# Convergence of limit shapes for 2D near-critical first-passage percolation

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# Outline

1. Background
2. Rescaled limit shape converges to a Euclidean ball as $p \uparrow p_c$
3. Asymptotics for Bernoulli FPP at $p_c$
4. Asymptotics for Bernoulli FPP as $p \downarrow p_c$

# Bond percolation

Let $G(V,E)$ be an infinite connected graph and let $p \in [0,1]$.

In Bernoulli($p$) **bond percolation**, we let each **bond** (edge) of $G$ be independently **open** with probability $p$ and **closed** with probability $1-p$.

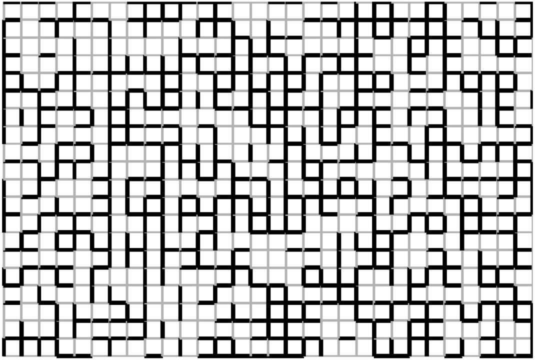


Figure: Bond percolation on the **square lattice** $\mathbb{Z}^2$

# Site percolation

Let $G(V, E)$ be an infinite connected graph and let $p \in [0, 1]$.

In Bernoulli($p$) **site percolation**, we let each **site** (vertex) of $G$ be independently **open** with probability $p$ and **closed** with probability $1 - p$.

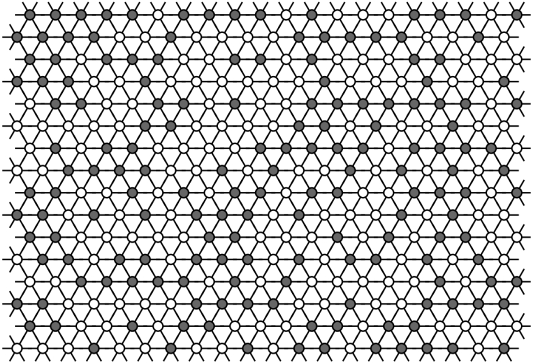


Figure: Site percolation on the **triangular lattice** $\mathbb{T}$

# Site percolation

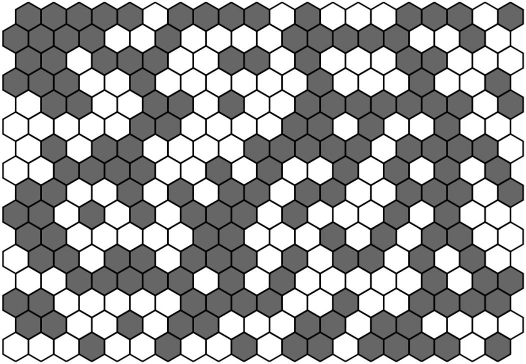


Figure: Site percolation on the triangular lattice can be considered as a random **two-coloring** of the faces of the dual **hexagonal lattice**.

# First-passage percolation (FPP)

Assign to each bond (edge) $e$ of the lattice $\mathbb{Z}^d$ i.i.d. **nonnegative passage time** $t(e)$. **Site version** of FPP is defined analogously. Given a path $\gamma$, define its passage time as $T(\gamma) = \sum_{e \in \gamma} t(e)$. Define the **point-to-point** passage times by

$$T(x, y) = \inf\{T(\gamma) : \gamma \text{ is a path from } x \text{ to } y\}.$$

**Triangular inequality**: $T(x, y) \leq T(x, z) + T(z, y)$

If $\mathbf{E}[t(e)] < \infty$, then **subadditive ergodic theorem** gives

$$\lim_{n \to \infty} \frac{T(0, n)}{n} = \mu \quad \text{a.s. and in } L^1. \quad \text{[Smythe-Wierman '78]}$$

The constant $\mu = \mu(F)$ is called the **time constant**.

$\mu = 0$ if $\mathbf{P}[t(e) = 0] \geq p_c(d)$; $\mu > 0$ if $\mathbf{P}[t(e) = 0] < p_c(d)$. [Kesten '86]

**Remarks**:

- **One dimension**, FPP on $\mathbb{Z}$, $\mathbf{E}[t(e)^2] < \infty$

  $T(0,n) = \sum_{i=1}^{n} t(e_i) \approx n\mathbf{E}[t(e_1)] + n^{1/2}\sqrt{\mathrm{Var}[t(e_1)]}Z, \quad Z \sim \mathcal{N}(0,1)$

  So $\mu = \mathbf{E}[t(e)]$.

- **Two dimensions**, FPP on general 2D lattice, $\mathbf{P}[t(e) = 0] < p_c$, ...

  It is expected that

  $T(0,n) \approx n\mu + n^{1/3}\sigma Z, \quad Z \sim$ **Tracy-Widom distribution**

- **Directed last-passage percolation** on $\mathbb{Z}^2$ (paths have non-decreasing coordinates) is **exactly solvable** when $t(v) \sim$ exponential or geometric distribution ("**memoryless property**").

  Related to interacting particle systems, random matrices, ...

- **KPZ (Kardar-Parisi-Zhang) universality class**, KPZ equation describing **surface growth models** (Parisi, 2021 Nobel Prize in Physics)

The set of points reached from the origin 0 within a time $t \geq 0$ is

$$B(t) := \{z \in \mathbb{R}^d : T(0, z) \leq t\}.$$

We are interested in the limiting behavior of this set.

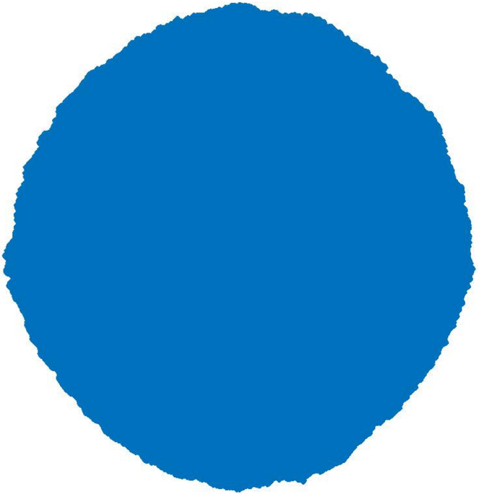
**Theorem** (**Shape Theorem**, Cox-Durrett '81)

*Consider FPP on $\mathbb{Z}^d$ (or $\mathbb{T}$). Suppose that $\mathbf{P}[t(e) = 0] < p_c(d)$ and $\mathbf{E}[t(e)^d] < \infty$. There exists a* **deterministic, convex, compact set** $\mathcal{B}$ *(depending on the distribution of $t(e)$ and the lattice) with non-empty interior, such that for each $\epsilon \in (0, 1)$,*

$$\mathbf{P}\left[(1-\epsilon)\mathcal{B} \subset \frac{B(t)}{t} \subset (1+\epsilon)\mathcal{B} \textit{ for all large } t\right] = 1.$$

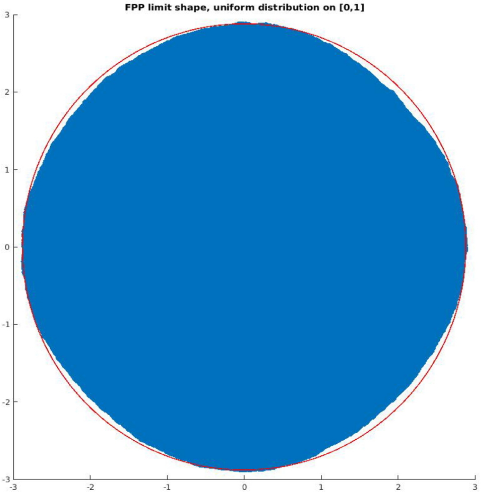
What does the limit shape look like? This is **one of the main questions for FPP**. Little is known about the geometry of the limit shape.

Simulation of $B(t)$ on $\mathbb{Z}^2$, with $t(e) \sim$ **uniform distribution** on $[0, 1]$



Is the limit shape a Euclidean ball?

Simulation of $B(t)/t$, with $t(e) \sim$ **uniform distribution** on $[0, 1]$




**Problem**: Prove that the time constant $\mu(\theta)$ is **strictly increasing** in $\theta \in [0, \pi/4]$, perhaps for some suitable class of passage time distributions.

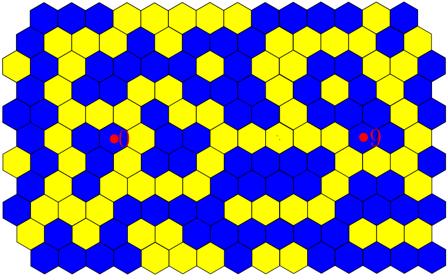
**Bernoulli first-passage percolation** on the triangular lattice $\mathbb{T}$:

Consider Bernoulli($p$) site percolation on $\mathbb{T}$. For each $v \in V(\mathbb{T})$,

$$t(v) = \begin{cases} 0 & \text{if } v \text{ is open,} \\ 1 & \text{if } v \text{ is closed.} \end{cases}$$

$$\mathbf{P}_p[t(v) = 0] = p, \quad \mathbf{P}_p[t(v) = 1] = 1 - p.$$

We represent it as a **random coloring** of the faces of the dual hexagonal lattice, each face centered at $v$ being **blue** ($t(v) = 0$) or **yellow** ($t(v) = 1$).




$T(0, 9) = 2$

**Bernoulli Percolation**: **study connectivity properties**, such as

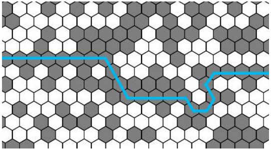
**box-crossing and annulus-crossing (i.e., 1-arm) events**

**Bernoulli FPP**: **study first-passage times**, such as

**box-crossing and annulus-crossing times**

A path realizing the box-crossing time has minimal number of **closed sites**, which can be seen as a crossing with minimal number of **defects**.

Consider Bernoulli site percolation on $\delta\mathbb{T}$, where $\delta$ is the mesh size.



$$\lim_{\delta\to 0} \mathbf{P}_{p,\delta}[\text{there is an open left-right crossing in } [0,1]\times[0,r]]$$

$$= \begin{cases} 0 & p < p_c, \\ f(r) & p = p_c, \quad \text{(Cardy's formula, proved by Smirnov '01)} \\ 1 & p > p_c. \end{cases}$$

For $p < p_c$, let $\mathcal{B}(p)$ denote the limit shape for Bernoulli($p$) FPP on $\mathbb{T}$.

- **Little is known about the geometry of** $\mathcal{B}(p)$ for fixed $p$. It is believed that $\mathcal{B}(p)$ is **not a Euclidean ball**, since the anisotropy of $\mathbb{T}$ may persist in the limit. It is easy to show that for small $p$, $\mathcal{B}(p)$ approximates the regular hexagon $\mathcal{B}(0)$. We shall show that as $p \uparrow p_c$, appropriately re-scaled $\mathcal{B}(p)$ **varies from a regular hexagon to a Euclidean ball**.
- FPP on $\mathbb{Z}^d$ for large $d$.

  [Auffinger-Tang '16]: Limit shapes are not Euclidean balls for very general distributions, including uniform distribution on $[0, 1]$, exponential distribution.

Consider Bernoulli($p$) percolation on the triangular lattice.

**Correlation length** $L(p)$ is a scale:

- below it the system **"looks like" critical percolation**. For instance, the crossing probabilities remain bounded away from 0 and 1.
- above it **"notable" super/sub-critical behavior** emerges.

**Power law**: $L(p) = |p - p_c|^{-4/3+o(1)}$ as $p \to p_c$. [Smirnov-Werner '01]

**Definition:** Fix $\epsilon \in (0, 1/2)$. For $p < p_c$, let

$$L_\epsilon(p) := \inf\{R \geq 1 : \mathbf{P}_p[\text{there is an } \textbf{open left-right crossing} \text{ of } [0,R]^2] \leq \epsilon\}.$$

**Another version:**

$$L(p) := \inf\left\{R \geq 1 : R^2 \mathbf{P}_{p_c}[\mathcal{A}_4(1,R)] \geq \frac{1}{p_c - p}\right\} \text{ for } p < p_c.$$

Let $\theta \in [0, 2\pi]$. There exists a **time constant** $\mu(p,\theta)$, such that

$$\lim_{n\to\infty} \frac{T(0, ne^{i\theta})}{n} = \mu(p,\theta) \qquad \mathbf{P}_p\text{-a.s. and in } L^1.$$

[Chayes-Chayes-Durrett '86]: $\mu(p,\theta) \asymp L(p)^{-1}$ as $p \uparrow p_c$.

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \le r\}$ denote the Euclidean ball of radius $r$.

The following theorem says that $\mathcal{B}(p)$ is **asymptotically circular** as $p \uparrow p_c$.

## Theorem (Y. 2021)

*There exists a constant $\nu > 0$, such that*

$$\lim_{p\uparrow p_c} L(p)\mu(p,\theta) = \nu \quad \textit{uniformly in } \theta \in [0, 2\pi].$$

*When $p \uparrow p_c$, the* **re-scaled limit shape** *$L(p)^{-1}\mathcal{B}(p)$ tends to the* **Euclidean ball** *$\mathbb{D}_{1/\nu}$ in the Hausdorff metric.*

The **Hausdorff distance** between two subsets $A, B$ of $\mathbb{C}$ is defined by

$d_H(A,B) := \inf\{\epsilon > 0 : \forall x \in A, \exists y \in B$ such that $|x-y| \le \epsilon$ and vice versa$\}$.

**Related works in the near-critical case** (**near-critical scaling limit** in [Garban-Pete-Schramm '18] was used to **extract geometric information** about the discrete model and **construct scaling limits of other objects**):

- Duminil-Copin '13: The Wulff crystal for subcritical percolation converges to a Euclidean disk as $p \uparrow p_c$. Roughly speaking, the typical shape of a cluster conditioned to be large becomes round as $p \uparrow p_c$.

  **Key ingredient: rotational invariance** of the near-critical scaling limit

- Garban-Pete-Schramm '18 (AOP): Construct the scaling limits of the minimal spanning tree and the invasion percolation tree.

  **In our proof, we will construct the scaling limit of the collection of clusters, and then define continuum FPP on this scaling limit.**

**Step 1. Construction of scaling limit for cluster ensemble as $p \uparrow p_c$**

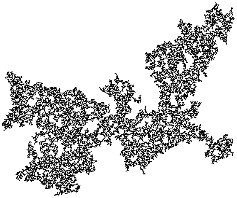
(1) Scaling limit of **critical percolation**:

- Camia-Newman '06: collection of cluster boundary loops
- Schramm-Smirnov '11: collection of quad-crossings
- **Camia-Conijn-Kiss '19: collection of open clusters**

(2) Scaling limit of **near-critical percolation**:

- **Garban-Pete-Schramm '18: collection of quad-crossings**

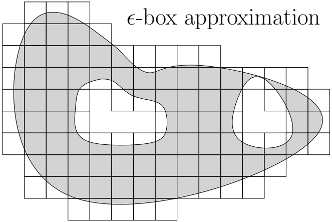
Using [GPS18] and [CCK19], we show that the collection of open clusters on the re-scaled lattice $L(p)^{-1}\mathbb{T}$ has a scaling limit as $p \uparrow p_c$, which is a collection of **random compact sets** called "**continuum clusters**".



**Idea of the construction** of continuum clusters

**Two approximations of macroscopic open clusters**:

- Use the union of $\epsilon$-boxes which intersect the cluster




- Use **arm events** related to $\epsilon$-boxes to characterize the cluster

  **Advantage**: it can also be defined in the near-critical **quad-crossing scaling limit** $\omega^\infty$. Letting $\epsilon \to 0$, one can construct the collection of continuum clusters $\mathscr{C}$ which is a measurable function of $\omega^\infty$.

For $L(p)^{-1}\mathbb{T}$ with $p$ close to $p_c$, the **two approximations coincide** with high probability. Under a coupling s.t. $L(p)^{-1}\omega_p \to \omega^\infty$ a.s. as $p \uparrow p_c$, the collection of open clusters on $L(p)^{-1}\mathbb{T}$ converges in probability to $\mathscr{C}$ (in some metric).

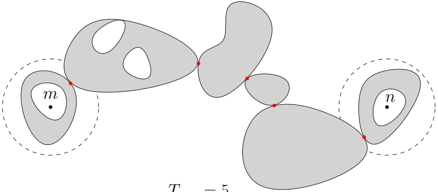
**Step 2. First-passage percolation for the continuum cluster ensemble**

**Continuum chain**: a sequence $\Gamma$ of continuum clusters with any two consecutive clusters in $\Gamma$ touching each other.

Write $T(\Gamma) := \#\{\text{clusters in } \Gamma\} - 1$.

Let $\mathcal{C}(n)$ be the outermost cluster surrounding the point $n$ in the unit ball centered at $n$. Define the **"point-to-point" passage time**:

$T_{m,n} := \inf\{T(\Gamma) : \Gamma \text{ is a continuum chain from } \mathcal{C}(m) \text{ to } \mathcal{C}(n)\}$.




**Discrete chain**: a sequence $\Gamma^p$ of blue clusters in $L(p)^{-1}\mathbb{T}$, s.t. for any two consecutive clusters $\exists$ a yellow hexagon touching both of them.

$T^p_{m,n} := T(\mathcal{C}^p(m),\mathcal{C}^p(n))$. For $p$ close to $p_c$, **with high probability**

$T^p_{m,n} = \inf\{T(\Gamma^p) : \Gamma^p \text{ is a discrete chain from } \mathcal{C}^p(m) \text{ to } \mathcal{C}^p(n)\}$.

For **FPP on the cluster-ensemble scaling limit**, there is a constant $\nu > 0$ such that

$$\lim_{n\to\infty} \frac{T_{0,n}}{n} = \nu \quad \text{a.s.}$$

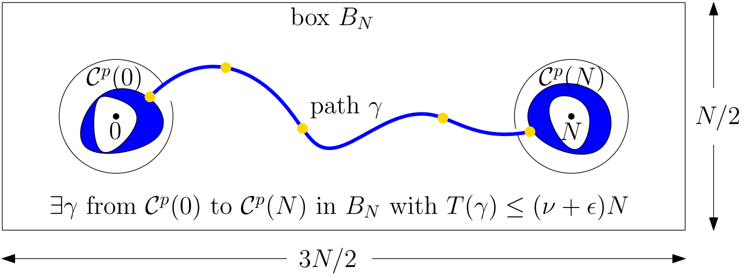
**Verify** that $(T_{m,n})_{0\le m<n}$ satisfies conditions of **subadditive ergodic theorem**:

- **Triangle inequality**: $T_{0,n} \le T_{0,m} + T_{m,n}$ for all $0 < m < n$.
- The distributions of $(T_{m,m+j})_{j\ge1}$ and $(T_{m+1,m+j+1})_{j\ge1}$ are the same.
- $(T_{nj,(n+1)j})_{n\ge1}$ is a stationary ergodic sequence for each $j \ge 1$.
- $\mathbb{E}[T_{0,1}] < \infty$. **Proof**: Show that $\mathbf{P}_p[T^p_{0,1} \ge k] \le C_1 \exp(-C_2 k)$ uniformly in $p \in (p_0, p_c)$ and $T^p_{0,1} \to T_{0,1}$ in distribution as $p \uparrow p_c$.

For FPP **on the scaled lattice $L(p)^{-1}\mathbb{T}$**, control passage times in **fixed boxes** uniformly in $p$ by using results for FPP **on the continuum cluster ensemble**:

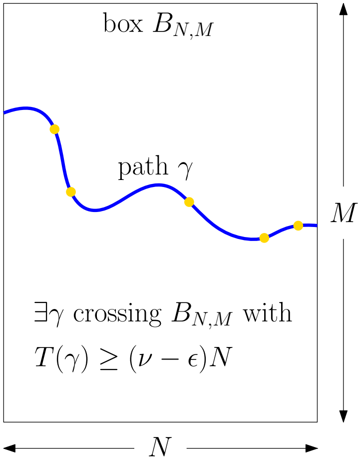
(1) "**point-to-point**" passage time in a box:

For all $p \in (p_0, p_c)$ and fixed large $N$, with **high probability** (in $\mathbf{P}_p$),



(2) **box-crossing time**:

For all $p \in (p_0, p_c)$ and fixed large $N, M$ with $M \geq N$, with **high probability** (in $\mathbf{P}_p$),

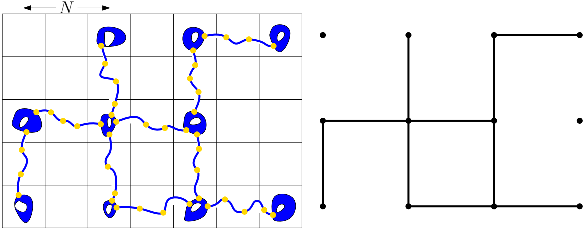


**Step 3. Convergence of $L(p)\mu(p,\theta)$ to $\nu$.**

By Step 2, for any **fixed** $n$, with high probability $T^p_{0,n} \approx T_{0,n} \approx \nu n$ for $p$ close to $p_c$. We need a **uniform "global" control** of $T^p_{0,n}$ for all large $n$ and $p$ close to $p_c$.

(1) **Upper bound** of $T^p_{0,n}$: use a **renormalization argument**

Consider box $B_N(e)$ corresponding to bond $e$ of $\mathbb{Z}^2$. Say $e$ is **good** if there is a path $\gamma$ in $B_N(e)$ joining the two corresponding clusters with $T(\gamma) \leq (1+\epsilon)\nu N$.



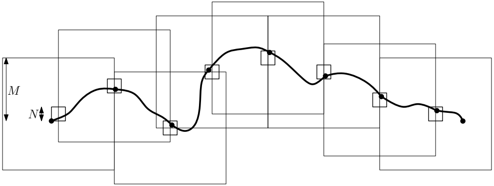

$T^p_{0,Nn} \leq (1+\epsilon)\nu N \cdot D(0,n) \leq (1+\epsilon)\nu N \cdot (1+\epsilon)n$ (with high probability)

$D(0,n)$ is the **graph distance** in the bond percolation configuration on $\mathbb{Z}^2$, and $\mathbf{P}^{bond}_{\mathbb{Z}^2,p}[D(0,n) \leq (1+\epsilon)n] \geq 1-\epsilon$ for $p$ close to 1.

This implies $\limsup_{p\uparrow p_c} L(p)\mu(p) \leq \nu$ since $\frac{T^p_{0,Nn}}{Nn} \to L(p)\mu(p)$ a.s.

(2) **Lower bound** of $T^p_{0,n}$: also by a type of **renormalization method**

[Grimmett-Kesten '84] used a "**block approach**" to obtain exponential large deviation bounds for passage times for a single FPP model. We apply it to the family of Bernoulli FPP on $L(p)^{-1}\mathbb{T}$ for all $p$ close to $p_c$.




**Idea: A path joining 0 to a point far away from 0 with a "very short" passage time should cross "many" annuli which have "very short" annulus-crossing times**, and by **estimates of box-crossing times** and a **counting argument** we show that it is unlikely that such a path exists.

So $T^p_{0,n} \geq (1-\epsilon)\nu n$ with high probability for large $n$ and $p$ close to $p_c$.

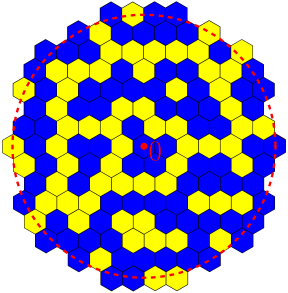
This implies $\liminf_{p\uparrow p_c} L(p)\mu(p) \geq \nu$ since $\frac{T^p_{0,n}}{n} \to L(p)\mu(p)$ a.s.

These arguments works for any direction $\theta \in [0, 2\pi]$ by rotating $\mathbb{Z}^2$.

$\mathbb{D}_n$ : the disk of radius $n$ centered at 0.

**Point-to-circle** passage times:

$$c_n = \inf\{T(\gamma) : \gamma \text{ is a path from 0 to } \partial\mathbb{D}_n\}.$$




$c_6 = 1$

At $p_c$, with high probability $c_n \approx T(0, n)/2$ for large $n$.

# Critical first-passage percolation in 2D

- Bernoulli critical FPP: $\mathbf{E}c_n \asymp \log n$ [Chayes-Chayes-Durrett '86]
- General critical FPP: $c_n/\mathbf{E}c_n \to 1 \;\; a.s.$ [Kesten-Zhang '97]
- General critical FPP: **Central limit theorem** for $c_n$

  [Kesten-Zhang '97], [Damron-Lam-Wang '17]:

$$\frac{c_n - \mathbf{E}c_n}{\sqrt{\mathrm{Var}[c_n]}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

**Remark**: It is expected that for general subcritical FPP the limiting distribution is **Tracy-Widom**.

Can we say anything finer on $c_n/\log n$?

## Theorem (Y. '18)

$$\lim_{n\to\infty}\frac{c_n}{\log n}=\frac{1}{2\sqrt{3}\pi}\ a.s.,\ \lim_{n\to\infty}\frac{\mathbf{E}c_n}{\log n}=\frac{1}{2\sqrt{3}\pi},\ \lim_{n\to\infty}\frac{\mathrm{Var}[c_n]}{\log n}=\frac{2}{3\sqrt{3}\pi}-\frac{1}{2\pi^2}.$$

This confirms a conjecture made by Kesten-Zhang '97 in the Bernoulli case.

**Generalization** [Damron-Hanson-Lam '22]: For **general critical FPP** on $\mathbb{T}$ (i.e., the distribution function $F$ of $t(v)$ satisfies $F(0)=p_c$), let $I:=\inf\{x>0:F(x)>p_c\}$. Then

$$\lim_{n\to\infty}\frac{c_n}{\log n}=\frac{I}{2\sqrt{3}\pi}\quad a.s.$$

So the time constant is **universal** for site version of critical FPP on $\mathbb{T}$ and depends only on the value of $I$. It is expected that similar result holds for bond version of critical FPP on $\mathbb{Z}^2$ with the same limiting value as above.

# Critical Bernoulli FPP in high dimensions

- Bramson '78: **$k$-ary tree**, $k \geq 2$ (every vertex has $k$ children).
  **"minimal displacement of branching random walk"**
  $\lim_{n\to\infty} c_n/\log_2 \log n = 1 \quad a.s.$
- Chayes '91: $\mathbb{Z}^d, d \geq 3$. If $\epsilon > 0$, then $\lim_{n\to\infty} c_n/n^\epsilon = 0 \quad a.s.$
- Addario-Berry, Hanson (**in preparation**): $\mathbb{Z}^d, d \geq 7$. Under assumption $\mathbf{P}[0 \leftrightarrow x] \asymp |x|^{2-d}$ (proved for $d \geq 11$ in [Fitzner-Hofstad '17]):
  - $d \geq 8: \quad \lim_{n\to\infty} c_n/\log_2 \log n = 1 \quad a.s.$
  - $d = 7: \quad \lim_{n\to\infty} c_n/\log_{3/2} \log n = 1 \quad a.s.$

## Sketch of proof for point-to-circle passage time on the triangular lattice

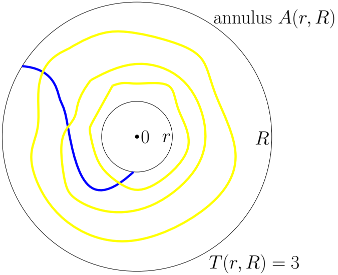
Consider annulus $A(r,R)$ of inner radius $r$ and outer radius $R$.

Define the **annulus-crossing time**:

$$T(r,R) := \inf\{T(\gamma) : \gamma \text{ is a path connecting the two boundary pieces of } A(r,R)\}.$$

**Topological observation**:

$$T(r,R) = \text{maximal number of disjoint } \textbf{yellow circuits} \text{ surrounding 0 in } A(r,R).$$

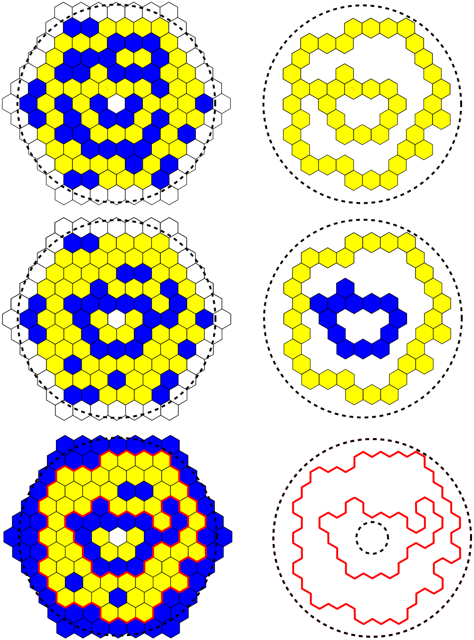


**The following three random variables have the same distribution.**

- maximal number of disjoint **yellow circuits** surrounding 0 in $A(r, R)$
- maximal number of disjoint circuits with color sequence **yellow, blue, yellow, blue, ...**, surrounding 0 in $A(r, R)$ from outside to inside
- number of **cluster boundary loops** surrounding 0 in $A(r, R)$ with boundary condition: the discrete outer site-boundary is blue

**Idea of proof**: "**color switching trick**". Analogous trick appeared in [Aizenman-Duplantier-Aharony '99] and [Smirnov '01].

## Color Switching



For $0 < \epsilon < 1$, let

$N(\epsilon, 1) :=$ **number of CLE$_6$ loops** surrounding 0 in the annulus $A(\epsilon, 1)$.

Using the moment generating function of the **conformal radius** about CLE$_6$ loops [Schramm-Sheffield-Wilson '09] and **renewal process** theory, we obtain

$$\lim_{\epsilon \to 0} \frac{\mathbb{E}[N(\epsilon, 1)]}{\log(1/\epsilon)} = \frac{1}{2\sqrt{3}\pi}, \quad \lim_{\epsilon \to 0} \frac{N(\epsilon, 1)}{\log(1/\epsilon)} = \frac{1}{2\sqrt{3}\pi} \quad a.s.,$$

$$\lim_{\epsilon \to 0} \frac{\mathrm{Var}[N(\epsilon, 1)]}{\log(1/\epsilon)} = \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}.$$

Based on [Camia-Newman '06]'s theorem for the scaling limit of cluster boundary loops, by **dividing the ball $\mathbb{D}_n$ into nested annuli**, it is easy to derive the limit theorem for $c_n$ and $\mathbf{E}[c_n]$ from the above result.

It is more involved for the variance. We use a **martingale method** from [Kesten-Zhang '97].

Combining our explicit limit theorem for Bernoulli critical FPP and the CLT of [Kesten-Zhang '97] gives:

## Corollary

*There is a function* $\delta(n)$ *with* $\delta(n) \to 0$ *as* $n \to \infty$, *such that*

$$\frac{c_n - (1+\delta(n))\frac{\log n}{2\sqrt{3}\pi}}{\sqrt{\left(\frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}\right)\log n}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

**Problem**: Show that one can choose $\delta(n) \equiv 0$.

For this, one needs to show that $\mathbf{E}c_n = \frac{1}{2\sqrt{3}\pi}\log n + o(\sqrt{\log n})$.

## Theorem (Y. '19)

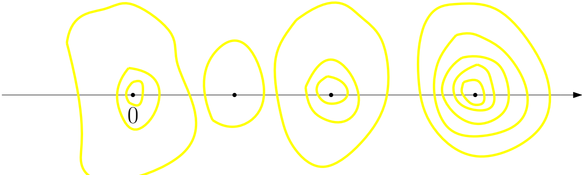
$$\lim_{n\to\infty} \frac{T(0,n)}{\log n} = \frac{1}{\sqrt{3}\pi} \quad \textbf{in probability but not a.s.}$$

*Let $C_0 > 1/(2\sqrt{3}\pi)$ be a constant from [Miller-Watson-Wilson '16]. Almost surely, for each $C \in [0, C_0]$, there is a* **random subsequence** $\{n_i : i \geq 1\}$ *such that*

$$\lim_{i\to\infty} \frac{T(0,n_i)}{\log n_i} = \frac{1}{2\sqrt{3}\pi} + C.$$

Loosely speaking, we can find many subsequences of sites with different growth rate, **growing unusually quickly or slowly**.

**Idea of proof**: Use a **large deviation estimate** on extreme nesting in $\mathrm{CLE}_6$.



**Problem:** Show that for all functions $\delta(n) \downarrow 0$ sufficiently slowly as $n \to \infty$,

$$\begin{cases} \mathbf{P}\left[\nu \le \dfrac{c_n}{\log n} \le \nu + \delta(n)\right] = n^{-\gamma(\nu)+o(1)}, \quad \text{for } \nu > 0, \\ \mathbf{P}\left[\dfrac{\delta(n)}{2} \le \dfrac{c_n}{\log n} \le \delta(n)\right] = n^{-5/48+o(1)}, \end{cases}$$

where $\gamma(\nu)$ is the function from [Miller-Watson-Wilson '16].

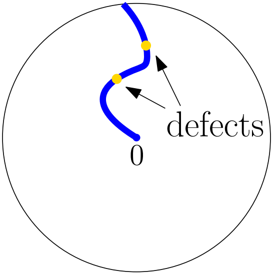
- We have obtained the lower bounds for these probabilities.

  It remains to get the upper bounds.

- $5/48$ is the 1-arm exponent. So $\mathbf{P}[c_n = 0] = n^{-5/48+o(1)}$.

Nolin '08 studied **arms with "defects"** (i.e. sites of the opposite color).

$\{c_n \leq j\} = \{\exists$ 1-arm from 0 to $\partial \mathbb{D}_n$ with $\#$ yellow defects $\leq j\}$.




[Nolin '08]: $\mathbf{P}[c_n \leq j] \asymp (\log n)^j \mathbf{P}[c_n = 0]$ for fixed $j \geq 0$.

Note that $\mathbf{P}[c_n = 0] = n^{-5/48+o(1)}$.

**Problem**: Show that $\mathbf{P}[c_n \leq j] = C(j)(\log n)^j n^{-5/48}(1 + o(1))$ for fixed $j \geq 0$.

**Related works**: By using the power law convergence rate of percolation exploration path towards $\text{SLE}_6$ [Binder-Richards '21], [Du-Gao-Li-Zhuang '22] get **sharp asymptotics for arm probabilities**, partly answering a problem in [Schramm '06, ICM].

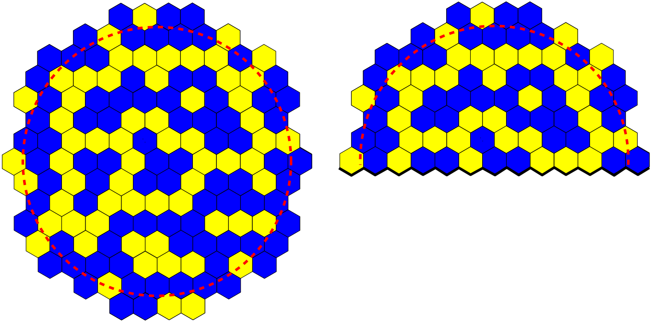


Figure: $c_n = 1, c_n^+ = 2$

Let $\mathbb{D}_n^+$ denote the upper half-disk of radius $n$ centered at 0. Define the **point to half-circle** passage times:

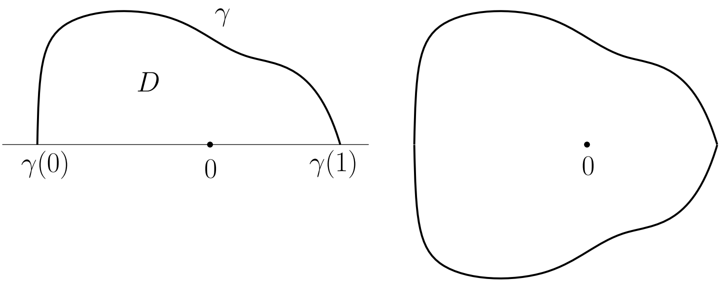
$$c_n^+ = \inf\{T(\gamma) : \gamma \text{ is a path from 0 to } \partial\mathbb{D}_n \text{ in } \mathbb{D}_n^+\}.$$

## Theorem (Jiang-Y. '19)

$$\lim_{n\to\infty} \frac{c_n^+}{\log n} = \frac{\sqrt{3}}{2\pi} \; a.s., \; \lim_{n\to\infty} \frac{\mathbf{E}[c_n^+]}{\log n} = \frac{\sqrt{3}}{2\pi}, \; \lim_{n\to\infty} \frac{\mathrm{Var}[c_n^+]}{\log n} = \frac{2\sqrt{3}}{\pi} - \frac{9}{\pi^2}.$$

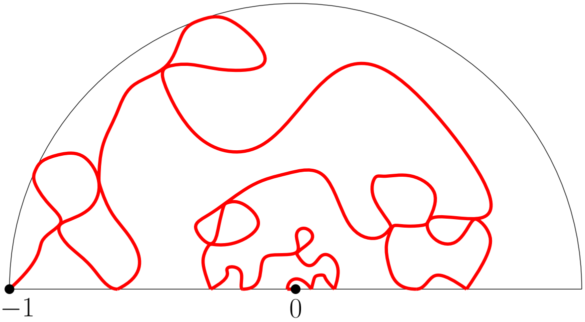
Therefore, with high probability $c_n^+ \approx 3c_n$ for large $n$.

The proof is similar to that for $c_n$ by using "half-circuits", "half-loops" and a color switching trick. We just need to study the **reflected conformal radius** related to half-loops.

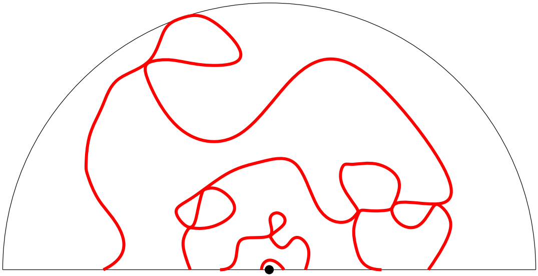



Let $\gamma = \gamma[0,1]$ be a path in the half-plane, starting and ending at $x$-axis, and surrounding 0. Let $D$ be the domain that is surrounded by $\gamma$ and $x$-axis. The **reflected conformal radius** of $D$ viewed from 0 is the conformal radius of $\{D\} \cup \{D \text{ reflected }\} \cup \{\overline{\gamma(0)\gamma(1)}\}$ viewed from 0.

(Suppose $D$ is a simply connected domain and $z \in D$. The **conformal radius** of $D$ viewed from $z$ is $|f'(z)|^{-1}$, where $f$ is any conformal map from $D$ to the unit disk $\mathbb{D}$ that sends $z$ to 0.)




We run a chordal $\mathrm{SLE}_6$ from $-1$ to 0 in $\overline{\mathbb{D}^+}$ to explore the **half-loops** surrounding 0 in the half-disk $\overline{\mathbb{D}^+}$.



We just need to calculate the **reflected conformal radius** of the domain that is surrounded by the **outermost half-loop** together with the $x$-axis. Use the SDE in [Lawler '15] and the result in [Schramm-Sheffield-Wilson '09] to derive its distribution.

# Coupling of percolation measures

There is a standard **coupling** of the percolation measures $\mathbf{P}_p, 0 \leq p \leq 1$: Take i.i.d. random variables $U_v$ for each site $v$, with $U_v$ **uniformly distributed** on $[0, 1]$. For each $p$, we obtain $\mathbf{P}_p$ by declaring each site $v$ to be $p$-open ($t(v) = 0$) if $U_v \leq p$, and $p$-closed ($t(v) = 1$) otherwise.

Let $\mathcal{C}_\infty(p)$ denote the **infinite** $p$**-open cluster** when $p > p_c$.

Almost surely the passage time $T_p(0, \infty) := T_p(0, \mathcal{C}_\infty(p))$ **is finite**.

How does $T_p(0, \infty)$ vary as $p \downarrow p_c$ ?

## Theorem (Y. 19)

$$\lim_{p\downarrow p_c} \frac{T_p(0,\infty)}{-\frac{4}{3}\log(p-p_c)} = \frac{1}{2\sqrt{3}\pi} \quad \textit{a.s.},$$

$$\lim_{p\downarrow p_c} \frac{\mathbf{E}_p[T_p(0,\infty)]}{-\frac{4}{3}\log(p-p_c)} = \frac{1}{2\sqrt{3}\pi},$$

$$\lim_{p\downarrow p_c} \frac{\mathrm{Var}_p[T_p(0,\infty)]}{-\frac{4}{3}\log(p-p_c)} = \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}.$$

Note that $L(p) = (p-p_c)^{-4/3+o(1)}$ as $p \downarrow p_c$ (Smirnov-Werner '01).

So $T_p(0,\infty) \approx T_{p_c}(0,\partial\mathbb{D}_{L(p)})$ as $p \downarrow p_c$.

**Idea of proof.**

- $T_p(0,\infty) \approx T_p(0,\partial\mathbb{D}_{L(p)})$ as $p \downarrow p_c$.
- Use **Russo's formula** to show that $T_p(0,\partial\mathbb{D}_{L(p)}) \approx T_{p_c}(0,\partial\mathbb{D}_{L(p)})$ as $p \downarrow p_c$. This combined with the limit theorem for the critical case gives the result.

**Key ingredient**:

**Lemma**

*There is a universal constant $C > 0$ such that for all $p > p_c$,*

$$\left|\mathbf{E}_{p_c}[T_{p_c}(0,\partial\mathbb{D}_{L(p)})] - \mathbf{E}_p[T_p(0,\partial\mathbb{D}_{L(p)})]\right| \leq C,$$

$$\left|\mathrm{Var}_{p_c}[T_{p_c}(0,\partial\mathbb{D}_{L(p)})] - \mathrm{Var}_p[T_p(0,\partial\mathbb{D}_{L(p)})]\right| \leq C.$$

Related papers:

- Chang-Long Yao: Convergence of limit shapes for 2D near-critical first-passage percolation. arXiv:2104.01211 (2021)

  (We will give a **simplified proof** in an updated version of this article. In particular, we **will not use** the scaling limit of the collection of **portions of clusters in a region**, avoiding some technical arguments.)

- Chang-Long Yao: Asymptotics for 2D critical and near-critical first-passage percolation. PTRF (2019)
- Jianping Jiang, Chang-Long Yao: Critical first-passage percolation starting on the boundary. SPA (2019)
- Chang-Long Yao: Limit theorems for critical first-passage percolation on the triangular lattice. SPA (2018)

# Thanks!