COMPACT LIE GROUPS

1. Lie Groups and Lie Algebras

Till the last section, we had made no assumptions on the group G, except that G was assumed to be a compact topological group. Later in this section, we will deduce, as a consequence of the Peter-Weyl theorem, that a compact Lie group is linear. In the first part of this section, we will define Lie groups and Lie algebras and study the relationship between them.

1.1. **Manifolds.**

Definition 1. A smooth manifold M of dimension n is a Hausdorff topological space M, together with a collection of open sets (called an atlas) U_I which cover M such that for each i, there is a homeomorphism $\phi_i : U_i \to V_i$ where $V_i \subset \mathbb{R}^n$ is an open set, and such that for each i, j the map (homeomorphism) $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is a smooth map of open sets in $\mathbb{R}^n$. The open sets U_i are called coordinate charts.

Examples: the vector space $\mathbb{R}^n$, open sets in $\mathbb{R}^n$, the unit sphere S^n in $\mathbb{R}^{n+1}$.

If $f : M \to N$ is a map between manifolds M, N it is said to be smooth if the associated maps between coordinate charts U_i, V_j of M, N, containing $m, f(m)$ of M, N respectively, the map $\psi_j \circ f \circ \phi_i^{-1}\phi_i^{-1}(U_i) \to \psi_j(V_j)$ are smooth. (We may shrink the charts so that $f(U_i) \subset V_j$). If M, N are manifolds (with cordinate charts U_i, V_j respectively, then on the product topological product space $M \times N$, there is a natural structure of a manifold, with $U_i \times V_j$ being the coordinate charts on $M \times N$. The coordinate projections are then smooth.

1.2. **Lie Groups.**

Suppose G is a topological group. If the multiplication map $m : G \times G \to G$ and the inverse map $i : G \to G$ are smooth maps of manifolds, we say that G is a *Lie Group.*

The main example of a Lie group is $GL_n(\mathbb{R})$. It is clear that the group maps are polynomial functions in the matrix entries of the relevant matrices.

If G, G' are Lie groups, then a morphism $f : G \to G'$ is a homomorphism of groups which is also smooth. A morphism of Lie groups $f : H \to G$ is said to be a Lie subgroup if f is injective as a map of manifolds. If $H \subset G$ is a closed subgroup of a Lie group, then H is a Lie subgroup. We assume this non-trivial result from Lie theory .

If H is a closed subgroup of a Lie group G, then G/H has the natural structure of a manifold so that the action map $G \times G/H \to G/H$ given by $(g, xH) \mapsto gxH$ for $g, x \in G$, is smooth. Moreover, $G \to G/H$ is a locally trivial fibration.

1.3. **Lie Algebras.** The tangent space to the identity element e in G is denoted $\mathfrak{g} = Lie(G)$. It can be verified that $\mathfrak{g}$ is naturally the dual to the space $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of smooth functions on G which vanish at $e \in G$.

If $G = GL_n(\mathbb{R})$, then $Lie(G)$ is the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices. Denote by $[X, Y]$ the commutator of X and Y in $M_n(\mathbb{R})$. If $H \subset GL_n(\mathbb{R})$ is a Lie subgroup, then $\mathfrak{h} = Lie(H)$ is closed under the bracket $[X, Y]$. A real subspace $\mathfrak{g}$ of $M_n(\mathbb{R})$ is called a (real) *Lie algebra*, if for $X, Y \in \mathfrak{g}$ the bracket $[X, Y]$ also lies in $\mathfrak{g}$.

1.4. **The Exponential Map: linear case.** Given $X \in M_n(\mathbb{R})$, consider the exponential $exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$. It can easily be shown that this series converges in $M_n(\mathbb{R})$ and defines an element in $GL_n(\mathbb{R})$ (with positive determinant).

Suppose $G \subset GL_n(\mathbb{R})$ is a Lie subgroup. Given $X \in LieG$, for all $t \in \mathbb{R}$, we have $exp(tX) \in G$. Further,

$$\mathfrak{g} = \{X \in M_n(\mathbb{R}) : \forall t \in \mathbb{R}, \quad exp(tX) \in G\}.$$

We may replace $GL_n(\mathbb{R})$ by $GL_n(\mathbb{C})$ and replace $M_n(\mathbb{R})$ by $M_n(\mathbb{C})$; the exponential map is then a holomorphic map from $M_n(\mathbb{C})$ into $GL_n(\mathbb{C})$. $M_n(\mathbb{C})$ is then a "complex Lie algebra" under the bracket operation $(X, Y) \mapsto [X, Y]$; this is a complex bilinear map.

1.5. **The Exponential Map in general.** Let G be a Lie group and $X\mathfrak{g}$ an element in the Lie algebra. We then get a vector field X on G by setting $X_g = l_g^*(X)$ for $g \in G$. Then, from the theory of ordinary

differential equations, we get, for small $\varepsilon > 0$ a unique differentiable curve $\gamma : [0, \varepsilon] \to G$ such that $\gamma'(t) = l^*_{\gamma(t)}(X)$ and $\gamma(0) = 1$.
The uniqueness immediately implies that $\gamma(s+t) = \gamma(s)\gamma(t)$ for small s, t and by continuation, we get a group homomorphism $\gamma : \mathbb{R} \to G$, with $\gamma(0) = 1$, and $\gamma'(0) = X$. We write $exp_G(X) = \gamma(1)$. This is called the exponential of X. Thus $exp : \mathfrak{g} \to G$.

The derivative of the exponential map at $0 \in \mathfrak{g}$ is the identity map on $\mathfrak{g}$. Then the inverse function theorem says that the map $exp : \mathfrak{g} \to G$ is a diffeomorphism from a small neighbourhood of $0 \in \mathfrak{g}$ into a small neighbourhood of $1 \in G$.

1.6. **The Adjoint Representation.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The conjugation action of G on itself preserves the identity element. Hence it yields, by differentiation, an action of G on its tangent space at 1, namely $\mathfrak{g}$. This action is linear and is called the *adjoint representation*: we have a representation $Ad : G \to GL(\mathfrak{g})$. By differentiation, we get a linear representation $ad : \mathfrak{g} \to End(\mathfrak{g})$, also called the adjoint representation of the Lie algebra $\mathfrak{g}$. By differentiation with respect to y, of the equation $Ad(xyx^{-1}) = Ad(x)Ad(y)Ad(x^{-1})$, it is clear that $ad(Ad(x)(Y)) = Ad(x)ad(Y)Ad(x^{-1})$ as operators on $\mathfrak{g}$. Denote, for $X \in \mathfrak{g}, Y \in \mathfrak{g}$, by $[X, Y] = adX(Y)$. If $G \subset GL_n(\mathbb{R})$, then $\mathfrak{g} \subset M_n(\mathbb{R})$ and $[X, Y]$ coincides with $XY - YX$ in $M_n(\mathbb{R})$.

1.7. **The Killing Form.** The *Killing form* $(X, Y) \mapsto trace(adXadY) = \kappa(X, Y)$ is easily seen to be invariant under the action of $Ad(G)$, using the observations of the preceding paragraph.

If G is a *compact* Lie group and $g \in G$, then for any representation ρ of G, the image $\rho(g)$ is semi-simple (diagonalisable) and the eigenvalues of $g \in G$ in any representation are of the form λ with λ of modulus 1. Hence for $X \in \mathfrak{g}$, $\rho(X)$ is also diagonalisable and the eigenvalues of $X \in \mathfrak{g}$ are purely imaginary and real. Consequently, $\kappa(X, X)$ is a sum of squares of purely imaginary numbers and is hence negative or zero. It is zero if and only if the semisimple operator adX is zero; that is, X is in the centre of $\mathfrak{g}$ and hence the connected component of identity of the centre of G is non-trivial.

1.8. **The Closed Subgroup Theorem.** Suppose $H \subset G$ is a submanifold and is a subgroup. Then H is itself a Lie group and the inclusion $H \subset G$ is a morphism of Lie groups. One calls H a Lie subgroup of G. The topology on H need not coincide with the topology on H viewed as a subspace of G.

The map $H \to G$ yields, by differentiation, a morphism $\mathfrak{h} \to \mathfrak{g}$ of Lie algebras. The following two fundamental theorems provide a converse.

Theorem 1. *If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, then there exists a Lie subgroup $H \subset G$ such that the Lie algebra of H is $\mathfrak{h}$.*

This is a consequence of a theorem of Frobenius on involutive distributions.

Theorem 2. *Given a homomorphism $p : \mathfrak{h} \to \mathfrak{g}$ of the Lie algebras $\mathfrak{h}, \mathfrak{g}$ of two connected Lie groups H and G, there exists a homomorphism $q : H^* \to G$ whose differential at the identity $1 \in H^*$ yields the map p (where H^* is the simply connected covering of the LIe group H).*

Theorem 3. *(E.Cartan) A closed subgroup of a Lie group is a Lie subgroup.*

If $H \subset G$ is a closed subgroup of a Lie group G, equip the quotient G/H with the quotient topology. Then G/H gets a natural structure of a manifold with the action map $G \times G/H \to G/H$ being smooth. If H is in addition, a normal subgroup, then the map $G \to G/H$ is a morphism of Lie groups with kernel H.

Theorem 4. *(Ado's theorem) Every Lie algebra over $\mathbb{R}$ is linear.*

In contrast, not every Lie group is linear. One can show that the universal cover of $SL_2(\mathbb{R})$ is not linear. For $n \geq 3$, the group $SL_n(\mathbb{R})$ admits a two sheeted connected covering group G^* which is not linear.

Suppose $\mathfrak{g}$ is a real Lie algebra; it is a sub-algebra of $M_n(\mathbb{R})$ and the latter is the Lie algebra of $GL_n(\mathbb{R})$. It follows from the above theorems that there exists a connected Lie group $G \subset GL_n(\mathbb{R})$ with Lie algebra $\mathfrak{g}$. Let G^* denote the simply connected covering of G. It is immediate that G^* is a Lie group.

Now suppose $\rho : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras, and let G' be a connected Lie group with Lie algebra $\mathfrak{g}'$ (such a G' exists by the observations of the preceding paragraph). Consider the graph $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}'$ of ρ, and denote by $\mathfrak{h}$ the image of this diagonal map $X \mapsto (X, \rho(X)$. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}'$ and hence there exists (by the above theorem) a connected subgroup $H \subset G^* \times G'$ with Lie algebra $\mathfrak{h}$.

The first projection of H onto G is an isomorphism of Lie algebras and hence the projection $H \to G^*$ is a covering map of connected spaces. Since G^* is simply connected, it follows that $H = G^*$. The differential of the second projection $G^* = H \to G'$ is ρ. Therefore, every representation $\rho : \mathfrak{g} \to \mathfrak{g}'$ "integrates" to a representation of G^* into G'.

2. Compact Lie Groups

We prove as a consequence of the Peter-Weyl Theorem :

Theorem 5. *A compact Lie group is a closed subgroup of* $U(n)$ *for some* n.

Proof. We argue by induction on the dimension of G. If $dim(G) = 0$, then G is discrete. But G is compact and hence is finite. But then the Cayley theorem says $G \subset S_n$ for some n, where S_n is the symmetric group on n letters . Therefore, $G \subset S_n \subset GL_n(\mathbb{C})$, where S_n may be thought of as the group of permutation matrices.

Suppose the theorem holds when G is connected. Then we prove that it holds in general: suppose the connected component G^0 of identity is linear (i.e. has a faithful finite dimensional linear representation ρ^0), and G/G^0 is linear (via a faithful representation τ say). Then pick a representation ρ of G whose restriction to G^0 contains ρ^0 (Corollary **??**). Then the direct sum $\rho \oplus \tau$ is a faithful representation of G.

We assume then that G is connected. By the Peter-Weyl theorem, given $x \neq 1$ in G, there exists a representation ρ of G such that $\rho(x) \neq \rho(1) = 1$ (since representation functions separate points). Since ρ is non-trivial and G is non-trivial and connected, so is the image $\rho(G)$. Therefore, the kernel K of ρ has dimension strictly smaller than that of G. By induction assumption, K has a faithful representation τ, say. Let θ be a representation of G whose restriction to K contains τ (such a representation exists by Corollary **??**). The representation $\theta \oplus \rho$ is then easily seen to be faithful on G. □

2.1. **Properties of Tori.**

Definition 2. A compact connected abelian Lie group is called a *torus*.

For example, $S^1 \times \cdots \times S^1$ is a torus. We will see that every torus is of this form.

We now consider representations of $S^1 \times S^1 \times \cdots S^1$. Consider the l-fold product $T = (S^1)^l$ of the group S^1 with itself. Given integers $m_1, \cdots, m_l$

and $z = (z_1, \cdots, z_l) \in T$ we have the one dimensional representation i.e. character $\chi_m(z) = z_1^{m_1} z_2^{m_2} \cdots z_l^{m_l}$ on T. The span R of these characters χ_m as m varies through l-tuples of integers, is a subring of functions on T which contains 1, closed under complex conjugation, and separates points. Hence by the Weirstrass-Stone theorem, R is dense in the space of continuous functions on T and hence also in $L^2(T)$. By the orthogonality relations, it follows that if ρ is an irreducible representation of T then ρ is of the form χ_m for some m. Thus the ring of representation functions is the span of the characters χ_m. This span is also called the algebra of *trigonometric polynomials* on T. Thus the Peter-Weyl theorem is equivalent to saying that the space of trigonometric polynomials is dense in the space of continuous functions on the torus T.

Fix an l tuple m of integers m_i and consider the function $\chi_m : T = (S^1)^l \to S^1$. If $\mathbb{R}^l \to \mathbb{R}^l/\mathbb{Z}^l = T$ is the quotient map, then χ_m lifts to the linear map $\mathbb{R}^l \to \mathbb{R}$ given by $(x_1, x_2, \cdots, x_l) \in \mathbb{R}^l \mapsto m_1x_1 + \cdots m_lx_l$. The pre-image of the kernel of the map $\chi_m : T \to S^1$ in $\mathbb{R}^l$ is the set of points $(x_1, \cdots, x_l) \in \mathbb{R}^l$ such that $m_1x_1 + \cdots + m_lx_l$ is an integer k, as k varies. Therefore, this pre-image is a countable union of the hyperplanes $\sum m_ix_i = k$. Hence the pre-image of the complement of the kernel of χ_m is the complement of a countable number of hyperplanes and is clearly a dense open subset of $\mathbb{R}^l$.

The union of the kernels of all these χ_m therefore has pre-image which is a countable union (over m and k) of the hyperplanes $\sum m_ix_i = k$. Therefore, the intersection of the complement of the kernels of the characters χ_m is a dense subset of T (e.g. by the Baire Category theorem).

Proposition 6. *(Kronecker's theorem) Let $T = (S^1)^l$ be a torus. The set E of points $t \in T$ such that $\chi_m(t) \neq 1$ for* any *l-tuple of integers m is dense in T. Consequently, the set of points $t \in T$ such that the group generated by the element t is dense in T, is a dense set.*

Proof. Fix $t \in E$. Now, as was already noted, E is dense in T by the Baire category theorem. Let S be the closed subgroup of T generated by t and consider the quotient T/S. This is a connected abelian group and hence if $T \neq S$, T/S has a non-trivial character χ which is a character on T. But then $\chi(t) = 1$ contradicting the choice of t; therefore, $T = S$ and hence every element of E generates a dense subgroup of T. □

Proposition 7. *A compact connected abelian group T of dimension k is $(S^1)^k$.*

Proof. By the corollary to Peter-Weyl theorem, a torus T is linear i.e. has a faithful (finite dimensional) unitary representation ρ. By complete reducibility, ρ is a direct sum of irreducible representations of T. Since T is abelian, these irreducible representations are one dimensional. Hence T is a subgroup of the group D_n of diagonals for some unitary group $U(n)$. By a result in the previous Chapter, every character on T extends to a character on D_n i.e. $res : \widehat{D_n} \to \widehat{T}$ is surjective where $\widehat{T}$ is the group of characters on T. Since the character group of $D_n = (S^1)^n$ is $\mathbb{Z}^n$ it follows that the character group of T is a finitely generated abelian group.

Since T is connected, it follows that every non-trivial character $T \to S^1$ is surjective, which shows that $\widehat{T}$ is torsion free (and finitely generated by the preceding paragraph). Thus $\widehat{\phi} : \widehat{T} \simeq \mathbb{Z}^k$ whence, $p : T \to (S^1)^k$ induced by $\widehat{\phi}$ is an isomorphism. □

Corollary 1. *(Kronecker's Theorem) Given a torus T, the set of points $t \in T$ such that $\chi(t) \neq 1$ for any nontrivial character χ of T, is a dense set (say D). If $t \in D$, then the closed subgroup generated by t is all of T.*

The first part is simply a consequence of the Proposition and Proposition 6. If H is the closed subgroup generated by an element t in the dense set D, then consider a character χ on the connected abelian Lie group T/H. Then χ is a character on T and $\chi(t) = 1$. Since $t \in D$, this means that χ is trivial. That is, $H = T$. (One says T is *topologically generated* by every $t \in D$).

2.2. **Compact Semi-simple Groups.** A compact connected Lie group with finite centre is called a compact *semi-simple* Lie group.

This means that the centre of the Lie algebra $\mathfrak{g}$ of G is $\{0\}$. Let κ be the Killing form on $\mathfrak{g}$. Since $\mathfrak{g}$ consists of skew symmetric matrices, it follows that κ is negative semi-definite. Moreover, on some $Z \in \mathfrak{g}$, $\kappa(Z, Z) = 0$ if and only if the skew symmetric matrix $adZ = 0$; that is, Z lies in the centre of $\mathfrak{g}$.Therefore, $Z = 0$ and hence κ is negative definite.

Lemma 8. *(Bourbaki) Let G^* be a locally compact topological group with $\mathbb{Z}$ a central discrete subgroup of G^* such that the quotient $G = G^*/\mathbb{Z}$ is compact. Then any homomorphism χ of $\mathbb{Z}$ into the multiplicative group $\mathbb{R}_{>0}$ of positive real numbers extends to a continuous homomorphism of G^* into $\mathbb{R}_{>0}$.*

Proof. Let F be a compact subset of G^* mapping onto the compact quotient $G = G^*/\mathbb{Z}$. Let f^* be a positive compactly supported function on G^* which is strictly positive on F. Consider the " Mellin transform" of f along $\mathbb{Z}$, given, for $x \in G^*$ by

$$f(x) = \int_{\mathbb{Z}} dh f^*(xh)\chi(h)^{-1}.$$

Then f is strictly positive everywhere on G^*. Moreover, f is equivariant: $f(xh) = f(x)\chi(h)$ for $h \in \mathbb{Z}$ and $x \in G^*$.

If $x, y \in G^*$ the function $f(xy)f(x)^{-1}f(y)^{-1}$ is strictly positive and is therefore of the form $e^{\Omega(x,y)}$ for some real valued function on $G^* \times G^*$. By the equivariance of f, it follows that Ω descends to $G \times G$ i.e. is actually a function on $G \times G$.

Since $f(xy)f(x)^{-1}f(y)^{-1}$ is a coboundary on G^* with values on $R_{>0}$ it follows that $\Omega(x, y)$ is a cocycle on $G \times G$. That is, for $x, y, z \in G$ we have the equation

$$\Omega(x, y) + \Omega(xy, z) = \Omega(y, z) + \Omega(x, yz).$$

Since G is compact, we can integrate with respect to z. We then find that, for all $x, y \in G$,

$$\Omega(x, y) + \phi(xy) = \phi(x) + \phi(y),$$

where $\phi(x) = \int_G dg(z)\Omega(x, z)$. Put $g(x) = e^{-\phi(x)}$ for $x \in G$. Then for $x, y \in G^*$ we have

$$f(xy)f(x)^{-1}f(y)^{-1} = e^{\Omega(x,y)} = g(xy)g(x)^{-1}g(y)^{-1},$$

showing that the function $\theta(x) = f(x)g(x)^{-1}$ is a homomorphism of G^* into $\mathbb{R}_{>0}$. The equivariance of f and the *invariance* of g then show that θ coincides with χ on the subgroup $H = \mathbb{Z}$. This proves the lemma.

□

Theorem 9. *(H.Weyl) The fundamental group of a compact semi-simple Lie group G is finite.*

Proof. Since G is a compact manifold, its fundamental group is a finitely generated abelian group. If it is infinite, then by the structure theorem for finitely generated abelian groups, the fundamental group of G has $\mathbb{Z}$ as a quotient. Let G^* be the connected covering of G corresponding to this quotient $\mathbb{Z}$. By lemma 8, G^* has a nontrivial map into $\mathbb{R}_{>0}$; since G^* is connected, the image of G^* is open and hence is all of $R_{>0}$. Therefore, the Lie algebra $\mathfrak{g}$ of G has an abelian quotient, and hence by complete reducibility of G action on $\mathfrak{g}$, the centre of $\mathfrak{g}$ is non-zero.

Therefore, the centre of G has positive dimension and hence G cannot be semi-simple. □

Theorem 10. *Every compact connected Lie group G may be written in the form ZK where Z is the connected component of identity of the centre of G and K is a compact connected semi-simple subgroup of G. In fact K is the commutator subgroup of G.*

Proof. The group Z is closed and hence $Q = G/Z$ is a compact connected quotient of G. If $\mathfrak{q} = LieQ$ has non-zero centre, then $\mathfrak{q}$ has G invariants. The complete reducibility of the representation $\mathfrak{g}$ of the compact group G shows that $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{q}$ as G modules and hence as $\mathfrak{g}$ modules. It follows that $\mathfrak{q}$ cannot have G invariants.

Therefore, Q is semi-simple. By Weyl's theorem (Theorem 9) , Q has finite fundamental group. But since $\mathfrak{q}$ is an ideal in G, and hence a subalgebra, there exists a connected subgroup K of G with Lie algebra $\mathfrak{q}$. Therefore, K is a covering of Q and is hence compact by Hermann Weyl's theorem.

Since K contains the commutator subgroup $[K, K] = [G, G]$ and the latter is open in K, it follows that $[G, G] = K$. □

Corollary 2. *The classification of irreducible representations of a compact connected Lie group follows from the classification of the irreducible representations of a compact connected simply connected group.*

Theorem 11. *(H. Weyl) The representations of a complex semi-simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ are completely reducible.*

Proof. A representation of $\mathfrak{g}_{\mathbb{C}}$ is a complex representation of $\mathfrak{g}$. Let G be a simply connected semi-simple group whose Lie algebra is $\mathfrak{g}$. Then representations $\mathfrak{g}$ are equivalent to those of G. By Hermann Weyl's theorem (Theorem 9), G is compact. Hence G representations are completely reducible. The theorem follows.

□

References

[A] J. F. Adams, Lectures on Compact Lie groups,

[BD] Brocker and Dieck, Compact Lie groups.

[Bou] Bourbaki, Lie Groups and Lie Algebras