Singular perturbation method for stability of infinite-dimensional systems

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Recent advances on control theory of PDE systems
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This course is based on joint works with:

- Vincent Andrieu (Lyon, France) - KdV equation
- Gonzalo Arias (Santiago, Chile) - Heat-wave system
- Jean-Michel Coron (Paris, France) - KdV equation
- Miroslav Krsitc (San Diego, USA) - Wave equation
- Swann Marx (Nantes, France) - KdV equation
- Christophe Prieur (Grenoble, France) - Wave equation
- Andrey Smyshlyaev (San Diego, USA) - Wave equation

What is the plan?

Part I:
Introduction

Control

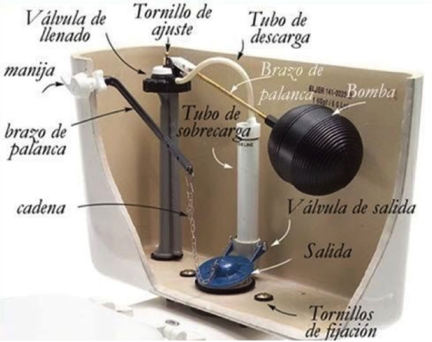
Very simple task: To fill a glass (recipient) with water!

Solution: By means of acting on the system (turning the faucet on/off) we can drive it from an initial state (empty) to a final state (full).

Control

Problem: Very boring task, not robust with respect to phone calls!

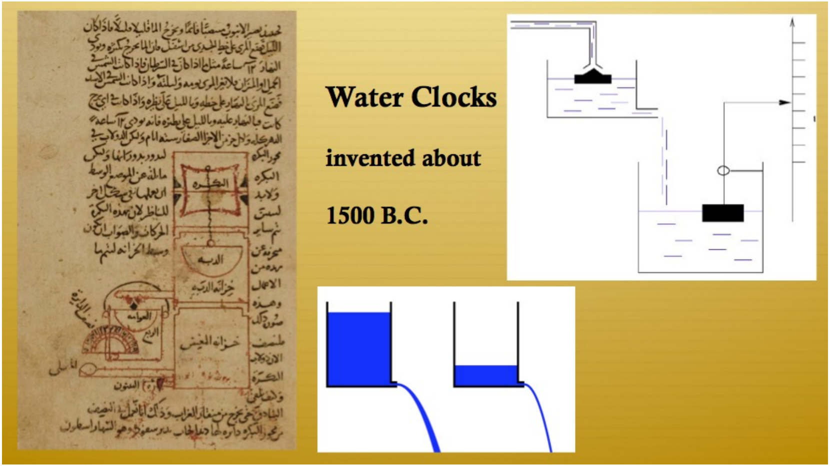
Solution: To introduce a device (feedback control) in order to do the task in an automatic way!



Application of Control

Task: To measure time.

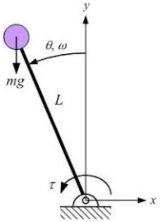
Solution: Same idea as before!



Stable?

Less simple task: To keep the inverted pendulum in up-right position.

Unstable configuration: Without forcing the pendulum, this will go away from its equilibrium position. There always are perturbations in real life!



$$\boxed{mL\ddot{\theta}(t) = mg\sin\theta(t)}$$

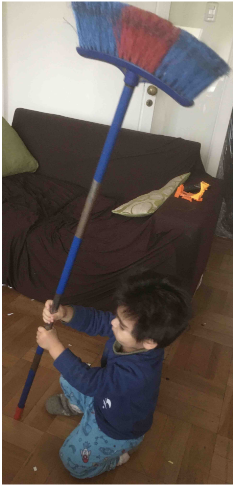


Figure: Technical support by Martín Cerpa

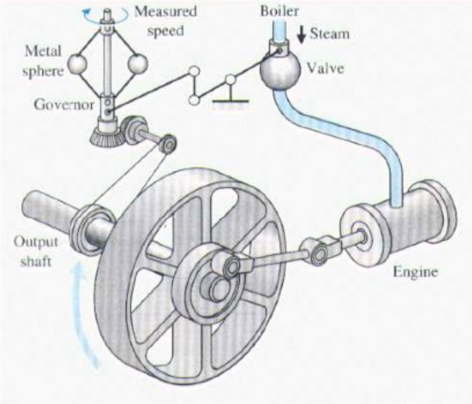
A fundamental case: the Watt regulator

A fundamental case: the Watt regulator

During Industrial Revolution steam engines were very important.

Arise the need to regulate rotation speeds.

The Scottish engineer James Watt invented in 1788 a device to solve that problem and regulate the rotation speeds.



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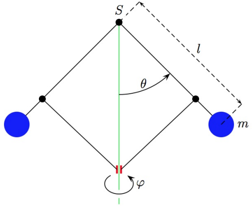
A fundamental case: the Watt regulator

The Scottish Physician and Mathematician James Clerk Maxwell performed in 1868 the first mathematical analysis of stability for an ODE system. He studied the Watt regualtor and other ones.



Consider the new variables $x_1 = \theta$, $x_2 = \dot{\theta}$ y $x_3 = \dot{\varphi}$ to obtain

$$\boxed{\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \sin(x_1)\cos(x_1){x_3}^2 - \frac{g}{\ell}\sin(x_1) - \frac{C}{2m\ell^2}x_2 \\ \frac{dx_3}{dt} &= -\frac{\Gamma_r}{J} + \frac{\Gamma_0}{J} - \frac{k}{J}(1-\cos(x_1)) \end{aligned}}$$

Remark

People had tried with no success to solve nonlinear equations. Maxwell linearized!

A fundamental case: the Watt regulator

$$
\boxed{
\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= \sin(x_1)\cos(x_1){x_3}^2 - \frac{g}{\ell}\sin(x_1) - \frac{C}{2m\ell^2}x_2 \\
\frac{dx_3}{dt} &= -\frac{\Gamma_r}{J} + \frac{\Gamma_0}{J} - \frac{k}{J}(1-\cos(x_1))
\end{aligned}
}
$$

- Find equilibrium point (x_1^e, x_2^e, x_2^e) such that $x_2^e = 0$ and $x_3^e = \omega_0$ is the desired rotation speed (choosing k, Γ_0 associated to the device).
- Center the system around the equilibrium point $y(t) = x(t) - x^e$.
- Linearize around (x_1^e, x_2^e, x_2^e) to get $A \in M_{3\times 3}$ such that

$$\dot{y}(t) = Ay(t)$$

- Are the solutions satisfying $y(t) \to 0$ si $t \to \infty$ for any initial data?

A fundamental case: the Watt regulator

$$\boxed{\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \sin(x_1)\cos(x_1){x_3}^2 - \frac{g}{\ell}\sin(x_1) - \frac{C}{2m\ell^2}x_2 \\ \frac{dx_3}{dt} &= -\frac{\Gamma_r}{J} + \frac{\Gamma_0}{J} - \frac{k}{J}(1-\cos(x_1)) \end{aligned}}$$

- The answer depends on the location of the eigenvalues of A (real part stric. negative).
- The eigenvalues are the solutions of

$$p(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0.$$

- Are there conditions on the coefficients of $p(\lambda)$ ensuring what we want?
- Maxwell solved the cases when $p(\lambda)$ has degree 1, 2 or 3.

A fundamental case: the Watt regulator

Pictures from Carahue in Chile


And the general case of a polynomial of degre n?

- In 1877 Edward John Routh was the winner of the Adams Prize by U. of Cambridge for The Criterion of Dynamical Stability.
- Routh gave a simple criterion, depending only on the coefficients of $p(\lambda)$, to determine if the roots of the polynomial have real part stric. negative.
- Tools: Cauchy Index and Sturm Sequences.
- In 1854 Hermite was using quadratic forms to study a similar problem: number of roots of algebraic equations between two limits.
- In 1894 Aurel Stodola (from Slovenia, professor in Zurich) published with no proof other criterion that was studied later by Adolf Hurwitz
- All those works were not known by the engineers community at that time!

And what about nonlinear systems?

An important name: Alexandre Lyapunov (S. Petersburg), who got his PhD in 1892. Huge influence in the topic.

- Let us start with $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$, and an eq. point x_e.
- Linearize and arrive to $\dot{y} = Ay$ with $A = \frac{\partial f}{\partial x}(x_e)$.
- If $A \in M_{n\times n(\mathbb{R})}$ has all the eigenvalues on the left-hand side of the Complex plane, then there exists $P \in M_{n\times n(\mathbb{R})}$ symmetric and definite positive such that

$$PA + A^*P = -\mu I_d$$

- The function

$$V(y(t)) = y(t)^* P y(t)$$

  satisfies

$$\dot{V}(y(t)) \leq -\mu V(y(t))$$

  what ensure V exponentially decreases to 0. The same with $\|y(t)\|$.

And what about nonlinear systems?

Now we look the nonlinear dynamics

$$V(x(t)) = x(t)^* P x(t),$$

and we can prove (under technical hypotheses) that

$$\dot{V}(x(t)) \leq -\mu V(x(t)) + \text{Garbage}(V(x(t))) \leq -\frac{\mu}{2} V(x(t))$$

if the inital data is close to the eq. point.

Remark

Thus, we can prove a local result about the stability of the nonlinear system.

Remark

The same can be proven if we directly get a function V such that

$$\dot{V}(x(t)) \leq -\mu V(x(t))$$

or a weaker version as

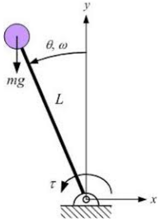
$$\dot{V}(x(t)) \leq 0.$$

Back to the pendulum example

Back to the pendulum example

Nonlinear system:

$$\boxed{mL\ddot{\theta}(t) - mg\sin\theta(t) = 0}$$



Linearize around $(\theta = 0, \dot{\theta} = 0)$:

$$\boxed{mL\ddot{\theta}(t) - mg\theta(t) = 0}$$

The characteristics polynomial, roots, and solutions are

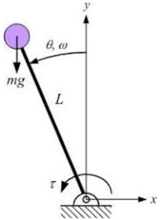
$$L\lambda^2 - g = 0, \quad \lambda_{\pm} = \pm g/L, \quad \theta(t) = Ae^{gt/L} + Be^{-gt/L}.$$

The eq. point $(\theta = 0, \dot{\theta} = 0)$ is not stable!

Back to the pendulum example

Nonlinear system:

$$\boxed{mL\ddot{\theta}(t) - mg\sin\theta(t) = 0}$$



Linearize around $(\theta = \pi, \dot{\theta} = 0)$:

$$\boxed{mL\ddot{\theta}(t) + mg\theta(t) = 0}$$

The characteristics polynomial, roots, and solutions are

$$L\lambda^2 + g = 0, \quad \lambda_{\pm} = \pm ig/L, \quad \theta(t) = A\cos(gt/L) + B\sin(gt/L).$$

The eq. point $(\theta = 0, \dot{\theta} = 0)$ is stable!

Back to the pendulum example

Even for a simple system as the pendulum (with damping), if we want to use a Lyapunov method to get stability, we will need go beyond the classical energy and instead consider

$$V(\theta,\dot{\theta}) = \frac{1}{2}|\dot{\theta}|^2 + \frac{1}{2}|\theta|^2 + \delta\theta\dot{\theta}$$

to prove that there exists $\mu > 0$ such that

$$\dot{V}(\theta,\dot{\theta}) \le -\mu V(\theta,\dot{\theta})$$

and thus to obtain

$$V(\theta(t),\dot{\theta}(t)) \le e^{-\mu t} V(\theta(0),\dot{\theta}(0))$$

for initial data close to the eq. point.

Remark

This method is very robust to consider perturbations (non constant coefficients and nonlinearities). However, you get bad estimations of μ!

Other difficulties for stability analysis

Many difficulties as delays

The system

$$\dot{x}(t) = -x(t)$$

is clearly asymptotically stable with $x(t) = x(0)e^{-t}$.

However, what happens with $\dot{x}(t) = -x(t - D)$?

Answer: the system becomes unstable if $D > \pi/2$.

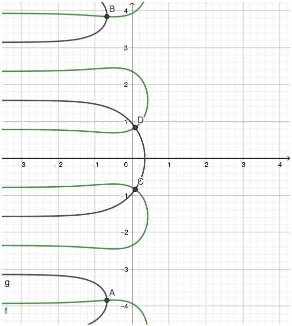
We can find $a > 0$ and b giving solutions as

$$e^{at}\cos(bt) \quad \text{and} \quad e^{at}\sin(bt)$$

coming from

$$a = -e^{-aD}\cos(bD)$$

$$b = e^{-aD}\sin(bD)$$



... or coming from saturations of some components

The origin is unstable for

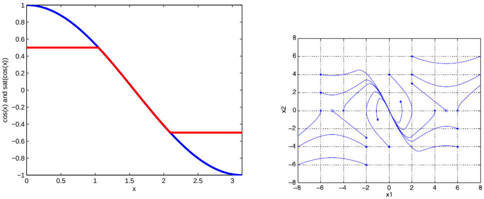
$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

but we can make it stable by considering the control

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 13z_1(t) + 7z_2(t) \end{bmatrix}$$

However, the origin is unstable again when we saturate

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathtt{Sat}(13z_1(t) + 7z_2(t)) \end{bmatrix}$$



... or coming from time-dependent coefficients

If the coefficients depend on time (non autonomous systems), the eigenvalues analysis becomes useless.

The system $\dot{x} = A(t)x$, with $x \in \mathbb{R}^2$ and

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(t) & 1 - \frac{3}{2}\sin(t)\cos(t) \\ -1 - \frac{3}{2}\sin(t)\cos(t) & -1 + \frac{3}{2}\sin^2(t) \end{pmatrix}$$

is such that for any t the eigenvalues of $A(t)$ are $-\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$. However, the resolvent is

$$\Phi(t,0) = \begin{pmatrix} e^{t/2}\cos(t) & e^{-t}\sin(t) \\ -e^{t/2}\sin(t) & e^{-t}\cos(t) \end{pmatrix}$$

and thus we can see that there are diverging solutions.

... or coming from the infinite-dimensional character of the phenomenon

The eigenvalue analysis is not always enough to get the stability.

Fact 1: All the eigenvalues can belong to the left-hand side of the complex plane but asymptotically converge to the imaginary axis.

Fact 2: There is not a basis of the space composed by eigenfunctions. Thus, the spectral analysis does not capture all the possible behaviors.

A first PDE example - 1D Transport equation

Let us take a look to the simplest PDE equation.

We consider on the domain $[0, L]$ the following equation

$$y_t(t,x) + y_x(t,x) = 0$$

with the boundary condition for some $k \in \mathbb{R}$

$$y(t,0) = ky(t,L).$$

Is this system stable? We can look at explicit formulae for the solution but let us follow instead a Lyapunov approach with the functional

$$V(y) = \frac{1}{2}\int_0^L e^{-\mu x}|y|^2 dx$$

where we can pick $\mu > 0$ later.

Remark

We consider $k \neq 1$ to have a unique stationary solution: $y(x) = 0$ everywhere.

A first PDE example - 1D Transport equation

It is important to note that $V(y)^{1/2}$ is a norm equivalent to the L^2-norm! Thus, it is enough to prove that V decreases exponentially to zero in order to have that y will converge exponentially in L^2 to the origin.

Computing the time derivative of V along the solutions of the transport equation we get

$$\dot{V}(y) = \int_0^L e^{-\mu x} y y_t dx = -\int_0^L e^{-\mu x} y y_x dx = -\mu V(t) - \frac{1}{2}|y(t,L)|^2 \left(e^{-\mu L} - k^2\right)$$

Notice that for any $|k| < 1$ we can pick $\mu > 0$ such that $e^{-\mu L} \geq k^2$ and thus

$$\dot{V}(y(t)) \leq -\mu V(y(0))$$

In consequence, for any $|k| < 1$ we obtain the desired convergence and thus the stability of the equation.

Stability of systems with different time scales

Systems with different time scales

- Many natural phenomena feature interaction of processes on different times scales
- Example: a quantity grows in the span of years, influenced by something happening in seconds
- A lot of difficulties appear. For instance, huge cost in numerical simulations since the fastest time scale sub-system must be fully solved over a timespan of the slowest scales' order
- Desirable: we want instead solve a limit system, describing approximately the full behavior when some parameters (representing the scales) go to zero (or infinity)

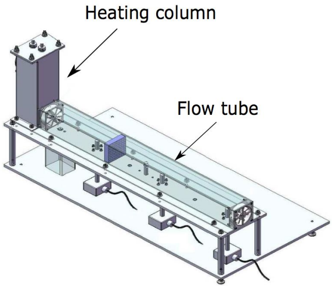
Example 1: a coupled PDE/ODE system

- Gaz density in the heating column $\rho_o(t)$.
- Gaz density in the flow tube $\rho(x,t)$.
- Assumption: transport velocity of gas in the heating column is much smaller than that in the tube (a parameter ε very small)
- Physical constants: μ and c.

(ODE) $\dot{\rho}_o(t) + \mu\rho_o(t) = c\rho(1,t),$

(PDE) $\varepsilon\rho_t(x,t) + \mu\rho_x(x,t) = 0,$

(BC) $\rho(0,t) = \rho_o(t).$



We call this case a Slow ODE coupled to a Fast PDE.
Example taken from *[Tang Mazanti, Automatica 2017]*.

Example 2: a Fast PDE with a Slow ODE

We consider

- $\varepsilon > 0$ points-out the different time scales
- c, b are coupling terms
- $d > 0, a$ are associated to the string and the ODE, respectively

$$\begin{cases} \varepsilon^2 w_{tt} - w_{xx} = 0, & t \geq 0,\ 0 < x < 1, \\ w(t,0) = cz(t), & t \geq 0, \\ w_x(t,1) = -d\varepsilon w_t(t,1), & t \geq 0, \\ \dot{z}(t) = az(t) + bw(t,1), & t \geq 0. \end{cases} \tag{1}$$

(+ initial condition given by some w^0, w^1 and z^0.)

Questions?

- The equilibrium solution is asymptotically stable?
- If so, how this convergence is when ε is small?

Idea:

- The system with ε small should behave like a limit system, obtained (roughly speaking) doing $\varepsilon = 0$.
- We have to study the link between the stability of that limit system with the stability of the full-system (case $\varepsilon > 0$ small).
- We have to study the asymptotic dynamic of the approximation of the full-system by the limit system. How the convergence to zero is? (Tikhonov-like theorems)
- This kind of approaches are known as Singular Perturbation Methods, very well understood in a finite-dimensional framework.

Finite-dimensional case

Singular Perturbation Method

After Hassan Khalil *[Khalil, Nonlinear Systems 2002]*:

"... the difficulties caused by singular perturbations can be avoided if analyzed in separate time scales."

We consider the Full-System

$$\begin{aligned} \dot{x}(t) &= f(t,x,z,\varepsilon), \quad x(0) = x_0 \\ \varepsilon\dot{z}(t) &= g(t,x,z,\varepsilon), \quad z(0) = z_0 \end{aligned} \tag{2}$$

Step 1. Take $\varepsilon = 0$.
From $0 = g(t,x,z,0)$, we obtain $z = h(t,x)$ and then

$$\dot{x}(t) = f(t,x,h(t,x),0), \quad x(0) = x_0 \tag{3}$$

whose solution is called $\bar{x}(t)$, which defines $\bar{z}(t) = h(t,\bar{x}(t))$.

- The Reduced System is (3) with solution $\bar{x}(t)$
- The quasi-static solution is $\bar{z}(t) = h(t,\bar{x}(t))$

Singular Perturbation Method

Step 2. Define $y = z - h(t, x)$.
We obtain

$$\varepsilon \dot{y}(t) = g(t, x, y + h(t, x), \varepsilon) - \varepsilon \frac{\partial h}{\partial t} - \varepsilon \frac{\partial h}{\partial x} f(t, x, y + h(t, x), \varepsilon)$$

Introduce the time variable $\tau = t/\varepsilon$. Thus, $\varepsilon \frac{d}{dt} = \frac{d}{d\tau}$.

Using τ and taking $\varepsilon = 0$ we get the system

$$\frac{dy}{d\tau}(\tau) = g(0, x_0, y + h(0, x_0), 0), \quad y(0) = z_0 - h(0, x_0) \tag{4}$$

whose solution is called $\bar{y}(\tau)$ or $\bar{y}(t/\varepsilon)$.

- The Boundary Layer is (4) with solution $\bar{y}(\tau)$

Theorem (Short version of a result in Khalil's book)

Under technical hypothesis. If the Reduced System has a solution in $[0, T]$ and the Boundary Layer is exponentially stable, then the Full System satisfies for small enough ε

$$x(t) - \bar{x}(t) = O(\varepsilon), \quad z(t) - \bar{z}(t) - \bar{y}(t/\varepsilon) = O(\varepsilon),$$

uniformly in $[0, T]$.

Singular Perturbation Method - Example

We consider the Full-System

$$\begin{aligned} \dot{x}(t) &= z(t), & x(0) &= x_0 \\ \varepsilon \dot{z}(t) &= -x(t) - z(t) + t, & z(0) &= z_0 \end{aligned} \tag{5}$$

Step 1. Take $\varepsilon = 0$.

- We obtain $z = h(t, x) = (-x + t)$
- The Reduced System is

$$\dot{x}(t) = -x + t, \quad x(0) = x_0 \tag{6}$$

  whose solution is $\bar{x}(t) = t - 1 + (1 + x_0)e^{-t}$.
- The quasi-static solution is $\bar{z}(t) = 1 - (1 + x_0)e^{-t}$.

Singular Perturbation Method - Example

Step 2. Define $y = z - h(t,x)$ and $\tau = t/\varepsilon$.

The Bounday Layer is

$$\frac{dy}{d\tau}(\tau) = -y(\tau), \quad y(0) = x_0 + z_0 \tag{7}$$

whose solution is $\bar{y}(\tau) = e^{-\tau}(x_0 + z_0)$, that is exponentially stable.

Conclusion from Theorem:

$$x(t) - \underbrace{\left[t - 1 + (1 + x_0)e^{-t}\right]}_{\text{reduced system solution}} = O(\varepsilon)$$

and

$$z(t) - \underbrace{\left[1 - (1 + x_0)e^{-t}\right]}_{\text{quasi-static solution}} - \underbrace{e^{-t/\varepsilon}(x_0 + z_0)}_{\text{boundary layer solution}} = O(\varepsilon).$$

Singular Perturbation Method - Long time

To obtain stability properties (as we want here) for the original Full-System

$$\begin{aligned} \dot{x}(t) &= f(t, x, z, \varepsilon), \quad x(0) = x_0 \\ \varepsilon \dot{z}(t) &= g(t, x, z, \varepsilon), \quad z(0) = z_0 \end{aligned} \tag{8}$$

by using the Reduced order system

$$\dot{x}(t) = f(t, x, h(t, x), 0), \quad x(0) = x_0 \tag{9}$$

and the Boundary layer system

$$\frac{dy}{d\tau}(\tau) = g(0, x_0, y + h(0, x_0), 0), \quad y(0) = z_0 - h(0, x_0) \tag{10}$$

we can use the following result.

Theorem (Short version of a second result in Khalil's book)

Under technical hypothesis. If the Reduced System and the Boundary Layer are exponentially stable, then the Full System is exponentially stable for small enough ε. Moreover,

$$x(t) - \bar{x}(t) = O(\varepsilon), \quad z(t) - \bar{z}(t) - \bar{y}(t/\varepsilon) = O(\varepsilon)$$

Warning: not always work

Slow PDE/Fast ODE

We consider the Full-System

$$\begin{cases} w_{tt}(t,x) - w_{xx}(t,x) = 0, & t \geq 0,\ 0 < x < 1, \\ w(t,0) = cz(t), & t \geq 0, \\ w_x(t,1) = -dw_t(t,1), & t \geq 0, \\ \varepsilon \dot{z}(t) = az(t) + \varepsilon b w_t(t,1), & t \geq 0. \end{cases} \tag{11}$$

By applying Singular Perturbation Method we arrive to the Reduced-System

$$\begin{cases} \bar{w}_{tt}(t,x) - \bar{w}_{xx}(t,x) = 0, & t \geq 0,\ 0 < x < 1, \\ \bar{w}(t,0) = 0, & t \geq 0, \\ \bar{w}_x(t,1) = -d\bar{w}_t(t,1), & t \geq 0, \end{cases} \tag{12}$$

with the quasi-static solution $\bar{z} = 0$.

The Boundary Layer for $\tau = t/\varepsilon$ is

$$\frac{d}{d\tau}\bar{y}(\tau) = a\bar{y}(\tau). \tag{13}$$

If ($d > 0$ and $a < 0$), both sub-systems (12) and (13) are stable.

Slow PDE/Fast ODE

Even if both sub-systems are stable, the Full-System can be unstable if couplings are well-chosen.

Theorem *[EC Prieur, CDC 2017]*

For any $d > 0$ and for any $a < 0$, there exist $b, c \in \mathbb{R}$ such that the Full-System is not asymptotically stable.

Idea of the proof:

- We pass to Riemann coordinates $v_1 = w_x + w_t$ and $v_2 = w_x - w_t$.
- $\gamma(t) = (v_1(t,1), v_2(t,0), z(t))^T$ satisfies (for some matrices)

$$\frac{d}{dt}\Big(M\gamma(t) + N\gamma(t-1)\Big) = P\gamma(t) + Q\gamma(t-1) \tag{14}$$

- After [Henry, 1974]: If (14) is stable, then the discrete system

$$M\gamma(t) + N\gamma(t-1) = 0 \tag{15}$$

  is stable.
- We found values of b, c for which system (15) is unstable. Therefore (14) is unstable.

Slow PDE/Fast ODE

Numerical simulations for the case

$$\varepsilon = 0.1, \quad a = -1, \quad d = 1, \quad b = 1, \quad c = 2$$

with initial data $w(0,x) = 2 + \sin(2\pi x), \quad z(0) = 2.$

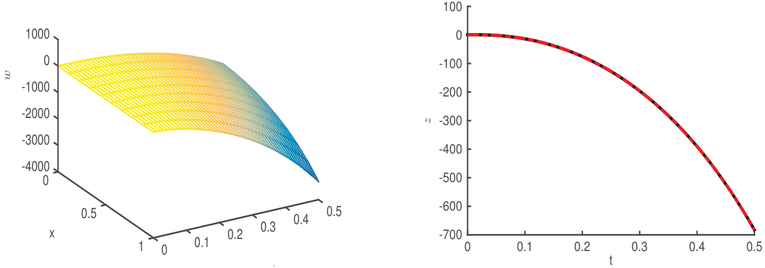
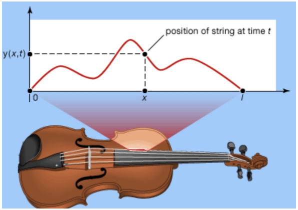


Figure: Coupled System. Evolution of w and z, respectively

Part II:
Fast wave equation coupled to a Slow ODE

String with internal damping



Let us consider a string $y = y(x,t)$ that is fixed at its extremes

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t) - 2dy_t(x,t) - cy(x,t), \\ y(0,t) = 0, \quad y(1,t) = 0, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \end{cases}$$

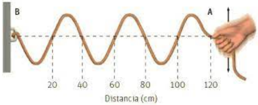
As the functions $\sin(k\pi x)$ are eigenfunctions of the operator $-\Delta$ with these boundary conditions and they compose a basis for $L^2(0,1)$, then all the solutions can be written as

$$y(x,t) = \sum_{k\in\mathbb{N}} e^{-dt}\left\{ w_0^k \cos(t\alpha_k) + \left(w_1^k + \frac{dw_0^k}{k\pi}\right)\sin(t\alpha_k)\right\} \frac{\sqrt{2}}{k\pi}\sin(k\pi x)$$

with $\alpha_k = \sqrt{c - d^2 + k^2\pi^2}$ when d, c are such that $0 < c - d^2 + \pi^2$.
The simplest case in *[Cox Zuazua, CPDE 1994]*.

String with boundary damping

The position of a string $y = y(x,t)$ that is fixed at one of its extremes and is moved from the other side to release energy (passive damper) satisfies



$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0, \\ y(0,t) = 0, \quad y_x(t,1) = -\tanh(d) y_t(t,1), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \end{cases}$$

We get exponential stability

$$\|(y(\cdot,t), y_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)} \leq C \mathrm{e}^{-dt} \|(y_0, y_1)\|_{H^1(0,1)\times L^2(0,1)}$$

which is proven using the Lyapunov function

$$V(y, y_t) = \frac{1}{4}\int_0^1 e^{2dx}(y_t + y_x)^2 dx + \frac{1}{4}\int_0^1 e^{-2dx}(y_t - y_x)^2 dx$$

[Smyshlyaev EC Krstic, SICON 2010], *[Coron d'Andrea-Novel Bastin, TAC 2007]*.

Backstepping Method

Backstepping Method

See the book [Krstic Smyshlyaev, SIAM 2008]

The goal in *[Smylshlyaev EC Krstic, SICON 2010]* was to stabilize the system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + \lambda u_t(x,t), \\ u(0,t) = U(t), \quad u(1,t) = 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x). \end{cases}$$

To do so, we look for a transformation

$$w(x,t) = h(x)u(x,t) - \int_x^1 k(x,y)u(y,t)dy - \int_x^1 s(x,y)u_t(y,t)dy\,,$$

in such a way that w is solution of

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - d w_t(x,t), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x) \end{cases}$$

Backstepping Method

Why doing that?

Because the transformation $u \to w$ from $H^1 \times L^2(0,L)$ to $H^1 \times L^2(0,L)$ is continuos, invertible, and the solution w of

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - d w_t(x,t), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x) \end{cases}$$

is exponentially stable:

$$\|(w(\cdot,t), w_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)} \leq C\mathrm{e}^{-\omega t}\|(w_0, w_1)\|_{H^1(0,1)\times L^2(0,1)}.$$

Notice that $w(0,t) = 0$ and

$$w(x,t) = h(x)u(x,t) - \int_x^1 k(x,y)u(y,t)dy - \int_x^1 s(x,y)u_t(y,t)dy\,.$$

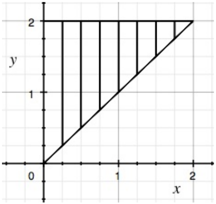
Thus, it gives the feedback control

$$U(t) = \frac{1}{h(0)}\left\{\int_0^1 k(0,y)u(y,t)dy + \int_0^1 s(0,y)u_t(y,t)dy\right\}$$

Backstepping Method

We obtain $h(x) = \cosh((\lambda + d)x)$ and the following coupled equations:

$$\left\{\begin{array}{rcl} k_{xx}(x,y) - k_{yy}(x,y) &=& (\lambda + d)s_{yy}(x,y) \\ k(x,x) &=& m(x), \\ k(x,0) &=& 0, \end{array}\right.$$



$$\left\{\begin{array}{rcl} s_{xx}(x,y) - s_{yy}(x,y) &=& (\lambda + d)k(x,y) + 2\lambda(\lambda + d)s(x,y), \\ s(x,x) &=& -\sinh\left((\lambda + d)x\right), \\ s(x,0) &=& 0. \end{array}\right.$$

Wave-type equations but on a triangle domain!

By applying Successive Approximation Methods, we obtain the existence and uniqueness for kernels k and s.

Remark

If we do not want to change the velocity term, then $\lambda + d = 0$ and the equations get decoupled, which is much simpler!

Backstepping Method

The system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) + \lambda u_t(x,t), \\ u(0,t) = U(t), \quad u(1,t) = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{cases}$$

with control

$$U(t) = \frac{1}{h(0)} \left\{ \int_0^1 k(0,y)u(y,t)dy + \int_0^1 s(0,y)u_t(y,t)dy \right\}$$

is proven to be well-posed in $H^1 \times L^2(0,L)$ and exponentially stable:

$$\|(u(\cdot,t), u_t(\cdot,t))\|_{H^1(0,1)\times L^2(0,1)} \leq C\mathrm{e}^{-\omega t} \|(u_0,u_1)\|_{H^1(0,1)\times L^2(0,1)}.$$

The key in the proof is that the transformation is invertible and the system for w is well-posed (classical equation) and exponentially stable (explicit solutions and eigenvalues on the left hand side of the complex plane).

Backstepping Method

- The well-posedness of the target system can be easily obtained. It is a classical system where we even get a closed formula by using Fourier Series. Also semigroup approach is possible.
- The stability of the target system can be also easily obtained by looking at the eigenvalues of the system. If we deal with non constant coefficient (no Fourier Series available), then the stability is found by means of Lyapunov analysis.
- The well-posedness of the full system is obtained thanks to the WP of the target system. There is a biyective link between trajectories of the controlled system and trajectories of the target system
- The stability of the controlled system is given by the decaying of the solutions of the target system and the correspondence with the solutions of the target system.

Backstepping Method

Some results in *[Smyshlyaev EC Krstic, SICON 2010]* for the classic (pure) wave equation

$$u_{tt} - u_{xx} = 0.$$

- With boundary conditions $u(t,0) = U(t),\ u(t,1) = 0$, the feedback law is explicitly defined:

$$U(t) = -\int_0^1 \frac{\sinh(dy)}{\cosh(d)} [u_t(y,t) - du(y,t)] dy$$

- With boundary conditions $u(t,0) = 0,\ u_x(t,1) = -\tanh(d) u_t(t,1)$, we get a new Lyapunov function

$$V = \frac{1}{4}\int_0^1 e^{2dx}(u_t + u_x)^2 dx + \frac{1}{4}\int_0^1 e^{-2dx}(u_t - u_x)^2 dx$$

by transforming the passive damper system to

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - 2d w_t(x,t) - d^2 w(x,t), \\ w(0,t) = 0, \quad w_x(1,t) = 0. \end{cases}$$

Backstepping Method

In variable w, we can find the Lyapunov function

$$V(w) = \frac{1}{2}\int_0^1 (w_t + dw)^2 dx + \frac{1}{2}\int_0^1 (w_x)^2 dx$$

and going back to the variable u we finally get

$$V(u) = \frac{1}{4}\int_0^1 e^{2dx}(u_t + u_x)^2 dx + \frac{1}{4}\int_0^1 e^{-2dx}(u_t - u_x)^2 dx,$$

which is related to the one for first-order hyperbolic equations found in *[Coron d'Andrea-Novel Bastin, TAC 2007]*.

Stability for the wave-ODE system

Fast PDE/Slow ODE

The Full-System is

$$\begin{cases} \varepsilon^2 w_{tt} - w_{xx} = 0, & t \geq 0,\ 0 < x < 1, \\ w(t,0) = cz(t), & t \geq 0, \\ w_x(t,1) = -d\varepsilon w_t(t,1), & t \geq 0, \\ \dot{z}(t) = az(t) + bw(t,1), & t \geq 0. \end{cases} \tag{16}$$

By applying Singular Perturbation Method we arrive to the Reduced-System

$$\frac{d}{dt}\bar{z} = (a+bc)\bar{z}, \quad t \geq 0, \tag{17}$$

with the quasi-static solution $\bar{w}(t,x) = c\bar{z}(t)$

The Boundary Layer for $\tau = t/\varepsilon$ is

$$\begin{cases} \bar{y}_{\tau\tau}(\tau,x) - \bar{y}_{xx}(\tau,x) = 0, & \tau \geq 0,\ 0 < x < 1, \\ \bar{y}(\tau,0) = 0, & \tau \geq 0, \\ \bar{y}_x(\tau,1) = -d\bar{y}_\tau(\tau,1), & \tau \geq 0. \end{cases} \tag{18}$$

If ($d > 0$ and $a + bc < 0$), both sub-systems (17) and (18) are stable.

Fast PDE/Slow ODE

Theorem *[EC Prieur, CDC 2017]*

Let $d > 0$ and a, b, c such that $a + bc < 0$. There exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ the Full-System is exponentially stable.

Idea of the proof:

We define $\tilde{w}(t,x) = w(t,x) - cz(t)$, which is solution of

$$\begin{cases} \varepsilon^2 \tilde{w}_{tt} - \tilde{w}_{xx} = O(\varepsilon^2), & t \geq 0,\ 0 < x < 1, \\ \tilde{w}(t,0) = 0, & t \geq 0, \\ \tilde{w}_x(t,1) = O(\varepsilon), & t \geq 0, \\ \dot{z}(t) = (a + bc) z(t) + b\tilde{w}(t,1), & t \geq 0. \end{cases} \tag{19}$$

With appropriate μ we have the Lyapunov function

$$V(z, \tilde{w}) = z^2 + \int_0^1 e^{\mu x} (\tilde{w}_x + \varepsilon \tilde{w}_t)^2 dx + \int_0^1 e^{-\mu x} (\tilde{w}_x - \varepsilon \tilde{w}_t)^2 dx$$

If ε is small enough, then we obtain $\dot{V}(z, \tilde{w}) \leq -|a + bc| V(z, \tilde{w})$ and therefore

$$|V(z, \tilde{w})| \leq C e^{-|a+bc|\, t} |V(z_0, \tilde{w}_0)|$$

Fast PDE/Slow ODE

The following is a Tikhonov-like theorem telling us how the Full System can be approached by the subsystems when the parameter ε is small enough.

Theorem *[EC Prieur, TAC 2020]*

Let $d > 0$ and a, b, c such that $\delta := a + bc + \sqrt{3}|bc| < 0$. There exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ we have for all $t \geq 0$,

$$\|w(t) - c\bar{z}(t) - \bar{w}(t/\varepsilon)\|_{H^1(0,1)} + \|w_t(t) - c(a+bc)\bar{z}(t) - \bar{w}_\tau(t/\varepsilon)\|_{L^2(0,1)} = e^{\delta t} O(\varepsilon),$$

and

$$|z(t) - \bar{z}(t)| = e^{\delta t} O(\varepsilon^{3/2})$$

under the hypothesis

$$\|w^0 - c\bar{z}^0 - \bar{w}^0\|_{H^1(0,1)} + \|w^1 - (a+bc)c\bar{z}^0 - \bar{w}^1\|_{L^2(0,1)} + |z^0 - \bar{z}^0| = O(\varepsilon^2)$$

$$\|\bar{w}^0\|_{H^1(0,1)} + \|\bar{w}^1\|_{L^2(0,1)} = O(\varepsilon^{3/2}), \quad |\bar{z}^0| = O(\varepsilon^{3/2}).$$

Fast PDE/Slow ODE

Idea of the proof:

We define $\alpha(t) = z(t) - \bar{z}(t)$ and $\beta(t,x) = w(t,x) - c\bar{z}(t) - \bar{w}(t/\varepsilon, x)$, which is solution of

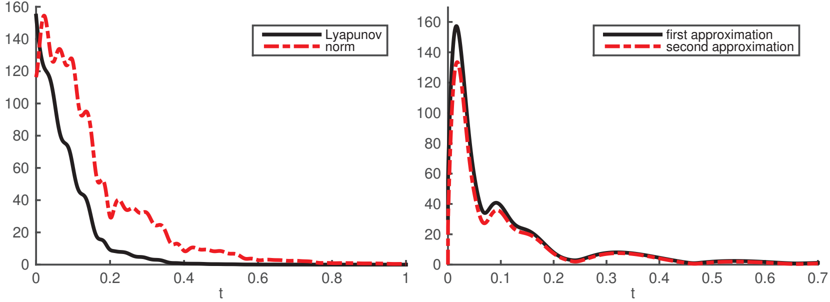
$$\begin{cases} \varepsilon^2 \beta_{tt} - \beta_{xx} &= -\varepsilon^2 c(a+bc)^2 \bar{z}(t) \\ \beta(t,0) &= c\alpha(t) \\ \beta_x(t,1) &= -d\varepsilon\beta_t(t,1) - d\varepsilon c(a+bc)\bar{z}(t) \\ \dot{\alpha}(t) &= (a+bc)\alpha(t) + b\beta(t,1) + b\bar{w}(\frac{t}{\varepsilon},1). \end{cases}$$

With the same Lyapunov function as before, we carefully track the role of ε to get

$$V(\alpha,\beta) \le e^{(a+bc+\sqrt{3}|bc|)t} O(\varepsilon^3)$$

As V depends on ε (it is a problem) we finally get for ε small enough the theorem.

Fast PDE/Slow ODE - Simulations



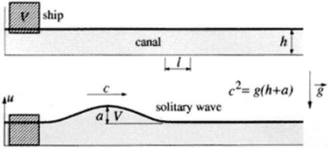
Stability (left): we solve full system with $\varepsilon = 0.1$, $a = -2$, $b = 1$, $c = -2$, and $d = 0.5$ and the i.c. $w^0(x) = 2\pi\sin(2\pi x)$, $w^1(x) = 2$, and $z^0 = w^0(1)/c$.

Tykhonov (right): we compute the solutions to the reduced system and to the boundary layer with the i.c. $\bar{z}^0 = z^0$, $\bar{w}^0 = w^0 - c\bar{z}^0$, and $\bar{w}^1(x) = w^1 - c(a+bc)\bar{z}^0$.

Part III:
A Korteweg-de Vries equation coupled to an ODE

Control System for the KdV equation

Korteweg-de Vries equation 1895



Function $u = u(t, x)$ models for a time t the amplitude of the water wave at position x. The nonlinear dispersive partial differential equation, named Korteweg-de Vries equation and abbreviated as KdV, describes approximately long waves in water of relatively shallow depth

$$\boxed{u_t + u_{xxx} + uu_x = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}}$$

Korteweg-de Vries equation on a bounded domain

On a bounded interval, the extra term u_x should be incorporated in the equation in order to obtain an appropriate model for water waves in a uniform channel when coordinates x is taken with respect to a fixed frame. Thus, for $L > 0$ the equation considered here is

$$\boxed{u_t + u_x + u_{xxx} + uu_x = 0, \quad x \in [0, L], t \geq 0}$$

+ Boundary conditions, for instance posed on

$$\boxed{u(t,0) = u(t,L) = u_x(t,L) = 0, \quad t \geq 0}$$

+ Initial data

$$\boxed{u(0,x) = u_0 \in L^2(0,L)}$$

Asymptotic behaviour

We are interested in the long-time behavior of the energy

$$E(t) = \int_0^L |u(t,x)|^2 \, dx.$$

More precisely we want to prove the exponential decay of $E(t)$ as t goes to infinity.

$$E(t) \leq Ce^{-\omega t} E(0), \quad \forall t \in [0, \infty)$$

Let us start considering the linear equation

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0 \end{array}}$$

Asymptotic behaviour

By performing integration by parts in the equation

$$\int_0^L (u_t + u_x + u_{xxx})u \, dx = 0$$

we get

$$\frac{d}{dt}\int_0^L |u(t,x)|^2 \, dx = -|u_x(t,0)|^2 \leq 0.$$

The energy is non-increasing, but is it strictly decreasing?

Remember we are looking for an exponential decay.

Solutions with constant energy

The energy is not decreasing. In fact there are solutions with constant energy!

For instance, if $L = 2\pi$ and

$$u_0 = (1 - \cos(x)),$$

the solution of the linear KdV $u_t + u_x + u_{xxx} = 0$ is stationary

$$u(t,x) = (1 - \cos(x))$$

which satisfies $u_x(t,0) = 0$ for any $t \geq 0$ and then

$$\boxed{\dot{E}(t) = \frac{d}{dt}\int_0^L |u(t,x)|^2 dx = 0}$$

Critical domains

For the linear KdV equation there exist constant energy solutions if and only if

$$L \in \mathcal{N} := \left\{ 2\pi\sqrt{\frac{k^2 + k\ell + \ell^2}{3}};\, k, \ell \in \mathbb{N}^* \right\}.$$

This phenomena is linked to the controllability of a linear KdV from the boundary.

(Controllability)
Take a look at the linear control system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0 \\ u(t,0) = u(t,L) = 0, \quad u_x(t,L) = \kappa(t), \\ u(0,\cdot) = 0 \end{array}}$$

(Controllability)

- Linear KdV is controllable ⇔ the following map is onto

$$B : \kappa \in L^2(0,T) \mapsto u(T,\cdot) \in L^2(0,L)\,.$$

- The map B is onto ⇔ the following inequality holds

$$\text{(Obs)} \qquad \|B^*(\phi_T)\|_{L^2(0,T)} \geq C\|\phi_T\|_{L^2(0,L)}$$

- The map B is onto ⇔ its adjoint system is observable, i.e.

$$\text{(Obs)} \qquad \|\phi_x(t,L)\|_{L^2(0,T)} \geq C\|\phi_T\|_{L^2(0,L)}$$

where $\phi = \phi(t,x)$ satisfies,

$$\text{(Adj)} \quad \begin{cases} \phi_t + \phi_x + \phi_{xxx} = 0, \\ \phi(t,0) = \phi(t,L) = \phi_x(t,0) = 0, \\ \phi(T,\cdot) = \phi_T. \end{cases}$$

(Controllability)

Theorem *[Rosier, COCV 1997]*

- The linear KdV system is controllable iff $L \notin \mathcal{N}$.
- If $L \notin \mathcal{N}$, the nonlinear system (KdV) is locally exactly controllable.

Theorem *[Coron Crépeau, JEMS 2004; EC, SICON 2007; EC Crépeau, Annales IHP-C 2009]*

Let $L \in \mathcal{N}$, there exists $T_L \geq 0$ such that (KdV) is locally exactly controllable in $L^2(0, L)$ if $T \geq T_L$.

Theorem *[Coron Koenig Nguyen, JEMS 2024]*

There are some $L \in \mathcal{N}$ for which (KdV) is not small-time locally controlable.

Back to stabilization

We will design some feedback control laws in order to get

$$E(t) \leq Ce^{-\omega t}E(0), \quad \forall t \geq 0.$$

Internal control:

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = F(u), \;\; u(0,\cdot) = u_0, \\ u(t,0) = 0, \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}}$$

Boundary control from the left:

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \;\; u(0,\cdot) = u_0, \\ u(t,0) = K_\omega(u), \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}}$$

Internal Control

Internal Control

Equation with internal control

$$\boxed{u_t + u_x + u_{xxx} + uu_x = F}$$

We consider a feedback law in the form

$$\boxed{F(u) = -au}$$

where $a \in L^\infty(0, L; \mathbb{R}^+)$ satisfies

$$\begin{cases} a(x) \geq a_0 > 0, \quad \forall x \in \mathcal{O}, \\ \text{where } \mathcal{O} \text{ is nonempty open subset of } (0, L). \end{cases}$$

Closed-loop system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + a(x)u + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot). \end{array}}$$

Internal Control - Linear

A natural strategy is to consider first the linearized equation around the origin

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}} \tag{20}$$

and prove the exponential decay of its solutions.

Theorem *[Perla Vasconcellos Zuazua, QAM 2002]*

Let $L > 0$ and $a = a(x)$ as before. There exist $C, \omega > 0$:

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (20) with $u_0 \in L^2(0,L)$.

Internal Control - Nonlinear

Nonlinear system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot) \end{array}} \tag{21}$$

Using a perturbative argument, a local version of this theorem is proven by adding a smallness condition on the initial data.

Theorem *[Perla Vasconcellos Zuazua, QAP 2002]*

Let $L > 0$ and $a = a(x)$ as before. There exist $C, r > 0$ and $\omega > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (21) with $\|u_0\|_{L^2(0,L)} \leq r$.

Internal Control - Semiglobal

Nonlinear system

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot). \end{array}} \tag{22}$$

Theorem *[Pazoto, COCV 2005]*

Let $L > 0$, $a = a(x)$ as before and $R > 0$. There exist $C = C(R) > 0$ and $\omega = \omega(R) > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of (22) with $\|u_0\|_{L^2(0,L)} \leq R$.

This result was proved in *[Perla Vasconcellos Zuazua, QAP 2002]* by assuming

$$\exists \delta > 0, \quad (0,\delta) \cup (L-\delta, L) \subset \mathcal{O}$$

which has been removed by Pazoto.

Linear System on a noncritical case

No damping ($a(x) = 0$) and $L \notin \mathcal{N}$.
We have the observability inequality for $T = 1$

$$\forall u_0 \in L^2(0, L), \quad C\|u_x(\cdot, 0)\|_{L^2(0,T)} \geq \|u_0\|_{L^2(0,L)}$$

Integrating with respect to time

$$\frac{d}{dt}\int_0^L |u(t,x)|^2\,dx = -|u_x(t,0)|^2$$

from $t = 0$ to $t = 1$ we get

$$\int_0^L |u(1,x)|^2\,dx - \int_0^L |u_0(x)|^2\,dx \;=\; -\int_0^1 |u_x(s,0)|^2\,ds \;\leq\; -\frac{1}{C^2}\int_0^L |u_0(x)|^2\,dx,$$

that implies

$$\int_0^L |u(1,x)|^2\,dx \leq \frac{C^2-1}{C^2}\int_0^L |u_0(x)|^2\,dx.$$

Linear System on a noncritical case

Of course we also have

$$\int_0^L |u(t+1,x)|^2\,dx \le \frac{C^2-1}{C^2}\int_0^L |u(t,x)|^2\,dx,$$

that implies the exponential decay.

Indeed, let $k \le t \le k+1$. Denoting $\gamma := \frac{C^2-1}{C^2} < 1$, we have

$$\begin{aligned} E(t) &\le E(k) \le \gamma E(k-1) \le \gamma^2 E(k-2) \le \dots \\ &\le \gamma^k E(0) = \frac{\gamma^{k+1}}{\gamma}E(0) = \frac{1}{\gamma}e^{(k+1)\ln(\gamma)}E(0) \\ &\le \frac{1}{\gamma}e^{-t|\ln(\gamma)|}E(0) \end{aligned}$$

Linear System on a critical case

With damping $a(x)u$ active in $\mathcal{O}$ and $L \in \mathcal{N}$. From

$$\int_0^L (u_t + u_x + u_{xxx} + au)u\,dx = 0$$

we get

$$\frac{d}{ds}\int_0^L |u(s,x)|^2\,dx = -|u_x(s,0)|^2 - \int_0^L a(x)|u(s,x)|^2\,dx \leq 0$$

and then by integrating on $(0,1)$ we obtain

$$\int_0^L |u(1,x)|^2\,dx - \int_0^L |u_0(x)|^2\,dx$$
$$= -\int_0^1 |u_x(s,0)|^2\,ds - \int_0^1\int_0^L a(x)|u(s,x)|^2\,dxds.$$

Linear System on a critical case

$$\int_0^L |u(1,x)|^2\,dx - \int_0^L |u_0(x)|^2\,dx$$
$$= -\int_0^1 |u_x(s,0)|^2\,ds - \int_0^1\int_0^L a(x)|u(s,x)|^2\,dxds.$$

Same proof as before runs if we are able to prove that $\exists C > 0$ such that

$$-\int_0^1 |u_x(s,0)|^2\,ds - \int_0^1\int_0^L a(x)|u(s,x)|^2\,dxds \leq -C^2\int_0^L |u_0(x)|^2\,dx.$$

Let us prove that for any $T, L > 0$, there exists $C > 0$:

$$\forall u_0 \in L^2(0,L), \quad \|u_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|u(t,x)|^2\,dxdt \geq C^2\|u_0\|^2_{L^2(0,L)}$$

Linear System on a critical case

By integrating by parts

$$\int_0^L (u_t + u_x + u_{xxx} + au)(T-t)u\,dx = 0$$

we obtain

$$\|u_0\|^2_{L^2(0,L)} \leq \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + \|u_x(\cdot,0)\|^2_{L^2(0,T)} + 2\int_0^T\int_0^L a(x)|u(t,x)|^2\,dxdt$$

and therefore we will be done if we prove that there exists a constant $K>0$ such that

$$K\|u\|^2_{L^2(0,T;L^2(0,L))} \leq \|u_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|u(t,x)|^2\,dxdt$$

Linear System on a critical case

We proceed by contradiction by supposing that

$$\boxed{\begin{array}{l}\forall\, K>0, \exists\, u=u(t,x), \text{ such that} \\ K\|u\|^2_{L^2(0,T;L^2(0,L))} > \|u_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|u(t,x)|^2\,dxdt\end{array}}$$

By using this successively with $K=1/n$, we obtain a sequence $\{u^n\}_{n\in\mathbb{N}}$ of solutions such that $\|u^n\|_{L^2(0,T;L^2(0,L))}=1$ (if not, we normalize. This is due to the linearity of the equation) and

$$\frac{1}{n} > \|u^n_x(\cdot,0)\|^2_{L^2(0,T)} + \int_0^T\int_0^L a(x)|u^n(t,x)|^2\,dxdt$$

Then, as n goes to ∞

$$u^n_x(t,0)\to 0, \text{ in } L^2(0,T), \quad au^n(t,x)\to 0, \text{ in } L^2(0,T,L^2(0,L))$$

Linear System on a critical case

We can pass to the limit in the equation

$$u_t^n + u_x^n + u_{xxx}^n + au^n = 0.$$

and get a solution u of

$$u_t + u_x + u_{xxx} = 0.$$

with

$$a(x)u(t,x) = 0 \quad \forall x \in [0,L], \forall t \in (0,T)$$

From the properties of the damping (active in $\mathcal{O}$), we get

$$u(t,x) = 0, \quad \forall x \in \mathcal{O}, \forall t \in (0,T).$$

A unique continuation principle (Holmgrem's Theorem) implies that $u = 0$, which contradicts the fact that

$$\|u\|_{L^2(0,T;L^2(0,L))} = 1$$

Stabilization of the Linear System

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}}$$

Theorem *[Perla Vasconcellos Zuazua, QAP 2002]*

Let $L > 0$ and $a = a(x)$ as before. There exist $C, \omega > 0$:

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of linear KdV with $u_0 \in L^2(0,L)$.

Nonlinear System

The solution u of

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}}$$

can be written as $u = u^1 + u^2$ where u^1 is the solution of

$$\boxed{\begin{array}{l} u^1_t + u^1_x + u^1_{xxx} + au^1 = 0, \\ u^1(t,0) = u^1(t,L) = u^1_x(t,L) = 0, \\ u^1(0,x) = u_0 \end{array}}$$

and u^2 is the solution of

$$\boxed{\begin{array}{l} u^2_t + u^2_x + u^2_{xxx} + au^2 = -uu_x, \\ u^2(t,0) = u^2(t,L) = u^2_x(t,L) = 0, \\ u^2(0,x) = 0 \end{array}}$$

Nonlinear System

- From some linear estimates of the system

$$\begin{aligned}\|u(t,\cdot)\|_{L^2(0,L)} &\leq \|u^1(t,\cdot)\|_{L^2(0,L)} + \|u^2(t,\cdot)\|_{L^2(0,L)}\\ &\leq \gamma\|u_0\|_{L^2(0,L)} + C\|uu_x\|_{L^1(0,T;L^2(0,L))}\\ &\leq \gamma\|u_0\|_{L^2(0,L)} + C\|u\|^2_{L^2(0,T;H^1(0,L))}\end{aligned}$$

where $\gamma < 1$.

- Here we need a nonlinear estimate

$$\int_0^L (u_t + u_x + u_{xxx} + au + uu_x)xu\,dx = 0$$

we get

$$\begin{aligned}3\int_0^T\int_0^L |u_x|^2 dxdt + \int_0^L x|u(T,\cdot)|^2 dx + 2\int_0^T\int_0^L xa|u|^2 dxdt\\ = \int_0^T\int_0^L |u|^2 dxdt + \int_0^L x|u_0|^2 dx + \frac{2}{3}\int_0^T\int_0^L |u|^3 dxdt\end{aligned}$$

Nonlinear System

We obtain

$$\|u\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{(3T+L)}{3}\|u_0\|^2_{L^2(0,L)} + \frac{2}{9}\int_0^T\int_0^L |u|^3 dxdt$$

As $u \in L^2(0,T;H^1(0,L))$ and $H^1(0,L)$ embeds into $C([0,L])$:

$$\begin{aligned}
\int_0^T\int_0^L |u|^3 dxdt &\leq \int_0^T \|u\|_{L^\infty(0,L)} \int_0^L |u|^2 dxdt \\
&\leq C\int_0^T \|u\|_{H^1(0,L)} \int_0^L |u|^2 dxdt \\
&\leq C\|u_0\|^2_{L^2(0,L)} \int_0^T \|u\|_{H^1(0,L)} dt \\
&\leq CT^{1/2}\|u_0\|^2_{L^2(0,L)}\|u\|_{L^2(0,T;H^1(0,L))}
\end{aligned}$$

Nonlinear System

We obtain

$$\|u\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{(8T+2L)}{3}\|u_0\|^2_{L^2(0,L)} + \frac{TC}{27}\|u_0\|^4_{L^2(0,L)}$$

which gives the existence of $C > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq \|u_0\|_{L^2(0,L)} \left\{\gamma + C\|u_0\|_{L^2(0,L)} + C\|u_0\|^3_{L^2(0,L)}\right\}$$

Given $\epsilon > 0$ small enough such that $(\gamma+\epsilon) < 1$, we can take r small enough so that $r + r^3 < \frac{\epsilon}{C}$, in order to have

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq (\gamma+\epsilon)\|u_0\|_{L^2(0,L)}$$

The rest of the proof runs as before thanks to the fact that $(\gamma+\epsilon) < 1$.

Stabilization of the Nonlinear System

We have introduced an internal damping mechanism in order to be sure the energy of the system decreases to zero in an exponential way. We have proved a local result for the KdV equation.

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + au + uu_x = 0, \\ u(t,0) = u(t,L) = u_x(t,L) = 0, \\ u(0,\cdot) = u_0(\cdot), \end{array}}$$

Theorem *[Perla Vasconcellos Zuazua, QAP 2002]*

Let $L > 0$ and $a = a(x)$ as before. There exist $C, r, \omega > 0$:

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

for any solution of KdV with $\|u_0\|_{L^2(0,L)} \leq r$.

Remark

- In the papers *[Tang Chu Shang Coron, Adv. NA 2018]* and *[Nguyen, JDE 2021]* the authors proved that for some critical lengths the nonlinear KdV system (with no internal damping) is asymptotically stable.
- Other stability results have been proven recently for coupled systems of KdV equations. See *[Capistrano-Fihlo Komornik Pazoto, CCM 2014]*, *[Pazoto Souza, QAM 2014 and Adv. DF 2013]*, *[Massarolo Perla-Mezala Pazoto, QAM 2011]*, *[Nina Pazoto Rosier, MCRF 2011]*, *[Pazoto Rosier, DCDS-B 2010]*.

Saturated inputs

What is saturation for a function? Different choices

$$\mathtt{sat}_{loc}(f)(x) = \begin{cases} -u_0 & \text{if } f(x) \leq -u_0, \\ f(x) & \text{if } -u_0 \leq f(x) \leq u_0, \\ u_0 & \text{if } f(x) \geq u_0, \end{cases}$$

$$\mathtt{sat}_2(f)(x) = \begin{cases} f(x) & \text{if } \|f(x)\|_{L^2(0,L)} \leq u_0, \\ \frac{f(x)u_0}{\|f(x)\|_{L^2(0,L)}} & \text{if } \|f(x)\|_{L^2(0,L)} \geq u_0. \end{cases}$$

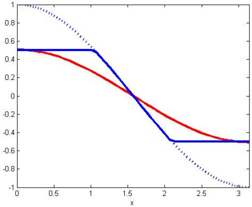


Figure: $x \in [0, \pi]$. Red: $\mathtt{sat}_2(\cos)(x)$ and $u_0 = 0.5$, Blue: $\mathtt{sat}_{loc}(\cos)(x)$ and $u_0 = 0.5$, Dotted lines: $\cos(x)$.

Saturated inputs

Let us consider the KdV equation controlled by a saturated distributed control as follows

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + \mathtt{sat}(ay) = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,x) = y_0(x), \end{cases}$$

where `sat` is any of previous saturations, and a is a localized function as in previous sections.

Theorem *[Marx EC Prieur Andrieu, SICON 2017]*

There exist a positive value $\mu^\star$ and a class $\mathcal{K}$ function $\alpha_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any $y_0 \in L^2(0,L)$, the mild solution y of saturated-KdV satisfies

$$\|y(t,.)\|_{L^2(0,L)} \leq \alpha_0(\|y_0\|_{L^2(0,L)})e^{-\mu^\star t}, \quad \forall t \geq 0. \tag{23}$$

Simulations: $\mathtt{sat}_2$, $u_0 = 0.5$, $\omega = [0, L]$, $a = 1$

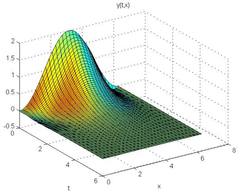


Figure: Solution with no sat.

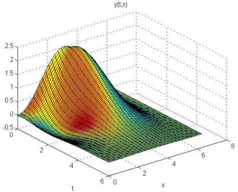


Figure: Saturated solution

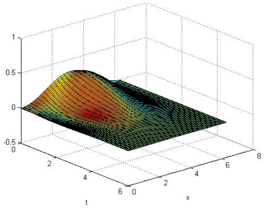


Figure: Saturated control

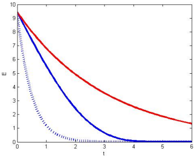


Figure: Blue: Saturated energy. Red: Theoretical energy. Dotted line: Energy with no sat

Simulations: $\texttt{sat}_{loc}$, $u_0 = 0.5$, $\omega = [\frac{L}{3}, \frac{2L}{3}]$, $a = 1$

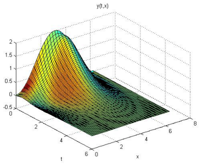


Figure: Solution with no sat.

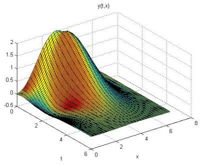


Figure: Saturated solution

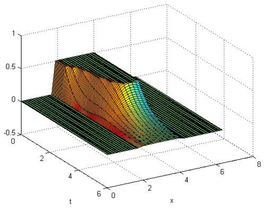


Figure: Saturated control

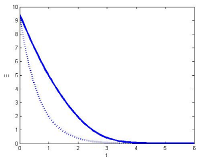


Figure: Blue: Saturated energy. Dotted line: Energy with no sat

Boundary Stabilization for KdV

Boundary Control from the left

Given $L > 0$, the linear control system is

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \ \ u(0,\cdot) = u_0, \\ u(t,0) = K_\omega, \quad u(t,L) = 0, \quad u_x(t,L) = 0, \end{array}}$$

and the nonlinear one is

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} + uu_x = 0, \ \ u(0,\cdot) = u_0, \\ u(t,0) = K_\omega, \quad u(t,L) = 0, \quad u_x(t,L) = 0. \end{array}}$$

Boundary Control from the left

We use the Backstepping method to get

Theorem *[EC Coron, TAC 2013]*

For any $\omega > 0$, there exist a feedback control law $K_\omega = K_\omega(u(t,\cdot))$ and $D > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq De^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0,$$

for any solution of linear KdV.

Theorem *[EC Coron, TAC 2013]*

For any $\omega > 0$, there exist a feedback control law $K_\omega = K_\omega(u(t,\cdot))$, $r > 0$ and $D > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq De^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0,$$

for any solution of nonlinear KdV satisfiying $\|u_0\|_{L^2(0,L)} \leq r$.

Boundary Control from the left

In both cases the feedback law K_ω is explicitly defined as follows

$$K_\omega(u(t,\cdot)) = \int_0^L k(0,y)u(t,y)dy,$$

where the function $k = k(x,y)$ will be characterized as the solution of a given partial differential equation depending on ω.

Unlike the cases of the wave and the heat equation, we have not found a closed formula for the gain $k = k(x,y)$.

Control Design

Let us consider the linearized system around the origin

$$\boxed{\begin{array}{l} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = K_\omega, \quad u(t,L) = 0, \quad u_x(t,L) = 0. \end{array}} \tag{24}$$

Given a positive parameter ω, we look for a transformation $\Pi : L^2(0,L) \to L^2(0,L)$ defined by

$$v(x) = \Pi(u(x)) := u(x) - \int_x^L k(x,y)u(y)dy,$$

such that a trajectory $u = u(t,x)$, solution of (24) with

$$K_\omega(t) = \int_0^L k(0,y)u(t,y)dy,$$

is map into a trajectory $v = v(t,x)$, solution of the linear system

$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} + \omega v = 0, \\ v(t,0) = 0, \quad v(t,L) = 0 \quad v_x(t,L) = 0. \end{array}}$$

Target System

Take a look at the target system

$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} + \omega v = 0, \\ v(t,0) = 0, \quad v(t,L) = 0 \quad v_x(t,L) = 0. \end{array}}$$

We have for any $t \geq 0$

$$\frac{d}{dt}\int_0^L |v(t,x)|^2 dx \;=\; -|v_x(t,0)|^2 - 2\omega \int_0^L |v(t,x)|^2 dx \;\leq\; -2\omega \int_0^L |v(t,x)|^2 dx$$

and therefore we easily obtain for $v = v(t,x)$ the exponential decay at rate ω

$$\|v(t,\cdot)\|_{L^2(0,L)} \leq e^{-\omega t}\|v(0,\cdot)\|_{L^2(0,L)}, \quad \forall t \geq 0.$$

Target System

Is this decay rate sharp? Let us notice that the eigenvalues of target system

$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} + \omega v = 0, \\ v(t,0) = 0, \quad v(t,L) = 0 \quad v_x(t,L) = 0. \end{array}}$$

are the eigenvalues of

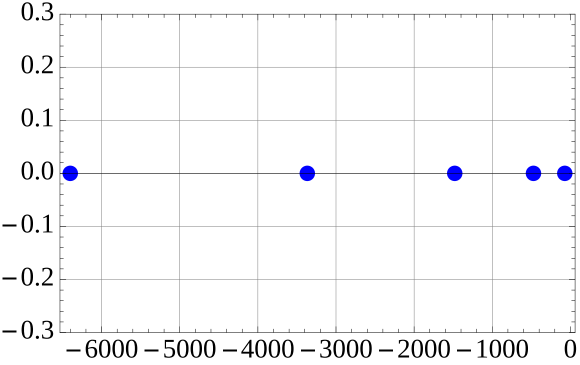
$$\boxed{\begin{array}{l} v_t + v_x + v_{xxx} = 0, \\ v(t,0) = 0, \quad v(t,L) = 0, \quad v_x(t,L) = 0, \end{array}}$$

shifted to the left ω units. Thus, we are lead to study the eigenvalues σ of

$$\begin{cases} -\phi'(x) - \phi'''(x) = \sigma\phi(x), \\ \phi(0) = 0, \quad \phi(L) = 0, \quad \phi'(L) = 0. \end{cases}$$

Eigenvalues

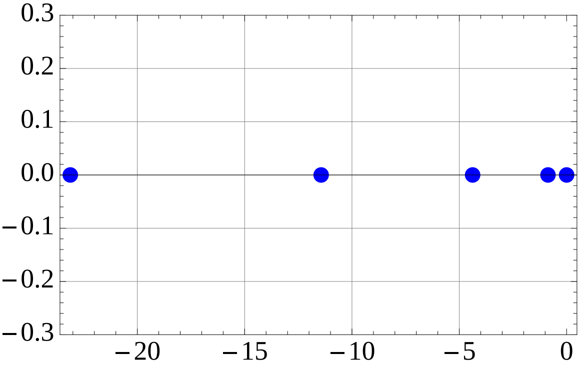
Surprisingly, the eigenvalues behavior depends on the length of the interval.



(a)

In case (a), $L = 1$ (non-critical) and the first eigenvalue σ_1 is approximately -72. The system behaves like a dissipative one.

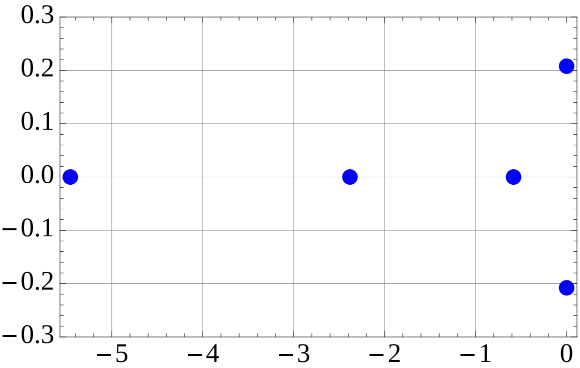
Eigenvalues



(b)

In (b), $L = 2\pi$ (critical) and we have $\sigma_1 = 0$. The system has one conservative component given by the eigenfunction $\phi(x) = 1 - \cos(x)$.

Eigenvalues



(c)

In (c), $L = 2\pi\sqrt{7/3}$ and the first two eigenvalues are imaginary numbers $\sigma_1 = 0.2i$ and $\sigma_2 = -0.2i$.

This examples show the different behaviors that the target system can have and the important role played by the parameter ω in our design.

Kernel function

Let us find the kernel $k = k(x,y)$ such that

$$\boxed{v(x) = u(x) - \int_x^L k(x,y)u(y)dy}$$

is sent into the target. For instance,

$$\begin{aligned}
v_t(t,x) &= u_t(t,x) - \int_x^L u_t(t,y)k(x,y)dy \\
&= u_t(t,x) + \int_x^L (u_y(t,y) + u_{yyy}(t,y))k(x,y)dy \\
&= u_t(t,x) - \int_x^L u(t,y)\left(k_y(x,y) + k_{yyy}(x,y)\right)dy \\
&\quad - k(x,x)(u(t,x) + u_{xx}(t,x)) \\
&\quad + k_y(x,x)u_x(t,x) - k_{yy}(x,x)u(t,x) + k(x,L)u(t,L) \\
&\quad + k(x,L)u_{xx}(t,L) - k_y(x,L)u_x(t,L) + k_{yy}(x,L)u(t,L)
\end{aligned}$$

Kernel function

$$\boxed{v(x) = u(x) - \int_x^L k(x,y)u(y)dy,}$$

$$v_x(t,x) = u_x(t,x) + k(x,x)u(t,x) - \int_x^L k_x(x,y)u(t,y)dy,$$

$$\begin{aligned} v_{xx}(t,x) &= u_{xx}(t,x) + u(t,x)\frac{d}{dx}k(x,x) + k(x,x)u_x(t,x) \\ &\quad + k_x(x,x)u(t,x) - \int_x^L k_{xx}(x,y)u(t,y)dy, \end{aligned}$$

and

$$\begin{aligned} v_{xxx}(t,x) &= u_{xxx}(t,x) + u(t,x)\frac{d^2}{dx^2}k(x,x) + 2u_x(t,x)\frac{d}{dx}k(x,x) \\ &\quad + k(x,x)u_{xx}(t,x) + u(t,x)\frac{d}{dx}k_x(x,x) + k_x(x,x)u_x(t,x) \\ &\quad + k_{xx}(x,x)u(t,x) - \int_x^L k_{xxx}(x,y)u(t,y)dy. \end{aligned}$$

Kernel function

Thus, given $\omega \in \mathbb{R}$ we have

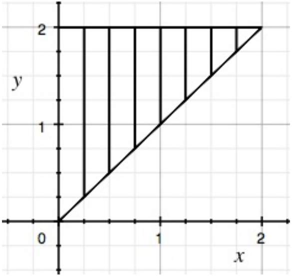
$$
\begin{aligned}
&v_t(t,x) + v_x(t,x) + v_{xxx}(t,x) + \omega v(t,x) = \\
&\quad - \int_x^L u(t,y)\Big(k_{xxx}(x,y) + k_x(x,y) + k_{yyy}(x,y) + k_y(x,y) + \omega k(x,y)\Big)dy \\
&\qquad + k(x,L)u_{xx}(t,L) + u_x(t,x)\Big(3\frac{d}{dx}k(x,x)\Big) \\
&\qquad + u(t,x)\Big(\omega + k_{xx}(x,x) - k_{yy}(x,x) + \frac{d}{dx}k_x(x,x) + \frac{d^2}{dx^2}k(x,x)\Big).
\end{aligned}
$$

Kernel function

Thus, we obtain that the kernel $k = k(x, y)$ defined in the triangle

$$\mathcal{T} = \{(x, y) \,/\, x \in [0, L],\ y \in [x, L]\}$$

must satisfy one third-order PDE with 3 boundary conditions



$$\boxed{\begin{array}{rcl} k_{xxx}(x,y) + k_{yyy}(x,y) + k_x(x,y) + k_y(x,y) & = & -\omega k(x,y) \\ k(x,L) & = & 0 \\ k(x,x) & = & 0 \\ k_x(x,x) & = & \dfrac{\omega}{3}(L-x) \end{array}}$$

Kernel function

Let us make the following change of variable

$$\boxed{t = y - x, \quad s = x + y,}$$

and define

$$G(s,t) := k(x,y)$$

We have

$$k(x,y) = G(x+y, y-x)$$

and therefore

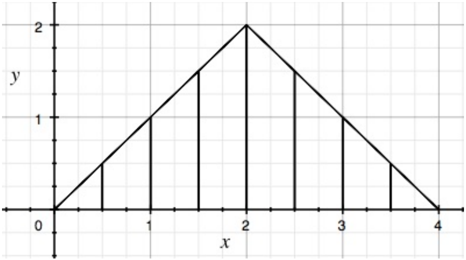
$$\begin{aligned} k_x = G_s - G_t, \quad k_y = G_s + G_t, \\ k_{xx} = G_{ss} - 2G_{st} + G_{tt}, \quad k_{yy} = G_{ss} + 2G_{st} + G_{tt}, \\ k_{xxx} = G_{sss} - 3G_{sst} + 3G_{stt} - G_{ttt}, \\ k_{yyy} = G_{sss} + 3G_{sst} + 3G_{stt} + G_{ttt} \end{aligned}$$

Kernel function

Now, the function $G = G(s,t)$, defined in

$$\mathcal{T}_0 = \{(s,t) \,/\, t \in [0,L], s \in [t, 2L-t]\}$$

satisfies



$$\boxed{\begin{array}{rcll} 6G_{tts}(s,t) + 2G_{sss}(s,t) + 2G_s(s,t) &=& -\omega G(s,t), & \text{in } \mathcal{T}_0, \\ G(s, 2L-s) &=& 0, & \text{in } [L, 2L], \\ G(s,0) &=& 0, & \text{in } [0, 2L], \\ G_t(s,0) &=& \dfrac{\omega}{6}(s-2L), & \text{in } [0, 2L]. \end{array}}$$

Kernel function

Let us transform this system into an integral one:

- We write the equation in variables (η, ξ), integrate ξ in $(0, \tau)$ and use that $6G_{ts}(\eta, 0) = \omega$.
- We integrate τ in $(0, t)$ and use that $G_s(\eta, 0) = 0$.
- We integrate η in $(s, 2L - t)$ and use that $G(2L - t, t) = 0$.

Thus, we can write the following integral form for $G = G(s, t)$

$$G(s,t) = -\frac{\omega t}{6}(2L - t - s) \\ + \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G_{sss}(\eta,\xi) + 2G_s(\eta,\xi) + \omega G(\eta,\xi)\Big)d\xi d\tau d\eta.$$

Kernel function

To prove that such a function $G = G(s,t)$ exists, we use the method of successive approximations. We take as an initial guess

$$G^1(s,t) = -\frac{\omega t}{6}(2L - t - s)$$

and define the recursive formula as follows,

$$G^{n+1}(s,t) \;=\; \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G^n_{sss}(\eta,\xi) \;+\; 2G^n_s(\eta,\xi) \;+\; \omega G^n(\eta,\xi)\Big)d\xi d\tau d\eta.$$

Performing some computations, we get for instance

$$G^2(s,t) \;=\; \frac{1}{108}\Big\{t^3\big(\omega \;-\; \omega^2 L \;+\; \frac{\omega^2 t}{4}\big)\big(2L \;-\; t \;-\; s\big) \;+\; \frac{t^3\omega^2}{4}\big[(2L \;-\; t)^2 \;-\; s^2\big]\Big\},$$

Kernel function

... and more generally the following formula

$$G^k(s,t) = \sum_{i=1}^{k} \left(a_k^i t^{2k-1} + b_k^i t^{2k}\right)\left[(2L-t)^i - s^i\right],$$

where the coefficients satisfy $b_k^k = 0$ and more importantly, there exist positive constants M, B such that, for any $k \geq 1$ and any $(s,t) \in \mathcal{T}_0$

$$\left|G^k(s,t)\right| \leq M \frac{B^k}{(2k)!}(t^{2k-1} + t^{2k}).$$

This implies that the series $\sum_{n=1}^{\infty} G^n(s,t)$ is uniformly convergent in $\mathcal{T}_0$.

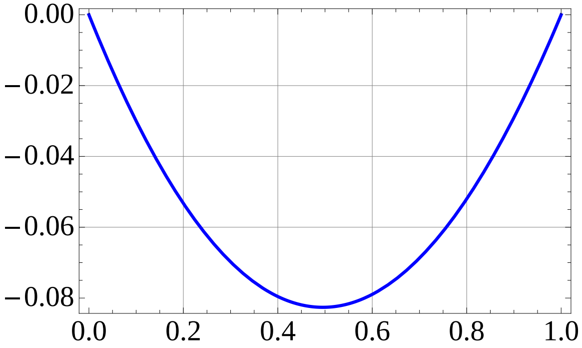
Kernel function

We get a solution of our integral equation. Indeed,

$$
\begin{aligned}
G &= G^1 + \sum_{n=1}^{\infty} G^{n+1} \\
&= G^1 + \frac{1}{6}\sum_{n=1}^{\infty}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G^n_{sss}(\eta,\xi) + 2G^n_s(\eta,\xi) + \omega G^n(\eta,\xi)\Big)d\xi d\tau d\eta \\
&= G^1 + \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2\sum_{n=1}^{\infty} G^n_{sss}(\eta,\xi) + 2\sum_{n=1}^{\infty} G^n_s(\eta,\xi) + \omega\sum_{n=1}^{\infty} G^n(\eta,\xi)\Big)d\xi d\tau d\eta \\
&= G^1 + \frac{1}{6}\int_s^{2L-t}\int_0^t\int_0^\tau \Big(2G_{sss}(\eta,\xi) + 2G_s(\eta,\xi) + \omega G(\eta,\xi)\Big)d\xi d\tau d\eta.
\end{aligned}
$$

Kernel function

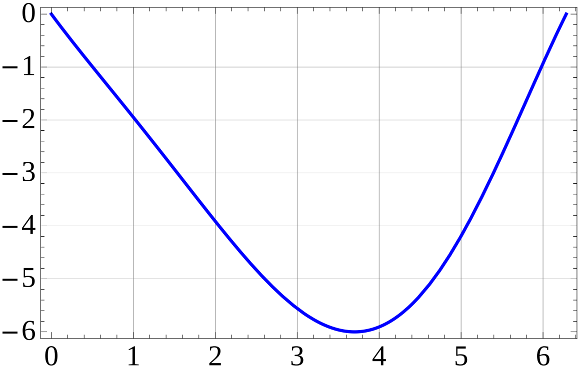
We plot the gain kernel $k(0, y)$ as a function of $y \in [0, L]$ for the length (a) $L = 1$ (non-critical). $\omega = 1$.



(a)

Kernel function

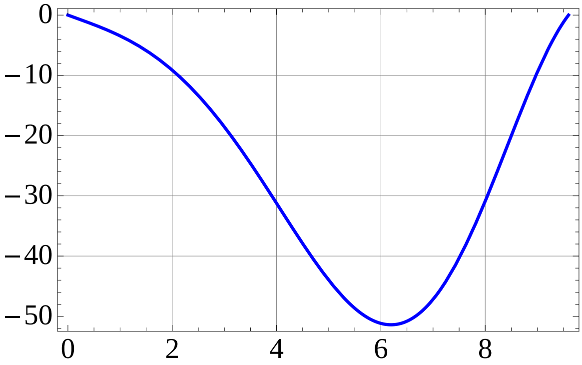
We plot the gain kernel $k(0, y)$ as a function of $y \in [0, L]$ for the length (b) $L = 2\pi$ (critical). $\omega = 1$.



(b)

Kernel function

We plot the gain kernel $k(0,y)$ as a function of $y \in [0,L]$ for the length (c) $L = 2\pi\sqrt{7/3}$ (critical). $\omega = 1$.



(c)

Stability Linear System

We know that the target system is exponentially stable. In order to get the same conclusion for the original linear system the method we are applying uses the inverse transformation Π^{-1}. For that, we introduce a kernel function $\ell(x, y)$ which satisfies

$$\boxed{\begin{array}{rcl} \ell_{xxx}(x,y) + \ell_{yyy}(x,y) + \ell_x(x,y) + \ell_y(x,y) & = & \omega \ell(x,y), \\ \ell(x,L) & = & 0, \\ \ell(x,x) & = & 0, \\ \ell_x(x,x) & = & \dfrac{\omega}{3}(L-x) \end{array}}$$

The existence and uniqueness of such a kernel $\ell = \ell(x, y)$ are proven in the same way than for the kernel $k = k(x, y)$ previously. Once we have defined $\ell = \ell(x, y)$, it is easy to see that the transformation Π^{-1} is characterized by

$$\boxed{u(x) = \Pi^{-1}(v(x)) := v(x) + \int_x^L \ell(x,y)v(y)dy.}$$

Stability Linear System

The operator $\Pi : L^2(0,L) \to L^2(0,L)$, is continuous and consequently we have the existence of a positive constant D_κ such that

$$\|\Pi(f)\|_{L^2(0,L)} \leq D_\kappa \|f\|_{L^2(0,L)}, \quad \forall f \in L^2(0,L).$$

The map $\Pi^{-1} : L^2(0,L) \to L^2(0,L)$ is also continuous and therefore we get the existence of a positive constant D_ℓ such that

$$\|\Pi^{-1}(f)\|_{L^2(0,L)} \leq D_\ell \|f\|_{L^2(0,L)}, \quad \forall f \in L^2(0,L).$$

Stability Linear System

Given $u_0 \in L^2(0,L)$, we define

$$v_0(x) = \Pi(u_0(x)) := u_0(x) - \int_x^L k(x,y)u_0(y)dy.$$

The solution of target system with initial condition $v(0,x) = v_0(x)$ satisfies

$$\|v(t,\cdot)\|_{L^2(0,L)} \leq e^{-\omega t}\|v_0(\cdot)\|_{L^2(0,L)}, \quad \forall t \geq 0.$$

Moreover, the solution of linear KdV is given by $u(t,x) = \Pi^{-1}(v(t,x))$. Thus,

$$\begin{aligned}\|u(t,\cdot)\|_{L^2(0,L)} \leq D_\ell\|v(t,\cdot)\|_{L^2(0,L)} \leq D_\ell e^{-\omega t}\|v_0(\cdot)\|_{L^2(0,L)}& \\ \leq D_\ell D_k e^{-\omega t}\|u_0(\cdot)\|_{L^2(0,L)}&\end{aligned}$$

Nonlinear System

Let $u = u(t,x)$ be a solution of the nonlinear KdV equation with the control given by

$$K(t) = \int_0^L k(0,y)u(t,y)dy,$$

Then, $v = \Pi(u(t,x))$ satisfies

$$\begin{aligned} &v_t(t,x) + v_x(t,x) + v_{xxx}(t,x) + \omega v(t,x) = \\ &\qquad - \left(v(t,x) + \int_x^L \ell(x,y)v(t,y)dy \right) \left(v_x(t,x) + \int_x^L \ell_x(x,y)v(t,y)dy \right) \end{aligned}$$

with homogeneous boundary conditions

$$v(t,0) = 0, \quad v(t,L) = 0, \quad v_x(t,L) = 0.$$

Nonlinear System

We multiply by v and integrate in $(0, L)$ to obtain

$$\frac{d}{dt}\int_0^L |v(t,x)|^2 dx \;=\; -|v_x(t,0)|^2 \;-\; 2\omega\int_0^L |v(t,x)|^2 dx \;-\; 2\int_0^L v(t,x)F(t,x)dx$$

where the term $F = F(t,x)$ is given by

$$\begin{aligned} F(t,x) = v(t,x)\int_x^L \ell_x(x,y)v(t,y)dy + v_x(t,x)\int_x^L \ell(x,y)v(t,y)dy \\ + \left(\int_x^L \ell(x,y)v(t,y)dy\right)\left(\int_x^L \ell_x(x,y)v(t,y)dy\right) \end{aligned}$$

Nonlinear System

We can prove that there exists a positive constant $C = C(\|\ell\|_{C^1(\mathcal{T})})$ such that

$$\left|2\int_0^L v(t,x)F(t,x)dx\right| \leq C\left(\int_0^L |v(t,x)|^2\right)^{3/2}$$

and therefore, if there exists $t_0 \geq 0$ such that

$$\|v(t_0,\cdot)\|_{L^2(0,L)} \leq \frac{\omega}{C},$$

then we obtain

$$\frac{d}{dt}\int_0^L |v(t,x)|^2 dx \leq -\omega\int_0^L |v(t,x)|^2 dx, \quad \forall t \geq t_0.$$

Nonlinear System

Thus, we get

Theorem *[EC Coron, TAC 2013]*

For any $\omega > 0$, there exist a feedback control law $K_\omega = K_\omega(u(t,\cdot))$, $r > 0$ and $D > 0$ such that

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq De^{-\omega t}\|u_0\|_{L^2(0,L)}, \quad \forall t \geq 0,$$

for any solution of nonlinear KdV satisfiying $\|u_0\|_{L^2(0,L)} \leq r$.

Remarks

- The backstepping method has been applied to build some boundary feedback laws, which locally stabilize the Korteweg-de Vries equation posed on a finite interval.
- Our control acts on the Dirichlet boundary condition at the left hand side of the interval where the system evolves.
- The closed-loop system is proven to be locally exponentially stable with a decay rate that can be chosen to be as large as we want.

Remarks

Let us consider one or two control inputs at the right hand side

$$u(t,0)=0, \quad u(t,L)=K_1(t), \quad u_x(t,L)=K_2(t)$$

- To impose $v_t+v_x+v_{xxx}+\omega v=0$, we have to vanish

$$\boxed{k(x,L)u_{xx}(t,L)}+k(x,L)u(t,L)+k_{yy}(x,L)u(t,L)-k_y(x,L)u_x(t,L)$$

  As we do not have to our disposal $u_{xx}(t,L)$, the first term above arises the condition $k(x,L)=0$.

  Moreover, to keep $w(t,0)=u(t,0)=0$, we have to impose $k(0,y)=0$ for any $y\in(0,L)$. We get four boundary restrictions (the other two are on $k(x,x)$), the third order kernel equation satisfied by $k=k(x,y)$ may become overdetermined.

Remarks

- A natural idea to deal with controls at $x = L$ is to use
$$v(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy,$$
If we do so, we deal now with the extra condition $k_y(x,0) = 0$ for any $x \in (0,L)$. This is due to the fact that when imposing $v_t + v_x + v_{xxx} + \omega v = 0$ on the target system, we get the extra term $u_x(t,0)k_y(x,0)$ to be cancelled. As previously, this fourth restriction may give an overdetermined kernel equation for $k = k(x,y)$.

  Moreover, the existence of critical lengths when only one control is considered at the right end-point suggests that either the existence of the kernel or the invertibility of the corresponding map Π should fail for some spatial domains.

- As mentioned before, *[Coron Lu, JMPA 2014]* solve this problem changing the structure of the transformation:
$$v(t,x) = u(t,x) - \int_0^L k(x,y)u(t,y)dy.$$
The kernel equation is now posed on $[0,L] \times [0,L]$ but there appears a δ function as internal source acting on the diagonal of the domain.

Output feedback control

GOAL: To design a controller $u = K(y(t))$ depending on some partial measurements $y(t)$ of the solution and not on the full state $u = u(t, x)$.

What measurements?

The natural choice for the KdV equation should be $y(t) = u_x(t, 0)$.

Unfortunately, the system is not observable with this choice. (Critical values)

In this paper we consider the output given by

$$\boxed{y(t) = u_{xx}(t, L).}$$

By using this measurement, we build an observer and apply the backstepping method to design an output feedback control which exponentially stabilizes the closed-loop system.

Output feedback control - A Classic Strategy in ODE framework

Full state feedback

Let $u = Kx$ be a feedback control stabilizing the system $\dot{x} = Ax + Bu$. Thus,

$$\dot{x} = (A + BK)x$$

is asymptotically stable: $x \to 0$ as time goes to ∞. This is equivalent to pick a matrix K such that $(A + BK)$ is Hurwitz. Such a matrix exist iff (A, B) is controllable.

Static output feedback

Now, assume our control has to depend only on a measure of the full solution $u = Ky$ where $y = Cx$. Even if (A, B) is controllable and (A, C) is observable, the existence of K is not guaranteed. For instance,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad K = k \in \mathbb{R}$$

but

$$(A + BKC) = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}$$

is never Hurwitz.

A Classic Strategy - ODE framework

Dynamic output feedback

IDEA: to build an auxiliar system $\hat{x}$, the observer, that converge to the solution x:

$$\dot{\hat{x}} = Ax + Bu + L(C\hat{x} - y).$$

To study the closed-loop system

$$\dot{x} = Ax + Bu$$

$$\dot{\hat{x}} = Ax + Bu + LC(\hat{x} - x)$$

$$u = K\hat{x}$$

we define $e = (\hat{x} - x)$ which is solution of

$$\dot{e} = (A + LC)e.$$

We have here a general result: if matrices K, L are such $(A + BK)$ and $(A + LC)$ are both Hurwitz, then the closed loop system converges to the origin. Such matrices exists iff (A, B) is controllable and (A, C) is observable.

Our strategy for the KdV equation is the same. For infinite-dimensional systems, there is no general result and we have to prove the convergence. An important tool to do that is the Lyapunov approach.

Going back to our KdV equation

Theorem *[Marx EC, CDC 2014]*

Let us consider system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = \kappa(t),\ u(t,L) = 0,\ u_x(t,L) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

where $u_0 \in H^3(0,L)$ and $\kappa(t) \in H^1(0,T)$. Then
$u \in C([0,T], H^3(0,L)) \cap L^2(0,T; H^4(0,L))$ and $u_{xx}(\cdot, L) \in C([0,T])$.

Definition

Let us introduce the new transformation Π_o defined by:

$$u(t,x) = \Pi_o(w(t,x)) = w(t,x) - \int_x^L p(x,y)w(t,y)dy$$

where an appropriate kernel function $p = p(x,y)$.

Output feedback control

By following the classical approach due to Luenberger, we construct the following observer:

$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} + p(x,L)[y(t) - \hat{u}_{xx}(t,L)] = 0, \\ \hat{u}(t,0) = \kappa(t),\ \hat{u}(t,L) = \hat{u}_x(t,L) = 0, \\ \hat{u}(0,x) = 0, \end{cases} \tag{25}$$

$$y(t) = u_{xx}(t,L).$$

Theorem *[Marx EC, CDC 2014]*

For any $\omega > 0$, there exist a feedback law $\kappa(t) := \kappa(\hat{u}(t,x))$, a function $p = p(x,y)$, and a constant $C > 0$ such that the coupled system (LKdV)-(25) is globally exponentially stable with a decay rate equals to ω, i.e., for any $u_0 \in H^3(0,L)$ we have

$$\|u(t,\cdot) - \hat{u}(t,\cdot)\|_{H^3(0,L)} + \|\hat{u}(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{H^3(0,L)}$$

Remark

We get the following estimate using different spaces:

$$\|u(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{H^3(0,L)}$$

Output feedback control

By using the output feedback control

$$\kappa(t) = \int_0^L k(0,y)\hat{u}(t,y)dy,$$

the transformations Π and Π_o, we can see that $(\tilde{u} = u - \hat{u}, \hat{u})$ are mapped into $(\tilde{w}, \hat{w}) = (\Pi_o^{-1}(\tilde{u}), \Pi(\hat{u}))$ solutions of the target system

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + \omega\hat{w} = \\ -\left\{ p(x,L) - \int_x^L k(x,y)p(y,L)dy \right\} \tilde{w}_{xx}(t,L), \\ \hat{w}(0) = \hat{w}(L) = \hat{w}_x(L) = 0, \\ \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \omega\tilde{w} = 0, \\ \tilde{w}(0) = \tilde{w}(L) = \tilde{w}_x(L) = 0. \end{cases}$$

Output feedback control

To prove the exponential stability of $(\tilde{w}, \hat{w})$, we use a Lyapunov argument

$$V(t) = \frac{A}{2}\int_0^L |\hat{w}(t,x)|^2\,dx + \frac{B}{2}\int_0^L |\tilde{w}(t,x)|^2\,dx + \frac{B}{2}\int_0^L |\tilde{w}_{xxx}(t,x)|^2\,dx,$$

with A, B to be chosen later.

In this way, by tuning A, B large enough, we get for any $\epsilon > 0$ that

$$\dot{V}(t) \leq 2\Big(-\omega + \epsilon\Big)V(t),$$

which gives an exponential stability with decay rate as close to ω as we want.

Output feedback control

Theorem *[Marx EC, CDC 2014]*

Let $\omega > 0$ given. $\exists C > 0$, $\forall u_0 \in H^3(0,L)$, the solution $(u,\hat{u})$ of

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \\ u(t,0) = \displaystyle\int_0^L k(0,y)\hat{u}(t,y)dy,\ u(t,L) = 0,\ u_x(t,L) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

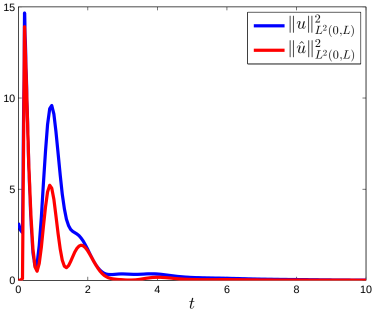
$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} + p(x,L)[u_{xx}(t,L) - \hat{u}_{xx}(t,L)] = 0, \\ \hat{u}(t,0) = \displaystyle\int_0^L k(0,y)\hat{u}(t,y)dy,\ \hat{u}(t,L) = \hat{u}_x(t,L) = 0, \\ \hat{u}(0,x) = 0, \end{cases}$$

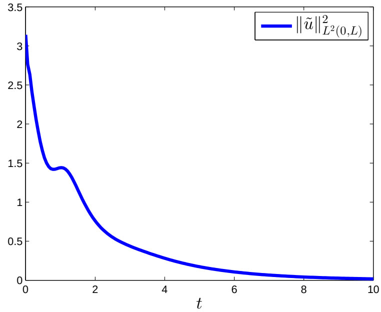
satisfies

$$\|u(t,\cdot) - \hat{u}(t,\cdot)\|_{H^3(0,L)} + \|\hat{u}(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{H^3(0,L)}$$

Simulations work fine, even for the nonlinear system

Good behavior of the observer:





Left: Evolution of the L^2-norm for the state (blue line) and the observer (red line).
Right: Time evolution of the L^2-norm for the observation error $u - \hat{u}$.

Remarks

- Not able to deal with the nonlinear system because regularity issues.
- In *[Marx EC, Automatica 2018]* other configurations inputs-outputs to overcome this mathematical difficulty.
- Other related works by *[Tang and Krstic, ACC 2013 and ACC 2015]*, *[Hassan, 2016]*.

Output feedback control - Other BC and output

Theorem *[Marx EC, Automatica 2018]*

Let $\omega > 0$ given. $\exists C, r > 0$, $\forall u_0 \in L^2(0,L)$ with $\|u_0\|_{L^2} \leq r$, the solution $(u, \hat{u})$ of

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \\ u(t,0) = \displaystyle\int_0^L k(0,y)\hat{u}(t,y)dy, \; u_x(t,L) = 0, \; u_{xx}(t,L) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} + \hat{u}\hat{u}_x + \Big(p_{yy}(x,L) + p(x,L)\Big)[u(t,L) - \hat{u}(t,L)] = 0, \\ \hat{u}(t,0) = \displaystyle\int_0^L k(0,y)\hat{u}(t,y)dy, \; \hat{u}_x(t,L) = \hat{u}_{xx}(t,L) = 0, \\ \hat{u}(0,x) = 0, \end{cases}$$

satisfies

$$\|u(t,\cdot) - \hat{u}(t,\cdot)\|_{L^2(0,L)} + \|\hat{u}(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}$$

Output feedback control - Other BC and output

Theorem *[Balogoun Marx Astolfi, SICON 2023]*

Let $\omega > 0$ given and a non-critical L. $\exists C, r > 0$, $\forall u_0 \in L^2(0,L)$ with $\|u_0\|_{L^2} \leq r$, the solution $(u, \hat{u})$ of

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \\ u(t,0) = \int_0^L k_x(L,y)\hat{u}(t,y)dy, \; u(t,L) = 0, \; u_x(t,L) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} + \hat{u}\hat{u}_x + p_y(x,0)[u_x(t,0) - \hat{u}_x(t,0)] = 0, \\ \hat{u}(t,0) = \int_0^L k_x(L,y)\hat{u}(t,y)dy, \; \hat{u}(t,L) = \hat{u}_x(t,L) = 0, \\ \hat{u}(0,x) = 0, \end{cases}$$

satisfies

$$\|u(t,\cdot) - \hat{u}(t,\cdot)\|_{L^2(0,L)} + \|\hat{u}(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|u_0\|_{L^2(0,L)}$$

The coupled KdV-ODE system

Problem statement

Consider the following coupled system with different **time-scales**

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0, \end{cases}$$

$a, b, c \in \mathbb{R}$, $\varepsilon > 0$ is supposed to be **small**.

Questions

1. What are the **conditions** on a, b, c such that the coupled system is **exponentially stable**? Do these conditions change when ε is small?
2. What is the **behavior** of the solutions with respect to small ε ?

A similar finite-dimensional example

Let us consider first the following system

$$\begin{cases} \varepsilon \dot{y} = ay(t) + bz(t) \\ \dot{z} = cz(t) + dy(t), \end{cases}$$

with $a < 0$ and $b, c, d \in \mathbb{R}$.

Lyapunov function

$$V(y,z) = \frac{1}{2}\varepsilon y^2 + |\varepsilon M y - z|^2,$$

with $M \in \mathbb{R}$ to be selected. This Lyapunov function is inspired by the **forwarding** approach *[Mazenc Praly, TAC 1996]*.

Conditions on a, b, c, d

We pick any $\varepsilon > 0$:

$$\dot{V}(y,z) = ay^2 + bzy + (May + Mbz - cz - dy)(\varepsilon My - z)$$

Let us choose M such that $Ma = d$. Hence,

$$\dot{V}(y,z) = ay^2 + bzy + \left(\left(\frac{bd}{a} - c\right)z\right)(\varepsilon My - z)$$

Then, using **Young's inequalities** with α_1, α_2 to be chosen:

$$\begin{aligned}\dot{V}(y,z) \leq &(a + \alpha_1 + M^2\alpha_2)y^2 \\ &+ \left(\frac{\varepsilon^2}{\alpha_2}\left(\frac{bd}{a} - c\right)^2 - \left(\frac{bd}{a} - c\right) + \frac{b^2}{\alpha_1}\right) z(t)^2\end{aligned}$$

Conditions on a, b, c, d

$$\begin{aligned}\dot{V}(y,z) =&(a+\alpha_1+M^2\alpha_2)y^2\\ &+\left(\frac{\varepsilon^2}{\alpha_2}\left(\frac{bd}{a}-c\right)^2-\left(\frac{bd}{a}-c\right)+\frac{b^2}{\alpha_1}\right)z^2\end{aligned}$$

Choice of a, b, c, d

1. $a < 0$ and α_1, α_2 sufficiently small so that $a + \alpha_1 + M^2\alpha_2 < 0$
2. b sufficiently small, (a, d, c) satisfying $k_1 < \frac{bd}{a} - c < k_2$, with suitable $k_1, k_2 > 0$ so that the polynomial $\frac{\varepsilon^2}{\alpha_2}X^2 - X + \frac{b^2}{\alpha_1}$ is **always negative**.

The singular perturbation method

Question

If one assumes ε **sufficiently small**, do the conditions change ?

As we have already seen, the singular perturbation method consists in decoupling the coupled system into two subsystems systems:

1. The **reduced order system** $\simeq$ **slower system**
2. The **boundary layer system** $\simeq$ **faster system**

Question

How can one compute these two systems ?

The singular perturbation method

Recall that:

$$\begin{cases} \varepsilon \dot{y} = ay(t) + bz(t) \\ \dot{z} = cz(t) + dy(t). \end{cases}$$

Reduced order system

Suppose that $\varepsilon = 0$.
Then, $ay + bz = 0 \Rightarrow y = -\frac{b}{a}z$, which is called the **quasi-static solution**.
Then, replacing y by the quasi-static solution in the z-dynamics, gives the **reduced order system** that reads

$$\dot{\bar{z}} = \left(c - \frac{bd}{a}\right)\bar{z}.$$

The singular perturbation method

Recall that:

$$\begin{cases} \varepsilon \dot{y} = ay(t) + bz(t) \\ \dot{z} = cz(t) + dy(t). \end{cases}$$

Boundary layer system

Set $\tau = \frac{t}{\varepsilon}$ and $\bar{y} = y + \frac{b}{a}z$. Then,

$$\frac{d}{d\tau}\bar{y} = \frac{d}{d\tau}y + \frac{b}{a}\varepsilon\frac{d}{dt}z = a\left(y + \frac{b}{a}z\right) + \frac{b}{a}\varepsilon\frac{d}{dt}z$$

Taking $\varepsilon = 0$, one obtains the boundary layer system

$$\frac{d}{d\tau}\bar{y} = a\bar{y}.$$

The singular perturbation method

Sub systems

The reduced order system is

$$\dot{\bar{z}} = \left(c - \frac{bd}{a}\right)\bar{z}.$$

The boundary layer system is

$$\frac{d}{d\tau}\bar{y} = a\bar{y}.$$

Stability conditions

If $a < 0$ and $\left(c - \frac{bd}{a}\right) < 0$, then both systems are stable.

Question

Let us consider ε small enough.
Do these conditions imply the stability of the **full-system** ?

The singular perturbation method

Consider the following change of coordinates:

$$\tilde{y} = y + \frac{b}{a} z,$$

where $(-\frac{b}{a} z)$ is the **quasi-static solution**.

Then, the full-system can be written as

$$\begin{cases} \varepsilon \dot{\tilde{y}} = a\tilde{y} + \varepsilon \frac{b}{a} \left(\left(c - \frac{bd}{a} \right) z + d\tilde{y} \right), \\ \dot{z} = \left(c - \frac{bd}{a} \right) z + d\tilde{y} \end{cases}$$

The singular perturbation method

Using the Lyapunov function

$$V(\tilde{y}, z) := \frac{1}{2}\varepsilon\tilde{y}^2 + |\varepsilon M\tilde{y} - z|^2$$

one can find ε^* such that, for any $\varepsilon \in (0, \varepsilon^*)$, and for any $a, b, c, d \in \mathbb{R}$ satisfying $a < 0$ and $c - \frac{bd}{a} < 0$, there exist $\mu_1, \mu_2 > 0$ such that

$$\dot{V}(\tilde{y}, z) \leq -\mu_1\tilde{y}^2 - \mu_2 z^2.$$

Thus, we obtain the conditions in order to have exponential stability.

Generalities

Consider general linear systems:

$$\begin{cases} \varepsilon \dot{y} = Ay + Bz, \\ \dot{z} = Cz + Dy, \end{cases}$$

with $y \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ and the matrices A, B, C, D of appropriate dimension.

Result

For sufficiently small ε, the conditions for the reduced order system and the boundary layer system to be stable are sufficient conditions to the stability of the full-system.

Such a result can be found for instance in *[Kokotović Khalil O'Reilly, Singular Perturbations Methods in Control 1986]*. Nonlinear versions can be found in *[Khalil, Nonlinear Systems 2000]*.

Going back to our KdV-ODE system

Let us go back to the KdV equation coupled with an ODE

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \; (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \; t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \; t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \; x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \; t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Here, the fast system is the KdV equation. We consider the case where KdV is **exponentially stable without coupling** as in the finite-dimensional case.

Hypothesis

We work on a non-critical domain $[0,L]$.

Lyapunov functional for the single KdV equation

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = 0, \ t \in \mathbb{R}_+, \end{cases}$$

We know that $\frac{d}{dt}\int_0^L |y(t,x)|^2 dx = -|y_x(t,0)|^2$ but we would like to have a strict Lyapunov functional.

Good news: the following paper gave us that tool using the strictification method introduced in *[Praly, SCL 2019]*.

Theorem *[Balogoun Marx Astolfi, SICON 2023]*

Suppose $L \notin \mathcal{N}$. Then, there exists a **Lyapunov functional** $W : L^2(0,L) \to L^2(0,L)$ for the KdV equation, i.e. there exist positive constants $\underline{c}, \overline{c}, \lambda, \kappa$ such that

$$\underline{c}\|y\|^2_{L^2(0,L)} \leq W(y) \leq \overline{c}\|y\|^2_{L^2(0,L)}$$

and

$$\dot{W}(y) \leq -\lambda\|y\|^2_{L^2(0,L)} - \kappa|y_x(t,0)|^2$$

Lyapunov functional for the single KdV equation

- Use the observer already shown in previous section

$$\begin{cases} y_t + u_x + y_{xxx} = 0, \\ y(t,0) = 0,\ y(t,L) = 0,\ y_x(t,L) = 0, \\ y(0,x) = y_0(x), \end{cases}$$

$$\begin{cases} \hat{y}_t + \hat{y}_x + \hat{y}_{xxx} + p_y(x,0)[y_x(t,0) - \hat{y}_x(t,0)] = 0, \\ \hat{y}(t,0) = 0,\ \hat{y}(t,L) = \hat{y}_x(t,L) = 0, \\ \hat{y}(0,x) = 0, \end{cases}$$

- Define

$$W(y) = \int_0^L |y(t,x)|^2 dx + c\|\Pi(y)\|_{L^2}^2$$

where the backstepping operator Π maps the solutions of

$$\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} - p_y(x,0)\tilde{y}_x(t,0) = 0$$

(where $\tilde{y} = y - \hat{y}$) to the solutions of

$$\gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = 0,$$

both systems with homogeneous boundary conditions.

Exponential stability for ANY values of ε

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Theorem *[Marx EC, TAC 2024]*

For any $\varepsilon > 0$, there exist a_*, k_1, k_2 such that if $a < a_*$ and b, c satisfy $k_1 < ac - b < k_2$, then the origin is globally exponentially stable.

This result can be seen as a sort of generalization of one of the results in *[Balogoun Marx Astolfi, SICON 2023]*, where $b = 0$ and $c = 1$.

Lyapunov stability for any ε

As for the finite-dimensional example, we follow a **forwarding approach**, i.e. we consider

$$V(y,z) = \varepsilon W(y) + \frac{1}{2}(\varepsilon \mathcal{M} y - z)^2,$$

which is the same Lyapunov functional as in *[Balogoun Marx Astolfi, SICON 2023]*.
The operator $\mathcal{M}$ is an **integral operator**, i.e.

$$\mathcal{M} y = \int_0^L M(x) y(x) dx,$$

where M is the solution to

$$\begin{cases} M'''(x) + M'(x) = 0, \\ M(0) = M(L) = 0, \\ M'(0) = -c, \end{cases}$$

with $M(x) = -cf(x)$ and $f(x) = \frac{1}{\sin\left(\frac{L}{2}\right)} \sin\left(\frac{x}{2}\right) \sin\left(\frac{L-x}{2}\right)$.

Time-derivative of the Lyapunov functional

Recall that:

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Using the Lyapunov functional

Seeing $az(t)$ as a perturbation, one has

$$\varepsilon \dot{W}(y) \leq -\lambda \|y\|^2_{L^2(0,L)} + \kappa a^2 z(t)^2$$

Time-derivative of the Lyapunov functional

Recall that:

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ y(0,x) = y_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Differentiating the other term

Integration by parts + M + Young's inequality

$$\frac{d}{dt}\frac{1}{2}\left(\varepsilon \int_0^L M(x)y(t,x)dx - z(t)\right)^2 \leq$$

$$(b-ac)z(t)^2 + \alpha\varepsilon^2 \|M\|^2_{L^2(0,L)} \|y\|^2_{L^2(0,L)} + \frac{(b-ac)^2}{\alpha} z(t)^2$$

with α to be chosen later.

Time-derivative of the Lyapunov functional

One finally has:

$$\dot{V}(y,z) \leq \left(-\lambda + \alpha\varepsilon^2 \|M\|^2_{L^2(0,L)}\right)\|y\|^2_{L^2(0,L)} + \left(\frac{(b-ac)^2}{\alpha} + (b-ac) + \kappa a^2\right) z(t)^2$$

Choose

- α so that $-\lambda + \alpha\varepsilon^2 \|M\|^2_{L^2(0,L)} < 0$.
- a sufficiently small and $k_1 < (ac-b) < k_2$, $k_1, k_2 > 0$, so that the polynomial $\frac{X^2}{\alpha} - X + \kappa_2 a^2$ is always negative.

Exponential stability for SMALL values of ε

$$
\begin{cases}
\varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\
y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\
y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\
y(0,x) = y_0(x), \ x \in [0,L], \\
\dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \\
z(0) = z_0.
\end{cases}
$$

Theorem *[Marx EC, TAC 2024]*

For any $a, b, c \in \mathbb{R}$ such that $(b - ac) < 0$, there exists ε^* such that, for any $\varepsilon \in (0, \varepsilon^*)$, the origin is globally exponentially stable.

We will see that the singular perturbation method applies for the coupled KdV-ODE system !

Singular Perturbation Method

Suppose that $\varepsilon = 0$. Then,

$$\begin{cases} h_x + h_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ h(t,0) = h(t,L) = 0, \ t \in \mathbb{R}_+, \\ h_x(t,L) = az(t), \ t \in \mathbb{R}_+. \end{cases}$$

There exists an explicit solution to this problem:

$$h(t,x) = -2az(t)f(x),$$

with $f(x) = \frac{1}{\sin\left(\frac{L}{2}\right)} \sin\left(\frac{x}{2}\right) \sin\left(\frac{L-x}{2}\right)$. Note that $h_x(t,0) = -az(t)$, then replacing $y_x(t,0)$ by $-az$ in $\dot{z} = bz(t,0) + cy_x(t,0)$, one obtains

Reduced order system

$$\dot{\bar{z}}(t) = (b - ac)\bar{z}(t).$$

If $(b - ac) < 0$, then this system is **exponentially stable** !

Singular Perturbation Method

Set $\tau = \frac{t}{\varepsilon}$. After some computations similar to the finite-dimensional case, one obtains:

Boundary layer system

$$\begin{cases} y_\tau + y_x + y_{xxx} = 0, \\ y(\tau, 0) = y(\tau, L) = 0, \\ y_x(\tau, L) = 0. \end{cases}$$

If $L \notin \mathcal{N}$, the system is **exponentially stable** !

Singular Perturbation Method

We consider

$$\tilde{y}(t,x) = y(t,x) + 2f(x)az(t)$$

One obtains

$$\begin{cases} \varepsilon\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = -\varepsilon((b-ac)z(t) + c\tilde{y}_x(t,0))f(x) \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0 \\ \tilde{y}_x(t,L) = 0 \\ \dot{z} = (b-ac)z + c\tilde{y}_x(t,0). \end{cases}$$

Using the same Lyapunov functional as before, one obtains the desired result !

Theorem *[Marx EC, TAC 2024]*

For any $a, b, c \in \mathbb{R}$ such that $(b-ac) < 0$, there exists ε^* such that, for any $\varepsilon \in (0, \varepsilon^*)$, the origin is globally exponentially stable.

Tikhonov theorem

$$\begin{cases} \varepsilon y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \end{cases}$$

Theorem *[Marx EC, TAC 2024]*

There exist $a_*, k_1, k_2, \varepsilon^*, \mu > 0$ such that if $a < a_*$, b, c satisfy $0 < k_1 < -(b - ac) < k_2$ and $\varepsilon < \varepsilon^*$, then with any initial condition satisfying

$$\|y_0 - \bar{y}_0 + f z_0\|_{L^2(0,L)} + |z_0 - \bar{z}_0| = O(\varepsilon^{\frac{3}{2}}), \qquad |\bar{z}_0| = O(\varepsilon^{\frac{1}{2}})$$

$$\|\bar{y}_0\|_{L^2(0,L)} = O(\varepsilon^{\frac{3}{2}}),$$

one has

$$\|y(t,\cdot) - \bar{y}(t/\varepsilon,\cdot) + f(\cdot)z(t)\|_{L^2(0,L)} + |z(t) - \bar{z}(t)| = O(\varepsilon)e^{-\mu t}.$$

What about if a slow KdV is coupled to a fast ODE?

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ y(t,0) = y(t,L) = 0, \ t \in \mathbb{R}_+, \\ y_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ \varepsilon \dot{z}(t) = bz(t) + cy_x(t,0), \ t \in \mathbb{R}_+, \end{cases}$$

Question

Does the singular perturbation method apply for this system ?

Answer: Yes, but some adjustments have to be done!

Singular Perturbation Method

Reduced order system

Set $\varepsilon = 0$, one has $z(t) = -\frac{c}{b} y_x(t, 0)$.

$$\begin{cases} \bar{y}_t + \bar{y}_x + \bar{y}_{xxx} = 0 \\ \bar{y}(t, 0) = \bar{y}(t, L) = 0 \\ \bar{y}_x(t, L) = -\dfrac{ac}{b} \bar{y}_x(t, 0). \end{cases}$$

Boundary layer system

$$\frac{d}{d\tau} \bar{z}(\tau) = b\bar{z}(\tau),$$

with $\tau = \frac{t}{\varepsilon}$ and $\bar{z} = z + \frac{c}{b} y_x(t, 0)$.

Stability conditions: $b < 0$ (obvious) and $\left|\frac{ac}{b}\right| < 1$ (not at all obvious, but known from *[Zhang, 1994]*).

Singular Perturbation Method

We have to use now a different Lyapunov functional (notice that $b < 0$)

$$V(y,z) = W(y) - ca^2 b|z|^2$$

Theorem *[Marx EC, TAC 2024]*

$\exists \delta, \varepsilon^* > 0$ such that $\forall a, b, c \in \mathbb{R}$ with $\frac{|ac|}{|b|} < \delta$, and $\forall \varepsilon \in (0, \varepsilon^*)$, the origin is exponentially stable with initial conditions $(y_0, z_0) \in H^3(0,L) \times \mathbb{R}$ such that

$$y_0(0) = y_0(L) = 0, \quad y_0'(L) = ab(z_0 + \frac{c}{b} y_0'(0)).$$

Question

Why the initial conditions need to be **so regular** ? As we use $\tilde{z} = z + \frac{c}{b} y_x(t,0)$, we differentiate $y_x(t,0)$ with respect to time, which requires higher regularity !

Remark

A Tychonov-type theorem can also be obtained.

Singular perturbation method for other systems

Wave-heat system

In *[Arias EC Marx, 2023]* we have studied the following wave-heat system

$$\begin{cases} u_{tt} = u_{xx}, & (x,t) \in (0,1) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u_x(1,t) = -a u_t(1,t) + bp(0,t), & t \in (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1), \\ \varepsilon p_t = p_{xx}, & (x,t) \in (0,1) \times (0,\infty), \\ p_x(0,t) = cp(0,t), & t \in (0,\infty), \\ p_x(1,t) = du(1,t), & t \in (0,\infty), \\ p(x,0) = p_0(x), & x \in (0,1). \end{cases} \tag{26}$$

We study well-posedness, stability for any ε, and SPM to consider the case of small ε. The proofs are done using Lyapunov approach.

First-order hyperbolic system coupled to an ODE

In *[Arias Marx Mazanti, CDC 2023]* they have studied the following first-order hyperbolic equation coupled to a ODE

$$\begin{cases} \dot{z}(t) = Az(t) + By(1,t), & t \in (0,\infty), \\ \varepsilon y_t(x,t) + \Lambda y_x(x,t) = 0, & (x,t \in (0,1) \times (0,\infty), \\ y(0,t) = G_1 y(1,t) + G_2 z(t), & t \in (0,\infty), \\ z(0) = z_0, & \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{27}$$

They study the SPM using a frequency approach. They improve the result in *[Tang Mazanti, Automatica 2017]* with respect to the conditions on the parameter to have stability.

Thank you for your attention!