COMQUI QHE Lecture III Part 2 K-Matrix CS Theory

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So far we have looked at a single emergent U(1) CS field. It turns out that this can be generalized to an arbitrary number of Abelian fields.

To do so, we remind ourselves of the fact that the CS action should be invariant under small and large gauge transformations on any manifold.

Including the external EM vector potential A_μ

$$\boxed{\mathcal{L} = \frac{1}{4\pi} K_{IJ} \epsilon^{\mu\nu\lambda} a_{I\mu} \partial_\nu a_{J\lambda} + \frac{e}{2\pi} A_\mu \epsilon^{\mu\nu\lambda} t_I \partial_\nu a_{I\lambda}} \quad (1)$$

We have set $\boxed{\hbar = 1}$ (2) and I, J run from 1 to N, the number of emergent gauge fields.

Let us choose the manifold $S^2 \times S^1$ where S^2 is space. Because this is a quantum Hall system, there is a constant external $\bar{B}$ field. On a sphere this is implemented by putting in a electromagnetic monopole of strength $N_\phi \phi_0$ where

$$\boxed{\phi_0 = \frac{h}{e} = \frac{2\pi}{e}} \quad (3)$$ is the flux quantum

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We also allow a fundamental monopole of $a_{\mu I}$ for some particular I to exist within the S^2

$$\Rightarrow \boxed{\int_{S^2} d^2x \, f_{J,12} = 2\pi \delta_{IJ}} \quad (4)$$

$$\Rightarrow \boxed{S = \int_0^\beta d\tau \left\{ \left(\sum_J K_{IJ} a_{J0} \right) + \frac{e}{2\pi} a_{I0} t_I N_\phi \cdot \frac{2\pi}{e} \right\}} \quad (5)$$

Now we make a large gauge transformation of a_{J0}

$$\boxed{a_{J0} \to a_{J0} + \frac{2\pi}{\beta} m_J} \quad (6)$$

such that the observable Wilson loop

$$\boxed{W = e^{i\int_0^\beta d\tau \, a_0}} \quad (7)$$

remains unchanged.

Since we want e^{iS} to remain invariant we conclude

$$\boxed{K_{IJ} = \text{integer} \quad (8) \quad t_I = \text{integer.}}$$

Next, let us find the filling ν of this effective theory.

$$\boxed{J^0_e = -\frac{\delta \mathcal{L}}{\delta A_0} = -t_I \frac{e}{2\pi} f_{I,12}} \quad (9)$$

First, we integrate out $a_{I\mu}$. For a quadratic theory, this means solving the EOM and plugging it in.

$$\text{EOM for } a_{I\mu} \Rightarrow \boxed{\frac{1}{2\pi} K_{IJ} \epsilon^{\mu\nu\lambda} \partial_\nu a_{J\lambda} + \frac{t_I e}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = 0} \quad (10)$$

$$\Rightarrow \boxed{a_{I\mu} = -e\, K^{-1}_{IJ}\, t_J\, A_\mu} \quad (11)$$

Plugging this in,

$$\boxed{\begin{aligned} J^0_e &= -\frac{e}{2\pi} t_I f_{I,12} = \frac{e^2}{2\pi} t_I K^{-1}_{IJ} t_J F_{12} \\ F_{12} &= \partial_x A_y - \partial_y A_x \equiv B \end{aligned}} \quad (12.)$$

Recall that the magnetic length is

$$\boxed{\begin{aligned} \ell^2 &= \frac{\hbar}{eB} = \frac{1}{eB} \\ \Rightarrow \frac{e^2 B}{2\pi} &= \frac{e}{2\pi \ell^2} \end{aligned}} \quad (13) \qquad (\hbar = 1)$$

$\Rightarrow$ electron charge density $= e \dfrac{t^T K^{-1} t}{2\pi \ell^2} = \dfrac{e\nu}{2\pi\ell^2}$

$$\Rightarrow \boxed{\nu = t^T K^{-1} t} \quad (14)$$

Now we can introduce a quasiparticle that couples to all the emergent gauge fields, labeled by a vector of integers $\boxed{\vec{\ell} = \{\ell_I\}}$ (15)

$$\boxed{\mathcal{L}_{j_{\vec{\ell}}} = j^\mu_{\vec{\ell}} \left(\ell_I a_{I\mu}\right)} \quad (16)$$

What is the charge of a stationary qp of this type?

$$\boxed{j^\mu_{\vec{\ell}} = \delta^\mu{}_0 \, \delta(\vec{x} - \vec{x}_0)} \quad (17)$$

We find the EOM of $a_{I\mu}$ in the presence of this $j^\mu_{\vec{\ell}}$.

$$\frac{1}{2\pi} K_{IJ} \epsilon^{\mu\nu\lambda} \partial_\nu a_{J\lambda} + \ell_I j^\mu_{\vec{\ell}} + \frac{e}{2\pi} t_I \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = 0 \quad (18)$$

$$\Rightarrow \epsilon^{\mu\nu\lambda} \partial_\nu a_{I\lambda} = -e K^{-1}_{IJ} t_J \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda - 2\pi K^{-1}_{IJ} \ell_J j^\mu_{\vec{\ell}}$$

Now plug this into Eq (12) to find the total charge density in the presence of $j^\mu_{\vec{\ell}}$

$$\boxed{J^0_e = \frac{e^2}{2\pi} t^T K^{-1} t \, F_{12} + e \, t^T K^{-1} \ell \; j^0_{\vec{\ell}}} \quad (19)$$

Clearly, the excess charge due to the qp is

$$\boxed{q_{\vec{l}} = e\, t^T K^{-1} l = e\, t_I K^{-1}_{IJ} l_J} \quad (20)$$

Last, but not least, if two identical qps of this type are exchanged adiabatically the phase accumulated is

$$\boxed{\theta = \pi\, l^T K^{-1} l} \quad (21)$$

To make all this more concrete, let us consider two important examples of two-component $(N=2)$ fractional quantum Hall fluids.

Example 1:

$$K = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \qquad t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (22)$$

$$\det K = 5 \quad \Rightarrow \quad K^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

$$\nu = t^T K^{-1} t = \frac{2}{5}$$

$$5$$

This is an example of a "hierarchical" FQH state, originally explored by Halperin and Haldane.

The motivation for this K-matrix is as follows. We start with the Laughlin state, with a single emergent gauge field $a_{1\mu}$. The electron current is

$$\boxed{J^0_e = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_{1\lambda}} \quad (23)$$

The Lagrangian is (24)

$$\boxed{\mathcal{L}_1 = \frac{3}{4\pi}\epsilon^{\mu\nu\lambda}a_{1\mu}\partial_\nu a_{1\lambda} + \frac{e}{2\pi}A_\mu\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda}$$

To verify that this corresponds to $\nu = \frac{1}{3}$ we find the EOM of $a_{1\mu}$.

$$\frac{3}{2\pi}\partial_\mu a_{1\nu} = -\frac{e}{2\pi}\partial_\mu A_\nu \quad (25)$$

$$\Rightarrow \quad J^0_e = \frac{e}{6\pi}\vec{\nabla}\times\vec{A} = \frac{e}{6\pi}B$$

which implies $\nu = \frac{1}{3}$

Now we introduce quasiparticles by coupling a current to $a_{1\mu}$.

$$\boxed{\mathcal{L}_2 = \frac{3}{4\pi}\epsilon^{\mu\nu\lambda} a_{1\mu}\partial_\nu a_{1\lambda} - j^\mu a_{1\mu} + \frac{e}{2\pi} A_\mu \epsilon^{\mu\nu\lambda}\partial_\nu a_{1\lambda}} \quad (26)$$

Any current coupled to a vector potential must be conserved. So we express

$$\boxed{j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_{2\lambda}} \quad (27)$$

where we have introduced a second emergent gauge field $a_{2\lambda}$.

Now we imagine increasing the density of quasiparticles. We can achieve this by introducing a CS term for $a_{2\lambda}$. Finally

$$\boxed{\begin{aligned}\mathcal{L} = {} & \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} a_{2\mu}\partial_\nu a_{2\lambda} + \frac{3}{4\pi}\epsilon^{\mu\nu\lambda} a_{1\mu}\partial_\nu a_{1\lambda} \\ & - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} a_{1\mu}\partial_\nu a_{2\lambda} + \frac{e}{2\pi} A_\mu \epsilon^{\mu\nu\lambda}\partial_\nu a_{1\lambda}\end{aligned}} \quad (28)$$

By the gauge invariance arguments we reviewed earlier k has to be an integer.

The filling factor and the density of qps are determined by the EOM of a_1 and a_2.

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$$\boxed{\begin{array}{ll} \frac{3}{2\pi}\partial_\mu a_{1\nu} - \frac{1}{2\pi}\partial_\mu a_{2\nu} = -\frac{e}{2\pi}\partial_\mu A_\nu & \text{EOM of } a_1 \\ \frac{k}{2\pi}\partial_\mu a_{2\nu} - \frac{1}{2\pi}\partial_\mu a_{1\mu} = 0 & \text{EOM of } a_2 \end{array}} \quad (29)$$

$$3J_{0,1} - J_{02} = \frac{eB}{2\pi}$$

$$-J_{01} + kJ_{02} = 0 \Rightarrow J_{02} = \frac{1}{k}J_{01}$$

$$3J_{01} - \frac{1}{k}J_{01} = \frac{eB}{2\pi} \Rightarrow J_{01} = \frac{k}{3k-1}\frac{eB}{2\pi}$$

$$\Rightarrow \boxed{\nu = \frac{k}{3k-1} \qquad \nu_{qp} = \frac{1}{3k-1}} \quad (30)$$

The K matrix is

$$K = \begin{bmatrix} 3 & -1 \\ -1 & k \end{bmatrix} \qquad \det K = 3k-1$$

$$K^{-1} = \begin{bmatrix} \frac{k}{3k-1} & \frac{1}{3k-1} \\ \frac{1}{3k-1} & \frac{3}{3k-1} \end{bmatrix} \qquad \nu = \frac{k}{3k-1}$$

Clearly $k=2$ represents $\nu = \frac{2}{5}$.

From Eq (20), reproduced here for convenience

$$\boxed{q_{\vec{l}} = e\, t^T K^{-1} l = e\, t_I K^{-1}_{IJ} l_J} \quad (20)$$

We can obtain the possible charges of the qps.

$$\boxed{q_{\vec{l}} = e\,[1,0]\begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \frac{2l_1+l_2}{5}\, e} \quad (31)$$

$\Rightarrow$ the charges are multiples of $\frac{e}{5}$.

Example 2: $\nu = \frac{2}{3}$. Now the K matrix is

$$\boxed{K = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \qquad t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{Det } K = -3 \qquad K^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}} \quad (32)$$

$$q_{\vec{l}} = e\,[1, 0]\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \quad (33)$$

$$= e\, 2l_1 - l_2$$

$$= e \, \frac{2l_1 - l_2}{3} \qquad (33)$$

Basically multiples of $\pm \frac{e}{3}$

Now we go to the edge theory. As we