Classical vs quantum field treatments of axions

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Less Travelled Path of Dark Matter
ICTS-TIFR, India

November 9-13, 2020

- QCD axions or axion-like particles are highly motivated candidates for CDM.

- QCD axions have very high phase-space degeneracy:

$$\mathcal{N} = n(t)\frac{(2\pi)^3}{\left(\frac{4\pi}{3}\ \delta p(t)^3\right)} \sim 10^{61}\left(\frac{6\times10^{-6}\ \mathrm{eV}}{m}\right)^{\frac{8}{3}}$$

(Sikivie and Yang; 2009)

$\mathcal{N}$ is time-independent because $n(t) \propto a(t)^{-3}$ and $\delta p(t) \propto a(t)^{-1}$.

Generally, in cosmology, highly degenerate scalar fields are assumed to follow the classical field equations.

- Thermalization:

  Classical fields → Maxwell-Boltzmann distribution (UV catastrophe)

  Bosonic quantum fields → Bose-Einstein distribution

There must be time scale after which the quantum evolution of a highly degenerate scalar field differs from its classical evolution.

Formalism to calculate "duration of classicality"

- Non-relativistic limit: *(we are interested in cold dark matter)*

$$\phi(\vec{x},t) = \frac{1}{\sqrt{2m}}\left[\psi(\vec{x},t)\, e^{-imt} + \psi^\dagger(\vec{x},t)\, e^{imt}\right]$$

- Klein-Gordon equation reduces to a Schrödinger-like equation:

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + V(\psi)\psi$$

- Classical: Quantum field $\psi(\vec{x},t)$ ⟹ Wave-function $\Psi(\vec{x},t)$

- Quantum: $\psi(\vec{x},t)$ is treated as an operator.

$$[\psi(\vec{x},t), \psi(\vec{y},t)] = 0, \quad \left[\psi(\vec{x},t), \psi^\dagger(\vec{y},t)\right] = \delta^3(\vec{x}-\vec{y})$$

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

Classical Treatment

Wavefunction: $\Psi(\vec{x},t) = A(\vec{x},t)e^{i\beta(\vec{x},t)}$

Number density: $n(\vec{x},t) = \mathrm{A}^2(\vec{x},t)$

Velocity: $\vec{v}(\vec{x},t) = \frac{1}{m}\vec{\nabla}\beta(\vec{x},t)$

- Continuity equation: $\partial_t n + \vec{\nabla}\cdot(n\vec{v}) = 0$
- Euler-like equation: $\partial_t \vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{m}\vec{\nabla}V - \vec{\nabla}q$

Potential: $V(\vec{x},t) = \frac{\lambda}{8m^2}n(\vec{x},t)$

Quantum pressure: $q(\vec{x},t) = -\frac{1}{2m^2}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}$

- Equations for a fluid of classical particles except for the quantum pressure term.

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

Quantum Treatment

- In the quantum description: $\psi(\vec{x},t) = \sum_{\vec{k}} a_{\vec{k}}(t) u^{\vec{k}}(\vec{x},t)$

- $a_{\vec{k}}(t)$ and $a_{\vec{k}}^{\dagger}(t)$ are annihilation and creation operators for mode $\vec{k}$

$$\left[a_{\vec{k}}(t), a_{\vec{k}'}(t)\right] = 0 \quad \text{and} \quad \left[a_{\vec{k}}(t), a_{\vec{k}'}^{\dagger}(t)\right] = \delta_{\vec{k}}^{\vec{k}'}$$

- $u^{\vec{k}}(\vec{x},t)$ form orthonormal complete set

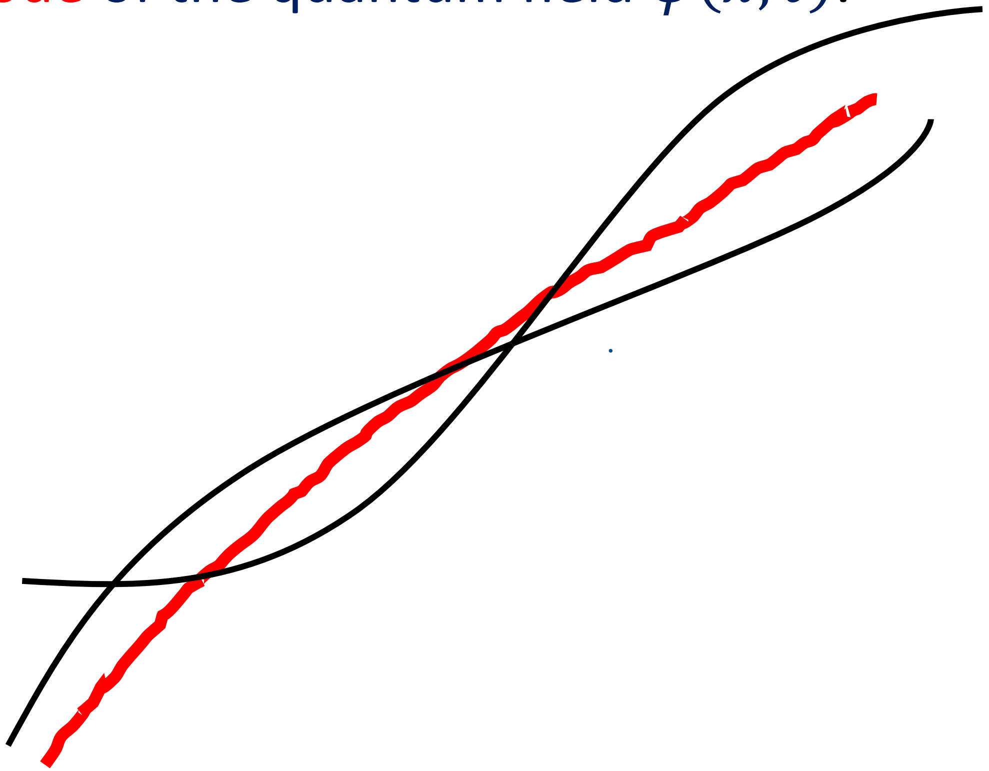
$$\int d^3x\, u^{\vec{k}}(\vec{x},t)^* u^{\vec{k}'}(\vec{x},t) = \delta_{\vec{k}}^{\vec{k}'} \qquad \sum_{\vec{k}} u^{\vec{k}}(\vec{x},t)^*\, u^{\vec{k}}(\vec{x}',t) = \delta^3(\vec{x} - \vec{x}')$$

We treat the classical solution $\Psi(\vec{x},t)$ as a specific mode of the quantum field $\psi(\vec{x},t)$.

$$u^{\vec{k}}(\vec{x},t) = \frac{1}{\sqrt{N}}\,\Psi(\vec{x},t)\,e^{i\vec{k}.\vec{\chi}(\vec{x},t)}$$

(N : Occupation number of $\vec{k} = \vec{0}$ mode)

Similar to classical solution $\Psi(\vec{x},t)$ but differ by long wavelength modulations



$\vec{\chi}\,(\vec{x},t)$: Co-moving coordinates.
Density in $\vec{\chi}$ -space is constant in space and time

$$\frac{d^3N}{d\chi^3} = \frac{d^3N}{dx^3}\left|\det\left(\frac{\partial\vec{x}}{\partial\vec{\chi}}\right)\right| = |\Psi(\vec{x},t)|^2\left|\det\left(\frac{\partial\vec{x}}{\partial\vec{\chi}}\right)\right| = n_0$$

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

If *and only if* the mode corresponding to the classical solution $\left(\vec{k} = \vec{0}\right)$ is highly occupied, quantum corrections are suppressed by $1/N$.

Do the quanta stay in this mode?

- The total number of quanta that have left $\vec{k} = \vec{0}$ in time t :

$$N_{ev}(t) = \sum_{\vec{k} \neq \vec{0}} \langle N_{\vec{k}}(t) \rangle$$

- The classical theory is valid until a time-scale of t_{cl} : $N_{ev}(t_{cl}) \approx N$

Examples

- Initially homogeneous condensate with attractive self-interactions
  - Attractive $\lambda\phi^4$ interactions
  - Gravitational self-interactions

- Initially inhomogeneous condensate with repulsive $\lambda\phi^4$ interactions

Attractive $\lambda\phi^4$-interactions

- Classical homogeneous solution:

$$\Psi_0(t) = \sqrt{n_0}\, e^{-i\delta\omega t} \quad \text{with} \quad \delta\omega = -\frac{|\lambda| n_0}{8m^2}$$

In the classical evolution, the homogeneous condensate always remains homogeneous.

- Quantum evolution of a homogeneous condensate:

$$u^{\vec{k}}(\vec{x}, t) = \frac{1}{\sqrt{N}}\, \Psi_0(t)\, e^{i\vec{k}.\vec{x}}$$

Quanta jump out of $\vec{k} = \vec{0}$ mode in pairs and all modes with wavevector $k < k_J = \sqrt{\frac{|\lambda| n_0}{2m}}$ become populated.

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

- Average occupation number for $k < k_J = \sqrt{\frac{|\lambda| n_0}{2m}}$ grows with time t :

$$\langle N_{\vec{k}}(t) \rangle \approx e^{2\gamma(k)t} \qquad\qquad \gamma(k) = \frac{k}{2m} \sqrt{k_J^2 - k^2}$$

- $\vec{k} = \vec{0}$ mode (classical solution $\Psi_0(\mathrm{t})$) is almost entirely depleted after:

$$\tau_c \sim \frac{2m}{k_J^2} \ln\left(\frac{32\pi\sqrt{\pi} n_0}{k_J^3}\right) = \frac{4m^2}{|\lambda| n_0} \ln\left(\frac{64\pi m\sqrt{2\pi m}}{|\lambda|\sqrt{|\lambda| n_0}}\right)$$

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

Gravitational self-interactions

- Studied homogeneous condensate in a matter-dominated expanding universe.
- In classical field treatment, the homogeneous condensate persists.
- In quantum field treatment, instability occurs.

Quanta jump out of $\vec{k} = \vec{0}$ mode in pairs and all modes with wavevector $k < k_J = (16\pi G m^3 n_0)^{\frac{1}{4}}$ become populated.

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

- The instability is similar to *Jean's instability in astrophysics*.

$k < k_J$ (wavelength $> k_J^{-1}$): attractive forces dominate

$k > k_J$ (wavelength $< k_J^{-1}$): quantum pressure dominates

- $\vec{k} = \vec{0}$ mode (classical solution $\Psi_0(\mathrm{t})$) is almost entirely depleted after:

$$t_c \sim t_* \frac{1}{(Gm^2\sqrt{mt_*})^{\frac{1}{2}}}$$

(SSC, Enomoto, Han, Sikivie, Todarello; 2018)

- If small density perturbation is added to the classical homogeneous condensate, quanta tend to move towards regions of higher density due to attractive nature of the interactions. Hence crowded places tend to become more crowded.

- In quantum evolution, small quantum fluctuations are *seeded* in the homogeneous condensate.

Are quantum and classical evolutions also different when self-interactions are repulsive?

Repulsive $\lambda\phi^4$ interactions

Classical theory

- Density contrast in the Fourier space:

$$n(\vec{x},t) = n_0[1+\delta(\vec{x},t)]$$

$$\delta(\vec{x},t) = \sum_{\vec{k}\neq\vec{0}} s_{\vec{k}}(t)\, e^{i\vec{k}.\vec{x}}$$

$\Rightarrow$

$$s_{\vec{k}} \sim \left(\frac{\delta n_{\vec{k}}}{n_0}\right)$$

- In the first order of density perturbations, $s_{\vec{k}}(t)$ are stable for all modes and oscillates with angular frequency $\omega\left(\vec{k}\right) = \sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m}+2\delta\omega\right)}$.

$$s_{\vec{k}}(t) = s_{\vec{k}}^{+}\, e^{i\omega(\vec{k})t} + s_{\vec{k}}^{-}\, e^{-i\omega(\vec{k})t}$$

$$\delta\omega = \frac{\lambda n_0}{8m^2}$$

(Arza, SSC, Enomoto, Han, Todarello; 2019)

To study the difference between quantum and classical evolution, we consider a simple case where only the mode with $\vec{k} = \vec{P}$ is initially perturbed in a homogeneous background.

Classical theory

- First order: Only $\vec{k} = \vec{P}$ mode is present and oscillates with $\omega(\vec{P})$.

$$s_{\vec{P}}(t) \sim \left(\frac{\delta n_{\vec{P}}}{n_0}\right) e^{i\omega(\vec{P})t}$$

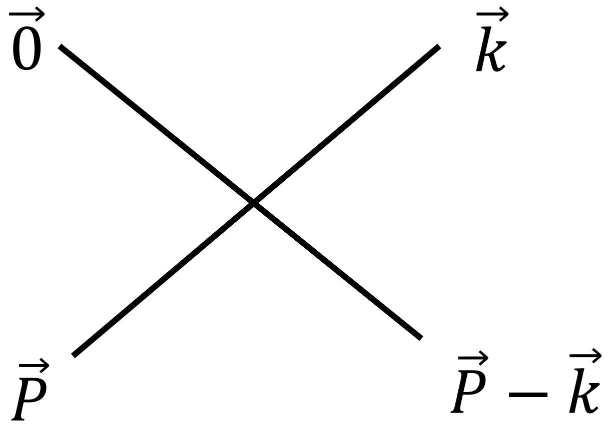
- Second order: Mode with $\vec{k} = 2\vec{P}$ appears and oscillates with $2\omega(\vec{P})$.

$$s_{2\vec{P}}(t) \sim \left(\frac{\delta n_{\vec{P}}}{n_0}\right)^2 e^{i2\omega(\vec{P})t}$$

- In general, only modes with $\vec{k} = j\vec{P}$ $(j = 1,2,3,..)$ are present and oscillates.

(Arza, SSC, Enomoto, Han, Todarello; 2019)

Quantum theory



$\epsilon \sim 0$

$$\epsilon \equiv \omega\left(\vec{k}\right) + \omega\left(\vec{P}-\vec{k}\right) - \omega(\vec{P})$$

Occupation number of the modes with $\epsilon \approx 0$ grow exponentially!

- For the modes corresponding to $\epsilon \approx 0$:

$$\langle N_{\vec{k}}(t) \rangle \sim e^{2\gamma(\vec{k})t}$$

$$\gamma\left(\vec{k}\right) = \sqrt{\left(q\left(\vec{k},\vec{P}\right)\delta\omega\frac{\delta n_{\vec{P}}}{n_0}\right)^2 - \left(\frac{\epsilon}{2}\right)^2}$$

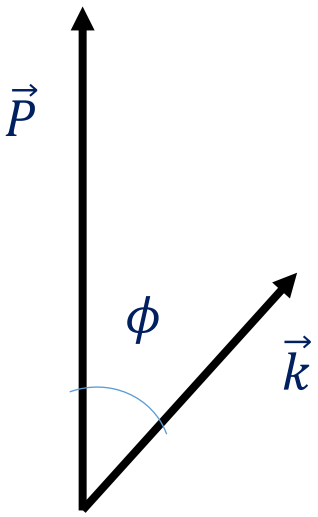
$$\delta\omega = \frac{\lambda n_0}{8m^2}$$

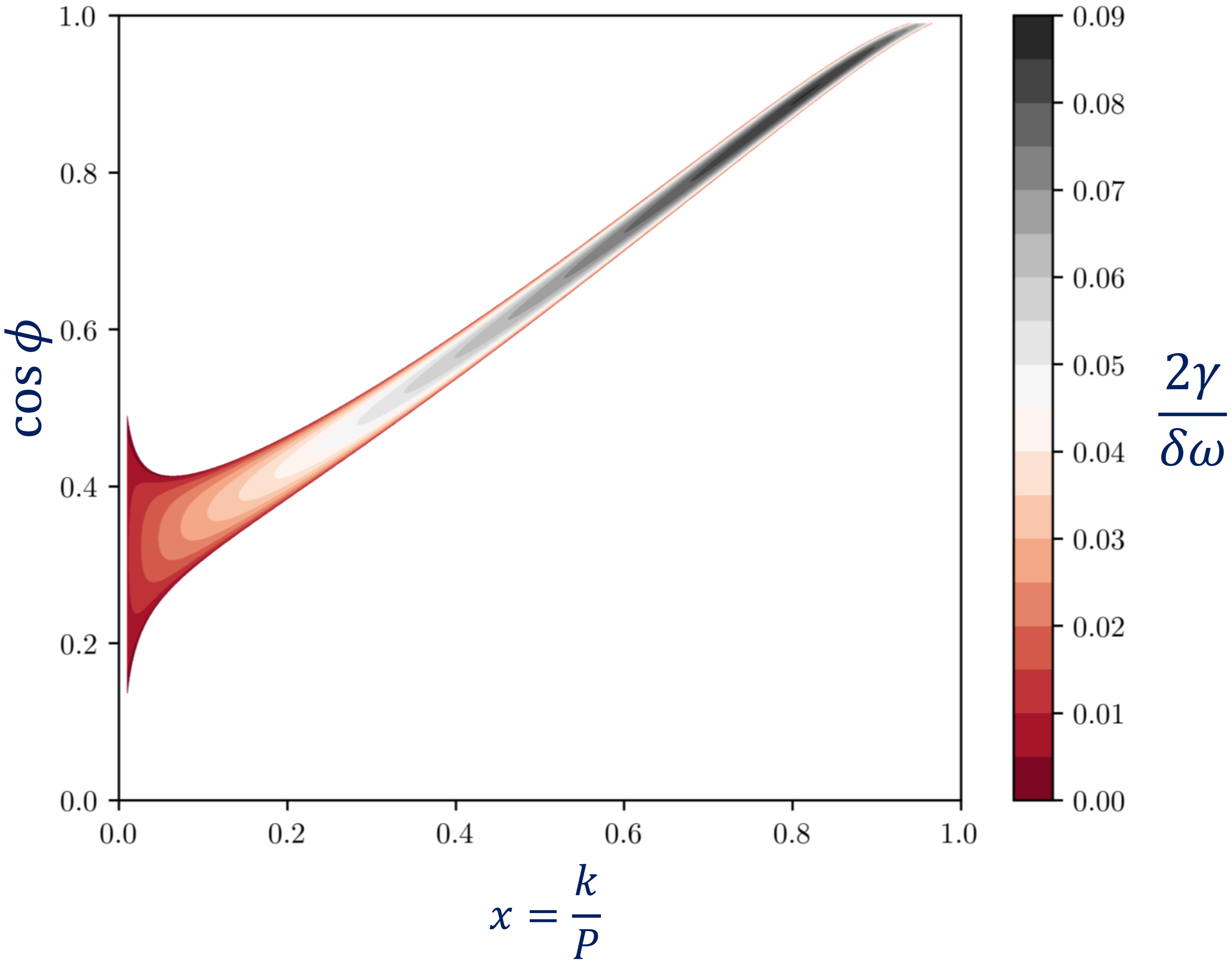
- The region of parametric instability: $\left|\frac{\epsilon}{2}\right| < \left|q\left(\vec{k},\vec{P}\right)\delta\omega\frac{\delta n_{\vec{P}}}{n_0}\right|$

(Arza, SSC, Enomoto, Han, Todarello; 2019)

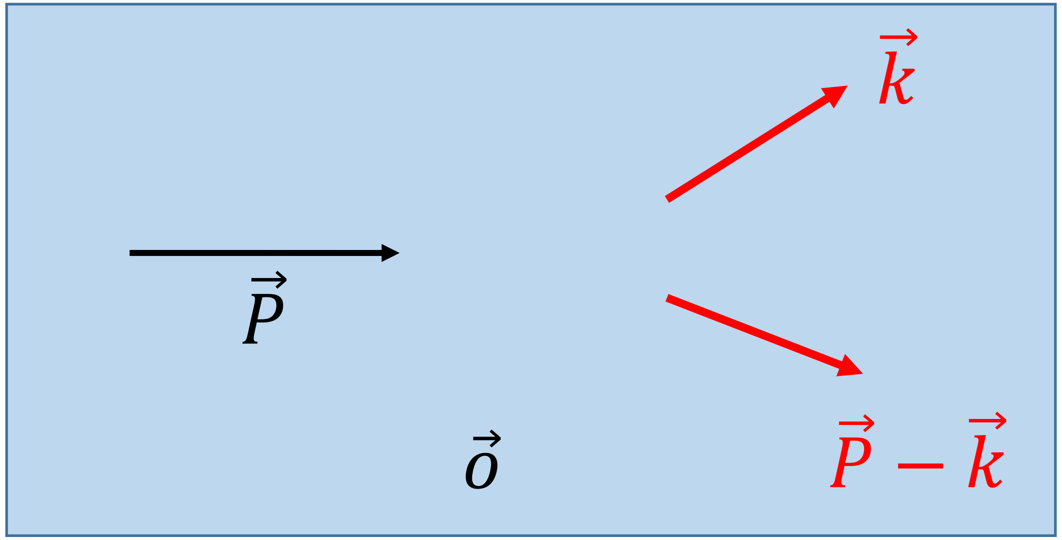
Window of instability:

$$\left|\frac{\epsilon}{2}\right| < \left|q\left(\vec{k},\vec{P}\right)\delta\omega\frac{\delta n_{\vec{P}}}{n_0}\right|$$





$$\delta\omega = \frac{\lambda n_0}{8m^2} \qquad \frac{\delta n_{\vec{P}}}{n_0} = 0.1 \qquad \frac{P^2}{2m\delta\omega} = 9$$



The time when almost all the quanta leave of modes with $\vec{k} = \vec{0}$ and $\vec{P}$:

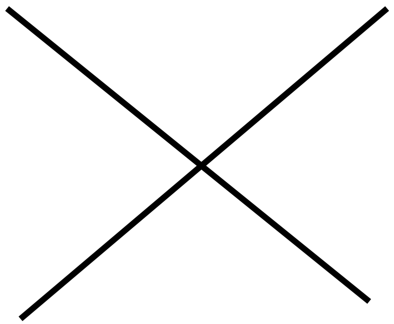
$$t_{cl} \sim \frac{8m^2}{\lambda n_0 \left(\frac{\delta n_{\vec{P}}}{n_0}\right)} \ln\left(\frac{(8m)^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}\, n_0^{\frac{1}{2}}}\right)$$

- Modes other than $\vec{k} = \vec{P}, 2\vec{P}, \ldots$ cannot be generated in the classical evolution. But, in the quantum evolution, they are present as quantum fluctuations.

- Parametric resonance occurs when a scattering of the quasiparticles satisfy both energy and momentum conservation e.g.

$$\vec{0} + \vec{P} \rightarrow \vec{k} + \left(\vec{P} - \vec{k}\right) \quad \text{and} \quad \omega(\vec{P}) = \omega\left(\vec{k}\right) + \omega\left(\vec{P} - \vec{k}\right)$$

(Arza, SSC, Enomoto, Han, Todarello; 2019)



$\epsilon \sim 0$

Consider a process: $i + j \rightarrow k + l$ with final states unoccupied i.e. $N_k = N_l = 0$

$$\frac{dN_l}{dt} \propto \left[N_i N_j (N_k + N_l) - N_k N_l (N_i + N_j)\right] = 0$$

Classical

$$\frac{dN_l}{dt} \propto \left[N_i N_j (N_k + 1)(N_l + 1) - N_k N_l (N_i + 1)(N_j + 1)\right] \neq 0$$

Quantum

(Banik and Sikivie; 2015)

Summary

- We develop a formalism to calculate the time scale after which quantum evolution of a degenerate scalar field differs from its classical evolution.
- Classical solution $\Psi(\vec{r},t)$ is considered as one mode $\left(\vec{k} = \vec{0}\right)$ of the quantum field.
- If this and only this mode is highly occupied, classical description is accurate. But if this mode is unstable and depleted in quantum theory, classical description is no longer valid.
- For attractive $\lambda\phi^4$ interactions and gravitational self-interactions, we have studied classical homogeneous solution.
- In quantum theory, the quanta jump out of $\vec{k} = \vec{0}$ mode in pairs and all modes with $k < k_J$ become populated.

- For repulsive $\lambda\phi^4$ interactions, the classical and quantum theories differ from each other.

- In all the examples we have studied, certain single-particle states are not occupied in the classical field theory, but occupation number of those states increase in quantum field treatment.

- Representing axions with a single classical wavefunction cannot accurately describe the transition between various single-particle states due to self-interactions.

- In the context of thermalizations of axions, it is necessary to treat axions as quantum field.

THANK YOU!