Group actions and power maps

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Based on a joint work with A. Mandal IIT Roorkee

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Theorem 1 [Wi-94]

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In general s is not continuous on Aut (G)

Group actions

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Proposition [MaR-20]

If $g_n, g \in G$ are such that $g_n^{k_n} \to g$ and $k_n \to \infty$ as $n \to \infty$, g fixes a compact open subgroup of X.
If P_k has dense image, then each $g \in G$ fixes a compact open subgroup of X.

Distal

A linear map α on a vector space V over a local field is called *distal* if 0 is not a limit point of $\{\alpha^n(v) \mid n \in \mathbb{Z}\}$ for any $v \in V \setminus \{0\}$.

Theorem [CoG-74]

Let G be a subgroup of $GL(V)$. Then the following are equivalent:

- each $\alpha \in G$ is distal on V;
- eigenvalues of each $\alpha \in G$ are of absolute value one;
- there is a G-invariant flag of subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = V$$

such that all orbits of G in V_i/V_{i-1} are bounded.

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V to be a finite-dimensional vector space over a non-Archimedean local field.
G be a tdlc group for which
$P_k \colon G \to G$ is dense for some $k > 1$ and
$\rho \colon G \to GL(V)$ is a linear action or representation of G over V.
In this situation we obtain the following necessary condition

Theorem [MaR-20]

There is a flag of subspaces with associated unipotent group U and compact subgroup L such that $\rho(G) \subset LU$ and the flag is L-invariant.

Sufficient condition

Sufficient condition

We first look at the following useful sufficient condition:

Theorem [DaM-17]

Let L be a compact totally disconnected group and N be a nilpotent locally compact group. Suppose L acts on N and the action is linear over a field $\mathbb{F}$. If P_k is surjective on L and k is coprime to the characteristic of $\mathbb{F}$, then P_k is surjective on $L \ltimes K$.

We obtain the following

Theorem [MaR-20]

Let $\mathbb{F}$ be a non-Archimedean local field and G be a group with linear representation $\rho\colon G \to GL(V)$. Suppose that P_k is dense in G for some $k > 1$. Then we have the following:

Suff condition contd.,

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Theorem [MaR-20]

Let $\mathbb{F}$ be a non-Archimedean local field and G be a group with linear representation $\rho\colon G \to GL(V)$. Suppose that P_k is dense in G for some $k > 1$. Then we have the following:

- there exists a compact subgroup L of $GL(V)$ and a split unipotent algebraic group $U \subset GL(V)$ normalized by L such that $L \cap U$ is trivial, $\rho(G) \subset LU$ and $\rho(G)U$ is dense in LU. Moreover, P_k is surjective on the compact group L.

Suff condition contd.,

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- If k is coprime to the characteristic of $\mathbb{F}$, then P_k is surjective on LU.

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- there exists a compact subgroup L of $GL(V)$ and a split unipotent algebraic group $U \subset GL(V)$ normalized by L such that $L \cap U$ is trivial, $\rho(G) \subset LU$ and $\rho(G)U$ is dense in LU. Moreover, P_k is surjective on the compact group L.
- If k is coprime to the characteristic of $\mathbb{F}$, then P_k is surjective on LU.
- If the characteristic p of $\mathbb{F}$ divides k, then $\rho(G)$ is finite.

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Theorem [MaR-20]

If p divides k, then L is finite, that is, $\rho(G)$ is contained in a finite extension of a split unipotent algebraic group U and P_k is dense in $\rho(G) \cap U$.
In addition if the characteristic of $\mathbb{F}$ is positive, $\rho(G)$ is finite.

Lie group

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- Similar results are proved for linear groups over Global fields: If H is a subgroup of $GL(d, \mathbb{E})$ and P_k is surjective on H for some $k > 1$, then H contains a unipotent normal subgroup of finite index.
- If G is a tdlc group acting on a tdlc group X by automorphisms. Suppose G has a finite co-volume or cocompact subgroup H and P_k is dense in H. Then every element of G fixes a compact open subgroup of X. Thus, the main results remain valid in this case also.

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Thanks for your attention!!!