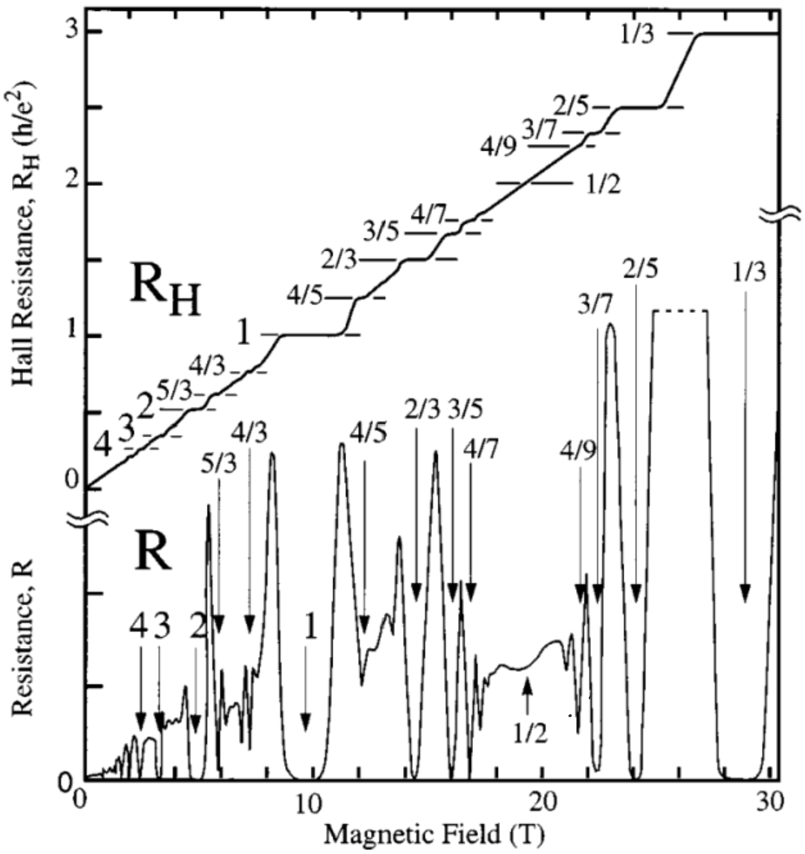
COMQUI Lecture II, Part 1: FQHE Wavefunctions

Thursday, September 21, 2023 12:04 PM

1. Laughlin fractions.

2. Jain's Composite Fermions.



From Störmer's Nobel lecture RMP 71, 875 (1999)

1. Laughlin fractions

Volume 48, Number 22 PHYSICAL REVIEW LETTERS 31 May 1982

Two-Dimensional Magnetotransport in the Extreme Quantum Limit

D. C. Tsui,[a],[b] H. L. Stormer,[a] and A. C. Gossard
Bell Laboratories, Murray Hill, New Jersey 07974
(Received 5 March 1982)

A quantized Hall plateau of $\rho_{xy} = 3h/e^2$, accompanied by a minimum in ρ_{xx}, was observed at $T < 5$ K in magnetotransport of high-mobility, two-dimensional electrons, when the lowest-energy, spin-polarized Landau level is $\frac{1}{3}$ filled. The formation of a Wigner solid or charge-density-wave state with triangular symmetry is suggested as a possible explanation.

Volume 50, Number 18 PHYSICAL REVIEW LETTERS 2 May 1983

Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations

R. B. Laughlin
Lawrence Livermore National Laboratory, University of California, Livermore, California 94550
(Received 22 February 1983)

This Letter presents variational ground-state and excited-state wave functions which describe the condensation of a two-dimensional electron gas into a new state of matter.

The first fractional quantum Hall state was $\nu = \frac{1}{3}$, observed by Tsui, Störmer, & Gossard in 1982. Within a year Laughlin had essentially solved the problem.

Recall that the filling factor is defined as

$$\boxed{\nu = \frac{N_e}{N_\phi}} \quad (1.1)$$

N_e = # of electrons
N_ϕ = degeneracy of a LL.

For any $\nu < 1$, there are an enormous number of ways of arranging N_e electrons in N_ϕ levels, if one assumes no disorder, as we will levels, if one assumes no disorder, as we will assume most of the time.

For example, assuming spin-polarized electrons at $\nu = \frac{1}{3}$, the number of many-body states is

$$\boxed{\frac{N_\phi!}{\left(\frac{N_\phi}{3}\right)!\,\left(\frac{2N_\phi}{3}\right)!}} \tag{1.2}$$

At the non-interacting level these many-body states are all degenerate. However, when the pairwise Coulomb interaction is turned on, a miracle occurs, and the system chooses a unique ground state. The easiest way to express this state is to write down the wavefunction, as Laughlin did.

Before we do this, a brief digression into symmetric gauge is needed.

Recall $$\boxed{\vec{B} = \vec{\nabla} \times \vec{A} = -B\,\hat{e}_z} \tag{1.3}$$

In symmetric gauge $$\boxed{\vec{A} = \frac{B}{2}\left(y\,\hat{e}_x - x\,\hat{e}_y\right)} \tag{1.4}$$

The cyclotron operators are

$$\eta_x = -\frac{\ell^2}{\hbar}\Pi_y = -\frac{\ell^2}{\hbar}\left(p_y - \frac{\hbar x}{2\ell^2}\right) = \frac{x}{2} - \frac{\ell^2}{\hbar}p_y$$

$$\eta_x = -\frac{\ell^2}{\hbar}\Pi_y = -\frac{\ell^2}{\hbar}\left(P_y - \frac{\hbar x}{2\ell^2}\right) = \frac{x}{2} - \frac{\ell^2}{\hbar}P_y$$

$$\eta_y = \frac{\ell^2}{\hbar}\Pi_x = \frac{y}{2} + \frac{\ell^2}{\hbar}P_x \qquad (1.5)$$

The guiding center operators are

$$R_x = x - \eta_x = \frac{x}{2} + \frac{\ell^2}{\hbar}P_y \qquad R_y = \frac{y}{2} - \frac{\ell^2}{\hbar}P_x$$

$$[R_x, R_y] = -i\ell^2 \qquad (1.6)$$

We can construct ladder operators

$$b = \frac{R_x - iR_y}{\ell\sqrt{2}} \qquad b^\dagger = \frac{R_x + iR_y}{\ell\sqrt{2}}$$

$$[b, b^\dagger] = 1 \qquad (1.7)$$

One can now construct the states

$$|m\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}}\,| \qquad (1.8) \qquad b|0\rangle = 0$$

Expressing the operators in differential form

$$b = \frac{1}{\ell\sqrt{2}}\left(\frac{x-iy}{2} + \frac{\ell^2}{\hbar}(-i\hbar)(\partial_y + i\partial_x)\right)$$

$$x + iy = z \qquad x - iy = \bar{z}$$

$$x + iy = z \qquad\qquad x - iy = \bar{z}$$

$$\frac{\partial}{\partial z} \equiv \partial = \frac{1}{2}\left(\partial_x - i\partial_y\right) \qquad\qquad \frac{\partial}{\partial \bar{z}} \equiv \bar{\partial} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$$

So $$\boxed{b = \frac{1}{\ell\sqrt{2}}\left(\frac{\bar{z}}{2} + \ell^2 \partial\right) \quad \text{and} \quad b^{+} = \frac{1}{\ell\sqrt{2}}\left(\frac{z}{2} - \ell^2 \bar{\partial}\right)} \qquad (1.9)$$

$$\langle x, y | 0 \rangle = \psi_0(x,y) \Rightarrow \frac{1}{\ell\sqrt{2}}\left(\frac{\bar{z}}{2} + \ell^2 \partial\right)\psi_0 = 0$$

$$\Rightarrow \psi_0(x,y) = \frac{e^{-\frac{|z|^2}{4\ell^2}}}{\ell\sqrt{2\pi}}. \qquad (1.10)$$

$$\langle x, y | m \rangle = \frac{(b^{+})^m}{\sqrt{m!}} \psi_0(x,y) = \frac{z^m}{\sqrt{2^m m!}} \frac{1}{\ell^m} \frac{e^{-\frac{|z|^2}{4\ell^2}}}{\ell\sqrt{2\pi}} \qquad (1.11)$$

Other than the ubiquitous $e^{-|z|^2/4\ell^2}$ factor, in the $n=0$ LL, also called the lowest Landau level, or LLL, the wavefunctions are analytic functions of z in symmetric gauge.

The size of the wavef^n $\psi_m(x,y)$ can be obtained from

$$\langle r^2 \rangle_m = \int_0^\infty 2\pi r\,dr\, r^2 |\psi_m|^2 = \frac{1}{m!}\int_0^\infty \frac{dr\, r}{\ell^2} \frac{r^{2m+2}}{2^m \ell^{2m}} e^{-\frac{r^2}{2\ell^2}}$$

$$= \quad \ell^2 \qquad (1.12)$$

$$= 2m\ell^2 \qquad (1.12)$$

So the size of Ψ_m is $\boxed{\ell\sqrt{2m}}$ (1.13)

Now we are ready to write the Laughlin wavef^n for $\nu = \frac{1}{3}$

(1.14)

$$\boxed{\Psi_{\frac{1}{3}}(\{z_i\}) = \left\{\prod_{i<j}(z_i - z_j)^3\right\} e^{-\sum_k \frac{|z_k|^2}{4\ell^2}}}$$

This is a superposition of products of 1-body wavef^{n}s all living in the LLL. Also the power 3 means it is antisymmetric w.r.t exchanging two electrons, as it should be.

A more basic question is, how do we know this corresponds to a filling of $\nu = \frac{1}{3}$?

To understand this we go to the disk geometry. Let the system have N_e particles. The highest power of any particular coordinate, say z_1, in $\Psi_{\frac{1}{3}}$ is

(1.15)

$$\boxed{z_1^{3(N_e-1)} \Rightarrow m_{max} = 3(N_e - 1)}$$

The radius of the droplet is

$$\boxed{r_{max} = \ell\sqrt{2m_{max}} \approx \ell\sqrt{6N_e}} \qquad (1.16)$$

$$\boxed{r_{max} = \ell\sqrt{2m_{max}} \approx \ell\sqrt{6N_e}} \quad (1.16)$$

$$\text{Area of droplet} = 6\pi\ell^2 N_e$$

$$\Rightarrow \boxed{N_\phi = \frac{\text{Area}}{2\pi\ell^2} = 3N_e \;\; (1.17) \quad \Rightarrow \quad \nu = \frac{1}{3}}$$

So far we know that this state has $\nu = \frac{1}{3}$ and lives in the lowest LL. Laughlin was able to compute the density correlations in the state and establish that it is a liquid, that is, it does not break translation or rotation symmetry in the thermodynamic limit.

He also exactly diagonalized systems with a small number of particles and showed that the exact ground state with the Coulomb interaction has excellent overlap with the Laughlin wavefunction.

In addition, he showed the important fact that the state is incompressible, that is, there is a gap to all excitations.

Last, but not least, the elementary charged excitations have fractional charge $\pm\frac{e}{3}$.

Consider an excitation on top of $\Psi_{\frac{1}{3}}$. The

Consider an excitation on top of $\Psi_{\frac{1}{3}}$. The simplest excitation is a quasihole at z_0

$$\boxed{\Psi_{\frac{1}{3},qh} = \left\{ \prod_{i=1}^{N_e} (z_i - z_0) \right\} \Psi_{\frac{1}{3}}} \qquad (1.18)$$

This corresponds to adiabatically inserting Φ_0 through an infinitely thin solenoid at z_0.

Exercise: Assuming that $\sigma_{xx} = 0$ and $\sigma_{xy} = \frac{1}{3}\frac{e^2}{h}$

show that the qh has charge $+\frac{e}{3}$

The Laughlin wavefunction can be extended to all the "Laughlin fractions" $\nu = \frac{1}{2n+1}$

2. Jain's Composite Fermions

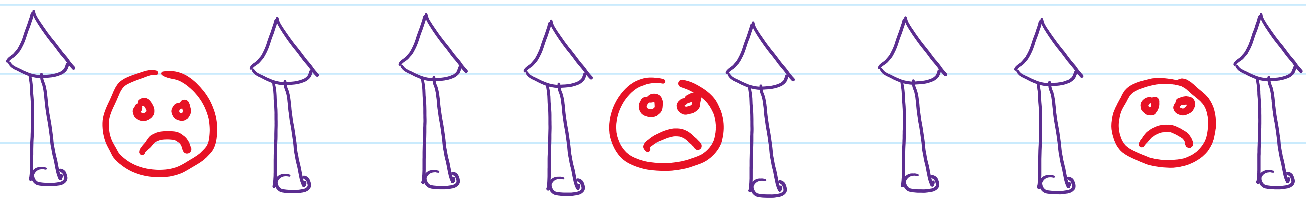
Experimentally, there are many plateaus seen beyond $\frac{1}{2n+1}$. A particularly prominent sequence

is $\frac{n}{2n+1} \rightarrow \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \cdots$

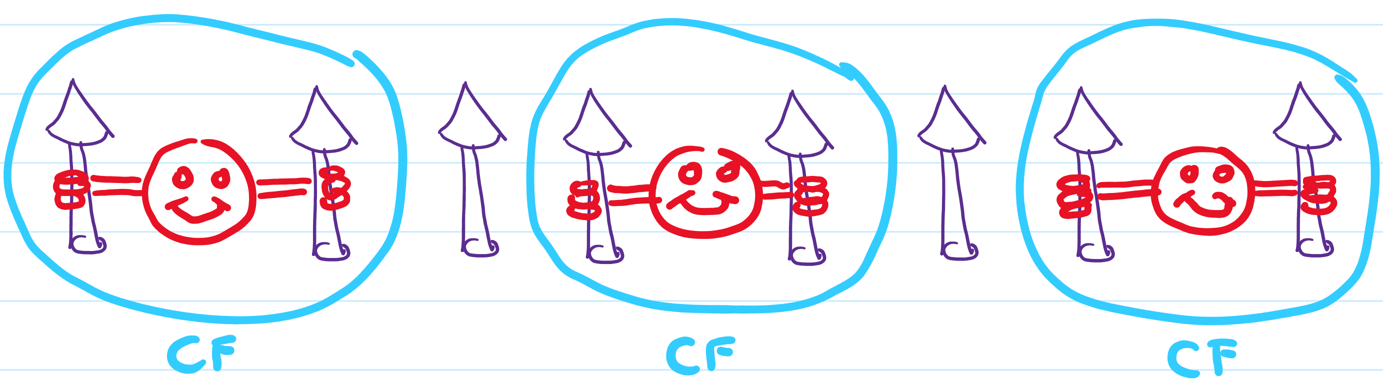
Several schemes were proposed to explain this so-called principal sequence. We will focus on Jain's construction of Composite Fermions (CFs) in 1989.

The key idea is flux attachment, which had been explored for $\nu = \frac{1}{3}$ by Girvin & MacDonald in 1987, and Zhang, Hansson & Kivelson in 1989 in the context of Composite Bosons.

Here is a cartoon due to Kwon Park (appearing in Jain's "Composite Fermions"). Say $\nu = \frac{1}{3}$. There are $3\phi_0$ per electron



The electrons are confused and unhappy because there is a lot of degeneracy. They decide to each grab two flux quanta.



They are still fermions because $2\phi_0$ are attached. But now, there is one flux quantum per CF, so they completely fill a CFLL.

Now they are happy!

$$N_{\phi,\text{eff}} = N_\phi - 2N_e \qquad (\text{each } e^- \text{ grabs } 2\phi_0)$$

$$\nu_{\text{eff}} = \frac{N_e}{N_\phi - 2N_e} = \frac{\nu}{1-2\nu}$$

(2.1)

$$\text{or} \quad \nu = \frac{\nu_{\text{eff}}}{1+2\nu_{\text{eff}}}$$

The natural incompressible states for CFs are when $\nu_{\text{eff}} = 1, 2, 3, \cdots$ all the integer QH states.

$$\nu_{\text{eff}} = 1 \Rightarrow \nu = \frac{1}{3}$$

(2.2)

$$\nu_{\text{eff}} = 2 \Rightarrow \nu = \frac{2}{5}$$

$$\nu_{\text{eff}} = 3 \Rightarrow \nu = \frac{3}{7}$$

FQHE of e^-

= IQHE of CFs

Exactly the principal fractions!

One can attach $4\phi_0$ to each e^- to get

$$\nu = \frac{\nu_{\text{eff}}}{1+4\nu_{\text{eff}}} = \frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \cdots$$

(2.3)

One can also explain $1-\frac{1}{3}$, $1-\frac{2}{5}$, $1-\frac{3}{7}$ etc

One can also explain $1-\frac{1}{3}$, $1-\frac{2}{5}$, $1-\frac{3}{7}$ etc

as hole CF states in a sea of $\nu=1$.

CFs provide a natural explanation for a vast number of observed fractions.

To flesh out this idea Jain constructed explicit wavefunctions. The basic relation is

$$\boxed{\Psi^e_\nu(\{\bar{r}_i\}) = \left\{\prod_{i<j}(z_i - z_j)^2\right\} \psi^{CF}_{\nu_{eff}}(\{\bar{r}_i\})} \quad (2\cdot4)$$

For $\nu_{eff} > 1$ the wavefunctions involve $\bar{z}_i$ as well as z_i, and need to be projected to the LLL.

Jain and co-workers carried out exact diagonalizations of system at many principal fractions and found excellent overlap with the Jain wavefunctions.

An interesting state arises at $\nu=\frac{1}{2}$, because

$$\boxed{N_{\phi, eff} = N_\phi - 2N_e = 0} \quad (2\cdot5)$$

The CFs see no B field on average, and thus form a Fermi Sea. This wave function thus form a Fermi sea. This wave function

$$\boxed{\Psi_{\frac{1}{2}} = \mathcal{P}_{LLL} \left\{ \prod_{i<j} (z_i - z_j)^2 \right\} \mathrm{Det}\left[e^{i\vec{k}_i \cdot \vec{r}_j} \right]} \quad (2.6)$$

was proposed by Read & Rezayi in 1994.

Another interesting wavefunction at $\nu = \frac{1}{2}$ is the Moore-Read state, which is a good candidate for $\nu = \frac{5}{2}$.

$$\boxed{\Psi_{MR} = \mathrm{Pf}\left\{ \frac{1}{z_i - z_j} \right\} \left\{ \prod_{k<l} (z_k - z_l)^2 \right\} \times \text{gaussian.}} \quad (2.7)$$

This state is a p+ip supercoductor living on the CF-Fermi liquid.