COMQUI QHE Lecture II, Part 2: CS Flux Attachment

Saturday, December 17, 2022 12:16 AM

1. Flux attachment via Chern-Simons

2. Mean-field theory, CF-LLs, Fluctuations

3. $\nu = \frac{1}{2}$ CFFL.

1. Chern-Simons Flux attachment

Historically, Girvin & MacDonald proposed in 1987 that one should attach flux to electrons to change their statistics to bosons. This idea was fully realized by Zhang, Hansson, and Kivelson in 1989, who understood the $\nu = \frac{1}{3}$ state as a superfluid of composite bosons.

Inspired by Jain's composite Fermion picture of 1989, Lopez and Fradkin (1991) developed a path integral formulation for principal fractions. Halperin, Lee, and Read (1993) took the CF-Fermi liquid at $\nu = \frac{1}{2}$ as the starting point and explained fractions above & below symmetrically.

Let us start with the interacting Hamiltonian, but without disorder in 1st quantization

$$\boxed{H = \sum_i \frac{\left(\vec{p}_i + e\vec{A}(\bar{r}_i)\right)^2}{2m} + \frac{1}{2}\sum_{i\neq j} v(\bar{r}_i - \bar{r}_j)} \quad (1.1)$$

In 2nd quantization we write H as

$$\boxed{\begin{aligned} H = & \int d^2r\, \Psi_e^\dagger(\bar{r}) \frac{\left(-i\hbar\vec{\nabla} + e\vec{A}(\bar{r})\right)^2}{2m} \Psi_e(\bar{r}) \\ & + \frac{1}{2}\int d^2r_1 d^2r_2\, v(\bar{r}_1 - \bar{r}_2) :\rho_e(\bar{r}_1)\rho_e(\bar{r}_2): \end{aligned}} \quad (1.2)$$

$\Psi_e(\bar{r})$ and $\Psi_e^\dagger(\bar{r})$ are annihilation and creation operators

$$\boxed{\{\Psi_e(\bar{r}_1), \Psi_e^\dagger(\bar{r}_2)\} = \delta^2(\bar{r}_1 - \bar{r}_2) \qquad \rho_e(\bar{r}) = \Psi_e^\dagger(\bar{r})\Psi_e(\bar{r})} \quad (1.3)$$

We have suppressed the spin index assuming states are spin polarized, but this can easily be remedied.

Now we make a singular gauge transformation

$$\boxed{\Psi^{\dagger}_{CS}(\bar r) = \Psi^{\dagger}_{e}(\bar r)\, e^{-2i\,Q(\bar r)} \qquad \Psi_{CS}(\bar r) = e^{2iQ(\bar r)}\,\Psi_e(\bar r) \qquad Q(\bar r) = \int d^2r' \arg(\bar r - \bar r')\,\rho_e(\bar r')} \quad (1.4)$$

$\arg(\bar r - \bar r')$ is the angle $\bar r - \bar r'$ makes with the x-axis.

(1.5)

An obvious fact is $\boxed{\Psi^{\dagger}_{CS}(\bar r)\,\Psi_{CS}(\bar r) = \Psi^{\dagger}_{e}(\bar r)\,\Psi_e(\bar r)}$

The effect of this on the many-body wave f^n is

$$\boxed{\Phi_e(\{\bar r_i\}) = \left\{\prod_{i<j}\left(\frac{z_i - z_j}{|z_i - z_j|}\right)^2\right\}\Phi_{CS}(\{\bar r_i\})} \quad (1.6)$$

In other words, each electron sees other electrons as having a solenoid with $2\phi_0$ attached, which is why this is called Flux attachment.

The 2 in $e^{\pm 2iQ}$ is the number of flux quanta attached to each electron. When this number is even, the resulting composite object is a fermion. If this number had been odd the resulting object would behave like a boson.

In our case it is the composite fermion

$$\{\Psi_{cs}(\bar{r}_1), \Psi^{+}_{cs}(\bar{r}_2)\} = \delta^2(\bar{r}_1 - \bar{r}_2) \tag{1.7}$$

We show this in Appendix A.

We rewrite H in terms of Ψ_{cs}, Ψ^{+}_{cs}

$$H = -\int d^2r \; \Psi^{+}_{cs}(\bar{r})\, e^{2iQ(\bar{r})} \left[-i\hbar\vec{\nabla} + e\vec{A} - \vec{a}\right]^2 e^{-2iQ(\bar{r})}\, \Psi_{cs}(\bar{r}) + H_{int} \tag{1.8}$$

where

$$2\hbar\vec{\nabla}Q(\bar{r}) = \vec{a}(\bar{r}) \tag{1.9}$$

CS vector potential

We will now show that $\vec{a}$ is transverse

$$\vec{\nabla}\cdot\vec{a} = 0 \tag{1.10}$$

and its field strength is proportional to ρ

$$\vec{\nabla}\times\vec{a} = \hbar 4\pi\rho \tag{1.11}$$

$$\frac{\partial}{\partial x_i}\arg(\bar{r}-\bar{r}') = \frac{\partial}{\partial x_i}\tan^{-1}\left(\frac{x_2-x_2'}{x_1-x_1'}\right) = -\epsilon_{ij}\,\frac{x_j - x_j'}{|\bar{r}-\bar{r}'|^2}$$

$$\vec{a}(\bar{r}) = 2\hbar\int d^2r'\; \rho(\bar{r}')\left[-\hat{e}_x\,\frac{y-y'}{|\bar{r}-\bar{r}'|^2} + \hat{e}_y\,\frac{x-x'}{|\bar{r}-\bar{r}'|^2}\right] \tag{1.12}$$

$$\boxed{\begin{aligned}\vec{\nabla}\cdot\vec{a} &= 2\hbar\int d^2r'\rho(\bar{r}')\left[-(y-y')\partial_x\frac{1}{|\bar{r}-\bar{r}'|^2}+(x-x')\partial_y\frac{1}{|\bar{r}-\bar{r}'|^2}\right]\\ &=0\end{aligned}} \quad (1.13)$$

$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{a} &= 2\hbar\int d^2r'\rho(\bar{r}')\left[\partial_x\frac{(x-x')}{|\bar{r}-\bar{r}'|^2}+\partial_y\frac{(y-y')}{|\bar{r}-\bar{r}'|^2}\right]\\ &= 2\hbar\int d^2r'\rho(\bar{r}')\ \vec{\nabla}\cdot\left(\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^2}\right)\end{aligned}} \quad (1.14)$$

Now $\boxed{\dfrac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^2} = \vec{\nabla}\ln|\bar{r}-\bar{r}'|}$ (1.15)

$\ln|\bar{r}-\bar{r}'|$ is the 2D electrostatics Green's f^n.

$$\nabla^2\ln|\bar{r}-\bar{r}'| = 2\pi\,\delta^2(\bar{r}-\bar{r}') \quad (1.14)$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{a} = 4\pi\hbar\,\rho(\bar{r})$$

So we have a new Hamiltonian

$$\boxed{H = \int d^2r\ \Psi^{\dagger}_{cs}(\bar{r})\,\frac{(-i\hbar\vec{\nabla}+e\vec{A}-\vec{a})^2}{2m}\,\Psi_{cs}(\bar{r}) + H_{int}} \quad (1.15)$$

supplemented by the constraint $\vec{\nabla}\times\vec{a} = 4\pi\rho$

At this point it is traditional to go to the path integral formulation. The advantage is that it is easy to incorporate the constraint by a Lagrange multiplier.

$$Z = \int \mathcal{D}\bar{\Psi}_{cs}\, \mathcal{D}\Psi_{cs}\, \mathcal{D}\vec{a}\, \mathcal{D}a_0 \; e^{-S} \qquad (1.16)$$

$$S = \int d\tau\, d^2r \left\{ \bar{\Psi} i\hbar\partial_0 \Psi - H(\bar{\Psi},\Psi) + a_0\left(\frac{\vec{\nabla}\times\vec{a}}{4\pi\hbar} - \rho\right)\right\}$$

$\Psi(\bar{r},\tau)$, $\bar{\Psi}(\bar{r},\tau)$ are Grassmann fields, and $a_0(\bar{r},\tau)$ is the Lagrange multiplier imposing the constraint.

$$S = \int d\tau \left\{ \int d^2r \left[\bar{\Psi}(i\hbar\partial_0 - a_0)\Psi - \bar{\Psi}\frac{(-i\hbar\vec{\nabla} + e\vec{A} + \vec{a})^2}{2m}\Psi\right] \qquad (1.17)\right.$$

$$\left. + \frac{a_0}{4\pi\hbar}\bar{\nabla}\times\vec{a} - \frac{1}{2}\int d^2r\, d^2r'\; v(\bar{r}-\bar{r}') :\rho(\bar{r})\rho(\bar{r}'): \right\}$$

Now one notices that $\frac{a_0}{4\pi\hbar}\bar{\nabla}\times\vec{a}$ is the Chern-Simons action in transverse gauge.

Consider the relativistically invariant CS action

$$\mathcal{L}_{cs} = \frac{1}{8\pi\hbar}\epsilon_{\mu\nu\lambda}\, a_\mu \partial_\nu a_\lambda \qquad (1.18) \qquad \mu,\nu,\lambda = 0,1,2$$

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$$= \frac{1}{8\pi\hbar}\left[a_0 \bar{\nabla}\times\vec{a} + a_1(\partial_2 a_0 - \partial_0 a_2) + a_2(\partial_0 a_1 - \partial_1 a_0)\right]$$

The $a_1\partial_2 a_0 - a_2\partial_1 a_0$ can be converted by an integration by parts into $a_0 \bar{\nabla}\times\vec{a}$, giving

$$\boxed{\mathcal{L}_{cs} = \frac{1}{4\pi\hbar} a_0 \bar{\nabla}\times\vec{a} - \frac{1}{8\pi\hbar}(a_1\dot{a}_2 - a_2\dot{a}_1)} \quad (1.19)$$

In transverse gauge

$$\boxed{a_i(\bar{r}) = \int \frac{d^2q}{(2\pi)^2} e^{i\bar{q}\cdot\bar{r}}\, i\epsilon_{ij}\frac{q_j}{q}\, \tilde{a}(\vec{q})} \quad (1.20)$$

You can easily check hat $a_1\dot{a}_2 - a_2\dot{a}_1$ vanishes. Therefore, not confining ourselves to any particular gauge we can say

$$\boxed{S = \int d\tau\, d^2r \left\{ \bar{\Psi}(i\hbar\partial_0 - a_0)\Psi - \bar{\Psi}\frac{(-i\hbar\vec{\nabla} + e\vec{A} - \vec{a})^2}{2m}\Psi + \frac{1}{8\pi\hbar}\epsilon_{\mu\nu\lambda} a_\mu\partial_\nu a_\lambda \right\} + S_{int}}$$

2. Mean-Field + Fluctuations

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The bosonic case was done by Zhang, Hansson, and Kivelson, PRL 62, 82 (1989).

The fermionic case was done Lopez + Fradkin, PRB 44, 5246 (1991). We will follow the Lopez-Fradkin treatment.

Now one can do a mean-field approximation. This could have been done at the Hamiltonian level as well, but the path integral makes it easier to analyze fluctuations around the mean-field solution.

Suppose one is at a filling $\boxed{\nu = \frac{p}{2p+1}}$ (2.1)

$$\boxed{\rho = \frac{p}{2p+1}\frac{1}{2\pi \ell^2} = \frac{\vec{\nabla}\times\vec{a}}{4\pi\hbar} \Rightarrow \vec{\nabla}\times\vec{a} = \frac{\hbar}{\ell^2}\frac{2p}{2p+1}} \quad (2.2)$$

$$\boxed{e\vec{\nabla}\times\vec{A} = eB = \frac{1}{\ell^2}\hbar} \quad (2.3)$$

$$\Rightarrow \boxed{e\vec{\nabla}\times\vec{A} - \vec{\nabla}\times\vec{a} = \frac{\hbar}{\ell^2}\frac{1}{2p+1} = e\vec{\nabla}\times\vec{A}_{eff}} \quad (2.4)$$

$$\Rightarrow \quad B_{eff} = \frac{B_0}{2p+1} \Rightarrow N_\phi^{eff} = \frac{1}{2p+1}N_\phi \Rightarrow \nu_{eff} = p$$

$$\Rightarrow \boxed{B_{eff} = \frac{B_0}{2p+1} \Rightarrow N_\phi^{eff} = \frac{1}{2p+1} N_\phi \Rightarrow \nu_{eff} = p} \quad (2.5)$$

This is exactly the right effective field for the CF's to fill p effective Landau levels.

An interesting and puzzling fact about this construction is that the interactions seem to play no role. In reality, it is the interactions that force the electrons to "attach" fluxes to themselves.

Next, consider the spectrum at the mean-field level. The CF's form a set of CFLL's or ΛL's, spaced by

$$\boxed{\Delta = \frac{e B_{eff}}{m}} \quad (2.6)$$

m here is the band mass. This leads to another puzzle. Suppose we take the limit when the cyclotron energy scale $\hbar\omega_c$ becomes much bigger than the Coulomb interactions

$$\boxed{\hbar\omega_c = \frac{\hbar e B}{m} = \frac{\hbar^2}{m\ell^2} \gg E_c = \frac{e^2}{\kappa\ell} \sim \sqrt{B}} \quad (2.7)$$

κ = dielectric constant

One can take this limit by keeping B

One can take this limit by keeping B fixed and letting $\boxed{m \to 0.}$ (2.8)

In this limit, we expect all the FQH states to exist and have a finite gap determined solely by interactions. The mean-field gap of (2.7) violates this requirement.

This is unfortunately a feature of the CS formulation of CFs: The LLL limit is not physically correct.

Let us continue and proceed to compute various physical quantities.

The most important object is the Hall conductance. To compute it we need to integrate out the fermions and obtain the effective action for the vector potential.

To understand the above statement, let us go back to the IQHE with $\nu = p$.

$$\boxed{\begin{gathered} S_p = \int d\tau\, d^2r \left\{ \bar{\Psi}(\bar{r},\tau)\left(i\hbar\partial_0 + eA_0\right)\Psi(\bar{r},\tau) - \bar{\Psi}\,\frac{\left(-i\hbar\vec{\nabla} + e\vec{A}\right)^2}{2m}\Psi \right\} \\ \vec{\nabla}\times\vec{A} = -B_0\,\hat{e}_z \qquad \rho = \frac{p}{2\pi\ell^2} \qquad (2.9) \end{gathered}}$$

$$\frac{\quad}{2\pi \ell^2}$$

The spectrum is $(n+\frac{1}{2})\hbar\omega_c$. Since p LL's are filled, the spectrum is gapped

What is the effective theory of A_μ when fermions are integrated out?

We can write it down without doing the calculation, because we know that in this state

$$\boxed{\sigma_{xx} = 0 \qquad \sigma_{xy} = p\frac{e^2}{h}} \quad (2.10)$$

The effective action must be gauge invariant, and not have any singularities so we should be able to expand it in a derivative expansion.

$$\boxed{S_{eff}[A] = \int d^2r\, dt \left\{ c_1\, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + c_2 F_{\mu\nu}^2 + \cdots \right\}} \quad (2.11)$$

$c_1, c_2 \cdots$ are constants with the appropriate dimensions.

$$\text{Current} = \boxed{j^\mu = \frac{\delta S_{eff}}{\delta A_\mu} = 2c_1 \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda + \cdots} \quad (2.12)$$

$$\boxed{j_x = 2c_1 (\partial_y A_0 - \partial_0 A_y) = 2c_1 E_y} \quad (2.13)$$

$$\Rightarrow \boxed{c_1 = \sigma_{xy} = \frac{e^2 p}{\quad} = \frac{e^2 p}{\quad}} \quad (2.14)$$

$$\Rightarrow \boxed{C_1 = \frac{\sigma_{xy}}{2} = \frac{e^2 p}{2h} = \frac{e^2 p}{4\pi\hbar}} \quad (2.14)$$

$\Rightarrow$ For p filled LL's of non-interacting electrons

$$\boxed{S_{eff}[A] = \int d^2r\, d\tau\, \frac{pe^2}{4\pi\hbar} A dA} \quad (2.15)$$

Now, we go back to our action, where at MF level the CFs fill p CFLL's. The gauge field they couple to is

$$\boxed{\vec{A} - \frac{\vec{a}}{e}} \quad (2.16)$$

$\Rightarrow$ after integrating out CFs we obtain

$$\boxed{\begin{aligned} S_{eff}[\vec{A},\vec{a}] &= \int d^2r\, d\tau \left\{ \frac{pe^2}{4\pi\hbar}\left(A - \frac{a}{e}\right) d\left(A - \frac{a}{e}\right) + \frac{1}{8\pi\hbar} a da \right\} \\ &= \int d^2r\, d\tau \left\{ \frac{pe^2}{4\pi\hbar} A dA - \frac{pe}{2\pi\hbar} A da + \frac{1}{8\pi\hbar}(2p+1)\, a da \right\} \end{aligned}} \quad (2.17)$$

Since a appears quadratically, integrating it out is the same as plugging in its value for the EOM. This is

$$\frac{pe}{2\pi\hbar} dA = \frac{(2p+1)}{4\pi\hbar} da \Rightarrow \boxed{a = \frac{2pe}{2p+1} A} \quad (2.18)$$

Plugging this value into (2.17), we get the coefficient of AdA in S_{eff} to be

$$\frac{pe^2}{4\pi\hbar} - \frac{2p^2e^2}{2\pi\hbar(2p+1)} + \frac{4p^2e^2}{8\pi\hbar(2p+1)} = \frac{pe^2}{4\pi\hbar}\left(1 - \frac{2p}{2p+1}\right)$$

$$= \boxed{\frac{p}{2p+1}\,\frac{e^2}{4\pi\hbar}} \qquad (2.19)$$

Ultimately, after integrating out $\psi, \bar{\psi}$ and a_μ we obtain

$$\boxed{S_{eff}^{\nu = p/2p+1} = \int d^2r\, d\tau\, \frac{p}{2p+1}\,\frac{e^2}{4\pi\hbar}\, A\, dA} \qquad (2.20)$$

This is the correct Hall conductance, which is encouraging!

One can go further, and find all the response functions. Instead, we will go directly to $\nu = \frac{1}{2}$.

$\nu = \frac{1}{2}$: Composite Fermion Fermi Liquid

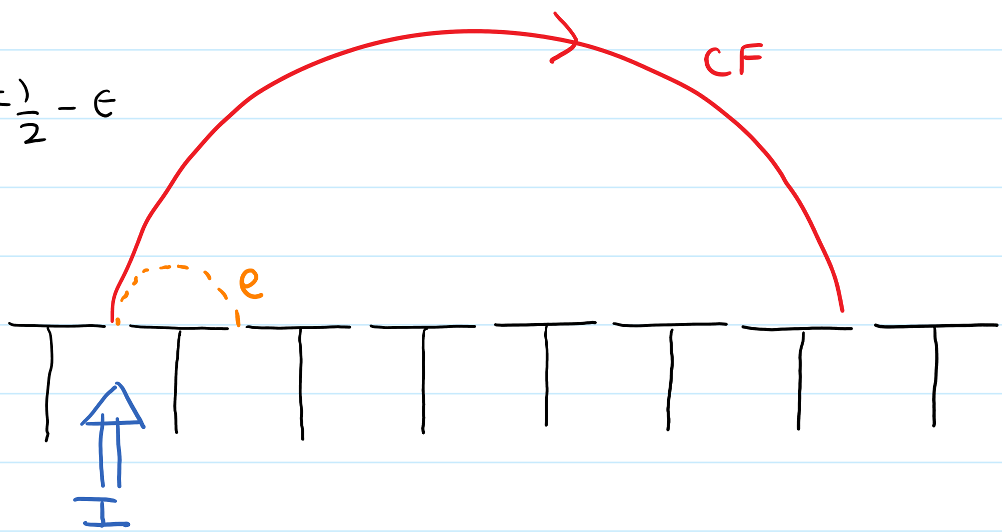
The sequence $\nu_p = \frac{p}{2p+1}$ are known as the principal fractions, because they are the most prominent plateaus. In Jain's picture this is easily understood as the $\nu_{CF} = p$ IQHE of CFs.

As p increases, the gaps get smaller. One can take the limit $p \to \infty$, which gives $\nu = \frac{1}{2}$. In Jain's picture, at $\nu = \frac{1}{2}$ the statistical flux quanta exactly cancel the external field leaving no average field. Since the CF's are fermions, they should form a gapless Fermi surface.

One may wonder whether CF's need a gap in order to exist. Today we believe the answer is no. CFs form at some intermediate energy, larger than any FQH gap. They continue to exist even when $\nu = \frac{1}{2}$. This is demonstrated by various experiments, the most striking of which is the detection of the CF's cyclotron radius.

Smet et al PRL 77, 2272 (1996)

$\nu = \frac{1}{2} - \epsilon$



The idea is to be slightly away from $\nu = \frac{1}{2}$, where there is no gap (at the temperature of the experiment).

The electrons see an enormous field and thus have a small cyclotron radius. The CF's see the effective field, which is very small, and thus have a large cyclotron radius. The experiment injects current into the leftmost port and looks for an output current at many places to the right. Most of the current is found to flow along the CF trajectory in red.

Theoretically, a big step was taken by Halperin, Lee, and Read PRB 47, 7312 (1993). They took the CFFL as the starting point, just as the usual FL is the starting point in talking about IQHE. However, there are some crucial differences, because the CFFL is coupled to a gauge field $\vec{a}$, which gets its own dynamics when the fermions are integrated out. This has many consequences, including an overdamped mode at very low energy

$$\boxed{\omega \simeq -i q^3 v(q)} \quad (3.1)$$

where $v(q)$ is the Fourier transform of the electron-electron interaction.

In parallel, Read, Semiconduct. Sci & Technol. 9, 1859 (1994) and Rezayi & Read PRL 72, 900 (1994) examined the wavefunctions at the $p \to \infty$ limit of the Jain sequence

$$\boxed{\Psi_{\frac{1}{2}}(\{\vec{r}_i\}) = \mathcal{P}_{LLL}\left\{\left[\prod_{i<j}(z_i - z_j)^2\right] \mathrm{Det}\left[e^{i\vec{k}_i \cdot \vec{r}_j}\right]\right\} e^{-\sum_k \frac{|z_k|^2}{4\ell^2}} \quad (3.2)}$$

and found excellent agreement with a CFFL.

Appendix A: $\{\Psi_{CF}(\bar r_1), \Psi^+_{CF}(\bar r_2)\} = \delta^2(\bar r_1 - \bar r_2)$

$$\boxed{\begin{array}{ll} \Psi^+_{CF}(\bar r) = \Psi^+_e(\bar r)\, e^{-2i Q(\bar r)} & \Psi_{CF}(\bar r) = e^{2iQ(\bar r)}\, \Psi_e(\bar r) \quad \text{A1} \\ Q(\bar r) = \int d^2 r' \arg(\bar r - \bar r')\, \rho(\bar r') & \rho_e(\bar r) = \rho_{CF}(\bar r) = \rho(\bar r) \end{array}}$$

Since $\boxed{\{\Psi_e(\bar r), \Psi_e^+(\bar r')\} = \delta^2(\bar r - \bar r')}$ A2

$$\boxed{\begin{array}{l} [\Psi_e(\bar r), \rho(\bar r')] = \delta^2(\bar r - \bar r')\, \Psi_e(\bar r) \\ [\Psi_e^+(\bar r), \rho(\bar r')] = -\delta^2(\bar r - \bar r')\, \Psi_e^+(\bar r) \end{array}} \quad \text{A3}$$

$$\Rightarrow \boxed{\begin{array}{l} [\Psi_e(\bar r_1), Q(\bar r_2)] = \int d^2 r' \arg(\bar r_2 - \bar r')\, \delta^2(\bar r_1 - \bar r')\, \Psi_e(\bar r_1) \\ \qquad = \arg(\bar r_2 - \bar r_1)\, \Psi_e(\bar r_1) \quad \text{A4} \\ [\Psi_e^+(\bar r_1), Q(\bar r_2)] = -\arg(\bar r_2 - \bar r_1)\, \Psi_e^+(\bar r_1) \end{array}}$$

$$\Rightarrow \boxed{\begin{aligned} [\Psi_e(\bar r_1), Q(\bar r_2)] &= \int d^2r' \arg(\bar r_2 - \bar r')\, \delta^2(\bar r_1 - \bar r')\, \Psi_e(\bar r_1) \\ &= \arg(\bar r_2 - \bar r_1)\, \Psi_e(\bar r_1) \qquad \text{A4} \\ [\Psi_e^+(\bar r_1), Q(\bar r_2)] &= -\arg(\bar r_2 - \bar r_1)\, \Psi_e^+(\bar r_1) \end{aligned}}$$

$\arg(\bar r_1 - \bar r_2)$ doesn't make sense if $\vec r_1 = \bar r_2$.

We can make sense of $\vec r_1 = \vec r_2$ by smearing out ρ in a symmetric way, obtaining

$$\boxed{[\Psi_e(\bar r), Q(\bar r)] = 0} \qquad \text{A5}$$

Consider

$$\boxed{\begin{aligned} \{\Psi_{CF}(\bar r_1), \Psi_{CF}^+(\bar r_2)\} &= e^{2iQ(\bar r_1)}\, \Psi_e(\bar r_1)\Psi_e^+(\bar r_2) e^{-2iQ(\bar r_2)} \qquad \text{A6} \\ &\quad + \Psi_e^+(\bar r_2) e^{-2iQ(\bar r_2)} e^{2iQ(\bar r_1)}\, \Psi_e(\bar r_1) \end{aligned}}$$

We need to exchange $e^{2iQ(\bar r)}$ and $\Psi_e^+(\bar r')$

Consider

$$\boxed{F(\phi) = e^{-i\phi Q(\vec r_1)}\, \Psi_e^+(\bar r_2)\, e^{i\phi Q(\vec r_1)}} \qquad \text{A7}$$

$$\boxed{\frac{dF}{d\phi} = i\, e^{-i\phi Q(\vec r_1)} [\Psi_e^+(\bar r_2), Q(\vec r_1)]\, e^{i\phi Q(\vec r_1)} = -i\arg(\vec r_1 - \bar r_2) F(\phi)} \qquad \text{A8}$$

$$\frac{dF}{d\phi} = i\, e^{\cdots} \left[\Psi_e(r_2), Q(r_1)\right] e^{\cdots} = -i \arg(\vec{r}_1 - \vec{r}_2)\, F(\phi)$$ A8

$$\Rightarrow \boxed{F(\phi) = e^{-i\phi \arg(\vec{r}_1 - \vec{r}_2)} F(0)}$$ A9

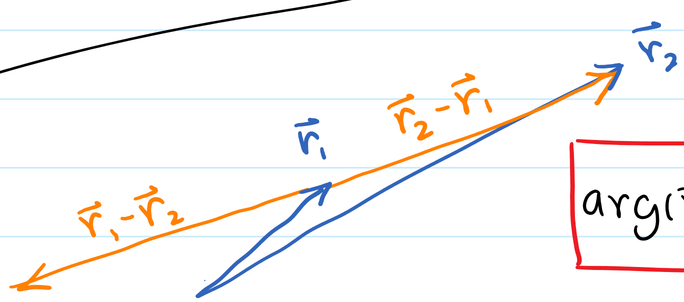
$$\Rightarrow \boxed{\Psi_e^+(\vec{r}_2)\, e^{2iQ(\vec{r}_1)} = e^{-2i\arg(\vec{r}_1-\vec{r}_2)}\, e^{2iQ(\vec{r}_1)}\, \Psi_e^+(\vec{r}_2)}$$ A10

Similarly $$\boxed{e^{-2iQ(\vec{r}_2)}\, \Psi_e(\vec{r}_1) = e^{2i\arg(\vec{r}_2-\vec{r}_1)}\, \Psi_e(\vec{r}_1)\, e^{-2iQ(\vec{r}_2)}}$$ A11

$$\boxed{\Psi_e^+(\vec{r}_2)\, e^{-2iQ(\vec{r}_2)}\, e^{2iQ(\vec{r}_1)}\, \Psi_e(\vec{r}_1) = e^{2i(\arg(\vec{r}_2-\vec{r}_1) - \arg(\vec{r}_1-\vec{r}_2))}\, e^{2iQ(\vec{r}_1)}\, \Psi_e^+(\vec{r}_2)\, \Psi_e(\vec{r}_1)\, e^{-2iQ(\vec{r}_2)}}$$ A12

So A6 becomes

$$\boxed{e^{2iQ(\vec{r}_1)} \underbrace{\left(\Psi_e(\vec{r}_1)\, \Psi_e^+(\vec{r}_2) + e^{2i(\arg(\vec{r}_2-\vec{r}_1) - \arg(\vec{r}_1-\vec{r}_2))}\, \Psi_e^+(\vec{r}_2)\, \Psi_e(\vec{r}_1) \right)} e^{-2iQ(\vec{r}_2)}}$$ A13



$$\boxed{\arg(\vec{r}_1 - \vec{r}_2) = \arg(\vec{r}_2 - \vec{r}_1) \pm \pi}$$ A14

$$2i(\arg(\vec{r}_2 - \vec{r}_1) - \arg(\vec{r}_1 - \vec{r}_2)) = \pm 2\pi i$$

So the ( ) becomes

$$\Psi_e(\vec{r}_1)\Psi_e^{+}(\vec{r}_2) + \Psi_e^{+}(\vec{r}_2)\Psi_e(\vec{r}_1) = \delta^2(\vec{r}_1 - \vec{r}_2)$$

$\Rightarrow$ Finally

$$\boxed{\{\Psi_{CF}(\vec{r}_1), \Psi_{CF}^{+}(\vec{r}_2)\} = \delta^2(\vec{r}_1 - \vec{r}_2)} \quad \text{A15}$$

Note how crucial the 2 in $e^{2iQ(\bar{r})}$ was. This 2 is the number of flux quanta attached to each electron.

In general, if we attach s flux quanta to form a composite particle

$$\Psi_{CP}^{+}(\vec{r}) = \Psi_e^{+}(\bar{r})\, e^{-isQ(\bar{r})} \quad \text{A16}$$

then $\Psi_{cp}(\bar{r})\Psi_{cp}^{+}(\bar{r}') + (-1)^s \Psi_{cp}^{+}(\bar{r}')\Psi_{cp}(\bar{r}) = \delta^2(\bar{r} - \bar{r}')$

S = even integer gives CF's while s = odd gives CB's.