COMQUI QHE Lecture III, Part 1

Basics of Chern-Simons Theories

I Gauge invariance, level quantization, σ_{xy}. Fractional charge and statistics of quasiparticles in the bulk with a single CS field.

II Theory of the edge with a single CS field.

I: Gauge Invariance, Level quantization, etc

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}a_\mu \; e^{\frac{i}{\hbar} S_{cs}} \qquad (1) \\ S_{cs} &= c \int d^3x \; \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \end{aligned}}$$

We are working in real time. Let us go to imaginary time $\boxed{\tau = it, \; t = -i\tau}$ (2)

$$\boxed{\begin{aligned} &a_0 \to \tilde{a}_0 = -ia_0 \qquad \text{because} \qquad \partial_t - ia_0 \to \partial_\tau - i\tilde{a}_0 \\ &a_i \to a_i \qquad i = 1,2 \end{aligned}} \quad (3)$$

$$\frac{i}{\hbar} S_{cs} = \frac{i}{\hbar} c \int dt \, d^2x \left\{ a_0 (\partial_1 a_2 - \partial_2 a_1) + a_1 (\partial_2 a_0 - \partial_0 a_2) + a_2 (\partial_0 a_1 - \partial_1 a_0) \right\}$$

$$= \frac{i}{\hbar} c \int_0^\beta (-i d\tau) \, d^2x \left\{ i\tilde{a}_0 (\partial_1 \tilde{a}_2 - \partial_2 \tilde{a}_1) + i a_1 (\partial_2 \tilde{a}_0 - \tilde{\partial}_0 a_2) + i a_2 (\tilde{\partial}_0 a_1 - \partial_1 \tilde{a}_0) \right\}$$

$$\boxed{= \frac{i}{\hbar} c \int_0^\beta d\tau \, d^2x \; \epsilon^{\mu\nu\lambda} \tilde{a}_\mu \tilde{\partial}_\nu \tilde{a}_\lambda} \quad (4) \qquad \text{Drop tildes}$$

(5)

Consider gauge transformations $\boxed{a_\mu \to a_\mu + \partial_\mu \chi}$

$$\boxed{\begin{aligned} S_{cs}[a + d\chi] &= c \int_0^\beta d\tau \int d^2x \; \epsilon^{\mu\nu\lambda} (a_\mu + \partial_\mu \chi) \, \partial_\nu (a_\lambda + \partial_\lambda \chi) \qquad (6) \\ &= S_{cs}[a] + c \int_0^\beta d\tau \int d^2x \; \epsilon^{\mu\nu\lambda} \left[\partial_\mu \chi \, \partial_\nu a_\lambda + \partial_\mu \chi \, \partial_\nu \partial_\lambda \chi + a_\mu \partial_\nu \partial_\lambda \chi \right] \end{aligned}}$$

We will assume that χ is well-defined. However, a_μ may not be, if there is a monopole involved. Let's integrate the last term by parts.

$$\boxed{\delta S = 2c \int_0^\beta d\tau \int d^2x \, \epsilon^{\mu\nu\lambda} \, \partial_\mu \chi \, \partial_\nu a_\lambda} \quad (7)$$

The gauge invariant quantities are the Wilson loops. Consider a timelike loop

$$\boxed{W(x,y) = e^{\frac{ie}{\hbar}\int_0^\beta a_0 d\tau} \quad (8)} \qquad \boxed{e = \text{fundamental charge} \quad (9)}$$

In addition to the usual "small" gauge transformations, $W(x,y)$ is invariant under

$$\boxed{a_0 \to a_0 + \frac{2\pi n}{\beta}\frac{\hbar}{e}} \quad (10) \qquad \text{"large" gauge tr.}$$

Let's apply this large gauge tr. to the action and see what happens.

$$\boxed{\begin{aligned} S_{cs}\left[a_0 + \frac{2\pi n}{\beta}\frac{\hbar}{e}, a_1, a_2\right] &= S_{cs}\left[a_0, a_1, a_2\right] \\ &\quad + 2c \, \frac{2\pi n \hbar}{\beta e} \int_0^\beta d\tau \int d^2x \, f_{12} \end{aligned}} \quad (11)$$

where $\boxed{f_{12} = \partial_1 a_2 - \partial_2 a_1}$ (12)

Now we imagine that space is S^2 and put a monopole "inside" the S^2.

For a monopole of strength g in 3D

$$\boxed{\vec{B} = \frac{g}{r^2}\,\hat{e}_r} \quad (13)$$

The Dirac quantization condition says

$$e^{\frac{ie}{\hbar} 4\pi g} = e^{i 2m\pi} \Rightarrow \boxed{g = \frac{m\hbar}{2e}} \quad (14)$$

$m \in \mathbb{Z}$ is an integer. The smallest g is

$$\boxed{g_{min} = \hbar/2e} \quad (15)$$

Put one of these inside the S^2

$$\boxed{\begin{gathered}\int d^2x \; f_{12} = 4\pi g = \frac{2\pi\hbar}{e} \\ e^{\frac{i}{\hbar} S S_{cs}} = e^{\frac{i}{\hbar} 2c \frac{2\pi\hbar}{e\beta} \beta \cdot \frac{2\pi\hbar}{e}}\end{gathered}} \quad (16)$$

For the theory to be gauge-invariant under large gauge tr. the exponent must be $2\pi \times$ integer

$$\Rightarrow \quad \frac{8\pi^2\hbar}{e^2} c = 2\pi k$$

$$\Rightarrow \quad \boxed{c = \frac{k}{4\pi} \cdot \frac{e^2}{\hbar}} \quad (17)$$

The coefficient of the CS Lagrangian must be quantized.

Now, let's couple a_μ to an external electromagnetic gauge field A_μ. The coupling must be via a conserved current. The only one we can define using only a_μ is

$$\boxed{J_\mu^{EM} \propto \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda} \quad (18)$$

For all smooth configurations of $f_{\nu\lambda}$ $\partial_\mu J_\mu^{EM} = 0$
Recall that even with the monopole "inside" spatial S^2 $f_{\nu\lambda}$ was smooth on S^2

So the full action must take the form (19)

$$\boxed{S_{cs} = \frac{k}{4\pi}\frac{e^2}{\hbar}\int d^3x \; \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + b\, A_\mu \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda}$$

Once again we appeal to gauge invariance in both a and A. This tells us that

$$\boxed{b = \frac{e^2}{2\pi\hbar}\cdot q} \quad (20) \qquad \text{where } q \in \mathbb{Z} \text{ is an integer}$$

Let us choose $q = 1$

Now we can integrate out a_μ and look at the EM response. Since a_μ appears quadratically, we can solve the eq^{n}s of motion and plug them back into S

$$\boxed{k f_{\mu\nu} = -F_{\mu\nu} \quad \Rightarrow \quad a_\mu = -\frac{1}{k} A_\mu} \quad (21)$$

$$\Rightarrow \boxed{S_{cs,eff} = -\frac{e^2}{\hbar}\int d^3x\left\{\frac{1}{4\pi k}A_\mu\,\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda\right\}} \quad (22)$$

$$\boxed{J_\mu = \frac{e^2}{h}\epsilon_{\mu\nu\lambda}f_{\nu\lambda} = -\frac{e^2}{kh}\epsilon_{\mu\nu\lambda}F_{\nu\lambda}} \quad (23)$$

Choose $A_1 = \dfrac{E\,e^{-i\omega t}}{i\omega}$ $\quad A_0 = A_2 = 0$

$F_{01} = -E\,e^{-i\omega t}$ $\quad$ rest $F_{\mu\nu} = 0$

$$\Rightarrow \boxed{J_2 = \frac{e^2}{kh}E\,e^{-i\omega t} \Rightarrow \sigma_{xy} = \frac{e^2}{kh}} \quad (24)$$

This represents a Laughlin state $\boxed{\nu = \frac{1}{k}}$ (25)

To see quasiparticles, we couple a_μ to an internal conserved current j_μ, not to be confused with J_μ. Since we have already used the fact that the fundamental unit of charge is e, for both a_μ and A_μ, we couple the current simply by adding

$$\boxed{iS = iS_{cs}[a] + ie\int_0^\beta d\tau\, d^2x\; j_\mu a_\mu} \quad (26)$$

The eq^n of motion (without an external A) is

$$\boxed{\frac{ek}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda = j_\mu} \quad (27)$$

If $\boxed{j_\mu \equiv j_0 = \delta^2(\bar{x}-\bar{x}_0) \quad \text{(static qp)} \quad \Rightarrow f_{12} = \frac{2\pi}{ke}\delta^2(\bar{x}-\bar{x}_0)}$ (28)

Each qp carries a $\frac{1}{k}$ flux of f.

From 18, 20 we know that the EM current is

$$J_\mu = \frac{e^2}{2\pi\hbar} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$$

$\Rightarrow$ The J_0 corresponding to j_0 is

$$\boxed{J_0 = \frac{e}{k} \delta^2(\bar{x}-\bar{x}_0)} \quad 29$$

Fractional charge!

Now consider two such qps. One sits still at the spatial origin while the other moves in a circle around it.

$$\boxed{\begin{aligned} j_0(\vec{x},t) &= \delta^2(\vec{x}) + \delta^2(\bar{x}-\vec{x}_0(t)) \\ \bar{x}_0(t) &= r_0 \cos\omega t\, \hat{e}_x + r_0 \sin\omega t\, \hat{e}_y \\ \vec{j}(\vec{x},t) &= \vec{v}_0(t)\, \delta^2(\bar{x}-\bar{x}_0(t)) \\ \vec{v}_0(t) &= r_0\omega(-\sin\omega t\, \hat{e}_x + \cos\omega t\, \hat{e}_y) \end{aligned}} \quad 30$$

We want to know the path integral in the presence of this source. We need to integrate out a_μ, which can be done by solving the equations of motion and plugging them back in S. The EOM are

$$\frac{ek}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda = j_\mu$$

$$\Rightarrow \quad \boxed{f_{12} = \frac{2\pi}{ek} J_0 \qquad f_{20} = \frac{2\pi}{ek} J_1 \qquad f_{01} = \frac{2\pi}{ek} J_2} \tag{31}$$

We can solve them in $a_0 = 0$ gauge.

$$\boxed{a_1(\bar{x},t) = -\frac{1}{ek}\left\{ \frac{y\,\hat{e}_x - x\,\hat{e}_y}{x^2+y^2} + \frac{(y-y_0(t))\,\hat{e}_x - (x-x_0(t))\,\hat{e}_y}{(x-x_0(t))^2+(y-y_0(t))^2} \right\}} \tag{32}$$

The two terms correspond to the two sources.

$$\boxed{\begin{aligned} \partial_x a_y^{(1)} - \partial_y a_x^{(1)} &= \frac{1}{ek}\left\{ \frac{\partial}{\partial x}\frac{x}{(x^2+y^2)} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2} \right\} \\ &= \frac{1}{ek}\,\bar{\nabla}\cdot\left(\frac{\hat{r}}{r}\right) \end{aligned}} \tag{33}$$

A naive differentiation produces zero. However

$$\boxed{\int_{r<R_0} d^2x\ \bar{\nabla}\cdot\left(\frac{\hat{r}}{r}\right) = \int_{r=R_0} r\,d\theta\ \hat{r}\cdot\left(\frac{\hat{r}}{r}\right) = 2\pi} \tag{34}$$

This means $\boxed{\frac{\partial}{\partial x} a_y^{(1)} - \frac{\partial}{\partial y} a_x^{(1)} = \frac{2\pi}{ek}\,\delta^2(\vec{x}) = J_0^{(1)}}$ (35)

Since $a_0 = 0$ the only source of Aharanov-Bohm phase is

$$ie \int_{2\pi/\omega} \vec{j}^{(2)}\cdot\vec{a}^{(1)}\, d^2x\, dt \tag{36}$$

$$= ie\left(\frac{-1}{ek}\right)\int d^2x \int_0 dt\ \delta^2(\bar{x}-\bar{x}_0(t))\ r_0\omega\left[-\sin\omega t\,\hat{e}_x + \cos\omega t\,\hat{e}_y\right] \cdot \frac{(y_0(t)\,\hat{e}_x - x_0(t)\,\hat{e}_y)}{x_0^2+y_0^2}$$

Plugging in for $x_0(t) = r_0\cos\omega t \quad y_0(t) = r_0 \sin\omega t$

We get

$$\boxed{i\int \vec{j}^{(2)}\cdot\vec{a}^{(1)}\, d^3x = \frac{2\pi i}{k}} \quad (37)$$

So, exchanging the qps should result in half this, which implies fractional statistics!

$$\boxed{B_{12}\,|2qp\text{ state}\rangle = e^{\frac{i\pi}{k}}\,|2qp\text{ state}\rangle} \quad (38)$$

This is strictly not an exchange but a braiding, because it is done adiabatically, and depends on the direction of the braiding (counter- or clockwise)

Now let us see what the CS theory looks like in Hamiltonian quantum mechanics.

We go back to real time. In order to quantize we must choose a gauge. The most convenient one is

$$\boxed{a_0 = 0} \quad (39)$$

However, one must be careful to impose the a_0 EOM as a constraint on the physical states.

$$\boxed{f_{12}\,|\Psi_{phys}\rangle = 0} \quad (40)$$

With $a_0 = 0$ the action is

$$S_{cs} = \frac{e^2}{4\pi k}\int d^2x\, dt \left\{ a_0(\partial_1 \cancel{a_2 - \partial_2} a_1) + a_1(\partial_2 \cancel{a_0} - \partial_0 a_2) + a_2 \cancel{(\partial_0 a_1} - \partial_1 a_0) \right\}$$

$$\boxed{S_{cs} = \frac{e^2}{4\pi k}\int d^2x\, dt \left\{ a_2 \dot{a}_1 - a_1 \dot{a}_2 \right\}} \quad (41)$$

This tells us that a_1 and a_2 are canonically conjugate variables. We can choose to think of a_1 as the "coordinate" and a_2 as its "momentum". Integrating by parts

$$\boxed{S_{cs} = \frac{e^2}{2\pi k}\int d^2x\, dt \;\; a_2 \dot{a}_1} \quad (42)$$

$$\boxed{a_1 \equiv q \qquad \qquad \pi = \frac{\partial \mathcal{L}}{\partial \dot{a}_1} = \frac{e^2}{2\pi k} a_2} \quad (43)$$

Since $\boxed{L = \int \pi \dot{q} \; d^2x - H}$ (44)

we see that $\boxed{H = 0}$! (45) The Hamiltonian vanishes. However, we recall the a_0 EOM $f_{12} = 0$, which should be imposed on all physical states.

This means a_i should be "flat". If a_μ were a pure gauge $a_\mu = \partial_\mu \chi$ this condition would be automatically satisfied. However, we are not interested in pure gauge configurations, because they are a redundancy in our description.

The question becomes : Are there a_i such that : (i) $f_{12}=0$ and (ii) They are not pure gauge.

On the plane we can put $a_i =$ constant indep of position. Since $H=0$ both Q and P can be constants of motion.

To get something more interesting we put the system on a torus. Let the two "circumferences" of the torus be L_1 & L_2.

We already know that $a_1 = \dfrac{2\pi n \hbar}{e L_1}$ $n \in \mathbb{Z}$ is a pure gauge. This is the "large" gauge tr.

Similarly $a_2 = \dfrac{2\pi n_2 \hbar}{e L_2}$ is pure gauge.

So $\left(a_1 + \dfrac{2\pi n_1 \hbar}{e L_1},\ a_2 + \dfrac{2\pi n_2 \hbar}{e L_2}\right)$ is gauge-equivalent to (a_1, a_2) and is not a separate point in phase space.

So the phase space is itself a torus!

The sides are $\dfrac{2\pi\hbar}{eL_1}, \dfrac{2\pi\hbar}{eL_2}$

Let us now specialize to constant a_1, a_2 (independent of position due to Eq. 40) lying on this phase space torus. Do the integral in Eq. 42

$$\boxed{\frac{i S_{cs}}{\hbar} = \frac{ik}{2\pi} \frac{e^2}{\hbar^2} \int dt \; L_1 L_2 \; a_2 \dot{a}_1} \quad (46)$$

CS Lagrangian

$$\boxed{L_{cs} = \frac{k}{2\pi\hbar} e^2 L_1 L_2 \quad a_2 \dot{a}_1} \quad (47)$$

$$\boxed{Q \equiv a_1 \qquad P = \frac{\partial L_{cs}}{\partial \dot{Q}} = \frac{k e^2}{2\pi\hbar} L_1 L_2 a_2} \quad (48)$$

How many states are in this phase space?

Recall that in going from classical mechanics to quantum states we have the rule that every cell of $\Delta P \Delta Q = 2\pi\hbar$ counts for one quantum state.

The area of this phase space is

$$\boxed{\frac{2\pi\hbar}{e L_1} \times \frac{2\pi\hbar}{e L_2} \cdot \frac{k e^2}{2\pi\hbar} L_1 L_2 = k \cdot 2\pi\hbar = \# \text{ of states} \times 2\pi\hbar} \quad (49)$$

$\Rightarrow$ There must be precisely k states in the phase space torus. This is the k-fold ground state degeneracy of the $\frac{1}{k}$ Laughlin state.

There is another way to see this. Let us construct the two gauge-invariant quantities given a a_1 and a_2 on a $L_1 \times L_2$ torus

$$\boxed{W_1 = \exp\left\{\frac{ie}{\hbar} \oint a_1 dx\right\} \qquad W_2 = \exp\left\{\frac{ie}{\hbar} \oint a_2 dy\right\}} \quad (50)$$

On the Hilbert space these become operators

$$\boxed{\begin{aligned} W_1 &= e^{\frac{ie}{\hbar} a_1 L_1} = e^{\frac{ie}{\hbar} L_1 Q} \\ W_2 &= e^{\frac{ie}{\hbar} a_2 L_2} = e^{\frac{ie}{\hbar} L_2 \frac{2\pi\hbar P}{ke^2 L_1 L_2}} = e^{\frac{2\pi i P}{keL_1}} \end{aligned}} \quad (51)$$

These two operators don't commute. Use (52)

$$\boxed{e^A e^B = e^B e^A e^{[A,B]} \quad \text{when } [[A,B],A] = 0 = [[A,B],B]}$$

$$\boxed{W_1 W_2 = W_2 W_1 e^{-\frac{2\pi}{k\hbar}[Q,P]} = W_2 W_1 e^{\frac{-2\pi i}{k}}} \quad (53)$$

As usual, let us assume that the wave f^ns are functions of Q. So a gauge-invariant basis can be constructed from eigenstates of W_1

$$\boxed{W_1 |\lambda\rangle = \lambda |\lambda\rangle} \quad (54)$$

Since W_1 is unitary

$$\boxed{W_1^\dagger W_1 = 1 \Rightarrow |\lambda| = 1} \quad (55)$$

Consider $W_2|\lambda\rangle$

$$\boxed{W_1 W_2 |\lambda\rangle = e^{\frac{-2\pi i}{k}} W_2 W_1 |\lambda\rangle = \lambda e^{\frac{-2\pi i}{k}} W_2 |\lambda\rangle} \quad (56)$$

$$\Rightarrow \boxed{W_2 |\lambda\rangle = |\lambda e^{\frac{-2\pi i}{k}}\rangle} \quad (57)$$

(58) $$\boxed{(W_2)^n |\lambda\rangle = |\lambda e^{\frac{-2\pi i n}{k}}\rangle} \Rightarrow n \text{ should be mod } k$$

The minimum dimension of the Hilbert space is k.

II : Edges

Once again we take a single CS field, but now we consider a semi-infinite system. The system is in the region $y \leq 0$, while there is vacuum in $y > 0$.

a_μ is an internal, emergent, gauge field so it is ill-defined for $y > 0$. We need to put some conditions on it. First consider the source-free action

$$\boxed{S_{cs} = \frac{ke^2}{4\pi} \int_{y \leq 0} d^3x \; a_\mu \partial_\nu a_\lambda \varepsilon_{\mu\nu\lambda}} \qquad (59)$$

set $\hbar = 1$

Under an arbitrary variation δa_μ

$$\delta S_{cs} = \frac{ke^2}{4\pi} \int_{y \leq 0} d^3x \; \varepsilon_{\mu\nu\lambda} \left\{ \delta a_\mu \partial_\nu a_\lambda + a_\mu \partial_\nu \delta a_\lambda \right\}$$

$$\boxed{= \frac{ke^2}{4\pi} \int_{y \leq 0} d^3x \; \varepsilon_{\mu\nu\lambda} \left[2\delta a_\mu \partial_\nu a_\lambda - \partial_\nu (a_\mu \delta a_\lambda) \right]} \qquad (60)$$

The last piece leads to a boundary term

$$\boxed{\delta S_{cs}\Big|_{y=0} = -\frac{ke^2}{4\pi} \int dx\, dt \; \varepsilon_{\mu\nu\lambda} \, a_\mu(x,0,t)\, \delta a_\lambda(x,0,t)} \qquad (61)$$

If we want to maintain the bulk eq^n of motion ($f = 0$) we are forced to make the boundary term vanish.

The simplest choice is $a_\mu(x,y,0)=0$, but this has a problem. In the quantum theory, in $a_0=0$ gauge a_1 and a_2 are conjugate variables, so we cannot force both to vanish.

We have two natural possibilities:

(i) Dirichlet b.c. $\quad a_1(x,0,t)=0 \Rightarrow \delta a_1(x,0,t)=0$

(ii) Neumann b.c $\quad a_2(x,0,t)=0 \Rightarrow \delta a_2(x,0,t)=0$

In either case $(a_0=0)$ $\delta S_{cs}\big|_{y=0}$ becomes

$$-\frac{ke^2}{4\pi}\int dx\,dt\,\left\{a_1(x,0,t)\,\delta a_2(x,0,t)-a_2(x,0,t)\,\delta a_1(x,0,t)\right\}$$

vanishes.

Now consider a gauge transformation of a_μ. We want the action to be gauge invariant.

$$a_\mu \to a_\mu+\partial_\mu \chi \tag{62}$$

$$\Rightarrow \quad S_{cs}[a+d\chi]=S_{cs}[a]+\frac{ke^2}{4\pi}\int_{-\infty}^{\infty}dx\,dt\,\epsilon_{\mu\nu\lambda}\int_{-\infty}^{0}dy\,\partial_\mu\chi\,\partial_\nu a_\lambda$$

We can demand that $\partial_\mu \chi$ vanishes at $y=0$ which restricts the set of gauge transformations we are allowed to make. This means some redundancies that used to be present will now be absent. Some degrees of freedom that used to be unphysical will become physical.

The coupling to the external EM A_μ also needs to be modified.

$$\boxed{\frac{e^2}{2\pi}\int d^3x\, A_\mu \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \rightarrow \frac{e^2}{2\pi}\int_{-\infty}^{\infty} dx\,dt \int_{-\infty}^{0} dy\; a_\mu \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda} \quad (63)$$

The final form is gauge-invariant w.r.t. the restricted gauge transformations of a_μ and arbitrary gauge transformations of A_μ.

Let us put periodic bc in x $\quad x=0 \equiv x=L$ and go to $a_0=0$ gauge.

The EOM of a_0 appears as a constraint

$$\boxed{f_{12}=0 \quad \Rightarrow \quad a_1 = \partial_x \phi \qquad a_2 = \partial_y \phi} \quad (64)$$

ϕ is an arbitrary smooth f^n of x, y, t

This appears to be pure gauge. But remember we here restricted our gauge transformations to those where $\partial_i \chi = 0$ at $y=0$. So ϕ will be a physical variable at $y=0$!

$$\boxed{\begin{aligned} S_{cs+A} = {} & \frac{ke^2}{4\pi}\int_{-\infty}^{\infty} dt \int_0^L dx \int_{-\infty}^{0} dy \left\{ \partial_y\phi\, \partial_t\partial_x\phi - \partial_x\phi\,\partial_t\partial_y\phi \right\} \quad (65) \\ & + \frac{e^2}{2\pi}\int_{-\infty}^{\infty} dt \int_0^L dx \int_{-\infty}^{0} dy \left\{ \partial_x\phi\,(\partial_y A_0 - \partial_0 A_y) + \partial_y\phi\,(\partial_0 A_x - \partial_x A_0) + \partial_t\phi\,(\partial_x A_y - \partial_y A_x) \right\} \end{aligned}}$$

In the 1st term of S_{cs} integrate by parts in x. Due to periodicity there are no boundary terms. In the second term integrate by parts in y to get a boundary term.

$$\boxed{\begin{aligned} S_{cs} &= \frac{e^2 k}{4\pi}\int_{-\infty}^{\infty} dt \int_0^L dx \int_{-\infty}^{0} dy \left\{\cancel{-\partial_x\partial_y\phi\,\partial_t\phi} + \cancel{\partial_y\partial_x\phi\,\partial_t\phi}\right\} \\ &\quad -\frac{ke^2}{4\pi}\int_{-\infty}^{\infty} dt \int_0^L dx\ \partial_x\phi\,\partial_t\phi \end{aligned}} \quad (66)$$

In S_A take all the terms containing A_0 and integrate by parts to bring A_0 out

$$\frac{e^2}{2\pi}\int_{-\infty}^{\infty} dt \int_0^L dx \int_{-\infty}^{0} dy \left\{\partial_x\phi\,\partial_y A_0 - \partial_y\phi\,\partial_x A_0\right\}$$

$$\boxed{= \frac{e^2}{2\pi}\int_{-\infty}^{\infty} dt \int_0^L dx \left\{ A_0\,\partial_x\phi - \int_{-\infty}^{0} dy\ A_0\left[\cancel{\partial_y\partial_x\phi - \partial_x\partial_y\phi}\right]\right\}} \quad (67)$$

So, at $y=0$ A_0 couples minimally to $\partial_x\phi\,\frac{e^2}{2\pi}$, which means $\boxed{\frac{e\,\partial_x\phi}{2\pi} = J_{0,\mathrm{edge}}}$ (68) is the electric charge density. Finally, the last term in S_A gives, upon integration by parts

$$\boxed{-\frac{e^2}{2\pi}\int_{-\infty}^{\infty} dt \int_0^L dx\ A_y\,\partial_t\phi} \quad (69) \quad \Rightarrow \quad \boxed{-\frac{e\,\partial_t\phi}{2\pi} = J_{x,\mathrm{edge}}.} \quad (70)$$

Note this automatically satisfies continuity.

So far we have (71)

$$\boxed{S_{CS+A} = -\frac{ke^2}{4\pi}\int_{-\infty}^{\infty} dt \int_0^L dx\, \partial_t\phi\,\partial_x\phi + \frac{e^2}{2\pi}\int_{-\infty}^{\infty} dt\int_0^L dx \left\{A_0\partial_x\phi - A_x\partial_t\phi\right\}}$$

As promised, ϕ has become dynamical at the edge, with $\partial_x\phi$ proportional to the density.

This action has no terms corresponding to density-density interactions, which we add by hand

$$\boxed{\begin{aligned} S = & -\frac{ke^2}{4\pi}\int_{-\infty}^{\infty} dt\int_0^L dx\,\partial_x\phi\,(\partial_t\phi + v\partial_x\phi) \\ & + \frac{e^2}{2\pi}\int_{-\infty}^{\infty} dt\int_0^L dx\,(A_0\partial_x\phi - A_x\partial_t\phi)\end{aligned}} \quad (72)$$

$$\boxed{\begin{aligned} \Pi_\phi &= -\frac{ke^2}{4\pi}\partial_x\phi \qquad (A_\mu = 0) \\ \mathcal{H} &= \frac{ke^2}{4\pi}\int_0^L dx\; v(\partial_x\phi)^2 \end{aligned}} \quad (73)$$

The canonical commutation relations are

$$\boxed{[\phi(x), \Pi_\phi(x')] = \frac{i}{2}\,\delta(x-x')} \quad (74)$$

The $\frac{1}{2}$ can be attributed to the fact that the chiral theory has $\frac{1}{2}$ the degrees of freedom

Let's expand $\phi(x)$

$$\boxed{\phi(x) = \phi_0 + \Pi_0 x + \sum_1^{\infty}\left(\phi_m e^{\frac{i2\pi m x}{L}} + \phi_{-m} e^{-\frac{i2\pi m x}{L}}\right)} \quad (75)$$

where I have used the periodicity in $x \to x+L$.

$$\boxed{\partial_x \phi = \sum_{1}^{\infty} \frac{i2\pi m}{L} \left(\phi_m e^{\frac{i2\pi mx}{L}} - \phi_{-m} e^{\frac{-2\pi imx}{L}} \right)} \quad (76)$$

$$[\phi(x), \Pi(x')] = -\frac{ke^2}{4\pi} \sum_{m,m'=1}^{\infty} \frac{i2\pi m'}{L} \Big\{ e^{\frac{i2\pi}{L}(mx+m'x')} [\phi_m, \phi_{m'}]$$

$$- e^{\frac{i2\pi}{L}(mx-m'x')} [\phi_m, \phi_{-m'}] + e^{\frac{i2\pi}{L}(-mx+m'x')} [\phi_{-m}, \phi_{m'}] \quad (77)$$

$$- e^{-\frac{i2\pi}{L}(mx+m'x')} [\phi_{-m}, \phi_{-m'}] \Big\} + [\phi_0, \Pi_0]$$

In order to obtain a translation-invariant result we must impose

$$m>0 \qquad [\phi_m, \phi_{m'}] = 0 = [\phi_{-m}, \phi_{-m'}] \quad (78)$$

$$[\phi_m, \phi_{-m'}] = \delta_{mm'} C_m$$

$$[\phi(x), \Pi(x')] = [\phi_0, \Pi_0] + \frac{ike^2}{2} \sum_{m=1}^{\infty} \frac{m}{L} C_m \left(e^{\frac{i2\pi}{L}m(x-x')} + e^{-\frac{i2\pi}{L}m(x-x')} \right) \quad (79)$$

Now

$$\boxed{\frac{1}{L} \sum_{-\infty}^{\infty} e^{\frac{i2\pi}{L}m(x-x')} = \delta(x-x')} \quad (80)$$

Check by integrating between 0 & L

$$\int_0^L \frac{dx}{L} \sum_{-\infty}^{\infty} e^{\frac{i2\pi}{L}m(x-x')} = \int_0^{2\pi} \frac{d\theta}{2\pi} \sum_{-\infty}^{\infty} e^{im(\theta-\theta')} = 1$$

$$\theta = \frac{2\pi x}{L}$$

$\Rightarrow$ we must impose $\boxed{[\phi_m, \phi_{-m'}] = \frac{1}{ke^2 m}\delta_{m-m'}}$

$\boxed{[\phi_0, \Pi_0] = 1}$ 82

An important point about ϕ is that it should be compact = periodic mod 2π. This is because ultimately, the CS theory should be able to describe tunneling of charges into the QHE sample from outside. Since

$J_0 = \frac{1}{2\pi} f_{12}$ in units of the fundamental charge

a change in charge of 1 corresponds to a monopole of a_μ. Monopoles exist only in compact theories, so a_μ must be a compact gauge field. In a compact U(1) theory a pure gauge is

$$e^{-i\phi}\partial_\mu e^{i\phi} = \partial_\mu \phi$$

hence ϕ must be compact.

This allows us to define operators such as $e^{i\phi(x)}$ $e^{2i\phi(x)}$ etc on the edge. More on this soon.

Appendix I

How to couple $a_\mu j^\mu$ in real & imaginary time

In real time, let's consider a particle obeying the Schrödinger eqn.

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \vec{\nabla}^2 \psi + V(\vec{x})\psi} \quad \text{AI.1}$$

We will follow the minimal coupling prescription

$$\boxed{\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - i a_0 \qquad \vec{\nabla} \to \vec{\nabla} - i\vec{a}} \quad \text{AI.2}$$

This maintains the gauge invariance of the spectrum.

The Schrödinger Eqn is the eqn of motion of the following action ($S_M =$ S of matter)

$$\boxed{S_M = \int d^dx\, dt \left\{ \psi^* i (\partial_t - i a_0)\psi - \frac{1}{2M}\left[(\vec{\nabla} + i\vec{a})\psi^*\right]\left[(\vec{\nabla} - i\vec{a})\psi\right] - V\psi^*\psi \right\}} \quad \text{AI.3}$$

One can compute the partition fn

$$\boxed{Z[a] = \int \mathcal{D}\psi^* \mathcal{D}\psi\, e^{\frac{i S_M}{\hbar}}} \quad \text{AI.4}$$

The current is defined as $\boxed{j^\mu = \frac{\partial \mathcal{L}_M}{\partial a_\mu}}$ AI.5

$$\boxed{j^0 = \psi^*\psi \qquad \vec{j} = -\frac{i}{2M}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right) + \frac{\vec{a}}{M}\psi^*\psi} \quad \text{AI.6}$$

$$\boxed{i S_M[a] = i S_M[a=0] + i \int dt\, d^dx\; j^\mu a_\mu + \cdots} \quad \text{AI.7}$$

In the limit $a_\mu \to 0$ we get the usual conserved probability current which we learn to get in QM101 as follows:

$$i\frac{\partial \psi}{\partial t} = -\frac{\vec{\nabla}^2\psi}{2M} + V\psi$$

$$-i\frac{\partial \psi^*}{\partial t} = -\frac{\vec{\nabla}^2\psi^*}{2M} + V\psi^*$$

$$\Rightarrow \quad i\left\{\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right\} = -\frac{1}{2M}\left\{\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*\right\}$$

or
$$\boxed{\frac{\partial(\psi^*\psi)}{\partial t} = \frac{i}{2M}\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = -\vec{\nabla}\cdot\vec{j}}$$

AI·8

Now we go to imaginary time.

$$\boxed{\begin{array}{lll} t = -i\tau & \tau = it & \partial_t = i\partial_\tau \\ a_0 = i\tilde{a}_0 & & \end{array}}$$

AI·9

$$\tilde{S}_M = i\int d^dx\,(-i)d\tau\left\{-\psi^*(\partial_\tau - i\tilde{a}_0)\psi - \frac{[(\bar{\nabla}+i\bar{a})\psi^*][(\bar{\nabla}-i\bar{a})\psi]}{2M} - V\psi^*\psi\right\}$$

AI·10

$$\boxed{\tilde{S}_M = \int d^dx\,d\tau\left\{-\psi^*(\partial_\tau - i\tilde{a}_0)\psi - \frac{[(\bar{\nabla}+i\bar{a})\psi^*(\bar{\nabla}-i\bar{a})\psi]}{2M} - V\psi^*\psi\right\}}$$

The eq^{n}s of motion arising from this for $a_\mu \to 0$ are

$$\boxed{\frac{\partial\psi}{\partial\tau} = \frac{\nabla^2\psi}{2M} - V\psi \qquad \frac{\partial\psi^*}{\partial\tau} = -\frac{\nabla^2\psi^*}{2M} + V\psi^*}$$

AI·11

Note that ψ and ψ^* are no longer complex conjugates!

Defining $$\boxed{\rho = \tilde{j}^0 = \psi^* \psi} \quad \text{AI.12}$$

$$\frac{\partial}{\partial t}(\psi^* \psi) = \frac{1}{2M}\left(\psi^* \vec{\nabla}^2 \psi - \psi \vec{\nabla}^2 \psi^*\right)$$

$$= -\vec{\nabla}\left\{-\frac{1}{2M}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)\right\}$$

$$\Rightarrow \boxed{\tilde{\vec{j}} = -\frac{1}{2M}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)} \quad \text{AI.13}$$

Thus, we can rewrite the imaginary time action as

$$\boxed{\tilde{S}_M[a] = \tilde{S}_M[a=0] + i\int d\tau\, d^2x\; j_\mu a_\mu + \cdots} \quad \text{AI.14}$$

So both in real and imaginary time the $j_\mu a_\mu$ term appears as

$$\boxed{\exp i\int j_\mu a_\mu\, d^{d+1}x} \quad \text{AI.15}$$