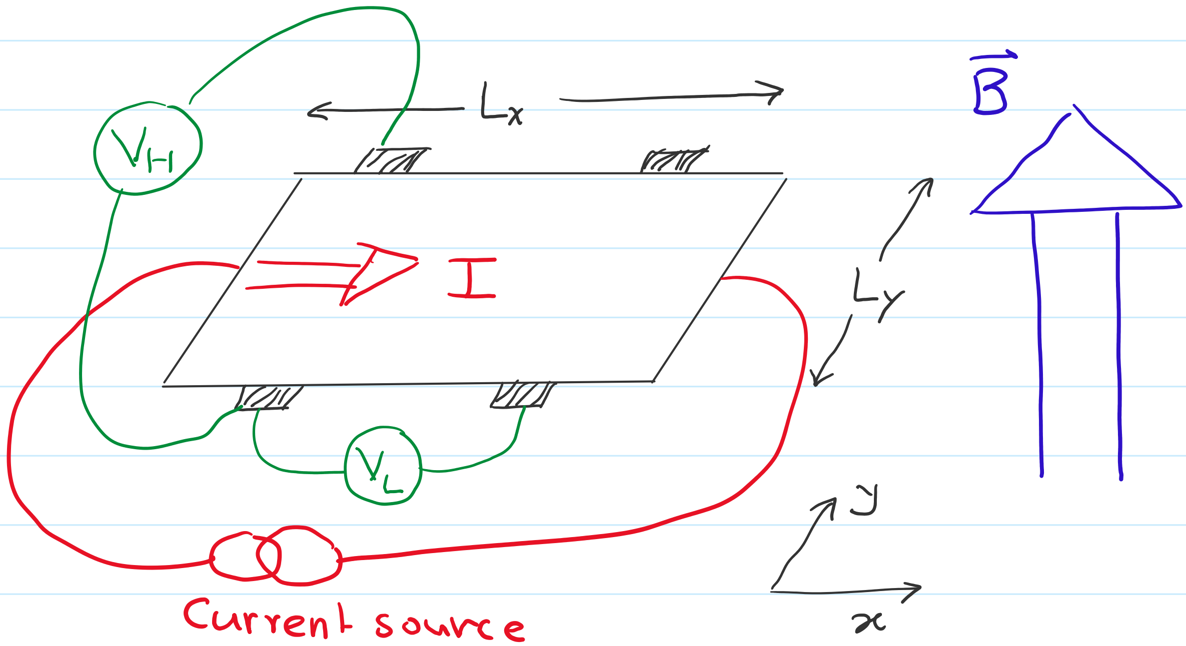
1. The Classical Hall Effect.

2. Phenomenology of the Quantum Hall Effects. Landau levels.

1. The Classical Hall Effect

The effect was discovered by Edwin Hall in 1879. Consider a two-dimensional conducting material subject to a $\perp$ B field.



A current I is driven through the material in the x-direction. Contacts are used to measure the longitudinal voltage V_L and the transverse, or Hall voltage V_H.

In solids, either electrons (charge $-e$) or holes (charge $+e$) can carry current.

Suppose electrons are the charge carriers. Then

$$\boxed{I\hat{e}_x = -en\vec{v}\,L_y} \quad (1.1)$$

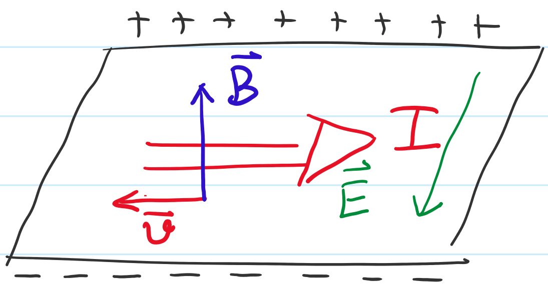
n = area density of electrons
$\vec{v}$ = drift velocity

$$\Rightarrow \boxed{\vec{v} = -\hat{e}_x \frac{I}{enL_y}} \quad (1.2)$$

The Lorentz force on the electron is

$$\boxed{\vec{F} = (-e)\,\vec{v} \times \vec{B} \qquad \vec{B} = B\hat{e}_z \qquad \vec{F} = -\hat{e}_y \frac{IB}{nL_y} \qquad (1.3)}$$

Electrons will be accelerated towards the lower edge of the sample. They pile up there and produce an electric field in the $-y$ direction



In the steady state, the force due to this $\vec{E}$ field will exactly cancel the Lorentz force, and the electrons travel straight through.

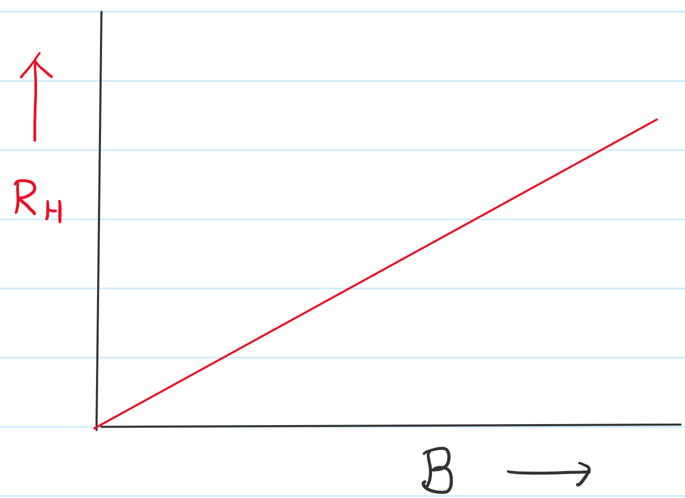
$$\Rightarrow \boxed{|E_y| = |v_x B_z| = \frac{IB}{enL_y} \qquad V_H = |E_y|\,L_y = \frac{IB}{en}} \qquad (1.4)$$

The Hall Resistance R_H is defined as

$$\boxed{R_H = \frac{V_H}{I} = \frac{B}{en}} \qquad (1.5)$$

Exercise: Show that if the charge carriers are holes, the sign of V_H changes.

If one plots R_H vs B the plot looks like this

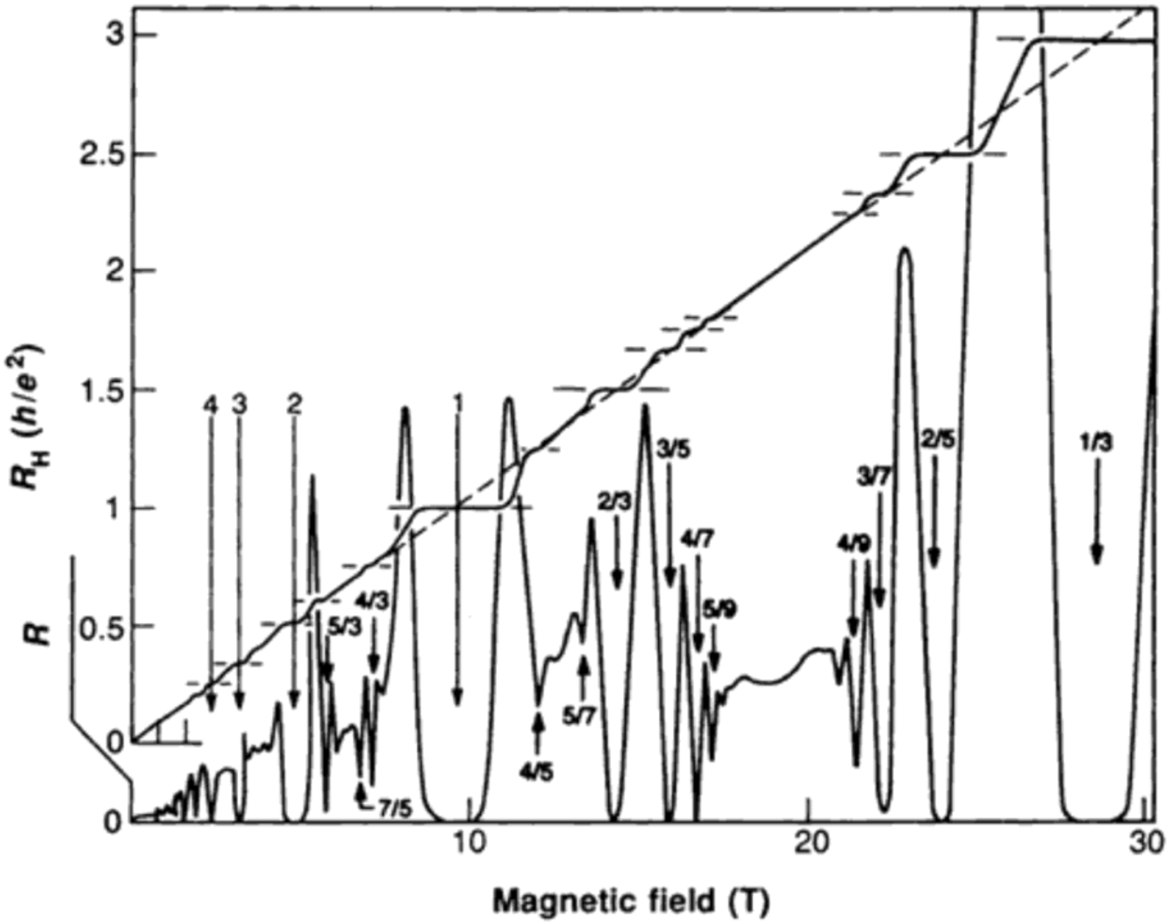


The sign of V_H tells us about the sign of the charge carriers in the system, while the slope of R_H vs B allows us to infer the density of charge carriers.

2. Phenomenology of the QHE

The integer QHE was discovered by von Klitzing, Dorda and Pepper in 1980, for which von Klitzing got the Nobel Prize in 1985.

Dorda and Pepper in 1980, for which von Klitzing got the Nobel Prize in 1985.



The above is from the press release for the 1998 Nobel Prize, awarded to Horst Störmer, Dan Tsui, and Bob Laughlin, for the fractional QHE.

R_H is the almost straight line, punctuated by plateaus. On the plateaus, R_H is constant to an amazing precision. Furthermore, the values of R_H on the plateaus are rational multiples of the quantum unit Ω resistance the quantum unit of resistance

$$\boxed{R_Q = \frac{h}{e^2}} \quad (2.1) \qquad h = \text{Planck's constant.}$$

The oscillating line which sometimes touches the x-axis is the longitudinal resistance

$$\boxed{R_L \equiv R_{xx} = \frac{V_L}{I}} \quad (2.2)$$

The key observation is that whenever $R_H \equiv R_{xy}$ is on a plateau, $R_{xx} = 0$ to very high precision.

The integer QHE, or IQHE, was understood completely by 1983 or so. This entire lecture will concentrate on the IQHE.

From the appearance of h in R_H we know that quantum mechanics is important to the QHE. So let's start with the non-interacting quantum Hamiltonian for electrons subject to a $\perp$ $\vec{B}$ field in two dimensions.

Recall that in the Hamiltonian formalism, a particle of charge q couples to E&M via the scalar and vector potentials

$$H = (\vec{p} - q\vec{A})^2 + q\Phi \quad (2.3)$$

$$\boxed{H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}} \quad (2.3)$$

Here there is only a $\vec{B}$ field so $\Phi = 0$, and $q = -e$ (electrons). For later convenience we take

$$\boxed{\vec{B} = -B\,\hat{e}_z} \quad (2.4)$$

As you all know, $\vec{E}$ and $\vec{B}$ do not uniquely determine $\vec{A}, \Phi$, because of the redundancy of gauge transformations.

$$\boxed{\vec{A}' = \vec{A} - \vec{\nabla}\chi(\vec{r},t) \qquad \Phi' = \Phi + \frac{\partial \chi}{\partial t}} \quad (2.5)$$

$\Phi', \vec{A}'$ describe the same fields as $\Phi, \vec{A}$.

For now, let us work in Landau gauge.

$$\boxed{\vec{A} = -Bx\,\hat{e}_y \qquad \vec{\nabla} \times \vec{A} = -B\,\hat{e}_z} \quad (2.6)$$

Going over to QM we promote $\vec{x}, \vec{p}$ to operators

$$\boxed{[x, p_x] = i\hbar = [y, p_y] \qquad [x, p_y] = 0 = [y, p_x]}$$

$$[x, y] = [p_x, p_y] = 0 \quad (2.7)$$

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2m}(p_y - eBx)^2 \quad (2.8)$$

We want to find the eigenstates & eigenvalues of H

$$H|\alpha\rangle = E_\alpha|\alpha\rangle \quad (2.9) \qquad \psi_\alpha(\vec{r}) = \langle \vec{r}|\alpha\rangle$$

In Landau gauge p_y commutes with H. Thus, we can simultaneously diagonalize H and p_y.

$$\Rightarrow \quad \psi_{nk}(x,y) = e^{iky}\,\Phi_{nk}(x) \quad (2.10)$$

Here n is another quantum number, as we will see soon.

$$\Rightarrow \quad H\psi_{nk}(x,y) = \frac{e^{iky}}{2m}\left(-\hbar^2\frac{d^2}{dx^2} + (\hbar k - eBx)^2\right)\Phi_{nk}(x)$$

$$= E_{nk}\,\psi_{nk}(x,y) \quad (2.11)$$

We recognize this as the Harmonic oscillator. The magnetic field defines a length scale called the magnetic length ℓ

$$\ell = \sqrt{\frac{\hbar}{eB}} \quad (2.12)$$

$$\ell = \sqrt{\frac{\hbar}{eB}} \qquad (2.12)$$

$$\Rightarrow \quad \frac{\hbar^2}{2m\ell^2}\left\{-\ell^2\frac{d^2}{dx^2} + (x-k\ell^2)^2\right\}\Phi_{nk}(x) = E_{nk}\,\Phi_{nk}(x)$$

$\boxed{X_k = k\ell^2}$ (2.13) defines the "guiding center" of the Harmonic oscillator potential given k.

The eigenvalues are

$$\boxed{E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega_c \qquad \omega_c = \frac{eB}{m}. \qquad n = 0,1,\cdots\infty} \qquad (2.14)$$

Note that E_n is independent of k, so all k states are degenerate.

To define the eigenstates in a mathematically well-defined way, let us impose periodic boundary conditions in the y-direction.

$$\psi(x, y+L_y) = \psi(x,y) \qquad (2.15)$$

$$\Rightarrow \quad k = \frac{2\pi j}{L_y} \qquad j \in \mathbb{Z}$$

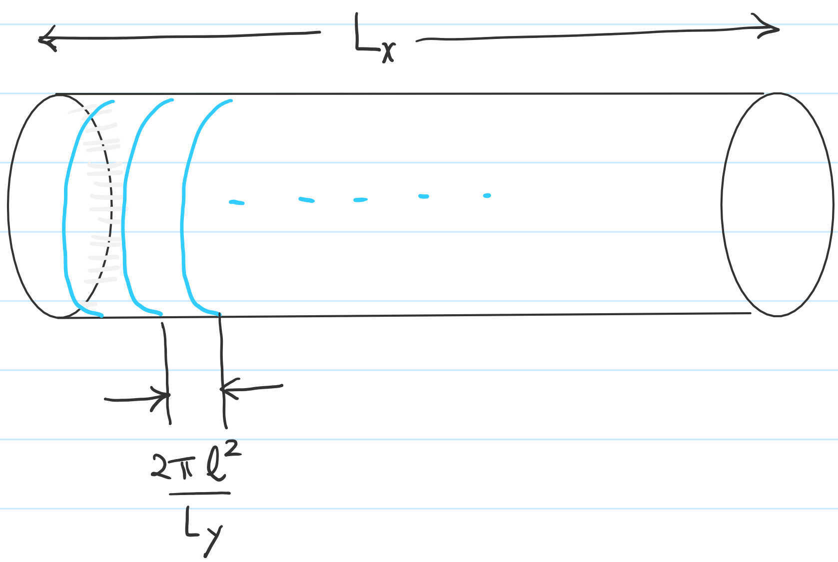
Then the properly normalized eigenstates are

$$\psi_{nk}(x,y) = e^{iky}\; e^{-\frac{(x-k\ell^2)^2}{2\ell^2}}\; H_n\!\left(\frac{x-k\ell^2}{\ell}\right) \qquad (2.16)$$

$$\boxed{\psi_{nk}(x,y) = \frac{e^{iky}}{\sqrt{L_y}}\;\frac{e^{-\frac{(x-kl^2)}{2l^2}}}{\sqrt{l\sqrt{\pi}}}\;\frac{H_n\left(\frac{x-kl^2}{l}\right)}{\sqrt{2^n\, n!}}} \qquad (2.16)$$

H_n are Hermite polynomials.

We noted before that the energy levels, called Landau levels, were highly degenerate. What exactly is the degeneracy?



L_y circumference.

The k's are spaced by $\frac{2\pi}{L_y}$, which means the guiding center positions are spaced by $\boxed{\Delta X = \frac{2\pi l^2}{L_y}}$ (2.17)

Within the sample of length L_x, there are

$$\boxed{N_\phi = \frac{L_x}{\Delta X} = \frac{L_x L_y}{2\pi l^2} = \frac{B L_x L_y}{h/e} = \frac{B L_x L_y}{\phi_0}} \qquad (2.18)$$

$\boxed{\phi_0 = \frac{h}{e}}$ is the flux quantum. An infinitely

$$\boxed{\Phi_0 = \frac{h}{e}} \quad (2.19)$$ is the flux quantum. An infinitely thin solenoid carrying an integer multiple of this flux is undetectable. Dirac used this property to quantize the monopole.

The degeneracy of each Landau level is the number of flux quanta going through the sample.

Exercise: Apply a static potential corresponding to $\vec{E} = E_x \hat{e}_x$ and compute the current. Thereby find the Hall conductance of each Landau level.

There is a general and beautiful way of obtaining the degeneracy of a Landau level.

Define the operators $\boxed{\vec{\Pi} = \vec{p} + e\vec{A}(x,y)}$ (2.20)

$$\boxed{[\Pi_x, \Pi_y] = -i\hbar e\left(\partial_x A_y - \partial_y A_x\right) = +i\hbar e B = \frac{i\hbar^2}{\ell^2}} \quad (2.21)$$

because $\vec{\nabla} \times \vec{A} = -B\hat{e}_z$

Now, we define the cyclotron coordinates η_x, η_y

$$\vec{\eta} = \frac{\ell^2}{\hbar}\hat{e}_z \times \vec{\Pi} \quad (2.22)$$

$$\boxed{\begin{gathered}\vec{\eta} = \frac{\ell^2}{\hbar}\hat{e}_z \times \vec{\Pi} \\ \eta_x = -\frac{\ell^2}{\hbar}\Pi_y \qquad \eta_y = \frac{\ell^2}{\hbar}\Pi_x \\ [\eta_x, \eta_y] = i\ell^2\end{gathered}} \qquad (2.22)$$

Alternatively, $\boxed{[\eta_i, \eta_j] = i\ell^2 \epsilon_{ij}}$ (2.23) $\quad \epsilon_{12} = 1 = -\epsilon_{21}$

You can also verify $\boxed{[x_i, \eta_j] = i\ell^2 \epsilon_{ij}}$ (2.24)

Next, we decompose the position coordinates into cyclotron and guiding center operators

$$\boxed{x = \eta_x + R_x \qquad y = \eta_y + R_y} \qquad (2.25)$$

$$R_x = x - \eta_x \qquad R_y = y - \eta_y \qquad (2.26)$$

$$\boxed{[R_x, R_y] = -[x, \eta_y] + [y, \eta_x] + [\eta_x, \eta_y] = -i\ell^2}$$

The truly interesting one is

$$\boxed{[R_i, \eta_j] = 0} \qquad (2.27)$$

The Hamiltonian is

$$\boxed{H = \hbar^2 \quad \vec{\pi}^2 \Rightarrow [R_i, H] = 0} \qquad (2.28)$$

$$\boxed{\mathbb{H} = \frac{\hbar^2}{2ml^4}\vec{\eta}^{\,2} \Rightarrow [\mathbb{R}_i, \mathbb{H}] = 0} \qquad (2.28)$$

Both guiding center coordinates commute with $\mathbb{H}$. However, they don't commute with each other. They form a phase-space like subspace because of

$$[\mathbb{R}_x, \mathbb{R}_y] = -il^2.$$

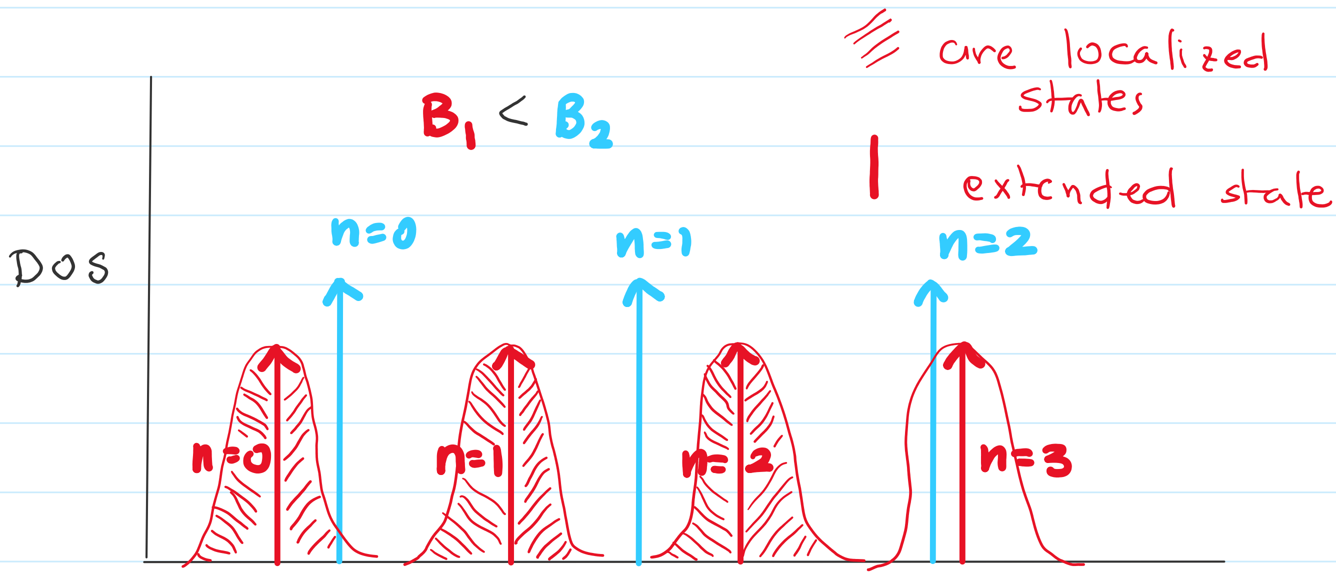
One thinks of $\mathbb{R}_x$ as a momentum and $\mathbb{R}_y$ as a coordinate, with l^2 playing the role of $\hbar$.

We know from the correspondence principle that each cell of phase space of "volume" $2\pi\hbar$ contains one quantum state. In our case "$\hbar$" $= l^2$, so each $2\pi l^2$ of the space of $\mathbb{R}_x, \mathbb{R}_y$, which is simply $L_x L_y$, contains one state.

$$\text{Degeneracy} = \frac{L_x L_y}{2\pi l^2} = N_\phi.$$

The final non-interacting picture is this. There are Landau levels (LLs) spaced by $\hbar\omega_c$, each with a degeneracy of N_ϕ. As B increases, the degeneracy of each LL increases while the spacing between LLs also increases.

are localized



As we will see shortly, we need disorder to quantize the Hall conductance. However, the general picture is that on a conductance plateau, the chemical potential is between LLs. There is a gap to extended, current-carrying states. However, the Hall current in this case is carried by the edge modes, which we now turn to.