral formula and the maximality of λ then ensures that no other character of the form $w(\nu)$ can be equal to $\lambda + \rho$ and hence $e = 1$ and the other terms are zero. Thus, we have

$$\chi_V(t) = \frac{\sum_{w \in W} sgn(w) t^{w(\lambda+\rho)}}{\sum_{w \in W} sgn(w) t^{w(\rho)}}.$$

We have thus proved the following theorem.

Theorem 11. (The Weyl Character Formula)

Given an irreducible representation V of a compact connected group G, and a character λ of V with respect to T, such that λ is maximal with respect to the partial order above (such a λ is called a highest weight of V), the dimension of the λ eigenspace is one; the representation V has a unique highest weight. We write $V = V(\lambda)$.

There is a vector v with T -eigenvalue λ and v is unique up to scalar multiples and is called the highest weight vector of V.

The trace $\chi_V(g)$ of $V = V(\lambda)$ is uniquely determined by its restriction to the maximal torus T and on $t \in T$, the trace is given by the "Weyl Character Formula"

$$\chi_V(t) = \frac{\sum_{w\in W} sgn(w)t^{w(\lambda+\rho)}}{\sum_{w\in W} sgn(w)t^{w(\rho)}}.$$

Corollary 1. *The representation V has a unique highest weight, namely λ. In particular, every weight μ of V is $\leq \lambda$ in the partial order.*

Proof. For, otherwise, suppose μ is another highest weight. The by comparing the Weyl character formula, we see that $A(\lambda+\rho) = A(\mu+\rho)$. This means that for some $w \in W$, we have $\lambda + \rho = w(\mu) + w(\rho)$; that is $w(\mu) = \lambda + \rho - w(\rho)$. Now, since μ is a weight of V, so is $w(\mu)$. Moreover, $\rho - w(\rho)$ is a sum of positive roots, and hence $w(\mu) \geq \lambda$ in the partial order. This means that $w(\mu) = \lambda$ and $w(\rho) = \rho$. But, by Lemma 9, $w(\rho) = \rho$ is and only if $w = 1$ in the Weyl group, and hence $\lambda = \mu$.

If there exist weights μ of V not comparable to λ, then pick one , call it μ, which is highest with respect to this partial order, among those which are not comparable to λ. Such a μ is necessarily a highest weight of V, and that is not possible by the preceding paragraph. This means that every other weight of the representation of V is comparable to λ and hence, is less than λ. □

2.7. **dominant integral weights.** Suppose λ is a weight of T (i.e. a character of T, which is sometimes, written additively as a linear form λ on the Lie algebra $\mathfrak{t}$ of T). The Weyl group acts on T and hence on its characters. We will say that λ is a dominant integral weight if for any $w \in W$, $\lambda - w(\lambda)$ is a sum of *positive roots* (or, is zero).

We can then form the following function on T_{reg}:

$$\chi_\lambda(t) = \frac{\sum_{w\in W} sgn(w)t^{w(\lambda+\rho)}}{\sum_{w\in W} sgn(w)t^{w(\rho)}}.$$

Note that this is a function on T_{reg} which is invariant under the conjugation action of W on T_{reg}. Hence it extends to a conjugate invariant function ϕ_λ on G_{reg} also, and G_{reg} has full measure in G. Let $V(\mu)$ be an arbitrary irreducible representation of G with highest weight μ. We then get, by the Weyl integral formula,

$$\int_G dx\phi_\lambda(x)\chi_\mu(x) = \frac{1}{\mid W \mid}\int_T dtA(\lambda+\rho)(t)A(\mu+\rho)(t).$$

Suppose λ is not the highest weight of any irreducible representation of G. Then $A(\lambda+\rho)$ is orthogonal to $A(\mu+\rho)$ for *all* μ which are highest weights of irreducible representations of G. Hence the class function ϕ_λ is orthogonal to all the χ_μ for all irreducible representations of G. This contradicts the Peter-Weyl theorem and hence λ is indeed the highest weight of an irreducible representation of G.

We will now show how to *realise* the representation $V(\lambda)$ with highest weight λ for every character λ of T which is *dominant*. This is the *Borel-Weil Theorem*. Write $\mathfrak{b} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}$ where $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and $(\alpha+\beta)(H) > \alpha(H)$ it follows that $\mathfrak{b}$ is a solvable subalgebra of $\mathfrak{g}$. Let B be the connected *complex analytic subgroup* of $G(\mathbb{C})$ with Lie algebra $\mathfrak{b}$. Then B is a connected solvable analytic subgroup of $G(\mathbb{C})$. Furthermore, $B \cap G = T$ and the real dimension of G/T is $2d$ where d is the complex dimension of N. Therefore, G/T is an open (and compact) submanifold of $G(\mathbb{C})/B$ and since $G(\mathbb{C})/B$ is connected, $G/T = G(\mathbb{C})/B$. Thus, $G(\mathbb{C})/B$ is a compact complex manifold. Further, $G(\mathbb{C}) = GT(\mathbb{C})N$ which shows that up to homotopy, $G(\mathbb{C})$ and G are the same. Thus $G(\mathbb{C})$ is simply connected as well.

For technical reasons, we replace $B(\mathbb{C})$ with $B^-(\mathbb{C})$ where $B^-(\mathbb{C}) = T(\mathbb{C})N^-$ where N^- is the analytic subgroup of $G(\mathbb{C})$ with Lie algebra $\oplus_{\alpha\in\Phi^+}\mathfrak{g}_{-\alpha}$ (N^- is the "opposite" of N). Thus, $B^-\backslash G(\mathbb{C}) = T\backslash G$.

Consider a holomorphic homomorphism $\lambda : B^-(\mathbb{C}) \to \mathbb{C}^*$. We can then form a line bundle $\mathscr{L}_\lambda$ on the compact complex manifold $B^-\backslash G(\mathbb{C})$. The space of holomorphic sections of this line bundle on $B^-\backslash G(\mathbb{C})$ is finite dimensional (Montel's theorem) and one can show that this representation is irreducible. The space of sections may be identified with holomorphic (algebraic) functions f on $G(\mathbb{C})$ which satisfy

$$f(bg) = \lambda(b)f(g), \quad \forall g \in G(\mathbb{C}), b \in B^-(\mathbb{C}).$$

Every irreducible representation of G arises this way and these are all the irreducible representations of G. This completes the classification of irreducible representations of G.

Theorem 12. *(Borel-Weil Theorem) Let* λ *be a dominant integral weight on* T*, which may then be extended to an algebraic homomorphism from* $T(\mathbb{C})$ *into* $\mathbb{C}^*$*, and an algebraic character on* $\lambda : B^-(\mathbb{C}) \to \mathbb{C}^*$*, by setting* λ *to be trivial on* N^-*. We can then form a holomorphic line bundle* $\mathscr{L}_\lambda$ *on the compact complex manifold* $B^-(\mathbb{C})\backslash G(\mathbb{C})$*. The space of holomorphic sections of this line bundle is a representation* $V(\lambda)$ *of* G *under the right action of* $G \subset G(\mathbb{C})$ *on* $B^-(\mathbb{C})\backslash G(\mathbb{C})$*, and is irreducible of highest weight* λ*.*

This space of holomorphic sections is the space of algebraic functions f *on* $G(\mathbb{C})$ *satisfying*

$$f(ntx) = \lambda(t)f(x) \quad for \quad all \quad n \in N^-(\mathbb{C}), t \in T(\mathbb{C}), x \in G(\mathbb{C}).$$

This realises the representation $V(\lambda)$ *explicitly.*