Periodic

Setting

Typical group $\mathbb{Z}/N\mathbb{Z} = \mathbb{Z}_N$

$$\hat{f}(\theta) = \frac{1}{N} \sum_{n \in \mathbb{Z}_n} f(n) \underbrace{e(-n\theta)}_{e^{-2\pi i n \theta}}$$

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$$f(n) = \sum_{\theta} \hat{f}(\theta)\, e(n\theta)$$

A few basic examples

Squares — Gauss Sums

Recall: $r^2 = s \mod n$

may not be solvable.

If s — s is a quadratic residue

A few basic examples

Squares — Gauss Sums

$$\underbrace{\frac{1}{n} \sum_{r \in \mathbb{Z}_n} e\left(r^2 a/n\right)}_{} = G(a, n)$$

$2 \times$ Fourier Transform

of $1\{r$ is quadratic residue$\}$

Periodic analog of

$$\frac{1}{N}\int_0^N e^{i\xi y^2}\,dy$$

$$= \frac{1}{N}\int_0^{N^2} e^{i\xi y}\,\frac{dy}{\sqrt{y}}$$

Gauss Theorem

- Define

$$\varepsilon_m = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4 \\ i & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

for every odd integer m. The values of Gauss sums with $b = 0$ and $\gcd(a, c) = 1$ are explicitly given by

$$G(a,c) = G(a,0,c) = \begin{cases} 0 & \text{if } c \equiv 2 \pmod 4 \\ \varepsilon_c \sqrt{c} \left(\frac{a}{c}\right) & \text{if } c \equiv 1 \pmod 2 \\ (1+i)\varepsilon_a^{-1} \sqrt{c} \left(\frac{c}{a}\right) & \text{if } c \equiv 0 \pmod 4. \end{cases}$$

Here $\left(\frac{a}{c}\right)$ is the Jacobi symbol. This is the famous formula of Carl Friedrich Gauss.

$$G(a,c) = \sum_{1}^{c} e(ar^2/c)$$

Ramanujan Function

$A_N = \{ a \in \mathbb{Z}_N : (a,n) = 1 \}$ "primes mod n"

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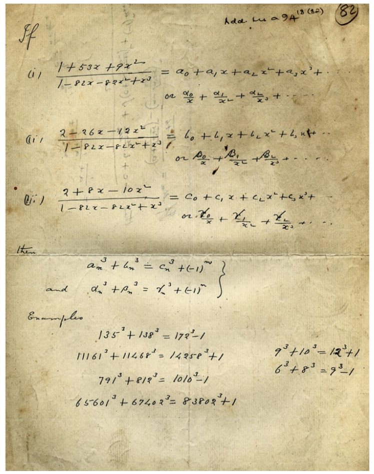
$$N \, \hat{A}_N(\theta) = \sum_{a \in A_N} e(ar/n) = c_n(r)$$

Ramanujan Function



If

(i) $\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ or $\frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{x^3} + \cdots$

(ii) $\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$ or $\frac{\beta_0}{x} + \frac{\beta_1}{x^2} + \frac{\beta_2}{x^3} + \cdots$

(iii) $\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$ or $\frac{\gamma_0}{x} + \frac{\gamma_1}{x^2} + \frac{\gamma_2}{x^3} + \cdots$

then

$$\left.\begin{aligned} a_n^3 + b_n^3 &= c_n^3 + (-1)^n \\ \text{and} \quad \alpha_n^3 + \beta_n^3 &= \gamma_n^3 + (-1)^n \end{aligned}\right\}$$

Examples

$135^3 + 138^3 = 172^3 - 1$

$11161^3 + 11468^3 = 14258^3 + 1$

$791^3 + 812^3 = 1010^3 - 1$

$65601^3 + 67402^3 = 83802^3 + 1$

$9^3 + 10^3 = 12^3 + 1$

$6^3 + 8^3 = 9^3 - 1$

$C_n(a)$ has very substantial cancellation

$$C_n(a) = \mu(a) \qquad (n, a) = 1$$

Mobius $f^{\underline{n}}$

$$\mu(n) = \begin{cases} 0 & p^2 \mid n \text{ for some prime } p \\ (-1)^k & n = p_1 \cdots p_k. \end{cases}$$

But $C_n(0) = \phi(n)$; $C_{p^k}(p^{k-1}) = -p^{k-1}$

S_λ is sphere of radius λ in $\mathbb{Z}_n^d$

$$\frac{1}{N^d} \sum_{\substack{s \in \mathbb{Z}_n^d \\ |s|^2 = \lambda}} e(s \cdot \theta) =$$

Use

$$\boxed{\delta_{|s|^2 = \lambda} = \frac{1}{N} \sum_{r \in \mathbb{Z}_N} e(r(|s|^2 - \lambda))}$$

S_λ is sphere of radius λ in $\mathbb{Z}_n^d$

$$\frac{1}{N^d} \sum_{\substack{s \in \mathbb{Z}_n^d \\ |s|^2 = \lambda}} e(s \cdot \theta) = \frac{1}{N^{d+1}} \sum_r \sum_s e\left(r(|s|^2 - \lambda) + \theta \cdot s\right)$$

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sum over $r = N \delta_{|s|^2 = \lambda}$

$$= \frac{1}{N} \sum_r e(-r\lambda) \underbrace{G_d(r, \theta, n)}_{\text{d-fold Gauss sum}}$$

$$G_d(r, \theta, n) = \prod_{1}^{d} G(r, \theta_j, n_j)$$

$$G(a,b,c) = \frac{1}{c} \sum_{s \in \mathbb{Z}_b} e\left((as^2 + bs)/c\right)$$

if c odd & $(a,c)=1$

$$= \frac{\varepsilon_c}{\sqrt{c}} \left(\frac{a}{c}\right) e\left(-\overline{4a}\, b^2/c\right)$$

$$\frac{1}{N} \sum_r e(-r\lambda)\, G_d(r, \theta, n)$$

becomes a Kloosterman sum

(discrete Bessel f^n)