# THE PETER-WEYL THEOREM

## 1. The Peter-Weyl Theorem

### 1.1. Representation Functions.

**Definition 1.** If $h$ is a continuous function on the group $G$, we say that $h$ is a *representation function* if the right translates of $h$ by elements of $G$ lie in a finite dimensional vector space.

Suppose $h$ is a representation function and let $E$ be the span of right $G$ translates of $h$. By assumption, $E$ is finite dimensional, and hence has an orthonormal basis $e_1, \cdots, e_n$ with respect to the $L^2$-inner product on $G$. Thus, for each $g \in G$, there exist scalars $\rho_{ij}(g)$ such that

$$e_i(xg) = \sum_j \rho_{ij}(g) e_j(x).$$

The orthonormality of the $e_j$ shows that $\rho_{ij}(g) = \int_G dx e_i(xg)\overline{e_j(x)}$; hence $\rho_{ij}(g)$ is continuous and $\rho : G \to GL_n(\mathbb{C})$ is a representation. Moreover, by taking $x = 1$ we see that each $e_i(g)$ is a linear combination of the $\rho_{ij}(g)$.

Suppose $\rho : G \to GL_n(\mathbb{C}) = GL(V)$ is a representation; with respect to the standard basis $\varepsilon_1, \cdots, \varepsilon_n$ of $\mathbb{C}^n$, and given $x \in G$, we have the functions $\rho_{ij}(x)$. We compute, for a fixed $g \in G$, the translated function $\rho_{ij}(xg)$. This is

$$\rho_{ij}(xg) = \sum_{k=1}^{n} \rho_{ik}(x)\rho_{kj}(g),$$

and hence is a linear combination of the finite set of functions $\rho_{ik}$ for $1 \le k \le n$. Therefore, $x \mapsto \rho_{ij}(x)$ is a representation function.

The last two paragraphs show that the space of representation functions is the $\mathbb{C}$-span of the matrix coefficient functions $\rho_{ij}$ as $\rho$ varies over representations of $G$. The equation of the preceding paragraph also shows that the span $E$ of the functions $\rho_{ij}(xg)$ as $g$ varies over $G$, is a representation of $G$, and is isomorphic to $\rho$.

If $\rho, \tau : G \to GL_n(\mathbb{C})$ are two equivalent representations, then there exists $A \in GL_n(\mathbb{C})$ such that $\tau(x) = A\rho(x)A^{-1}$ for all $x \in G$. This shows that the function $\tau_{ij}(x)$ is a linear combination of the functions $\rho_{kl}(x)$, and vice versa. Therefore the span $E_\rho = \sum_{1\le i,j\le n} \mathbb{C}\rho_{ij}(x)$ depends only on the *equivalence class* of $\rho$.

In particular, if $(\rho, V)$ is an irreducible representation of $G$, then the span of the matrix functions $\rho_{ij}$ depends only on the equivalence class of $\rho$; we may therefore assume that $\rho$ is *unitary.* Then the orthogonality relations show that the representation $E_\rho$ is $n^2$ dimensional, where $n$ is the dimension of $\rho$. We now get a homomorphism $\Phi_V : V^* \otimes V \longrightarrow E_\rho$ given by $\lambda \otimes v \mapsto f_{\lambda,v}$ where $f_{\lambda,v}(x) = \lambda(\rho(x)v)$. Let $v_1, \cdots, v_n$ be an orthonormal basis of $V$, and let $\lambda_1, \cdots, \lambda_n$ be the dual basis of $V^*$. The image of the vector $\lambda_i \otimes v_j$ is the function $\lambda_i(\rho(x)v_j) = \lambda_i(\sum_{k=1}^n \rho_{kj}(x)v_j) = \rho_{ij}(x)$ which shows that $V^* \otimes V \longrightarrow E_\rho$ is surjective. Since the dimensions of the two spaces are the same, it follows that this map is an isomorphism.

Therefore, if $\rho$ is an irreducible representation, then the span of $\rho_{ij}$ is isomorphic to $V^* \otimes V$ with $G$ acting only on $V$. However, we may also view $V^*$ as a representation of $G$ which under the isomorphism $\Phi_V$, converts the $G$ action on $V^*$ into left translations on $R$.

We then have : the space $R$ of representation functions on $G$ is a direct sum of the representations $V^* \otimes V$ where $V$ runs through equivalence classes of irreducible representations of $G$, and we write

$$R = \oplus_V V^* \otimes V.$$

This is a decomposition of $G \times G$ modules.

**Lemma 1.** *Let $\widehat{G}$ denote the set of equivalence classes of irreducible (unitary) representations of the compact group $G$, and pick a representative from each equivalence class. By an abuse of notation, we denote by $\widehat{G}$ the set of these representatives. Then we have the decomposition*

$$R = \oplus_{\rho \in \widehat{G}} \oplus_{i,j \le dim(\rho)} \mathbb{C}\rho_{ij}.$$

Note that the representation functions form a subring of the ring of continuous functions on $G$.

Start with a compact group $G$ with the Haar measure $\mu$ on $G$. We may form the space $L^2(G) = L^2(G, \mu)$. This is a Hilbert space, consisting of (equivalence classes of) measurable functions $f$ on $G$ such that the integral

$$\int_G d\mu(x) \mid f(x) \mid^2$$

is finite. Given $\phi, \psi \in L^2(G)$ the integral $\int_G d\mu(x)\phi(x)\overline{\psi(x)}$ makes sense (Cauchy-Schwarz inequality) and yields an inner product on $L^2(G)$. On the space $L^2(G)$ the group$G \times G$ operates by left and right translations and preserves the foregoing inner product.

Suppose $V$ is a finite dimensional complex vector space on which $G$ operates (we know from the preceding chapter that $G$ preserves an inner product on $V$). The group $G$ also operates on the dual $V^*$. Hence the group $G \times G$ operates on $V^* \otimes V$. If $V$ is irreducible for $G$, then $V^* \otimes V$ is irreducible for $G \times G$.

The goal of the present chapter is to prove the following

**Theorem 2.** (*The Peter-Weyl Theorem*) *Let $G$ be a compact topological group.*

*(A) If $V$ is an irreducible representation of $G$, then $V^* \otimes V$ embeds in $L^2(G)$. Moreover, the subspace $R = \oplus_V V^* \otimes V$ is a $G \times G$-invariant* **dense** *subspace of $L^2(G)$ where $V$ runs through equivalence classes of irreducible (finite dimensional) representations of the group $G$.*

*(B) The subspace $R$ is dense in the space $C(G)$ of continuous functions as well.*

1.2. **When $G$ is linear.** First observe that in an important special case, the Peter-Weyl Theorem is an easy consequence of the Weierstrass-Stone theorem. If $G \subset U(n)$ is a compact linear group, then take the embedding representation $\rho : G \to U(n) \subset GL_n(\mathbb{C})$ . Clearly, the algebra $R'$ generated by $\rho_{ij}$ and their complex conjugates separate points and contains the constant function 1. By construction $R'$ is closed under complex conjugation. Hence by the Weierstrass-Stone theorem, $R'(\subset R)$ is dense in the space of continuous functions on $G$ (and hence in $L^2(G)$). Therefore so is $R$.

**Corollary 1.** *Suppose $\rho : G \subset GL(V)$ is a faithful representation. Then every irreducible representation $\tau$ of $G$ is a sub-representation of $W_{k,l} = V^{\otimes k} \otimes (V^*)^{\otimes l}$ for some integers $k, l \geq 0$. The algebra $R$ is the algebra $R'$.*

*Proof.* Note that the matrix coefficients of the sum $\sum_{k,l} V^{\otimes k} \otimes (V^*)^{\otimes k}$ are $(k \times l)$ products of the matrix coefficients of the form $\rho_{ij}(x)$ and $\rho^*_{ij}(x)$. But since $\rho$ may be assumed to be unitary, $\rho^*_{ij}(x)$ is the complex conjugate $\overline{\rho}_{ji}(x)$. Since the algebra $R'$ generated by $\rho_{ij}$ and $\overline{\rho_{ij}(x)}$

separates points ($\rho$ is faithful), contains 1 , and is closed under complex conjugation, it follows that this algebra is dense in the space of continuous functions on $G$. It follows that if a continuous function is orthogonal to $R'$ in $L^2(G)$, then it is zero.

If an irreducible representation $\tau$ of $G$ does not occur in any of the representations $W_{k,l}$, then, by the orthogonality relations, the matrix coefficients $\tau_{ij}$ are all orthogonal to those of $W_{k,l}$ for all $k, l$. This contradicts the conclusion of the preceding paragraph and hence $\tau$ occurs in $W_{k,l}$ for some $k, l$.

This implies that $\tau_{ij}$ is a matrix coefficient of $W_{k,l}$ and therefore, $R = R'$. □

We prove the general theorem after recalling some preliminaries on compact operators.

1.3. **Compact Operators.** If $H$ is a Hilbert space an $T : H \to H$ is a bounded operator, $T$ is said to be a compact operator if $T$ takes bounded sets *into* compact sets (equivalently, takes the unit ball in $H$ into a compact set).

**Example.** If $T$ has finite dimensional range, then $T$ is compact.

**Lemma 3.** *If $T_n$ is a sequence of compact operators converging in the operator norm to an operator $T$, then $T$ is also compact.*

**Corollary 2.** *If $H = L^2(X)$ for a measure space $(X, m)$ and $K(x, y) \in L^2(X \times X)$, one can form the operator $T(\phi) = \int_X \phi(y)K(x, y)dm(y)$ on $H$. The operator $K$ is compact.*

This $K$ is called a kernel function.

*Proof.* If $K$ is a finite linear combination of characteristic functions of measurable rectangles, then the operator is compact since it has finite dimensional range. Consequently, any $K$, being an $L^2(X \times X)$ limit of such simple functions, is also a compact operator. □

**Definition 2.** If $T : H \to H$ is a continuous linear map from a Hilbert space $H$ into itself, and $w \in H$, the map $v \mapsto (Tv, w)$ is a continuous linear function on $H$; hence there exists a vector $w'$ such that $(Tv, w) = (v, w')$ for some $w'$. The vector $w'$ is uniquely defined and $S : w \mapsto w'$ is a linear map. We write $w' = S(w)$. The operator $S$ is called the *adjoint* of $T$, and denoted $S = T^* = \overline{{}^tT}$.

An continuous linear operator $T : H \to H$ on a Hilbert space $H$ is said to be *self adjoint* of for all $v, w \in H$ we have $(Tv, w) = (v, Tw)$.

1.4. **Compact Self Adjoint Operators.** We now state a fundamental theorem on a compact self adjoint operator $T$. It says that the orthogonal complement of the kernel of $T$ is a direct sum of finite dimensional non-zero eigen-spaces of $T$.

**Theorem 4.** *If $T$ is compact and self adjoint, then we have a direct sum decomposition $H = H_0 \oplus_{\lambda \neq 0} H_\lambda$ where $H_\lambda$ is the $\lambda$ eigen-space of the operator $T$, and each $H_\lambda$ is finite dimensional for $\lambda \neq 0$.*

Note also that the direct sum $H' = \oplus_{\lambda \neq 0} H_\lambda$ is dense in the image $T(H)$ of $T$. This is because $H'$ is stable under $T$ and the image of $H'$ is the same as $T(H)$ since $T = 0$ on $H_0$.

In the sequel, we assume this result on compact self adjoint operators. We will introduce some *convolution operators* which will be shown to be compact operators. Before establishing properties of convolutions, we will prove some preliminaries on continuous functions on compact groups.

1.5. **continuous functions on compact groups.**

**Definition 3.** A function $f : G \to \mathbb{C}$ is said to be *uniformly continuous* if, given $\varepsilon > 0$, there exists a neighbourhood $U = U_\varepsilon$ of identity in $G$ such that

$$| f(ux) - f(x) | < \varepsilon \quad \forall u \in U, \quad \forall x \in G.$$

**Lemma 5.** *A continuous function $f : G \to \mathbb{C}$ is uniformly continuous.*

*Proof.* Let $\varepsilon > 0$. Given $x \in G$, there exists an open set $U_x$ of $G$ containing the identity $1 \in G$ such that $| f(ux) - f(x) | < \varepsilon$. Since the multiplication map $m : G \times G \to G$ and the inverse map $i : G \to G$ are continuous, there exists a neighbourhood $V_x = V_x^{-1}$ of 1 such that $V_x.V_x \subset U_x$ (and, automatically, $V_x \subset U_x$).

Then $G = \bigcup_{x \in G} V_x x$ is an open cover of $G$. Since $G$ is compact, there is a finite subset $F \subset G$ such that $G = \bigcup_{y \in F} V_y y$. Write $V = \bigcap_{y \in F} V_y$; this is an open neighbourhood of 1, with $V = V^{-1}$.

Let $x \in G$. There exists $y \in F$ such that $x \in V_y y$. Then $Vx \subset VV_y y \subset U_y y$. We estimate, for $v \in V$, the difference $f(vx) - f(x)$ as follows. $f(vx) = f(uy)$ for some $u \in U_y$, and $f(x) = f(u'y)$ for some $u' \in U_y$. Therefore,

$$| f(vx) - f(x) | \leq | f(uy) - f(y) | + | f(y) - f(u'y) | < \varepsilon + \varepsilon.$$

This proves the uniform continuity of $f$. □

Given a neighbourhood $U$ of identity in $G$, there exists (by Urisohn's lemma), a continuous function $f \geq 0$ on $G$ whose support is contained in $U$. We may assume, by replacing $f$ by a positive scalar multiple, that $\int_G dgf(g) = 1$. Let $\{U_n\}_n$ be a decreasing sequence of open neighbourhoods of 1 decreasing to 1. Fix a continuous function $f_n \geq 0$ whose support is contained in $U_n$, such that $\int f_n(x)dx = 1$. We will call the sequence $\{f_n\}_{n\geq 1}$ an *approximate identity.*

**Lemma 6.** *The map* $G \times L^2(G) \to L^2(G)$ *given by* $(g, \phi) \mapsto g\phi(x) = \phi(g^{-1}x)$ *is continuous.*

*Proof.* Suppose that $\psi \in C(G)$. Given $\varepsilon > 0$ there exists a neighbourhood $U = U_\varepsilon$ of 1 in $G$ such that for $u \in U$, we have $| \psi(g^{-1}x) - \psi(x) | < \varepsilon$ for all $g \in U$, by Lemma **??**. Then it follows that $|| g\psi - \psi ||_2 < \varepsilon$ for all $g \in U$.

Suppose now that $\phi \in L^2(G)$ and fix $\varepsilon > 0$. We find a continuous function $\psi$ such that $|| \phi - \psi ||_2 < \varepsilon$. Then, we find a neighbourhood $U = U_\varepsilon$ as in the preceding paragraph, such that $|| g\psi - \psi ||_2 < \varepsilon$. It then follows that for all $g \in U$ and all , we have $|| g\phi - \phi ||_2 < 3\varepsilon$, proving the lemma. □

## 1.6. **Convolution Operators.**

**Definition 4.** Fix a continuous function $f \in C(G)$. Define the operator

$$\phi \mapsto \phi * f, \quad \phi * f(x) = \int_G dg\phi(g^{-1}x)f(g),$$

for $\phi \in L^2(G)$. The operator $T$ is called a *convolution by* $f$, and $T(\phi)$ is called *the convolution of* $\phi$ *and* $f$.

We verify that $T(\phi)$ is defined and that $T$ maps $L^2(G)$ into $L^2(G)$: let $M$ denote the supremum of the continuous function $| f |$ on the compact group $G$. Write $g' = g^{-1}x$. Then, $\phi * f(x) = \int_G dg'\phi(g')f(xg'^{-1})$ is bounded by $M\int_G dg \mid \phi(g) \mid$ which in turn is bounded by $M\int_G dg \mid \phi(g) \mid^2$ and the latter is finite since $\phi \in L^2(G)$.

Using the uniform continuity of $f$ (Lemma **??**) we will show that $\phi * f$ is uniformly continuous: let $\varepsilon > 0$. We can find a neighbourhood $U = U_\varepsilon$ such that $| f(uy) - f(y) | < \varepsilon$, $\forall y \in G$ and $\forall u \in U$. Now, $\phi * f(ux) - \phi * f(x) = \int_G dg'\phi(g')(f(uxg'^{-1} - f(xg'^{-1})$. Therefore, we get, for all $u \in U$, the estimate

$$| \phi * f(ux) - \phi * f(x) | \leq \varepsilon \int_G dg' \mid \phi(g') \mid \leq \varepsilon \mid\mid \phi \mid\mid_2 .$$

This proves the uniform continuity of $\phi * f$.

**Lemma 7.** *The convolution map $\phi \to \phi * f$ is (1) a continuous linear map from $L^2(G)$ into itself, and (2) a continuous linear map from $L^2(G)$ into $C(G)$.*

*Proof.* We first prove (1). By Cauchy-Schwarz,

$$\mid \phi * f(x) \mid^2 \leq (\int_G dg' \mid \phi(g') \mid^2)(\int_G dg' \mid f(xg'^{-1} \mid^2).$$

The last integral is bounded by $M^2$, where $M$ is the supremum of $\mid f \mid$ on $G$, and hence we see that $\int_G dx \mid (\phi * f)(x) \mid^2 \leq M^2 \int_G dg' \mid \phi(g') \mid^2$. This shows $\| \phi * f) \|_2 \leq M \| \phi \|_2$, proving the continuity of the convolution operator $*f : L^2(G) \to L^2(G)$. This is part (1).

The same estimate shows that $\| \phi * f \|_{sup} \leq M \| \phi \|_2$, proving the continuity of the convolution operator from $L^2(G)$ into $C(G)$; this is part (2).

□

**Lemma 8.** *The adjoint of $\phi \mapsto \phi * f$ is a convolution operator of the form of the form $\psi \mapsto \psi * f^*$ where $f^*(x) = \overline{f(x^{-1})}$.*

The operators $\phi \mapsto \phi * f = H(f)(\phi)$ where $f \in C(G)$ is continuous, has the property that

$$\phi * f(x) = \int \phi(g^{-1}x) f(g) d\mu(g) = \int \phi(g) f(xg^{-1}) d\mu(g),$$

is just integrating $\phi$ against the "kernel function " $K(x, g) = f(xg^{-1})$ for $\phi \in L^2(G)$. Consequently it is a compact operator. We may choose the function $f$ so that the kernel function becomes a self adjoint operator: we replace the arbitrary function $x \mapsto f(x)$ with the function $x \mapsto f(x) + \overline{f(x^{-1})}$. Moreover, since the right action commutes with the convolution operators, the eigenspaces $H(f)_\lambda$ are all $G$-stable.

**Lemma 9.** *For any $f$, the convolution $R * f$ is contained in $R$.*

*Proof.* Let $\rho_{ij}$ be a matrix coefficient of $\rho : G \to GL_n(\mathbb{C})$. Then $\rho_{ij} * f(x) = \int_G \rho_{ij}(g^{-1}x) f(g) dg = \sum_k \rho_{kj}(x) \int_G dg \rho_{ik}(g^{-1}) f(g)$, and is therefore a finite linear combination of the matrix coefficient functions $x \mapsto \rho_{kj}(x)$. This proves the lemma. □

**Lemma 10.** *Suppose $\phi \in L^2(G)$ and $\{f_n\}$ is an approximate identity. Then the sequence $\phi * f_n$ tends to $\phi$ in $L^2(G)$.*

*Proof.* Fix $x \in G$. Then $\phi * f_n(x) - \phi(x) = \int_G (\phi(g^{-1}x) - \phi(x)) f_n(g)dg$. We can use Cauchy Schwarz for the measure $f_n(g)dg$ on $G$ to conclude that $\mid \phi * f_n - \phi(x) \mid^2 \leq \int_G \mid \phi(g^{-1}x) - \phi(x) \mid^2 f_n(g)dg$. Integrating with respect to $x$, we get

$$|| \phi * f_n - \phi ||_2^2 \leq \int_G f_n(g)dg \int_G dx \mid \phi(g^{-1}x) - \phi(x) \mid^2 = \int_G f_n(g)dg \,|| g\phi - \phi ||_2^2 .$$

Fix $\varepsilon > 0$. Then for large $n$, $|| g\phi - \phi ||_2 < \varepsilon$ by Lemma **??**, for all $g \in U_n$. But since the support of $f_n$ lies in $U_n$, we may assume, in the above integral over $g$, that $g \in U_n$.Therefore, we get $|| \phi * f_n - \phi ||_2^2 \leq \int_{U_n} f_n(g)dg\varepsilon^2 = \varepsilon^2$. This proves the lemma.

□

## 1.7. **Proof of the Peter-Weyl Theorem.**

*Proof.* By choosing $\{f\}$ to be an approximate identity, we can ensure that every $\phi$ is approximated by $\phi * f$ . It follows that the *intersection* of all the $H_0(f)$ as $f$ varies is zero and hence $H$ is the sum of the *nonzero* eigen-spaces $H(f)_\lambda$ as $\lambda$ varies through *non-zero* eigenvalues of the convolution operator $*f$ and $f$ varies through all the continuous functions in $G$. Moreover, all these $H_\lambda(f)$ are $G$ stable and hence $H$ is a sum of finite dimensional $G$ stable subspaces of $H$. This proves that representation functions are dense in $L^2(G)$. This proves part (A) of the Peter-Weyl Theorem.

Let $\phi \in C(G)$ and let $\varepsilon > 0$. By choosing an approximate identity $\{f_k\}_k$ in $C(G)$, we can find $K = K(\varepsilon)$ large so that for $k \geq K$,

$$|| \phi - \phi * f_k || = sup_{x \in G} \mid \phi(x) - \phi * f_k(x) \mid < \varepsilon.$$

Fix $k \geq K$. Let $\phi_n$ be a sequence of representation functions tending to $\phi$ in $L^2(G)$. Such a sequence exists by part (A) of the theorem. Then for large $n$, $|| \phi_n * f_k - \phi * f_k || < \varepsilon$, since $\phi \mapsto \phi * f_k$ is a continuous linear map from $L^2(G)$ into $C(G)$. Hence $|| \phi_n * f_k - \phi || < 2\varepsilon$. Thus $\sum_{k \geq 1} R * f_k$ is dense in $C(G)$. But $\sum_k R * f_k \subset R$ by the above lemma; therefore, $R$ is dense in $C(G)$ as well. This proves part (B).

This completes the proof of the Peter-Weyl Theorem. □

Some consequences: In the regular representation, each irreducible representation $\rho$ of $G$ occurs $dim(\rho)$ times. The matrix coefficients of inequivalent irreducible representations are orthogonal to each other. The representation functions on $G$ are dense in $L^2(G)$.

**Corollary 3.** *Let $H \subset G$ be a closed subgroup of a compact group and let $R, R_H$ be the representation rings of $G$ and $H$ respectively. Consider the restriction homomorphism $res : C(G) \to C(H)$. Then $res(R) = R_H$.*

*Proof.* Since $R$ is dense in $C(G)$ by the Peter-Weyl Theorem, its restriction to $H$ separates points, contains constant functions and is closed under complex conjugation; by the Weierstrass-Stone theorem, $res(R)$ is dense in $C(H)$. Clearly $res(R) \subset R_H$. Therefore, $res(R)$ is dense in $R_H$ as well.

We now claim that given an representation $\tau$ of $H$, there is a representation $\rho$ of $G$ whose restriction to $H$ contains $\tau$. It is enough prove this claim, because of the complete reducibility of representations of $H$, when $\tau$ is irreducible. Let $R, R_H$ be the rings of representation functions on $G$ and $H$ respectively. If there is no representation $\rho$ as in the claim, then by the orthogonality relations, the restriction of the matrix coefficients of $\rho$ to $H$ are orthogonal to the matrix coefficients $\tau_{ij}$ of $\tau$, in $L^2(H)$. Therefore, the restriction of $R$ to $H$ is orthogonal to $\tau_{ij}$ for all $i, j$. But by the preceding paragraph, we have $res(R)$ is dense in $R_H$, contradicting the conclusion that $res(R)$ is orthogonal to $\tau_{ij}$. □

**Corollary 4.** *If $G$ is a compact group all of whose irreducible representations are one dimensional, then $G$ is abelian.*

*Proof.* Suppose $x, y \in G$ and let $z = xyx^{-1}y^{-1}$. Then for any irreducible representation $\rho$ of $G$, we have $\rho(z) = 1 = \rho(1)$ since $\rho$ is one dimensional. Since representation functions are dense in the space of continuous functions on $G$, we have $f(z) = f(1)$ for all continuous functions $f$. Therefore, $z = 1$ and hence $G$ is abelian. □