COMPACT GROUPS

1. Preliminaries

1.1. **Topological Groups.** A *topological group* is a (Hausdorff) topological space G which also has a group structure i.e. a multiplicaion map $m : G \times G \to G$ and an inverse map $i : G \to G$ satisfying the usual axioms of groups, such that the group structure and topology are *compatible*. That is, the maps m and i are continuous maps of topological spaces (we assume here that the space $G \times G$ is equipped with the product topology).

We assume that the topological group G is *locally compact*. That is, every point of the group G has a neighbourhood which is compact.

We will only deal with locally compact groups from now on.By a mild abuse of notation, we will call these *topological groups*.

Example. $\mathbb{R}$, $\mathbb{R}^n$ are topological groups under vector addition.

$GL_n(\mathbb{R})$ is a topological group under matrix multiplication.

The group $U(n)$ of unitary matrices is a *compact* topological group.

The group $SO(n)$ of orthogonal matrices of determinant one, is a compact topological group.

1.2. **Haar Measure on Topological Groups.** We assume the well known result that every locally compact group G comes equipped with a *left invariant measure*, which is a regular Borel measure, called a (left) Haar measure. "Regular" means that the measure of a Borel set is approximated by compact sets from below and open sets from above. Such a measure is unique up to a scalar multiple.

A compact topological group is automatically locally compact. It can easily be proved that a locally compact group is compact if and only if its Haar measure is finite. *The* Haar measure of G is then the unique Haar measure μ with $\mu(G) = 1$ (the total volume is one).

Example. (1) A group G with discrete topology is a locally compact group with Haar measure being the counting measure: the measure of any set is its cardinality.

(2) The group $\mathbb{R}^n$ with the Euclidean topology is a locally compact group (vector space) with Haar measure μ being the Lebesgue measure. For any $A \in GL_n(\mathbb{R})$, it follows that $d\mu(Ax)$ is also a left invariant meanure, and by the general uniqueness theorem, $d\mu(Ax) = cd\mu(x)$ for some scalar $c > 0$. Note that by the change of variables formula, we have, for $A \in GL_n(\mathbb{R})$ and $x \in \mathbb{R}^n$,

$$d\mu(Ax) =\mid detA) \mid d\mu(x).$$

Therefore, the scalar $c =\mid detA \mid$.

(3) The set $GL_n(\mathbb{R})$ of nonsingular $n \times n$ matrices is an open subset of the set $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ of $n \times n$ matrices. If dx denotes the Lebesgue measure on the vector space $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$, then the Haar measure on $GL_n(\mathbb{R})$ is $\frac{dx}{|detx)|^n}$.

(4) Suppose that G is a compact group and ν is a regular Borel measure on the quotient space G/H such that ν is G invariant. Suppose μ_H is a left invariant measure on H. Then the functional

$$f \mapsto \int_{G/H} d\nu(x) \int_H d\mu_H(h) f(xh),$$

gives a left invariant Haar measure on G.

Corollary: the groups $SO(n)$ have a left invariant Haar measure.

The proof is by induction. The group $SO(2) = S^1$ has a Haar measure, and $SO(n)/SO(n-1) = S^{n-1}$ has an $SO(n)$ invariant measure: consider $X = \mathbb{R}^n \setminus \{0\}$. The Lebesgue measure $d\mu_n$ on X is an $SO(n)$ invariant measure. Given a measurable set $E \subset S^{n-1}$, and the open interval $(0,1)$, consider the map $E \mapsto \mu_n(E \times (0,1))$. This gives a measure on S^{n-1} invariant under $SO(n)$. By the result (4), $SO(n)$ has a Haar measure.

Example. (Haar measure on Lie Groups) We will say that a topological group G is a *Lie Group* if G is a smooth manifold such that the group operations $m : G \times G \to G$ and $i : G \to G$ are smooth (i.e. infinitely differentiable).

If G is a *Lie group* of dimension d, let $\omega_1, \cdots, \omega_d$ denote linearly independent left invariant differential 1-forms on G. Such forms are obtained by taking a basis of cotangent vectors at the identity and left translating them to get differential one forms on the manifold. A left invariant Haar measure on G is got by fixing a left invariant top degree differential form $\omega = \omega_1 \wedge \cdots \wedge \omega_d$ and integrating continuous functions with respect to this differential form ω. We will see this in detail later.

Much of this course is concerned with *compact Lie groups*. Examples of compact Lie groups are the unitary group $U(n)$, the orthogonal group $O(n)$, the torus group $S^1 \times \cdots \times S^1$. Any closed subgroup of $U(n)$ may be shown to have the structure of a compact Lie group. One of the theorems proved in this course says that any compact Lie group may be realised as a closed subgroup of $U(n)$ for some n.

1.3. **Representations.** Given a topological group G, A continuous homomorphism

$$\rho : G \to GL_n(\mathbb{C}) = GL(\mathbb{C}^n) = GL(V) \quad (n \geq 1),$$

is called a (complex) representation where V is an n-dimensional complex vector space; we often refer to the *pair* (ρ, V) as a representation (or simply the vector space V when the action ρ is clear from the context, or simply the homomorphism ρ). We write $n = dim(V) = dim(\rho)$. Given a representation $\rho : G \to GL(V)$ on an n-dimensional vector space V, we fix a basis $v_1, \cdots, v_n$ of V. We then realise the representation ρ as a homomorphism $\rho : G \to GL_n(\mathbb{C})$, where now $\rho(g)$ denotes the *matrix* of the transformation $\rho(g)$ with respect to the basis $v_1, \cdots, v_n$. Different bases give conjugate homomorphisms from G into $GL_n(\mathbb{C})$.

A sub-representation W of V is a subspace W of V which is stable under the action of all the linear transformations $\rho(g) : g \in G$. A representation is *irreducible* if the only sub-representations W of V are V and $\{0\}$.

A representation (ρ, V) is *trivial* if V is one dimensional and for all $g \in G, \rho(g)v = v$ for $v \in V$. If the dimension of V is arbitrary, but $\rho(g) = 1$ for all g, then we say that V is a direct sum of trivial representations (or that G acts trivially on V).

If $W \subset V$ is a sub-representation of a representation (ρ, V), then we get a representation on the quotient vector space V/W as follows. If $v + W$ is a coset representative of the quotient V/W and $g \in G$, then write $g(v + W) = \rho(g)(v) + W$; it is routine to check that this defines a representation (called the *quotient representation*) of G on the quotient V/W.

Fix a basis $w_1, \cdots, w_m$ is a basis of W. Extend this to a basis

$$B : w_1, \cdots, w_m, e_1, \cdots, e_p$$

of V. Then the cosets $e_1 + W, \cdots, e_p + W$ form a basis of the quotient V/W. In matrix terms (with respect to the basis B), the matrices $\rho(g)$ are of the form

$$\begin{pmatrix} \tau(g) & z(g) \\ 0 & \overline{\rho}(g) \end{pmatrix}$$

where $\overline{\rho(g)}$ is the quotient representation on V/W. Here $z(g)$ is an $p \times m$ matrix depending on g.

Suppose (τ, W) and (ρ, V) are two representations and $T : W \to V$ a linear map such that for each $g \in G$, the equality $\rho(g)T = T\tau(g)$ holds. Then T is called a *morphism* of representations W and V; one also says that T is G-equivariant. Observe that the kernel of T is a sub-representation of (τ, W) (and the image of T is a sub-representation of (ρ, V)). The co-kernel of T, namely $V/Image(T)$, is a quotient representation of V.

Given two representations (τ, W) and (ρ, V), the vector space of linear maps $Hom(V, W)$ is also a representation of G defined, for $g \in G$ and $T \in Hom(V, W)$, by $g * T = \tau(g)T\rho(g^{-1})$ (if T is equivariant, $g * T = T$ for all $g \in G$). If W is the one dimensional trivial representation, then $Hom(V, W)$ is simply the dual (also denoted V^*) of the vector space V, and the foregoing representation on V^* is called the *contragredient* of V. If $v_1, \cdots, v_n$ is a basis of V let $v_1^*, \cdots, v_n^*$ be the *dual* basis of V^*, i.e. $< v_i^*, v_j >= \delta_{ij}$ for all i, j. Denote now by $\rho(g)$ the matrix of the transformation $\rho(g)$ with respect to the basis $v_1, \cdots, v_n$. The matrix of the contragredient $\rho^*(g)$ with respect to the dual basis $v_1^*, \cdots, v_n^*$ is easily seen to be ${}^t(\rho(g))^{-1}$, the transpose of the matrix $\rho(g)^{-1}$.

Two representations (τ, W) and (ρ, V) are *equivalent* of there is a morphism $T : W \to V$ of representations which is a linear isomorphism of vector spaces. If T is an isomorphism, then the inverse linear map $T^{-1} : V \to W$ is also a morphism of representations, called the inverse of T. If τ and ρ are equivalent, then let $w_1, \cdots, w_n$ be a basis of $W = \mathbb{C}^n$; identify $V = \mathbb{C}^n$. Now $T(w_1), \cdots, T(w_n)$ a basis of $V = \mathbb{C}^n$; let A denote the matrix of T. If T is a morphism of representations, then the matrix of $\rho(g)$ with respect to the basis $(w_1), \cdots, (w_n)$ is seen to be $A^{-1}\rho(g)A = \tau(g)$. Thus, if we view a representation $\tau : G \to GL(W)$ as a homomorphism $G \to GL_n(\mathbb{C})$, then $\rho(g) = A\tau(g)A^{-1}$ for all $g \in G$.

If $(\rho, V), (\tau, W)$ are representations, then on the direct sum $W \oplus V$ of vector spaces, the group G operates by the formula $(\tau \oplus \rho)(g)(v, w) = (\tau(g)v, \rho(g)w)$ and yields a representation denoted $(\tau \oplus \rho, W \oplus V)$.

If $w_1, \cdots, w_m$ is a basis of W, and $e_1, \cdots, e_p$ is a basis of V, then the sequence of vectors $w_1, \cdots, w_m; e_1, \cdots, e_p$ is a basis of $W \oplus V$; the matrix of $(\tau \oplus \rho)(g)$ with respect to this basis is clearly

$$\begin{pmatrix} \tau(g) & 0 \\ 0 & \rho(g) \end{pmatrix}.$$

The direct sum of several representations can similarly be defined.

A representation (ρ, V) is *completely reducible* if V is a direct sum of irreducible representations.

Lemma 1. (*Complete Reducibility*) *Every representation of a compact group G is completely reducible.*

Proof. Given any inner product $\langle v, v' \rangle$ on the complex vector space V, define the bilinear form

$$(v, v') = \int_G d\mu(g) \langle \rho(g)v, \rho(g)v' \rangle.$$

It is clear that (v, v') is an inner product on V, which is *preserved* by G i.e. is invariant under the action of G. Given a subrepresentation W of V, let W' be the orthogonal complement of W with respect to this invariant inner product. Then W' is easily seen to be a sub-representation of V. Moreover, $V = W \oplus W'$ as representations of G.

We now prove the lemma by induction on the dimension of V. If V is not irreducible, then there is a non-zero smaller invariant subspace W of V and we have proved that $V = W \oplus W'$; here W and W' have strictly smaller dimension than V. Therefore, by induction, W and W' are direct sums of irreducible representations, and hence, so is V. □

Define the *unitary group* $U(n)$ as the subgroup of $GL_n(\mathbb{C})$ which preserves the standard inner product $(v, w)_{std} = \sum v_i \overline{w}_i$ on $\mathbb{C}^n$. That is, for all $g \in U(n)$ and all vectors $v, w \in \mathbb{C}^n$, $(gv, gw)_{std} = (v, w)_{std}$. Also observe that if (v, w) is any inner product, there exists a non-singular linear transformation $T \in GL_n(\mathbb{C})$ such that $(v, w) = (Tv, Tw)_{std}$.

Note that elements of $U(n)$ may be viewed as orthonormal bases of $\mathbb{C}^n$ with respect to the standard inner product on $\mathbb{C}^n$. With respect to the standard basis of $\mathbb{C}^n$, $U(n)$ consists of matrices g in $GL_n(\mathbb{C})$ such that ${}^t(\overline{g})g = 1$ where $\overline{g}$ is the matrix whose entries are complex conjugates of the entries of g, and ${}^t(x)$ is the transpose of the matrix x. It is then clear that $U(n)$ is a closed and bounded subset of the set $M_n(\mathbb{C})$ of $n \times n$ complex matrices: the condition ${}^t(\overline{g})g = 1$ is equivalent to saying that if g_{ij} are the entries of the matrix g, then for each pair of integers i, k with $1 \le i, k \le n$, we have

$$\sum_{j=1}^{n} \overline{g}_{ji} g_{jk} = \delta_{ik}.$$

Since the matrix coefficients are continuous on $M_n(\mathbb{C})$ (even infinitely differentiable), it follows that $U(n)$ is the set of zeroes of a finite collection of smooth functions on $M_n(\mathbb{C})$ and hence $U(n)$ is closed in $M_n(\mathbb{C})$. On the other hand, taking the traces of both sides of ${}^t(\overline{g})g = 1$, we see that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{g}_{ji} g_{ji} = trace(1) = n.$$

This shows that the g_{ij} are bounded by n and hence $U(n)$ is a bounded subset of $M_n(\mathbb{C})$. By a theorem in general topology (the Heine-Borel Theorem), $U(n)$ is therefore compact. As a corollary, any closed subgroup of $U(n)$ is also compact.

Example. The group $U(n)$ acts irreducibly on $\mathbb{C}^n$.

Proof. For, suppose W is a non-zero $U(n)$-invariant subspace of $\mathbb{C}^n$ and $w \in W \setminus \{0\}$, of norm one. The vector w may then be completed to an orthonormal basis of $\mathbb{C}^n$; but every orthonormal basis is obtained by a translation of the standard orthonormal basis by a unitary transformation. Hence there exists a unitary transformation g which transforms w into the first element ε_1 of the standard basis of $\mathbb{C}^n$. This means that $\varepsilon_1 \in W$ and hence, by the same reasoning, any element of norm one in $\mathbb{C}^n$ lies in W; that is, $W = \mathbb{C}^n$. □

Corollary 1. *Every compact subgroup K of $GL_n(\mathbb{C})$ may be conjugated into $U(n)$. In particular, every representation of a compact group is unitary.*

Proof. By the proof of complete reducibility lemma, the compact group K preserves an inner product (v, w) on $\mathbb{C}^n$. Since the inner product (v, w) is of the form $(Tv, Tw)_{std}$ for some non-singular T, it follows that the conjugate group TKT^{-1} lies in $U(n)$. □

Lemma 2. *(Schur's Lemma) Suppose V is an irreducible representation f a topological group G and T is a linear transformation on V commuting with the operators $\rho(g)$ for all $g \in G$. Then T is a scalar.*

Proof. Let λ be an eigenvalue of T. Let W be the kernel of $T-\lambda$. Then W is a subrepresentation of V and is nonzero. By the irreducibility of V, it follows that $W = V$ i.e. T is the scalar transformation λ on the whole space V. □

Corollary 2. *The group $U(n)$ is a maximal compact subgroup of $GL_n(\mathbb{C})$.*

Proof. Suppose K is a compact subgroup of $GL_n(\mathbb{C})$ containing $U(n)$. By the preceding corollary, there exists a nonsingular T such that $TKT^{-1} \subset U(n)$. Hence $U(n) \subset TU(n)T^{-1}$. Therefore, $U(n)$ preserves two inner products $h(v,w) = (v,w)_{std}$ and $h'(v,w) = (Tv,Tw)_{std}$. We have thus two $U(n)$-equivariant isomorphisms $T_h, T_{h'} : \overline{V} \to V^*$. By Schur's Lemma, there exists a scalar λ such that $T_{h'} = \lambda T_h$; i.e. $(Tv,Tw)_{std} = \lambda(v,w)_{std}$ with $\lambda = \mu^{-2}$ for some positive scalar μ. Therefore $T\mu$ is unitary and hence $TU(n)T^{-1} = T\mu U(n)(T\mu)^{-1} = U(n)$. Hence $K = U(n)$ and $U(n)$ is a maximal compact subgroup of $GL_n(\mathbb{C})$. □

Corollary 3. *(Orthogonality relations) Let ρ, τ be irreducible unitary representations of a compact group G, with matrix coefficients ρ_{ij} and τ_{kl}. If ρ and τ are not equivalent, then their matrix coefficients are orthogonal; that is*

$$\int_G dg\rho_{ij}(g)\overline{\tau_{kl}(g)} = 0.$$

If $\tau = \rho$, then, we have

$$\int_G dg\rho_{ij}(g)\overline{\rho_{ij}(g)} = \frac{1}{dim\rho}$$

and

$$\int_G dg\rho_{ij}(g)\overline{\rho_{kl}(g)} = 0 \quad ((ij) \neq (kl)).$$

.

Proof. Let (ρ, V) and $\tau, W)$ be two irreducible representations as in the statement of the corollary, and let $T \in Hom(W,V)$ be an *arbitrary* linear transformation. Consider the linear map (it is an integral of a vector valued function)

$$T' = \int_G dg\rho(g)T\tau(g)^{-1}.$$

We then have $\rho(g)T'\tau(g)^{-1} = T'$ and hence T' is an equivariant map. If T' is non-zero, then by the irreducibility of W, the kernel of T' is zero (since it is a $(\rho(G)$ invariant subspace) and the image of T' is all of V.

Therefore, T' is an isomorphism and hence W, V are equivalent.

Consequently, if V, W are not equivalent, then $T' = 0$. Fix bases $v_1, \cdots, v_n$ of V and $w_1, \cdots, w_m$ of W. Let $T = (T_{jk})$ with T_{jk} arbitrary. Since $T' = 0$, we get

$$0 = T'_{il} = \sum_{j,k} \int_G dg\rho(g)_{ij} T_{jk} \tau(g^{-1})_{kl} = \sum_{j,k} \left(\int_G dg\rho(g)_{ij} \tau(g^{-1})_{kl} \right) T_{jk}.$$

The independence of the T_{jk} and the unitarity of the $\tau(g)$ (i.e. $\overline{\tau(g)_{lk}} = \tau(g^{-1})_{kl}$) now ensure that the functions $\rho_{ij}(g)$ and $\tau(g)_{lk}$ are orthogonal.

The remaining part is proved similarly. Suppose ρ, τ are equivalent. We may assume that $\rho = \tau$ and that $V = W$. Then the equivariant map T' is a scalar matrix λ and $T'_{ij} = 0$ if $i \neq j$ and $T'_{ii} = \lambda$. The trace of T' is $n\lambda$ with n the dimension of ρ. Since T' is the integral over G, of $\rho(g) T \tau(g)^{-1} = \rho(g) T \rho(g)^{-1}$, it follows that the trace of T' is the trace of T. We then get

$$\delta_{il} \frac{T_{11} + \cdots + T_{nn}}{n} = \delta_{il} \lambda = T'_{il} = \sum \int_G dg\rho(g)_{ij} T_{jk} \rho(g^{-1})_{kl}.$$

Since $\rho(g)$ is unitary , we have $\rho(g^{-1}_{kl}) = \overline{\rho(g)_{lk}}$ and therefore

$$\delta_{il} \frac{T_{jj}}{n} = \sum_k \int_G dg\rho(g)_{ij} T_{jk} \overline{\rho(g)_{lk}} = \sum_k \left(\int_G dg\rho(g)_{ij} \overline{\rho(g)_{lk}} \right) T_{jk}.$$

The independence of the linear forms T_{ij} gives the rest of the corollary.

□

Corollary 4. *Suppose ρ, τ are two irreducible representations of a compact group G. Then*

$$\int_G dg \chi_\rho(g) \overline{\chi_\tau(g)} = \delta_{\rho,\tau}$$

where δ is the Dirac delta function (i.e. $\delta_{\rho,\tau} = 0$ if ρ, τ are not equivalent and $\delta_{\rho,\tau} = 1$ if ρ, τ are equivalent).

Proof. Suppose ρ, τ are not equivalent. Then by the orthogonality relations, $\int_G dg\rho_{ij}(g) \overline{\tau_{kl}(g)} = 0$ for all i, j, k, l. Hence

$$\int_G dg \chi_\rho(g) \overline{\chi_\tau(g)} = \sum_{i,k} \int_G dg \rho_{ii}(g) \overline{\tau_{kk}(g)} = 0.$$

If ρ, τ are equal, then, again by the orthogonality relations,

$$\int_G dg \chi_\rho(g)\overline{\chi_\rho(g)} = \sum_i \int_G dg \rho_{ii}(g)\overline{\rho_{ii}(g)} = \sum_i \frac{1}{dim \rho} = 1.$$

□

Corollary 5. *Suppose ρ and τ are two representations of a compact group whose trace functions $g \mapsto \chi_\rho(g) = trace(\rho(g))$ and $g \mapsto \chi_\tau(g) = trace(\tau(g))$ are equal. Then the representations ρ, τ are equivalent.*

Proof. Suppose $\rho = \oplus m_i \rho_i$ and $\tau = \oplus n_j \tau_j$ is a decomposition into a direct sum of irreducibles ρ_i, τ_j with multiplicities m_i, n_j. Assume that $\chi_\rho = \chi_\tau$. Then the orthogonality relations say that for an irreducible representation θ, the integral

$$\int_G dg \chi_\rho(g)\overline{\chi_\theta(g)} = m_i,$$

if $\theta = \rho_i$ for some i and is zero otherwise. Since $\chi_\rho = \chi_\tau$, it follows that the sets $\{\rho_i\}_i$ and $\{\tau_j\}_j$ are the same, and hence that the multiplicities m_i, n_i are the same. □

Corollary 6. *Every irreducible representation of a compact* abelian *group S is one dimensional.*

Proof. Let (ρ, V) be a representation of S. We may assume that ρ is unitary. Fix an element $s \in S$ and consider $\rho(s)$; it has an eigenvector v with eigenvalue λ say. Consider the λ eigenspace V_λ of $\rho(s)$. Since S is abelian, all of $\rho(S)$ commutes with $\rho(s)$ and the λ eigenspace of $\rho(s)$ is stable under the action of S. By irreducibility of ρ, $V_\lambda = V$ and therefore, s acts by the scalar λ on all of V. But s was an arbitrary element of S. Therefore, all of S acts by scalar matrices, and hence every line in V is S-stable. By the irreducilbility of V, it follows that V is a line i.e. ρ is one dimensional.

□

2. Representations of $SU(2)$

2.1. **Representations of S^1.** We first note that if $S^1 = \{z \in \mathbb{C}^* : \mid z \mid = 1\}$ then, for a fixed m, the homomorphism $z \mapsto z^m \in GL(\mathbb{C})$ is a one dimensional irreducible representation of S^1. Denote by R the space of linear combinations of the functions $z \mapsto z^m$, as m varies. This is called the algebra of trigonometric polynomials. This algebra separates points, contains the constant function 1 and is closed under complex conjugation. Hence , by the Weierstrass-Stone Theorem, R is dense in the space of continuous functions on S^1.

If ρ is any irreducible representation of S^1 different from $z \mapsto z^m$ for any m, then by the orthogonality relations, the matrix coefficient ρ_{ij} is orthogonal to z^m for all m and hence to R. But, by the density of R, this means that $\rho_{ij} = 0$ i.e. ρ does not exist. We have thus proved that the only irreducible representations of S^1 are one dimensional representations of the form $z \mapsto z^m$.

Lemma 3. *Suppose V is a representation of the one dimensional "torus" S^1. Suppose $V = \oplus V_\chi$ is a direct sum of irreducible representations each occurring with multiplicity one. Let $v \in V$ be a vector such that $v = \sum v_\chi$ accordingly, with $v_\chi \neq 0$ for every χ. Then the S^1-translates of the vector v span V.*

Proof. Suppose W is the span of S^1 translates of the vector v. Under the projection $V \to V_\chi$ the image of W is non-zero, since this holds for the vector v. The space W is a direct sum of irreducibles W', and hence, by multiplicity one assumption, one of the W' is V_χ.This means that W contains all the V_χ and hence their direct sum, namely the whole space V. □

2.2. **Conjugacy Classes in the group $SU((2)$.** By definition, $SU(2)$ is the group of complex 2×2 matrices of the form $\begin{pmatrix} \alpha & \overline{-\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$, with determinant one, i.e. $\mid \alpha \mid^2 + \mid \beta \mid^2 = 1$. Clearly this is the unit ball in $\mathbb{C}^2$ and is hence compact (it is in fact, isomorphic to the unit sphere S^3 is $\mathbb{R}^4$). Now $SU(2)$ acts on $\mathbb{C}^2$ and hence acts on the space P_m of homogeneous polynomials of degree m in the two variables X, Y where X, Y are the coordinate functions on $\mathbb{C}^2$.

Given an element $g \in SU(2)$, let $v \in \mathbb{C}^2$ be an eigenvector for g, with eigenvalue λ, say. We may assume v has norm one. Let w be of norm one and generate the perpendicular of v in $\mathbb{C}^2$. Then w is also an eigenvector with eigenvalue λ^{-1}. We have thus proved that g can be conjugated into the diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, by an element of $SU(2)$. Note also that the matrix g and g^{-1} are conjugate by the element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $SU(2)$.

2.3. **Irreducible Representations of $SU(2)$.** We first exhibit a family of irreducible representations of $SU(2)$.

Lemma 4. *The space P_m is an irreducible representation of $SU(2)$.*

Proof. Suppose $W \subset P_m$ is a non-zero sub-representation of $SU(2)$. Decompose W as a direct sum of irreducible representations of the diagonal group T. The group T consists of matrices of the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ with $\theta \in \mathbb{R}$ arbitrary. Therefore, T is the group S^1. By the remark before Lemma 3, the irreducibles of S^1 are characters of the form $e^{i\theta} \mapsto e^{ik\theta}$ for some k. Now the only irreducibles of S^1 occurring in P_m are the lines $\mathbb{C}X^jY^{m-j}$ for j fixed; on this line, S^1 operates by the character $e^{i(j\theta-(m-j)\theta)} = e^{(2j-m)i\theta}$. Therefore, for some integer j, the representation W contains the vector X^jY^{m-j}.

We now apply the $SU(2)$ matrix $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ to the vector X^jY^{m-j} to obtain the following vector w in W:

$$w = \frac{(X+Y)^j(Y-X)^{m-j}}{\sqrt{2}^m}.$$

Note that the coefficient of X^m in the vector w, is non-zero: it is $\pm\frac{1}{\sqrt{2}^m}$. Since $W \subset P_m$ and the multiplicity of each irreducible representation of T in P_m is at most one, the same holds for W. Then, by Lemma 3, the vector X^m lies in W. Hence the translate of X^m by the matrix T also lies in W. But this translate is

$$v = \frac{(X+Y)^m}{\sqrt{2}^m},$$

and the coefficient of X^jY^{n-j} in v is nonzero for every $j \le m$. Therefore, by Lemma 3, the vector $X^jY^{m-j} \in W$ for all j. That is, $W = P_m$; the representation P_m is irreducible. $\square$

Since any element of $SU(2)$ may be conjugated into the diagonal subgroup T of $SU(2)$, it follows that the trace of any representation is determined by its restriction to T. We now compute the trace of $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ on the representation P_m. The space P_m has the vectors $v_j = X^j Y^{m-j}$ as basis, for $0 \leq j \leq m$. These vectors are eigenvectors for g with eigenvalue $\lambda^j(\lambda^{-1})^{m-j} = \lambda^{2j-m}$. Hence the trace of g is

$$S_m = \sum_{j=0}^{m} \lambda^{2j-m} = \frac{1}{\lambda^m}\frac{\lambda^{2m+2}-1}{\lambda^2-1} = \frac{\lambda^{m+1}-\lambda^{-m-1}}{\lambda-\lambda^{-1}}.$$

We also have

$$S_m = \sum_{j=0}^{m} \lambda^{2i-m} = (\lambda^m + \lambda^{-m}) + (\lambda^{m-2} + \lambda^{2-m}) + \cdots .$$

Theorem 5. *Every irreducible representation of $SU(2)$ is of the form P_m for some integer m.*

Proof. Let ρ be any representation of $SU(2)$. The restriction of ρ to the diagonal group $T = \{g \in G : g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\}$, is a direct sum of characters of the form λ^k. Moreover, if the character λ^k occurs, so does λ^{-k} since g, g^{-1} are conjugate. Consequently, the trace of ρ is a sum of terms of the form $\lambda^k + \lambda^{-k}$. The formula for the trace of P_m above shows that the trace of ρ is an integral linear combination of the traces S_m. In particular, ρ is the direct sum of the P_m's. This also proves that every irreducible representation of $SU(2)$ is of the form P_m for some m. □

2.4. **The Clebsch-Gordan Formula.**

Theorem 6. *(Clebsch-Gordan Formula) Let $m \geq n$. The tensor product of the irreducible representations P_m and P_n decomposes:*

$$P_m \otimes P_n = \bigoplus_{j=0}^{n} P_{m+n-2j}.$$

Proof. We need only compute the traces on both sides and show that they are equal, since the trace completely determines the representation for a compact group. Moreover, the trace function being conjugate invariant, the traces need only be proved equal on the diagonals T in $SU(2)$ since every element in $SU(2)$ can be conjugated into the diagonals.

Let $t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ be a diagonal matrix in $SU(2)$ with $\lambda \neq \pm 1$. Let S_m be the trace of the representation P_m evaluated at t. The trace of the tensor product of P_m and P_n is simply the product of the traces of P_m and P_n. Hence the product of $\lambda - \lambda^{-1}$ and the trace of the left hand side evaluated at the element t is

$$(\lambda - \lambda^{-1})S_m S_n = (\lambda^{m+1} - \lambda^{-m-1})S_n = (\lambda^{m+1} - \lambda^{-m-1})(\sum_{j=0}^{n} \lambda^{n-2j}) =$$

$$= \sum_{j=0}^{n} (\lambda^{m+n-2j+1} - \lambda^{-m-n+2j-1}) = \sum_{j=0}^{n} (\lambda - \lambda^{-1}) S_{m+n-2j}.$$

(Note that since $m \geq n$, and $0 \leq j \leq n$, we have $m+n-2j \geq m-n \geq 0$). We thus get

$$S_m S_n = \sum_{j=0}^{n} S_{m+n-2j},$$

proving the formula. □