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Arc spaces and vertex algebras

– Monograph –

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Contents

Preface

This book is addressed to researchers in mathematics or in physics eager to learn the basics on vertex algebras and related geometrical aspects, such as arc spaces and the concept of associated variety. It is based on several lecture courses given by authors in Pisa (Italy), Melbourne (Australia), Kyoto (Japan), Les Diablerets (Switzerland), Rio de Janeiro (Brazil), Amherst (US), Bengalore (India), etc.

Part I is devoted to the theory of arc spaces, the definition of vertex algebras, the first properties and basic examples of vertex algebras.

Part II is about Poisson vertex algebras, the notion of associated variety for vertex algebras and Zhu's functor, as well as the links between all these objects.

The BRST cohomology is the core of Part III. The $\mathcal{W}$-algebras which are the major motivations of the book are defined via the quantum Drinfeld–Sokolov reduction using the BRST cohomology. Their related algebraic/geometrical structures introduced in Part III can also be defined through the BRST cohomology: Slodowy slices, finite $\mathcal{W}$-algebras and arc spaces of Slodowy slices. All these structures are studied in this part as well.

We focus in Part IV on quasi-lisse vertex algebras. These are vertex algebras whose associated variety satisfies a certain finiteness condition. They form a large class of vertex algebras that enjoy remarkable properties similar to those sharing by lisse or rational vertex algebras. Part IV includes various examples of such vertex algebras, particularly in the context of $\mathcal{W}$-algebras, and many recent developments in the area.

Each part contains exercises with hints collected at the end of the book. They are five appendices, and an index.

The reader is supposed to be familiar with the basics in commutative algebra (for instance, some spectral sequence arguments are used in Part III) and in representation theory. Although there is an appendix giving a quick review on simple Lie algebras, the reading of the book is greatly facilitated by a good knowledge on Lie theory.

Introduction

The goal of this book is to introduce the theory of vertex algebras and affine $\mathcal{W}$-algebras, which are certain vertex algebras, with emphasis on their geometrical aspects, in particular through the study the arcs spaces.

Vertex algebras are algebraic structures introduced by Richard Borcherds in 1986 as a powerful tool in representation theory, motivated by the construction of an infinite-dimensional Lie algebra due to Igor Frenkel. They were notably used in the monstrous moonshine conjecture (phrased by Conway and Norton), which suggests an unexpected connection between the monster group and modular functions, to build representations of sporadic groups. Vertex algebras also play a prominent role in the representation theory of affine Kac–Moddy Lie algebras.

In the meantime, they appear in string theory in physics. They give the rigorous mathematical definition of the chiral part of a two-dimensional quantum field theory, intensively studied by Alexander Belavin, Alexander Polyakov and Alexander Zamolodchikov.

Since they were introduced by Borcherds, vertex algebras have turned out to be extremely useful in many areas of mathematics, such as algebraic geometry (moduli spaces), representation theory (modular representation theory, geometric Langlands correspondence), combinatorics, two-dimensional conformal field theory, string theory (mirror symmetry) and four-dimensional gauge theory (AGT conjecture). In fact, a recent discovery, which goes back to the works of Nakajima in 1994, reveals that vertex algebras appear in higher dimensional quantum field theories as well, in several ways. This gives new insights for the representation theory of vertex algebras.

Part I is about the general theory of arc spaces, the definition of a vertex algebras, the first properties and examples.

Roughly speaking, a vertex algebra is a vector space V, endowed with a distinguished vector, the vacuum vector, and the vertex operator map from V to the space of formal Laurent series with linear operators on V as coefficients. These data satisfy a number of axioms and have some fundamental properties as, for example, an analogue to the Jacobi identity, the locality and the associativity. The algebraic law of vertex algebras is governed by what is called the Operator Product Expansion

(OPE). This notion and the precise definition of a vertex algebra are at the heart of Chapter 2.

The simplest vertex algebras are the commutative ones (when the OPEs are trivial). The later are nothing but differential algebras, that is, unital commutative algebras equipped with derivations. The jet algebras of unital commutative algebras (of finite type) provide the typical examples of commutative vertex algebras. The schemes corresponding to jet algebras, called the arc spaces, are of great importance in the study of general vertex algebras. One of the reasons is that to any vertex algebra is attached in a canonical way a certain Poisson variety whose arc space is closely related to the graded vertex algebra. The geometry of the arc spaces arising in this way reflect important properties of the vertex algebra. On another note, arc spaces appear in the definition of the chiral de Rham complex on a complex manifold. The study of jet algebras and arc spaces is of independent interest, and the book starts with this topic (see Chapter 1).

Important basic examples of (noncommutative) vertex algebras are those, said affine, arising as representations of affine Kac–Moody Lie algebras or Virasoro Lie algebras. They are crucial in the representation theory of these Lie algebras, and that of $\mathcal{W}$-algebras, which are more sophisticated vertex algebras. Many vertex algebras that appear in the nature carry an action of the Virasoro algebra. This leads to the notions of conformal vertex algebras and vertex operator algebras. We refer to Chapter 3 for the study of these basic exemples, including the chiral differential operators on an algebraic group (a particular case of chiral de Rham complex).

Other important examples (not studied in this book) of noncommutative vertex algebras are the lattice vertex algebras and the module moonshine, constructed by Igor Frenkel, James Lepowsky, and Arne Meurman in 1988.

Part II concerns several functors from the category of vertex algebras to the categories of *simpler* objects (commutative and/or of finite type): Poisson algebras, Poisson vertex algebras, associated algebras. These functors are extremely helpful to study various aspects of the theory.

Vertex analogues of Poisson algebras are discussed in Chapter 4. In a summary way, Poisson vertex algebras are differential algebras equipped with a Poisson vertex λ-bracket $\{-_\lambda-\}$ compatible with the multiplication and the derivation. This theory allows to study the integrability of partial differential equations efficiently, as it was developed by Aliaa Barakat, Alberto De Sole, Victor Kac, Daniele Valeri among others. The following are two examples of Poisson vertex algebras. First, taking the arc space of a Poisson variety gives a Poisson vertex algebras (cf. §4.2). Second, as observed by Haisheng Li, any vertex algebra is filtered and the corresponding graded vertex algebra is naturally a Poisson vertex algebra (cf. §4.4). In fact, to each vertex algebra V one can naturally attach, using the Zhu C_2-functor R_-, a certain Poisson variety X_V called the associated variety of V, and the arc space of the associated variety is deeply connected the graded vertex algebra (cf. §4.5). The associated variety is a fundamental geometrical invariant that captures important properties of the vertex algebra.

Another functor, studied in Chapter 5, is the Zhu's functor $\mathrm{Zhu}(-)$. It attaches to any vertex algebra an associative filtered algebra which allows to parameterize the simple modules of the vertex algebra in the best situation. The graded Zhu's algebra of a vertex algebra V is a Poisson algebra deeply related to the associated X_V. Thus the knowledge of the associated variety is also important for the representation theory of vertex algebras. In fact, the associated varieties of vertex algebras, especially in the context of affine ones, play an analogue role to the associated varieties of primitive ideals of the enveloping algebra of a simple Lie algebra. Figure 0.1 summarizes the different functors. We warn that the diagram is not commutative in general.

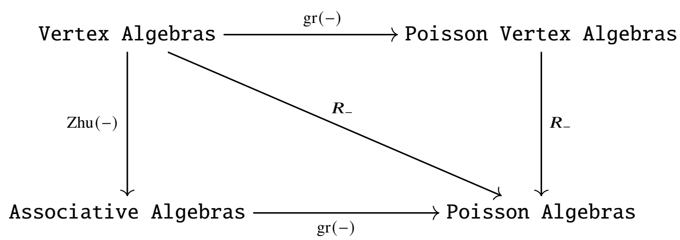


Fig. 0.1 Functors from the category of vertex algebras

The (affine) $\mathcal{W}$-algebras are more sophisticated vertex algebras associated with nilpotent elements of simple Lie algebras. They arise in geometric representation theory and mathematical physics. Their study is the major motivation of the book. They can be regarded as affinizations of finite $\mathcal{W}$-algebras (introduced by Alexander Premet), and can also be considered as generalizations of affine Kac-Moody algebras and Virasoro algebras. They quantize the arc space of the Slodowy slices associated with nilpotent elements. The study of $\mathcal{W}$-algebras began with the work of Zamolodchikov in 1985. Mathematically, $\mathcal{W}$-algebras are defined by the method of quantized Drinfeld–Sokolov reduction using the BRST cohomology that was discovered by Boris Feigin and Edward Frenkel in the 1990s. The general definition of affine $\mathcal{W}$-algebras were given by Victor Kac, Shi-Shyr Roan and Minoru Wakimoto in 2003. $\mathcal{W}$-algebras are related with integrable systems, the two-dimensional conformal field theory and the geometric Langlands program. The most recent developments in representation theory of affine $\mathcal{W}$-algebras were done by Kac–Wakimoto and the first author.

Since they are not finitely generated by Lie algebras, the formalism of vertex algebras is necessary to study them. The construction of affine $\mathcal{W}$-algebras is the main purpose of Part III. In this context, associated varieties of $\mathcal{W}$-algebras are important tools and they are related to the singularities of nilpotent Slodowy slices. In Part III is introduced the BRST cohomology for the different blocks of Figure 0.1:

– Poisson algebras (coordinate rings of Slodowy slices, studied in Chapter 7),
– Associative algebras (finite $\mathcal{W}$-algebras, studied in Chapter 8),
– Poisson vertex algebras (coordinate rings of the arc spaces of Slodowy slices, studied in Chapter 9),
– Vertex algebras (affine $\mathcal{W}$-algebras, studied in Chapters 9, 10 and **??**).

The nicest vertex algebras are those which are both rational and lisse. The rationality means the completely reducibility of modules. The lisse condition means that the associated variety has dimension zero. If a vertex algebra V is rational and lisse, then it gives rise to a rational conformal field theory. In particular, the characters of simple V-modules form a vector-valued modular functions and, moreover, the category of V-modules is a modular tensor category, so that one can associate with it an invariant of knots.

Lisse (or C_2-cofinite) vertex algebras are natural generalizations of finite-dimensional algebras The modular invariance of characters still holds without the rationality assumption. In fact the geometry of the associated variety often reflects algebraic properties of the vertex algebra. Vertex algebras whose associated variety has only finitely symplectic leaves are also of great interest for several reasons. They are referred to as quasi-lisse vertex algebras by the first author and Kazuya Kawazetsu, and have been intensively studied by both authors. Part IV is about this family of vertex algebras. Examples of lisse and quasi-lisse vertex algebras are presented in Chapters 11 and 12. Although the quasi-lisse condition is much weaker than the finiteness condition required for lisse or rational vertex algebras, quasi-lisse vertex are still nice for many reasons that are explained in Chapter 13. In particular their characters still satisfy certain modular invariant properties.

In the case where the vertex algebra is an affine vertex algebra, its associated variety is an invariant and conic subvariety of the corresponding simple Lie algebra. Such vertex algebras are quasi-lisse if the associated variety in contained in the nilpotent cone. For $\mathcal{W}$-algebras, it happens when the associated variety in contained in a nilpotent Slodowy slices. When the associated variety is symplectic (this is the case of chiral differential operators) it loosely implies the simplicity of the corresponding Poisson vertex algebras. All these observations are exploited to study more advanced aspects of the representation theory of $\mathcal{W}$-algebras, such as the rationality question or the obtention of collapsing levels, see Chapter 14.

Associated varieties not only capture some of the important properties of vertex algebras but also have interesting relationships with the Higgs branches of four-dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs) as discovered by Christopher Beem and Leonardo Rastelli and coauthors. For instance one can deduce the modular invariance of Schur indices of 4D $\mathcal{N} = 2$ SCFTs from the theory of vertex algebras. On the other hand, Jethro Van Ekeren et Reimundo Heluani recently observed that the arc space of the associated variety is involved in the chiral homology of elliptic curves with coefficients in a quasiconformal vertex algebra. It is beleived by physicists that vertex algebras coming from 4D $\mathcal{N} = 2$ SCFTs are quasi-lisse. This implies a number of interesting conjectures and expectations that are presented in Chapter 15 in connections with the 4D/2D duality in physics.

Part I
Vertex algebras: definitions and examples

This part aims to introduce the notion of vertex algebras. The heart of the matter is to give a definition. This is done in Chapter 2 after preliminary results on formal series. We establish also the first properties, and present basic examples (in Chapter 3), essentially generated by infinite-dimensional Lie algebras. Any finite-dimensional vertex algebra is commutative. Thus even the smallest examples of noncommutative vertex algebras require significant work.

Commutative vertex algebras are not, nevertheless, without significance. The structure sheaf of the arc spaces yields to interesting commutative vertex algebras. In fact, are spaces not only provide examples of commutative vertex algebras, but are powerful tools in the study of general vertex algebras. We start the book with their study in Chapter 1 from a slightly different point of view than the usual one, adopted for instance in the theory of singularities.

Chapter 1
Jet schemes and arc spaces

This chapter is devoted to the study of jet schemes and arc spaces associated with a scheme X defined over the field of complex numbers.

Roughly speaking, an arc on a scheme X is a formal path on X, that is, a morphism from the formal disc $D = \operatorname{Spec} \mathbb{C}[[t]]$ to X. Jets on X are obtained by truncation of such paths at a finite order.

The study of singularities via the space of arcs was initiated by Nash [224]. He conjectured a tight relationship between the geometry of the arc space and the singularities of X, see Ishii and Kollár [152]. More precisely, he suggested that the study of the images by the truncation morphisms of the space of arcs should give information about the fibers over the singular points in a resolution of singularities of X. The work of Mustaţă [221] supports these predictions; for example, rational singularities of a locally complete intersection variety can be detected by the irreducibility of all its jet schemes. The space of arcs also plays a key role in motivic integration, as the domain over which functions are integrated. We refer the reader to the the recent book by Chambert–Loir, Nicaise and Sebag [73], and the references given there, for more about this topic.

It turns out that arc spaces are also of great importance in the theory of vertex algebras. One of the main reasons is that the structure sheaf of the arc scheme over a scheme X has the structure of a sheaf of commutative vertex algebras ([53, 113]), see Chapter 2. Moreover, any vertex algebra is canonical filtered, and the associated graded space is a quotient of the space of the functions on the arc space of the associated scheme of the vertex algebra, see Chapter 4 for more details. The space of the functions on an arc space will be thus the most important example of commutative vertex algebras.

The chapter is structured as follows. Section 1.1 is about the jet construction of differential algebras. Section 1.2, Section 1.3, and Section 1.4 concerns first properties and examples related to arc schemes. We study in Section 1.5 geometrical properties of arc spaces. In the context of vertex algebras one needs also to consider the *loop space* $\mathscr{L}X$ of an affine scheme X. This is the topic of Section 1.6. Arc spaces of group schemes acting on a scheme is discussed in Section 1.7.

Throughout this chapter, the ground field will be the field $\mathbb{C}$ of complex numbers. We shall work with the Zariski topology, and by *variety* we mean a reduced and separated scheme of finite type over $\mathbb{C}$. We denote the smooth part of a variety X by X_{reg} and its complement by X_{sing}.

1.1 Jet construction of differential algebras

In this book, a *differential algebra* is a unital commutative $\mathbb{C}$-algebra A equipped with a derivation ∂, that is, a homomorphism of vector spaces $\partial\colon A \to A$ satisfying the Leibniz product rule $\partial(ab) = \partial(a)b + a\partial(b)$ for every $a, b \in A$.

A *differential algebra homomorphism* $f\colon A \to A'$ between two differential algebras (A, ∂) and (A', ∂') is a $\mathbb{C}$-algebra homomorphism which commutes with the derivations, that is, $\partial'(f(a)) = f(\partial(a))$ for every $a \in A$. We denote by `Dif.Alg` the category of differential algebras, and by `Alg` that of commutative $\mathbb{C}$ algebras.

Lemma 1.1 *For any finitely generated unital commutative algebra R, there exists a unique (up to an isomorphism) differential algebra $\mathcal{J}_\infty R$ such that*

$$\mathrm{Hom}_{\mathtt{Dif.Alg}}(\mathcal{J}_\infty R, A) \cong \mathrm{Hom}_{\mathtt{Alg}}(R, A) \tag{1.1}$$

for any differential algebra A.

Proof The uniqueness of $\mathcal{J}_\infty R$ follows from Yoneda's lemma.

Let us show the existence. First, let $R = \mathbb{C}[x_1, \ldots, x_N]$. We define $\mathcal{J}_\infty R$ to be the polynomial ring $\mathbb{C}[\partial^j x_i\colon i = 1, \ldots, N,\ j \geqslant 0]$ with infinitely many variables $\partial^j x_i$, $i = 1, \ldots, N$, $j \geqslant 0$, with the differential

$$\partial\colon \partial^j x_i \longmapsto \partial^{j+1} x_i, \tag{1.2}$$

We have the embedding

$$j\colon R \hookrightarrow \mathcal{J}_\infty R, \quad x_i \longmapsto \partial^0 x_i, \tag{1.3}$$

and $\mathcal{J}_\infty R$ is generated by R as a differential algebra. From now, we identify x_i with $\partial^0 x_i$. It is clear that $(\mathcal{J}_\infty R, \partial)$ satisfies the desired property.

Next, let R be general. We may assume that

$$R = \mathbb{C}[x_1, \ldots, x_N]/\langle f_1, f_2, \cdots, f_r\rangle$$

with $f_i \in \mathbb{C}[x_1, \ldots, x_N]$. We define

$$\mathcal{J}_\infty R = \mathbb{C}[\partial^j x_i\colon i = 1, \ldots, N,\ j \geqslant 0]/\langle \partial^j f_i\colon i = 1, \ldots, r,\ j \geqslant 0\rangle, \tag{1.4}$$

where f_i is considered as an element of $\mathbb{C}[\partial^j x_i\colon i = 1, \ldots, N,\ j \geqslant 0]$ by the embedding j. Since $\langle \partial^j f_i\colon i = 1, \ldots, r,\ j \geqslant 0\rangle$ is a differential ideal, $\mathcal{J}_\infty R$ is naturally a differential algebra with the derivation ∂. Because $\langle \partial^j f_i\colon i = 1, \ldots, r,\ j \geqslant 0\rangle$

is the smallest differential ideal of $\mathbb{C}[\partial^j x_i : i = 1, \dots, N,\ j \geqslant 0]$ containing $\langle f_1, f_2, \cdots, f_r \rangle$, $(\mathcal{J}_\infty R, \partial)$ satisfies the required property. □

The differential algebra $\mathcal{J}_\infty R$ is called the *jet algebra* of R.

By the proof of Lemma 1.1 we have the embedding $j : R \to \mathcal{J}_\infty R$ given by the correspondence (1.3). In particular, R can be regarded as a subalgebra of $\mathcal{J}_\infty R$ and the isomorphism (1.1) is given by restriction.

Observe that the correspondence $R \mapsto \mathcal{J}_\infty R$ is functorial. If $f \colon R \to R'$ is an algebra homomorphism, then we naturally obtain a morphism $\mathcal{J}_\infty f \colon \mathcal{J}_\infty R \to \mathcal{J}_\infty R'$ making the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{\ f\ } & R' \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ \mathcal{J}_\infty R & \xrightarrow[\mathcal{J}_\infty f]{} & \mathcal{J}_\infty R' \end{array}$$

Lemma 1.2 *Let R_1 and R_2 be finitely generated unital commutative algebras. Then*

$$\mathcal{J}_\infty(R_1 \otimes R_2) \cong \mathcal{J}_\infty R_1 \otimes \mathcal{J}_\infty R_2$$

as differential algebras, where the differential of $\mathcal{J}_\infty R_1 \otimes \mathcal{J}_\infty R_2$ is given by $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$.

Proof For any differential algebra A, we have

$$\begin{aligned} \mathrm{Hom}_{\mathtt{Alg}}(R_1 \otimes R_2, A) &\cong \mathrm{Hom}_{\mathtt{Alg}}(R_1, A) \otimes \mathrm{Hom}_{\mathtt{Alg}}(R_2, A) \\ &\cong \mathrm{Hom}_{\mathtt{Dif.Alg}}(\mathcal{J}_\infty R_1, A) \otimes \mathrm{Hom}_{\mathtt{Dif.Alg}}(\mathcal{J}_\infty R_2, A) \\ &\cong \mathrm{Hom}_{\mathtt{Dif.Alg}}(\mathcal{J}_\infty R_1 \otimes \mathcal{J}_\infty R_2, A). \end{aligned}$$

This concludes the proof. □

Corollary 1.1 *Let A be a finitely generated commutative Hopf algebra with counit $\epsilon \colon A \to \mathbb{C}$, coproduct $\Delta \colon A \to A \otimes A$ and antipode $S \colon A \to A$. Then $\mathcal{J}_\infty A$ is a commutative Hopf algebra with counit $\mathcal{J}_\infty \epsilon$, coproduct $\mathcal{J}_\infty \Delta$ and antipode $\mathcal{J}_\infty S$. Moreover, if M is a comodule over A with comodule map $\mu \colon M \to A \otimes M$, then $\mathcal{J}_\infty M$ is a comodule over $\mathcal{J}_\infty A$ with comodule map $\mathcal{J}_\infty \mu$.*

Proof Note that $\mathcal{J}_\infty \mathbb{C} = \mathbb{C}$. Hence $\mathcal{J}_\infty \epsilon$ defines an algebra homomorphism $\mathcal{J}_\infty A \to \mathbb{C}$. It is straightforward to check the assertion using Lemma 1.2. □

For any $\mathbb{C}$-algebra A, we set

$$A[[z]] = \left\{ \sum_{n \geqslant 0} a_n z^n : a_n \in A \right\}, \tag{1.5}$$

which is naturally an algebra. Note that $A \otimes \mathbb{C}[[z]] \subsetneq A[[z]]$ in general.

As an algebra, $\mathcal{J}_\infty R$ has the following characterization.

Proposition 1.1 *For a finitely generated unital commutative $\mathbb{C}$-algebra R, $\mathcal{J}_\infty R$ is the unique (up to an isomorphism) unital commutative $\mathbb{C}$-algebra such that*

$$\mathrm{Hom}_{\mathtt{Alg}}(\mathcal{J}_\infty R, A) \cong \mathrm{Hom}_{\mathtt{Alg}}(R, A[[z]])$$

for any unital commutative $\mathbb{C}$-algebra A.

Proof The uniqueness follows from Yoneda's lemma.

For $f \in \mathcal{J}_\infty R$, we set

$$e^{z\partial} f = \sum_{n \geqslant 0} \frac{1}{n!} (\partial^n f) z^n \in (\mathcal{J}_\infty R)[[z]].$$

Next, for $\alpha \in \mathrm{Hom}_{\mathtt{Alg}}(\mathcal{J}_\infty R, A)$, we define $\Phi(\alpha) \in \mathrm{Hom}_{\mathtt{Alg}}(R, A[[z]])$ by

$$\Phi(\alpha)(f) = \alpha(e^{z\partial} f) = \sum_{n \geqslant 0} \alpha(\partial^n f / n!) z^n, \quad f \in R.$$

Since $e^{z\partial}(fg) = e^{z\partial}(f) e^{z\partial}(g)$, $\Phi(\alpha)$ is an algebra homomorphism. Hence, Φ defines a map $\Phi\colon \mathrm{Hom}_{\mathtt{Alg}}(\mathcal{J}_\infty R, A) \to \mathrm{Hom}_{\mathtt{Alg}}(R, A[[z]])$.

Conversely, define a map $\Psi\colon \mathrm{Hom}_{\mathtt{Alg}}(R, A[[z]]) \to \mathrm{Hom}_{\mathtt{Alg}}(\mathcal{J}_\infty R, A)$ by

$$\Psi(\beta)(\partial^n f) = \lim_{z \to 0} \partial_z^n (\beta(f)), \quad f \in R, \; n \in \mathbb{Z}_{\geqslant 0}.$$

It is easy to see that Ψ is well-defined and we have $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{id}$. This completes the proof. □

We have

$$\mathcal{J}_\infty R = \varinjlim_m \mathcal{J}_m R, \tag{1.6}$$

where $\mathcal{J}_m R$ is the subalgebra of $\mathcal{J}_\infty R$ generated by $\partial^j x_i$ with $i = 1, \ldots, N$, $j = 0, \ldots, m$ in the presentation (1.4). The inductive limit is taken with respect to the natural inclusions $\mathcal{J}_n R \hookrightarrow \mathcal{J}_m R$, for $n \leqslant m$.

The proof of the following assertion is similar to that of Proposition 1.1 and is left to the reader.

Proposition 1.2 *Let R be a finitely generated unital commutative $\mathbb{C}$-algebra. For $m \geqslant 0$, $\mathcal{J}_m R$ is the unique (up to isomorphisms) commutative $\mathbb{C}$-algebra such that*

$$\mathrm{Hom}_{\mathtt{Alg}}(\mathcal{J}_m R, A) \cong \mathrm{Hom}_{\mathtt{Alg}}(R, A[z]/(z^{m+1}))$$

for any unital commutative $\mathbb{C}$-algebra A.

Exercise 1.1 Let K be a field of characteristic $p > 0$. Let us call a unital commutative K-algebra A a *differential algebra* if it is equipped with linear maps

$$\partial^{[n]}\colon A \to A, \quad n \geqslant 0,$$

(that corresponds to the divided power differential $\partial^n/n!$) such that

$$\partial^{[n]}(ab) = \sum_{j=0}^{n} \partial^{[j]}(a)\partial^{[n-j]}(b), \quad a, b \in A.$$

Show that statements of Lemma 1.1, Proposition 1.1 and Proposition 1.2 hold by replacing $\mathbb{C}$-algebra by K-algebra and defining $\mathcal{J}_m R$ to be the subalgebra generated by $\partial^{[j]} x_i$ with $i = 1, \dots, N$, $j = 0, \dots, m$.

1.2 Arc spaces for affine schemes

Let $\mathtt{Sch}$ be the category of schemes over $\mathbb{C}$.

Let X be an affine scheme of finite type, that is, $X = \operatorname{Spec} R$ for some finitely generated unital commutative $\mathbb{C}$-algebra R. Define

$$\mathcal{J}_\infty X := \operatorname{Spec}(\mathcal{J}_\infty R). \tag{1.7}$$

Then by Proposition 1.1,

$$\begin{aligned}\{\mathbb{C}\text{-points of } \mathcal{J}_\infty X\} = \operatorname{Hom}_{\mathtt{Sch}}(\operatorname{Spec}\mathbb{C}, \mathcal{J}_\infty X) &= \operatorname{Hom}_{\mathtt{Alg}}(\mathcal{J}_\infty R, \mathbb{C}) \\ &\cong \operatorname{Hom}_{\mathtt{Alg}}(R, \mathbb{C}[[z]]) = \operatorname{Hom}_{\mathtt{Sch}}(D, X),\end{aligned}$$

where D is the (formal) disc defined by

$$D = \operatorname{Spec}\mathbb{C}[[z]].$$

A morphism $\gamma \colon D \to X$ is called an *arc* of X. The scheme $\mathcal{J}_\infty X$, whose $\mathbb{C}$-points are arcs of X, is called the *arc space* of X. Note that $\mathcal{J}_\infty X$ is a scheme of infinite type in general.

By Proposition 1.1, we have

$$\operatorname{Hom}_{\mathtt{Sch}}(\operatorname{Spec} A, \mathcal{J}_\infty X) \cong \operatorname{Hom}_{\mathtt{Sch}}(\operatorname{Spec} A[[z]], X) \tag{1.8}$$

for any commutative $\mathbb{C}$-algebra A, and the arc space $\mathcal{J}_\infty X$ is characterized as the unique scheme satisfying this property (see e.g. [98, VI.1]).

We also define for $m \geqslant 0$

$$\mathcal{J}_m X = \operatorname{Spec}(\mathcal{J}_m R). \tag{1.9}$$

By the similar argument using Proposition 1.2, we find that the $\mathbb{C}$-points of $\mathcal{J}_m X$ are the *m-jets* of X, that is, the morphisms

$$\operatorname{Spec}(\mathbb{C}[z]/(z^{m+1})) \longrightarrow X.$$

The scheme $\mathcal{J}_m X$ is called the *m-th jet scheme* of X. It is a scheme of finite type.

By Proposition 1.2, the m-th jet scheme $\mathcal{J}_m X$ is characterized as the unique scheme satisfying

$$\mathrm{Hom}_{\mathtt{Sch}}(\operatorname{Spec} A, \mathcal{J}_m X) \cong \mathrm{Hom}_{\mathtt{Sch}}(\operatorname{Spec} A[z]/(z^{m+1}), X) \tag{1.10}$$

for any commutative $\mathbb{C}$-algebra A.

Denote by π_m the canonical morphism

$$\pi_m \colon \mathcal{J}_m(X) \longrightarrow \mathcal{J}_0(X) \cong X$$

induced by the projection

$$\mathbb{C}[z]/(z^{m+1}) \twoheadrightarrow \mathbb{C}[z]/(z) \cong \mathbb{C}.$$

More generally, we have truncation morphisms:

$$\pi_{m,n} \colon \mathcal{J}_m(X) \longrightarrow \mathcal{J}_n(X), \qquad m \geqslant n,$$

induced by the projection

$$\mathbb{C}[z]/(z^{m+1}) \twoheadrightarrow \mathbb{C}[z]/(z^{n+1}).$$

Namely, $\pi_{m,n}$ is the affine scheme morphism whose comorphism is the embedding of $\mathbb{C}$-algebras $\mathcal{J}_n R \hookrightarrow \mathcal{J}_m R$.

By (1.6), we have

$$\mathcal{J}_\infty X = \varprojlim_m \mathcal{J}_m X \tag{1.11}$$

in the category of affine schemes. For $m \in \mathbb{Z}_{\geqslant 0}$, we have the canonical truncation morphism

$$\pi_{\infty,m} \colon \mathcal{J}_\infty(X) \longrightarrow \mathcal{J}_m(X)$$

whose comorphism is the the embedding of $\mathbb{C}$-algebras $\mathcal{J}_m R \hookrightarrow \mathcal{J}_\infty R$.

The canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[z]/(z^{m+1})$ induces a morphism $\iota_m \colon X \to \mathcal{J}_m(X)$ whose comorphism is the canonical projection of $\mathbb{C}$-algebras $\mathcal{J}_m R \twoheadrightarrow R$. Since $\pi_m \circ \iota_m = \mathrm{id}_X$, we get that π_m is surjective and ι_m is injective.

Example 1.1 Let

$$X = \operatorname{Spec} \mathbb{C}[x, y, z]/(x^2 + yz) \subset \mathbb{A}^3.$$

The equations of the embedding of $\mathcal{J}_\infty(X)$ in $\mathcal{J}_\infty(\mathbb{A}^3)$ are given by the vanishing of the coefficients of the polynomial in the variable t,

$$(x+(\partial x)t+\frac{1}{2}(\partial^2 x)t^2+\cdots)^2+(y+(\partial y)t+\frac{1}{2}(\partial^2 y)t^2+\cdots)(z+(\partial y)t+\frac{1}{2}(\partial^2 z)t^2+\cdots),$$

in $\mathbb{C}[[t]]$ or, equivalently, by the following equations:

$$\begin{cases} x^2 + yz & = 0 \\ 2x(\partial x) + y(\partial z) + z(\partial y) & = 0 \\ x(\partial^2 x) + (\partial x)^2 + (\partial y)(\partial z) + \dfrac{1}{2}(y(\partial^2 z) + z(\partial^2 y)) & = 0 \\ \vdots & \vdots \end{cases}.$$

The truncation morphism $\pi_{\infty,m}\colon \mathcal{J}_\infty(X) \to \mathcal{J}_m(X)$ is given by forgetting the coordinates $\partial^i x, \partial^i y, \partial^i z$, for $i > m$.

In the context of vertex algebras, it will be sometimes convenient to set

$$x_{(-i-1)} := \frac{1}{i!}\partial^i x,$$

for $x \in R \subset \mathcal{J}_\infty R$.

1.3 Arc spaces for general schemes

The result of this section is not used for the rest of the book. The following results are due to Greenberg ([141, 142]). We omit the proofs.

Theorem 1.1 (Greenberg) *Let X be a scheme of finite type.*

(i) *For any $m \in \mathbb{Z}_{\geqslant 0}$ there exists a unique scheme $\mathcal{J}_m X$ such that*

$$\mathrm{Hom}_{\mathcal{S}ch}(\mathrm{Spec}\, A, \mathcal{J}_m X) \cong \mathrm{Hom}_{\mathcal{S}ch}(\mathrm{Spec}\, A[z]/(z^{m+1}), X)$$

for any commutative $\mathbb{C}$-algebra A. Equivalently,

$$\mathrm{Hom}_{\mathcal{S}ch}(Z, \mathcal{J}_m X) \cong \mathrm{Hom}_{\mathcal{S}ch}(Z \times_{\mathrm{Spec}\,\mathbb{C}} \mathbb{C}[z]/(z^{m+1}), X)$$

for any scheme Z.

(ii) *there exists a unique scheme $\mathcal{J}_\infty X$ such that*

$$\mathrm{Hom}_{\mathcal{S}ch}(\mathrm{Spec}\, A, \mathcal{J}_\infty X) \cong \mathrm{Hom}_{\mathcal{S}ch}(\mathrm{Spec}\, A[[z]], X)$$

for any commutative $\mathbb{C}$-algebra A. Equivalently,

$$\mathrm{Hom}_{\mathcal{S}ch}(Z, \mathcal{J}_\infty X) \cong \mathrm{Hom}_{\mathcal{S}ch}(Z \widehat{\times}_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\,\mathbb{C}[[z]], X)$$

for any scheme Z, where $Z \widehat{\times}_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\,\mathbb{C}[[z]]$ is the formal completion of $Z \times \mathrm{Spec}\,\mathbb{C}[[z]]$ with respect to $Z \times \{0\}$.

Thus, the $\mathbb{C}$-points of $\mathcal{J}_m(X)$ are the $\mathbb{C}[z]/(z^{m+1})$-points of X, and the the $\mathbb{C}$-points of $\mathcal{J}_\infty(X)$ are the $\mathbb{C}[[z]]$-points of X. From Theorem 1.1, we have for example that $\mathcal{J}_0(X) \simeq X$ and that $\mathcal{J}_1(X) \simeq TX$, where TX denotes the total tangent bundle of X.

We have a canonical projection $\pi_{m,n}\colon \mathcal{J}_m(X) \to \mathcal{J}_n(X)$ for $m \geqslant n$. It is defined at the level of the functor of points using Theorem 1.1 (i): the induced map

$$\mathrm{Hom}_{\mathsf{Sch}}(\mathrm{Spec}\, A[z]/(z^{m+1}), X) \longrightarrow \mathrm{Hom}_{\mathsf{Sch}}(\mathrm{Spec}\, A[z]/(z^{n+1}), X)$$

is induced from the truncation morphism $A[z]/(z^{m+1}) \to A[z]/(z^{n+1})$. Similarly, we have a canonical projection $\pi_{\infty,m}\colon \mathcal{J}_\infty(X) \to \mathcal{J}_m(X)$ for $m \in \mathbb{Z}_{\geqslant 0}$.

For an arbitrary scheme X of finite type, the following lemma allows to describe the jet schemes $\mathcal{J}_m X$, for $m \in \mathbb{Z}_{\geqslant 0}$, and the arc scheme $\mathcal{J}_\infty X$ from the affine case.

Lemma 1.3 *Given any $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and any open subset U of X, $\mathcal{J}_m(U) = \pi_m^{-1}(U)$.*

Proof We follow the arguments of [96]. Assume first $m \in \mathbb{Z}_{\geqslant 0}$. Let A be a $\mathbb{C}$-algebra and

$$j_m\colon\ \mathrm{Spec}\, A \to \mathrm{Spec}\, A[z]/(z^{m+1})$$

be the morphism induced by truncation. An A-valued point of $\mathcal{J}_m(X)$ is a morphism of schemes $\gamma\colon\ \mathrm{Spec}\, A[z]/(z^{m+1}) \to X$. Such a morphism is an A-valued point of $\pi_m^{-1}(U)$ if and only if $\gamma \circ j_m$ factors through U. Clearly, if γ is an A-valued point of $\mathcal{J}_m(U)$, that is, the image of γ lies in U, then γ is an A-valued point of $\pi_m^{-1}(U)$.

Conversely, assume that $\gamma\colon\ \mathrm{Spec}\, A[z]/(z^{m+1}) \to X$ is an A-valued point of $\pi_m^{-1}(U)$. Then $\gamma \circ j_m$ factors through U. Note that the set of prime ideals of $A[z]/(z^{m+1}) = A \otimes \mathbb{C}[z]/(z^{m+1})$ is in one-to-one correspondence with the set of prime ideals of A since $\mathrm{Spec}\,\mathbb{C}[z]/(z^{m+1})$ contains a unique element. Hence, γ induces a map from $\mathrm{Spec}\, A[z]/(z^{m+1})$ to U (just between sets). Because U is open in X, we have $\mathcal{O}_U \cong \mathcal{O}_X|_U$. Hence the map induced from the morphism of schemes γ is automatically a morphism, too. So γ induces a morphism $\mathrm{Spec}\, A[z]/(z^{m+1}) \to U$, that is, an A-valued point of $\mathcal{J}_m(U)$.

For $m = \infty$, the statement is obtained by taking the projective limit since $\pi_{\infty,0}^{-1}(U) = \varprojlim_m \pi_m^{-1}(U)$ and $\mathcal{J}_\infty(U) = \varprojlim_m \mathcal{J}_m(U)$. □

It follows from the lemma that for an arbitrary scheme X of finite type with an affine open covering $\{U_i\}_{i\in I}$, its jet scheme $\mathcal{J}_m(X)$ is obtained by glueing the jet schemes $\mathcal{J}_m(U_i)$ (see [96, 151]). Over an affine open subset $U_i \subset X$, the space of arcs is described by

$$\left(\pi_{\infty,*}\mathcal{O}_{\mathcal{J}_\infty X}\right)(U_i) = \mathscr{O}_{\mathcal{J}_\infty X}(\pi_{\infty,0}^{-1}(U_i)) = \varinjlim \mathscr{O}_{\mathcal{J}_m X}(\pi_m^{-1}(U_i)) = \mathscr{O}_{\mathcal{J}_\infty X}(\mathcal{J}_\infty U_i),$$

where $\pi_{\infty,*}\mathcal{O}_{\mathcal{J}_\infty X}$ denotes the pushforward sheaf of $\mathcal{O}_{\mathcal{J}_\infty X}$ induced by

$$\pi_{\infty,0}\colon \mathcal{J}_\infty \to X.$$

In particular, the structure sheaf $(\pi_{\infty,0})_*\mathscr{O}_{\mathcal{J}_\infty(X)}$ is a sheaf of differential algebras on X.

1.4 Functorial properties

The map from a scheme to its jet schemes, or its arc space, is functorial. If $f\colon X \to Y$ is a morphism of schemes, then we naturally obtain a morphism $\mathscr{J}_m f\colon \mathscr{J}_m(X) \to \mathscr{J}_m(Y)$ making the following diagram commutative,

$$\begin{array}{ccc} \mathscr{J}_m(X) & \xrightarrow{\mathscr{J}_m f} & \mathscr{J}_m(Y) \\ \Big\downarrow{\scriptstyle \pi_m} & & \Big\downarrow{\scriptstyle \pi_m} \\ X & \xrightarrow[f]{} & Y \end{array}$$

In terms of arcs, it means that $\mathscr{J}_m f(\alpha) = f \circ \alpha$ for $\alpha \in \mathscr{J}_m(X)$. This also holds for $m = \infty$.

In addition, we have the following results.

Lemma 1.4 *Let $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. For every schemes X, Y, we have a canonical isomorphism*

$$\mathscr{J}_m(X \times Y) \simeq \mathscr{J}_m(X) \times \mathscr{J}_m(Y).$$

For X, Y affines, and $m = \infty$, the lemma is just a reformulation of Lemma 1.2.

Proof Assume first that $m \in \mathbb{Z}_{\geqslant 0}$. Then for any affine scheme Z in Sch,

$$\begin{aligned} \operatorname{Hom}(Z, \mathscr{J}_m(X \times Y)) &\cong \operatorname{Hom}(Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1}), X \times Y) \\ &\cong \operatorname{Hom}(Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1}), X) \times \operatorname{Hom}(Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1}), Y) \\ &\cong \operatorname{Hom}(Z, \mathscr{J}_m(X)) \times \operatorname{Hom}(Z, \mathscr{J}_m(Y)) \\ &\cong \operatorname{Hom}(Z, \mathscr{J}_m(X) \times \mathscr{J}_m(Y)), \end{aligned}$$

whence the statement in this case. For $m = \infty$, just replace $\mathbb{C}[z]/(z^{m+1})$ with $\mathbb{C}[[z]]$ and take the completion $Z \widehat{\times}_{\operatorname{Spec} \mathbb{C}} \operatorname{Spec} \mathbb{C}[[z]]$ instead of $Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1})$. □

Let $f\colon X \to Y$ be a morphism between affine schemes $X, Y \in Sch$. Recall that f is called *formally smooth* (resp. *unramified*, *étale*) if for every $\mathbb{C}$-algebra A, every nilpotent ideal J of A, and every commutative square,

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{\varphi_0} & X \\ \Big\downarrow & \overset{\varphi}{\nearrow} & \Big\downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow[\psi]{} & Y, \end{array}$$

where $B = A/J$, there exists one (resp. at most one, one unique) diagonal arrow φ, called the *lifting*, making the two triangles commutatives, [144].

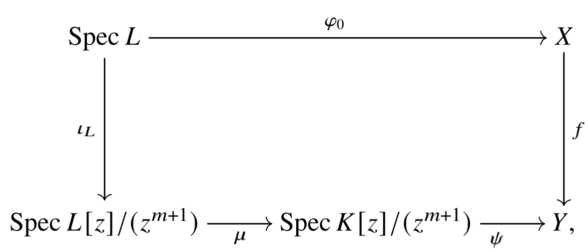
Since f is of finite type (the schemes X, Y are of finite type), the morphism f is formally smooth if and only if it is smooth. For the relation between formal and standard smoothness we refer for instance to [208, Chapter 10, Section 28].

Lemma 1.5 *If $f\colon X \to Y$ is a smooth surjective morphism between affine schemes $X, Y \in Sch$, then for every $m \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{J}_m f$ is also smooth and surjective. Moreover, $\mathcal{J}_\infty f$ is formally smooth and surjective.*

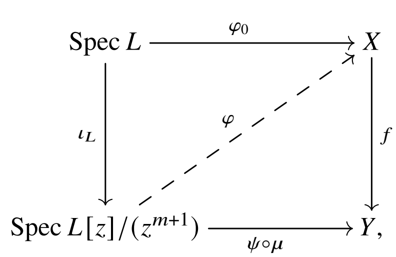
Recall [247, proposition 4.8] that a morphism of schemes $f\colon X \to Y$ is surjective if and only if for any field K and any $y \in Y(K)$ there is a field extension L/K and $x \in X(L)$ whose image by $X(L) \to Y(L)$ is the image of y under $Y(K) \to Y(L)$

Proof We follow the proof of [73, Proposition 3.7.1 and 3.7.4]. We prove at the same time the statements for $\mathcal{J}_m f$ and $\mathcal{J}_\infty f$. In the latter case, set $m = \infty$ and for every $\mathbb{C}$-algebra A, read $A[z]/(z^{m+1})$ as $A[[z]]$.

Let us first prove the surjectivity. Let K be a field. Given a K-valued point $\mathcal{J}_m Y(K)$ is the same as giving a morphism $\psi\colon \operatorname{Spec} K[z]/(z^{m+1}) \to Y$. Denoting by $\iota_K\colon \operatorname{Spec} K \to \operatorname{Spec} K[z]/(z^{m+1})$ the natural closed immersion, the composition map $\psi \circ \iota_K\colon \operatorname{Spec} K \to Y$ yields a K-valued point y. Since f is surjective, there is a field extension L/K and $x \in X(L)$ whose image by $X(L) \to Y(L)$ is the image of y under $Y(K) \to Y(L)$. Hence we get an L-valued point $\varphi_0\colon \operatorname{Spec} L \to X$ such that the following diagram commutes:

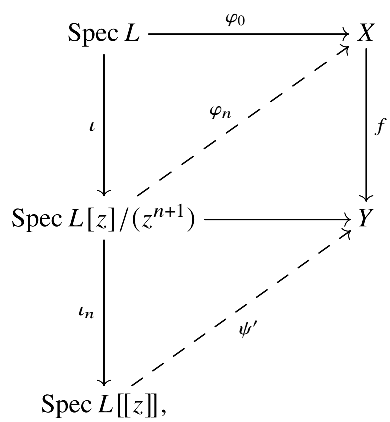


where $\mu\colon \operatorname{Spec} L[z]/(z^{m+1}) \to \operatorname{Spec} K[z]/(z^{m+1})$ is the natural morphism induced from $K \hookrightarrow L$. Assume first that $m < \infty$. Then the ideal of $\ker \iota_L^*$ is generated by (z), hence it is nilpotent in $L[z]/(z^{m+1})$. The morphism f being formally smooth, there exists a morphism $\varphi\colon \operatorname{Spec} L[z]/(z^{m+1}) \to X$, making the two triangles commutative:



that is, $\mathcal{J}_m f(\varphi) = \psi'$, with $\psi' := \psi \circ \mu$. This proves the surjectivity of $\mathcal{J}_m f$.

Assume now that $m = \infty$. Since $\mathcal{J}_\infty X$ is the projective limit of the $\mathcal{J}_m X$, in order to show the surjectivity of $\mathcal{J}_\infty f$ it is enough to check the compatibility of $\mathcal{J}_m f$ with the truncation morphisms $\pi_{m,n}\colon \mathcal{J}_m X \to \mathcal{J}_n X$. The foregoing shows that for every n, there exists a morphism $\varphi_n\colon \operatorname{Spec} L[z]/(z^{n+1}) \to X$ such that $\psi' \circ \iota_n = f \circ \varphi_n$, that is, $\pi_{\infty,n}(\psi) = \mathcal{J}_n f(\varphi_n)$:



where $\iota_n\colon \operatorname{Spec} L[z]/(z^{n+1}) \to \operatorname{Spec} L[[z]]$ is the canonical closed immersion. Moreover, $\pi_{m,n}(\varphi_m) = \varphi_n$ for every $m \geqslant n$. Therefore, the family $(\varphi_n)_n$ defines an L-valued point of $\varprojlim \mathcal{J}_n(X)$, hence, an L-arc φ of X. This proves the surjectivity of $\mathcal{J}_\infty f$.

We now show that $\mathcal{J}_m(X)$ is formally smooth, for $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. As before, we prove together the statement for $\mathcal{J}_m f$ and $\mathcal{J}_\infty f$. Notice that for $m < \infty$, $\mathcal{J}_m f$ is of finite type because f is so. Hence $\mathcal{J}_m f$ will be smooth if formally smooth.

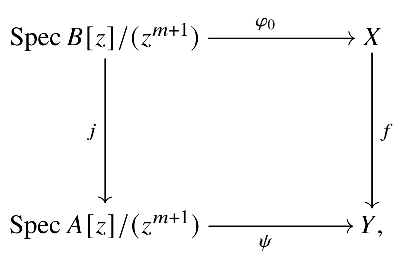
Let A be a $\mathbb{C}$-algebra, J a nilpotent ideal of A and set $B = A/J$. Let

$$\psi\colon \operatorname{Spec} A[z]/(z^{m+1}) \to Y$$

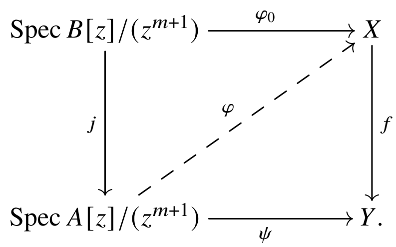
be an A-valued point of $\mathcal{J}_m Y$ and

$$\varphi_0\colon \operatorname{Spec} B[z]/(z^{m+1}) \to X$$

a B-valued point of $\mathcal{J}_m X$ such that $f \circ \varphi_0 = \psi \circ j$,



where $j\colon \operatorname{Spec} B[z]/(z^{m+1}) \to \operatorname{Spec} A[z]/(z^{m+1})$ is the canonical closed immersion. The ideal of $\ker j^*$ is generated by $J[z]/(z^{m+1})$, hence it is nilpotent. Since f is formally smooth, there exists a morphism $\varphi\colon \operatorname{Spec} A[z]/(z^{m+1}) \to X$ making the two triangles commutatives:



This shows that the morphism $\mathscr{J}_m f$ is formally smooth. Indeed, by definition, an A-valued point of $\mathscr{J}_m Y$ (respectively, a B-valued point of $\mathscr{J}_m X$) is an $A[z]/(z^{m+1})$-valued point of Y (respectively, $B[z]/(z^{m+1})$-valued point of X). □

Remark 1.1 Similarly, one can show that if $f\colon X \to Y$ is a formally étale morphism of affine schemes, then the canonical morphism $\mathscr{J}_m(X) \to \mathscr{J}_m(Y) \times_Y X$ induced by $\mathscr{J}_m(f)$ and $\pi_m\colon \mathscr{J}_m(Y) \to Y$ is an isomorphism. Hence Lemma 1.3 also follows from this fact applied to the open immersion $U \hookrightarrow X$ since $\pi_m^{-1}(U) \cong \mathscr{J}_m(X) \times_X U$.

1.5 Geometric properties of arc spaces

It is known that the geometry of the jet schemes $\mathscr{J}_m(X)$, for $m \geqslant 1$, is closely linked to that of X. More precisely, we can transport some geometrical properties from $\mathscr{J}_m(X)$ to X.

The following proposition gives examples of such phenomena.

Proposition 1.3 *Let $m \in \mathbb{Z}_{\geqslant 0}$, and let X be an affine scheme of finite type. If $\mathscr{J}_m(X)$ is smooth (respectively, irreducible, reduced, normal, locally a complete intersection) for some m, then so is X.*

For smoothness, the converse is true, even with “every m” instead of “for some m”. In fact, for smooth varieties, we have the following more precise statement, [96, Corollary 2.11].

Proposition 1.4 *If X is a smooth variety of dimension N, then the truncation morphism $\pi_{m,p}$, for $p \in \{0, \dots, m\}$, is a Zariski locally trivial projection with fiber isomorphic to $\mathbb{A}^{(m-p)N}$. In particular, $\mathscr{J}_m(X)$ is a smooth variety of dimension $(m+1)N$.*

Proof Around every point in X we can find an open subset U and an étale morphism $U \to \mathbb{A}^N$. Using Remark 1.1 the assertion reduced to the case where X is the affine space $\mathbb{A}^N$, in which case the statement is clear by Sections 1.1 and 1.2. □

For the other properties stated in Proposition 1.3, the converse is not true in general. We refer for instance to [151, §3] for counter-examples. See also [220] for counter-examples in the setting of nilpotent orbit closures in a simple Lie algebra.

The following lemma gives a necessary and sufficient condition for the converse of Proposition 1.3 to hold for irreducibility. Recall that the smooth part of a variety X is denoted by X_{reg}, and its complement by X_{sing}.

Lemma 1.6 *Assume that X is an irreducible reduced affine scheme of finite type over $\mathbb{C}$, and let $m \in \mathbb{Z}_{\geqslant 0}$. Then the Zariski closure of $\pi_{X,m}^{-1}(X_{\mathrm{reg}})$ is an irreducible component of $\mathcal{J}_m(X)$, and $\mathcal{J}_m(X)$ is irreducible if and only if $\pi_{X,m}^{-1}(X_{\mathrm{sing}})$ is contained in the Zariski closure of $\pi_{X,m}^{-1}(X_{\mathrm{reg}})$.*

Proof Since X_{reg} is smooth and irreducible, the Zariski closure $\overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$ of $\pi_{X,m}^{-1}(X_{\mathrm{reg}})$ is an irreducible closed subset of $\mathcal{J}_m(X)$ of dimension $(m+1)\dim X$ by Proposition 1.4. Then the lemma easily follows from the fact that we have the decomposition

$$\mathcal{J}_m(X) = \pi_{X,m}^{-1}(X_{\mathrm{sing}}) \cup \overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$$

of closed subsets, and that $\pi_{X,m}^{-1}(X_{\mathrm{sing}})$ does not contain $\overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$. □

It was suggested by Nash that the study of the geometry of $\mathcal{J}_m(X)$, or of $\pi_{\infty,m}(\mathcal{J}_\infty(X))$, should give information about the singularities of X. For example, we have the following result ([221, Corollary 4.2]).

Proposition 1.5 *If X is an integral curve, then for any $m \geqslant 1$, $\mathcal{J}_m(X)$ is irreducible if and only if X is smooth.*

Proof If $x \in X$ is a singular point, then $\dim T_x X \geqslant 2$ and by [221, Lemma 4.1], it follows that for every $m \geqslant 1$, $\dim \pi_{m,0}^{-1}(x) \geqslant m+1$. Therefore, $\pi_{m,0}^{-1}(x)$ gives an irreducible component of $\mathcal{J}_m(X)$ according to Lemma 1.6. □

In the same spirit, motivated by the case of the nilpotent cone of a reductive Lie algebra (see Example 1.3), Eisenbud and Frenkel conjectured the following result, proved by Mustaţă ([221]).

Theorem 1.2 *Let X be an irreducible affine variety over $\mathbb{C}$.*

(i) *If X is a complete intersection, then $\mathcal{J}_m(X)$ is irreducible for every $m \geqslant 1$ if and only if X has rational singularities.*
(ii) *If X is a complete intersection and if $\mathcal{J}_m(X)$ is irreducible for some $m \geqslant 1$, then $\mathcal{J}_m(X)$ is also reduced.*

We have seen that jet schemes and arc spaces share several functorial properties. For topological properties, they behave rather differently. The main reason is that $\mathbb{C}[[z]]$ is a domain, contrary to $\mathbb{C}[z]/(z^{m+1})$. Thereby, although $\mathcal{J}_\infty(X)$ is not of finite type in general, its geometric properties are somehow simpler than those of the finite jet schemes $\mathcal{J}_m(X)$.

Let us now turn to topological properties of the arc spaces. As a rule, for X a scheme, we shall denote by $X_{\mathrm{red}} \subset X$ the reduced scheme associated with X. When $X = \operatorname{Spec} R$ is affine, this is nothing but the affine scheme $X_{\mathrm{red}} = \operatorname{Spec} R/\sqrt{0}$, with $\sqrt{0}$ the radical of R.

Lemma 1.7 *Let X be a scheme. The natural morphism $X_{\mathrm{red}} \to X$ induces an isomorphism*

$$\mathcal{J}_\infty X_{\mathrm{red}} \xrightarrow{\sim} \mathcal{J}_\infty X$$

of topological spaces.

Proof We may assume that $X = \operatorname{Spec} R$, with R a ring. An arc α of X corresponds to a ring homomorphism $\alpha^* \colon R \to \mathbb{C}[[z]]$. Since $\mathbb{C}[[z]]$ is an integral domain, it decomposes as $\alpha^* \colon R \to R/\sqrt{0} \to \mathbb{C}[[z]]$. Thus, α is an arc of X_{red}. □

Note that Lemma 1.7 is false for the schemes $\mathcal{J}_m(X)$.

! Warning

If $\mathcal{J}_\infty X$ is reduced, then X is reduced, but $\mathcal{J}_\infty X$ no need to be reduced if X is reduced.

The following example was discovered by Julien Sebag [233]: let X be the hypersurface of $\mathbb{A}^2$ defined by equation $x^3 - y^2 = 0$. Then X is reduced, and one can verify that $3y(\partial x) - 2x(\partial y)$ is a nilpotent element of $\mathbb{C}[\mathcal{J}_\infty X]$.

Mustaţă's result (Theorem 1.2) furnishes a converse to the above "warning" in the case where X is a locally complete intersection with rational singularities ([221]).

Theorem 1.3 *If X is a locally complete intersection with rational singularities, then $\mathcal{J}_\infty$ is reduced (and irreducible).*

If X is a point (as topological space), then $\mathcal{J}_\infty(X)$ is also a point (as topological space), because $\operatorname{Hom}(D, X) = \operatorname{Hom}(\mathbb{C}, \mathbb{C}[[z]])$ consists of only one element. Thus, Lemma 1.7 implies the following.

Corollary 1.2 *If X is zero-dimensional, then $\mathcal{J}_\infty(X)$ is also zero-dimensional.*

In contrast to jet schemes, the irreducibility property is preserved for the space of arcs.

Theorem 1.4 (Kolchin) *The arc scheme $\mathcal{J}_\infty(X)$ is irreducible if and only if X is irreducible.*

Proof Since $\mathcal{J}_\infty X \cong \mathcal{J}_\infty X_{\mathrm{red}}$ as topological spaces, we may assume that X is reduced. Assume first that X is smooth. In this case, the result is easy. Indeed, by Proposition 1.4, the jet schemes $\mathcal{J}_m X$ are smooth for any m and the canonical projections $\mathcal{J}_\infty X \to \mathcal{J}_m X$ are all surjective. Therefore $\mathcal{J}_\infty X = \varprojlim_m \mathcal{J}_m X$ with the projective limit topology is irreducible, too.

Consider now the general case. We argue by induction on $d = \dim X$, the $d = 0$ case being trivial. By Hironaka's Theorem, there is a resolution of singularities $f\colon X' \to X$. In particular, f is a proper morphism and X' is smooth. Suppose that Z is a proper closed subset of X such that f is an isomorphism over $U = X \setminus Z$.

We claim that

$$\mathcal{J}_\infty(X) = \mathcal{J}_\infty(Z) \cup \mathrm{Im}(\mathcal{J}_\infty f). \tag{1.12}$$

Indeed, since f is proper, the Valuative Criterion for properness implies that an arc $\gamma\colon \operatorname{Spec}\mathbb{C}[[z]] \to X$ lies in the image of $\mathcal{J}_\infty(f)$ if and only if the induced morphism $\overline{\gamma}\colon \operatorname{Spec}\mathbb{C}((z)) \to X$ can be lifted to X' (moreover, if the lifting of $\overline{\gamma}$ is unique, then the lifting of γ is also unique). On the other hand, γ does not lie in $\mathcal{J}_\infty(Z)$ if and only if $\overline{\gamma}$ factors through $U \hookrightarrow X$. In this case, the lifting of $\overline{\gamma}$ exists and is unique since f is an isomorphism over U. This proves (1.12).

The smooth case implies that $\mathcal{J}_\infty(X')$ is irreducible and so is $\mathrm{Im}(\mathcal{J}_\infty f)$. Hence by (1.12) it only remains to prove that $\mathcal{J}_\infty(Z)$ is contained in the closure of $\mathrm{Im}(\mathcal{J}_\infty f)$.

Consider the irreducible decomposition $Z = Z_1 \cup \ldots \cup Z_r$, inducing by $\mathcal{J}_\infty(Z) = \mathcal{J}_\infty(Z_1) \cup \ldots \cup \mathcal{J}_\infty(Z_r)$. Since f is surjective and proper, for any i, there is an irreducible component Z_i' of $f^{-1}(Z_i)$ such that the induced map $Z_i' \to Z_i$ is surjective. By the Generic Smoothness Theorem ([185, Corollary 10.7]), one can find open subsets U_i' and U_i in Z_i' and Z_i, respectively, such that induced morphisms $g_i\colon U_i' \to U_i$ are smooth and surjective. In particular, we get

$$\mathcal{J}_\infty(U_i) = \mathrm{Im}(\mathcal{J}_\infty(g_i)) \subseteq \mathrm{Im}(\mathcal{J}_\infty f).$$

On the other hand, by induction, every $\mathcal{J}_\infty(Z_i)$ are irreducible. Since $\mathcal{J}_\infty(U_i)$ is a nonempty open subset of $\mathcal{J}_\infty(Z_i)$, it follows that

$$\mathcal{J}_\infty(Z_i) \subseteq \overline{\mathrm{Im}(\mathcal{J}_\infty f)}$$

for every i. This completes the proof of the theorem. □

This result is classically referred to as the Kolchin irreducibility theorem, and is an analogue for arc schemes of a theorem in differential algebra [181, IV.17, Prop. 10].

As a consequence of Kolchin's Irreducibility Theorem, if $X_1, \ldots, X_r$ are the irreducible components of X, then $\mathcal{J}_\infty(X_1), \ldots, \mathcal{J}_\infty(X_r)$ are the irreducible components of $\mathcal{J}_\infty X$.

Lemma 1.8 *Let Y be an irreducible affine scheme, and let $f\colon X \to Y$ be a morphism that restricts to a bijection between some open subsets $U \subset X$ and $V \subset Y$. Then $\mathcal{J}_\infty f\colon \mathcal{J}_\infty(X) \to \mathcal{J}_\infty(Y)$ is dominant.*

Proof The map $\mathcal{J}_\infty f$ restricts to the isomorphism $\mathcal{J}_\infty(U) \xrightarrow{\sim} \mathcal{J}_\infty(V)$, and the open subset $\mathcal{J}_\infty(V)$ is dense in $\mathcal{J}_\infty(Y)$ since $\mathcal{J}_\infty(Y)$ is irreducible. □

Remark 1.2 Note that all results of Sections 1.4 and 1.5 hold for any scheme of finite type (not necessarily affine).

1.6 Loop spaces

In the context of vertex algebras one needs also to consider the *loop space* $\mathscr{L}X$ of an affine scheme X. One of the reasons is that an $\mathcal{O}(\mathcal{J}_\infty X)$-module as a vertex algebra is the same as a *smooth module* over the topological ring $\mathcal{O}(\mathscr{L}X)$ (see Section 2.13).

Proposition 1.6 *Let X be an affine scheme of finite type over $\mathbb{C}$.*

(i) *There exists a unique, up to isomorphism, ind-scheme $\mathscr{L}X$ which is the inductive limit of affine schemes $\mathscr{L}_n X$ of infinite type such that for any commutative $\mathbb{C}$-algebra A,*

$$\mathrm{Hom}_{\mathtt{Alg}}(\mathcal{O}(X), A((z))) \cong \mathrm{Hom}_{\mathtt{Alg}}(\mathcal{O}(\mathscr{L}X), A),$$

where $A((z)) = \varinjlim_n z^{-n} A[[z]]$.

(ii) *If X is smooth, then $\mathscr{L}X$ is formally smooth.*

For a commutative $\mathbb{C}$-algebra A, by

$$\mathrm{Hom}_{\mathtt{Alg}}(\mathcal{O}(\mathscr{L}X), A)$$

we always mean the set of *continuous morphisms*, that is, the morphisms from $\mathcal{O}(\mathscr{L}X)$ to A which factorize through one of the quotients of the projective limit,

$$\rho_{\infty,n} \colon \mathcal{O}(\mathscr{L}X) \twoheadrightarrow \mathcal{O}(\mathscr{L}_n X).$$

Proof The unicity of the ind-scheme $\mathscr{L}X$ follows from Yoneda's lemma.

(i) Assume first that $X = \mathbb{A}^N = \mathrm{Spec}\, \mathbb{C}[x^i]_{i=1,\dots,N}$. Then set

$$\mathscr{L}X = \varinjlim_n \mathrm{Spec}\, \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n},$$

where the coordinate $x^i_{(-j-1)}$ is defined by sending a morphism $\gamma \colon \mathbb{C}[x^i]_i \to \mathbb{C}((z))$ giving by $\gamma(x^i) = \sum_{j\gg -\infty} \gamma^i_{(-j-1)} z^j$ to the scalar $\gamma^i_{(-j-1)}$. We have

$$\mathcal{O}(\mathscr{L}X) = \varprojlim_n \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}, \tag{1.13}$$

with respect to the surjective homomorphisms

$$\rho_{m,n} \colon \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -m} \twoheadrightarrow \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}, \qquad m \geqslant n.$$

A continuous algebra morphism from $\mathcal{O}(\mathscr{L}X)$ to a $\mathbb{C}$-algebra A is the same as a morphism from $\mathcal{O}(\mathscr{L}X)$ to A which factorizes through one the quotient morphisms

$$\rho_{\infty,n} \colon \mathcal{O}(\mathscr{L}X) \twoheadrightarrow \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}.$$

Hence we get that for every commutative $\mathbb{C}$-algebra A,

$$\begin{aligned}\mathrm{Hom}_{\mathtt{Alg}}(\mathcal{O}(\mathscr{L}X), A) &\cong \varinjlim_n \mathrm{Hom}_{\mathtt{Alg}}(\mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}, A)\\ &\cong \mathrm{Hom}_{\mathtt{Alg}}(\mathcal{O}(X), \varinjlim_n z^{-n}A[[z]])\\ &\cong \mathrm{Hom}_{\mathtt{Alg}}(\mathcal{O}(X), A((z))),\end{aligned}$$

and $\mathscr{L}X$ satisfies the required condition.

Suppose now that $X = \operatorname{Spec} R$ if an affine subscheme of $\mathbb{A}^N$ defined by equations $f_1, \ldots, f_r$. Any polynomial $f \in \mathbb{C}[x^i]_{i=1,\ldots,N}$ induces a morphism of ind-schemes $\tilde{f}\colon \mathscr{L}\mathbb{A}^N \to \mathscr{L}\mathbb{A}$ via base extension. Hence one may realize the loop space $\mathscr{L}X$ as the sub-ind-scheme of $\mathscr{L}\mathbb{A}^N$ defined by the equations $\tilde{f}_1, \ldots, \tilde{f}_r$. More concretely, replacing x^i by $x^i(z) = \sum_{j\geqslant -n} x^i_{(-j-1)} z^j$ in the equations f_k, we get, for each m, a system of equations in $\mathbb{C}[x^i_{(-j-1)} : i = 1, \ldots, N,\ j \geqslant -m]$ which defines a subscheme in $\mathscr{L}\mathbb{A}^N$. Our desired ind-scheme $\mathscr{L}X$ is the inductive limit of these schemes as $n \to \infty$.

(ii) Assume that $X = \operatorname{Spec} R$ is smooth. We need to prove that for any surjection of $\mathbb{C}$-algebras $B \to A$ whose kernel J satisfies $I^n = 0$ for some n, the map of sets $\mathrm{Hom}_{\mathtt{Alg}}(R, B((z))) \to \mathrm{Hom}_{alg}(R, A((z)))$ is surjective. But the kernel of $B((z)) \to A((z))$ is $J((z))$ which is also nilpotent of order n. So the smoothness of R implies that any morphism $R \to A((z))$ can be lifted to a morphism $R \to B((z))$. □

Since X is separated, the valuative criterion for separated morphisms gives an inclusion of the arc space $\mathcal{J}_\infty X$ into the loop space $\mathscr{L}X$. One could extend the definition to any scheme, and the inclusion

$$\mathcal{J}_\infty X \subset \mathscr{L}X$$

would still hold. The valuative criterion for properness guarantees that this inclusion is a bijection if and only if X is proper. In fact, if X is proper, let's say projective, then there is no difference between $A[[z]]$-points and $A((z))$-points of X. However, the category of $\mathcal{O}(\mathcal{J}X)$ is different than the category of $\mathcal{O}(\mathscr{L}X)$. In this book, we only need to consider the case of affine schemes. We refer the reader to [170] for an appropriate construction in a more general setting.

1.7 Arc spaces of group schemes acting on an algebraic variety

A *proalgebraic group* is an inverse limit of algebraic groups. As a consequence of Lemma 1.4 we get the following result.

Lemma 1.9 *Let $m \in \mathbb{Z}_{\geqslant 0}$ (respectively, $m = \infty$). If G is a group scheme over $\mathbb{C}$, then $\mathcal{J}_m(G)$ is also a group scheme (respectively, a proalgebraic group scheme) over $\mathbb{C}$. Moreover, if G acts on X, then $\mathcal{J}_m(G)$ acts on $\mathcal{J}_m(X)$.*

Proof According to Lemma 1.4, the multiplication morphism $\mu\colon G \times G \to G$ induces a morphism $\mathcal{J}_m\mu\colon \mathcal{J}_m(G\times G) \cong \mathcal{J}_m G\times \mathcal{J}_m G \to \mathcal{J}_m G$ for any m. Moreover, since the jet of a point is a point, the restrictions to $\{e\}\times G \to G$ and $G\times\{e\} \to G$ of μ induces morphisms $\{e\}\times \mathcal{J}_m G \to \mathcal{J}_m G$ and $\mathcal{J}_m G\times\{e\} \to \mathcal{J}_m G$ and, so, the neutral element e of G is still a neutral element for the operation $\mathcal{J}_m\mu$. From this, it is easy to verify that the operation $\mathcal{J}_m\mu$ gives to $\mathcal{J}_m G$ a group scheme (respectively, a proalgebraic group scheme) structure.

Suppose now that G acts on X. The above group scheme (respectively, a proalgebraic group scheme) structure shows that $\mathcal{J}_m G$ acts on $\mathcal{J}_m X$ using the map $\mathcal{J}_m(G\times X) \cong \mathcal{J}_m G \times \mathcal{J}_m X \to \mathcal{J}_m X$ induces from the action map $G\times X \to X$. ◻

Remark 1.3 As a consequence of Lemma 1.9, if G is an affine group scheme over $\mathbb{C}$ acting on an affine scheme X, the action comorphism,

$$\mathcal{O}(\mathcal{J}_\infty X) \to \mathcal{O}(\mathcal{J}_\infty G)\otimes\mathcal{O}(\mathcal{J}_\infty X),$$

is a morphism of differential algebras, hence is a morphism of commutative vertex algebras (see §2.10 for the notion of *commutative vertex algebra*).

Example 1.2 Let G be an linear algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$. By Lemma 1.9, $\mathcal{J}_\infty G$ is an affine proalgebraic group, whose $\mathbb{C}$-points are the $\mathbb{C}[[t]]$-points of G. We denote by $G[[t]]$ the set of $\mathbb{C}$-points of G. We have

$$\mathrm{Lie}(\mathcal{J}_\infty G) = \mathcal{J}_\infty\mathfrak{g} = \mathfrak{g}[[t]], \quad \mathrm{Lie}(\mathcal{J}_r G) = \mathcal{J}_r\mathfrak{g} = \mathfrak{g}[t]/(t^{r+1}),$$

with Lie bracket:

$$[xt^m, yt^n] = [x,y]t^{m+n}, \qquad x,y\in\mathfrak{g},\ m,n\in\mathbb{Z}_{\geqslant 0}. \tag{1.14}$$

Indeed, by definition, for $r \in \mathbb{Z}_{\geqslant 0}\sqcup\{\infty\}$, $\mathrm{Lie}(\mathcal{J}_r G)$ is the Lie algebra of the left invariant vector fields on $\mathcal{J}_r G$, that is,

$$\mathrm{Lie}(\mathcal{J}_r G) = \{D\in \mathrm{Der}(\mathcal{O}(\mathcal{J}_r G))\colon \Delta\circ D = (1\otimes D)\circ\Delta\}$$

(see Appendix C, Section C.3), where $\Delta\colon \mathcal{O}(\mathcal{J}_r G) \to \mathcal{O}(\mathcal{J}_r G)\otimes\mathcal{O}(\mathcal{J}_r G)$ is the coproduct induced by the coproduct of $\mathcal{O}(G)$, see Corollary 1.1. Note that if $\Delta(f) = \sum_i u_i\otimes v_i$ for $f\in\mathcal{O}(G)$, we have

$$\Delta(f_{(-n-1)}) = \sum_i\sum_{k=0}^{n}(u_i)_{(k-n-1)}\otimes(v_i)_{(-k-1)}, \tag{1.15}$$

where $f_{(-n-1)} = \partial^n f/n!$. This is clear for $r=\infty$ since Δ is the homomorphism of differential algebras, and the coproduct of $\mathcal{O}(\mathcal{J}_r G)$ is obtained by restricting the coproduct of $\mathcal{O}(\mathcal{J}_\infty G)$ to $\mathcal{O}(\mathcal{J}_r G)$. Let $r\in\mathbb{Z}_{\geqslant 0}$ and consider the Lie algebra homomorphism $\phi\colon \mathfrak{g}[t]/(t^{r+1}) \to \mathrm{Der}(\mathcal{O}(\mathcal{J}_r G))$ defined by

$$\phi(xt^m)f_{(-n-1)} = (x_L f)_{(m-n-1)}, \tag{1.16}$$

where x_L is the left invariant vector field on G corresponding to $x \in \mathfrak{g}$ and we have put $f_{(n)} = 0$ for $n \geqslant 0$. We find from (1.15) that the image of ϕ is contained in $\mathrm{Lie}(\mathcal{J}_r G)$, and thus, we have the Lie algebra homomorphism

$$\psi\colon \mathfrak{g}[t]/(t^{r+1}) \longrightarrow \mathrm{Lie}(\mathcal{J}_r G).$$

The map ψ is injective since $\mathfrak{g} = \mathrm{Lie}(G)$, and therefore, ϕ must be isomorphism since $\dim \mathfrak{g}[t]/(t^{r+1}) = \dim \mathrm{Lie}(\mathcal{J}_r G)$. As this is true for all $m \geqslant 0$, we find that $\mathfrak{g}[[t]] \cong \mathrm{Lie}(\mathcal{J}_\infty G)$, and that the action of $\mathfrak{g}[[t]]$ on $\mathcal{O}(\mathcal{J}_\infty G)$ as left invariant vector fields is given by the formula (1.16).

By Lemma 1.9, note that the adjoint action of G on $\mathfrak{g}$ induces an action of $\mathcal{J}_\infty(G)$ on $\mathcal{J}_\infty(\mathfrak{g})$, and the coadjoint action of G on $\mathfrak{g}^*$ induces an action of $\mathcal{J}_\infty(G)$ on $\mathcal{J}_\infty(\mathfrak{g}^*)$.

Example 1.3 Assume that $\mathfrak{g}$ is simple, and let $\mathcal{N}$ be the *nilpotent cone* of $\mathfrak{g}$ that is, the set of nilpotent elements of $\mathfrak{g}$. The reader is referred to Section D.4 for important properties of the nilpotent cone.

Let $p_1, \ldots, p_\ell$ be homogeneous generators of the polynomial algebra $\mathcal{O}(\mathfrak{g})^G$. By Theorem D.3 (i), $\mathcal{N}$ is the reduced scheme of $\mathfrak{g}$ defined by the augmentation ideal $\mathcal{O}(\mathfrak{g})^G_+ := (p_1, \ldots, p_\ell)$. Hence, $\mathcal{J}_\infty(\mathcal{N})$ is the subscheme of $\mathfrak{g}[[t]]$ defined by the equations $\partial^j p_i$, $i = 1 \ldots, \ell$ and $j \geqslant 0$. Moreover, still by Theorem D.3 (i), the nilpotent cone is a complete intersection, which is irreducible and reduced, and by Theorem D.3 (ii), it has rational (hence canonical) singularities. So by Mustaţă's result (Theorem 1.3), all jet schemes $\mathcal{J}_m(\mathcal{N})$, $m \geqslant 1$, are irreducible. Moreover, they are all reduced and complete intersection ([221, Proposition 1.5]).

On the other hand, it was shown that by Raïs–Tauvel [230] and, independently, Beilinson–Drinfeld [54] (see also [97]) that:

$$\mathcal{J}_\infty(\mathfrak{g}/\!\!/G) \cong \mathcal{J}_\infty\mathfrak{g}/\!\!/\mathcal{J}_\infty G,$$

where $\mathfrak{g}/\!\!/G := \mathrm{Spec}\, \mathcal{O}(\mathfrak{g})^G$ and $\mathcal{J}_\infty\mathfrak{g}//\mathcal{J}_\infty G := \mathrm{Spec}\, \mathcal{O}([\mathcal{J}_\infty\mathfrak{g})^{\mathcal{J}_\infty G}$. In other words, the invariant ring $\mathcal{O}(\mathcal{J}_\infty\mathfrak{g})^{\mathcal{J}_\infty G}$ is the polynomial ring

$$\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}/\!\!/G)) = \mathbb{C}[\partial^j p_i \colon i = 1, \ldots, \ell, j \geqslant 0],$$

since $\mathcal{O}(\mathfrak{g}/\!\!/G) = \mathbb{C}[p_1, \ldots, p_\ell]$. In particular,

$$\mathcal{J}_\infty(\mathcal{N}) = \mathrm{Spec}\, \mathcal{O}(\mathcal{J}_\infty\mathfrak{g})/\!\!/\mathcal{O}(\mathcal{J}_\infty\mathfrak{g})_+^{\mathcal{J}_\infty G},$$

where $\mathcal{O}(\mathcal{J}_\infty\mathfrak{g})_+^{\mathcal{J}_\infty G}$ is the augmentation ideal of $\mathcal{O}(\mathcal{J}_\infty\mathfrak{g})^{J_\infty G}$. In [97], Eisenbud and Frenkel used all these results to extend results of Kostant in the setting of jet schemes: they obtained that $\mathcal{O}(\mathcal{J}_\infty\mathfrak{g})$ is free over $\mathcal{O}(\mathcal{J}_\infty\mathfrak{g})^{\mathcal{J}_\infty G}$, which is an important result in representation theory.

Chapter 2
Operator product expansion and vertex algebras

In this chapter, we collect the basic definitions (see Section 2.7) and standard properties of vertex algebras. We give several equivalent characterisation of the locality axiom (cf. Section 2.3), which is the most important axiom of a vertex algebra, and derive from this the Borcherds identities (see Section 2.3 and 2.8). The easiest examples of vertex algebras are the commutatives vertex algebras which are discussed in Section 2.10.

The first interesting examples of noncommutative vertex algebras are given in the next chapter (Chapter 3). Other important examples of noncommutative vertex algebras will occur further in the book.

The best general references for this chapter are [113, 160, 87].

2.1 Notation

For R a $\mathbb{C}$-algebra and $n \in \mathbb{Z}_{>0}$, we denote by $R[[z_1^{\pm}, \dots, z_n^{\pm}]]$ the vector space of all R-valued *formal power series* (or *formal Laurent series*) in the variables $z_1, \dots, z_n$, that is, the elements of the form

$$\sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} a_{i_1,\dots,i_n} z_1^{i_1} \dots z_n^{i_n}, \tag{2.1}$$

where each $a_{i_1,\dots,i_n}$ is in R. If $a \in R[[z_1^{\pm}, \dots, z_n^{\pm}]]$ and $b \in R[[w_1^{\pm}, \dots, w_m^{\pm}]]$, $m, n > 0$, then the product ab is well-defined in $R[[z_1^{\pm}, \dots, z_n^{\pm}, w_1^{\pm}, \dots, w_m^{\pm}]]$. But if a, b are two elements of $R[[z_1^{\pm}, \dots, z_n^{\pm}]]$, then their product does not make sense in general since the coefficient in a given z_i^j, for $i = 1, \dots, n$ and $j \in \mathbb{Z}$, of the product may be an infinite sum. However, the product of $a \in R[[z_1^{\pm}, \dots, z_n^{\pm}]]$ by a *Laurent polynomial*, that is, a series as in (2.1) such that $a_{i_1,\dots,i_n} = 0$ for all but finitely many n-tuples $i_1, \dots, i_n$, is well-defined.

Given a formal power series in one variable $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in R[[z^{\pm}]]$, we define its *residue* at $z = 0$ as:

$$\mathrm{Res}_{z=0}\, a(z) := a_{-1}.$$

If $R = \mathbb{C}$ and if $a(z)$ is the Laurent series of a meromorphic function defined on a punctured disc at 0, having pole only at 0, then

$$\mathrm{Res}_{z=0}\, a(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} a(z)dz,$$

where the integral is taken over any closed curve γ winding once around 0.

2.2 Formal delta function

Define the *formal delta-function* by

$$\delta(z-w) = \frac{1}{z} \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \in \mathbb{C}[[z^{\pm}, w^{\pm}]].$$

We have

$$\delta(z-w) = \tau_{z,w}\left(\frac{1}{z-w}\right) - \tau_{w,z}\left(\frac{1}{z-w}\right), \tag{2.2}$$

where the two maps $\tau_{z,w}$ and $\tau_{w,z}$ are the embeddings of algebras defined by:

$$\tau_{z,w} : \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] \longrightarrow \mathbb{C}((z))((w)), \qquad \frac{1}{z-w} \longmapsto \frac{1}{z}\sum_{n\geqslant 0}\left(\frac{w}{z}\right)^n,$$
$$\tau_{w,z} : \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] \longrightarrow \mathbb{C}((w))((z)), \qquad \frac{1}{z-w} \longmapsto -\frac{1}{z}\sum_{n>0}\left(\frac{z}{w}\right)^n.$$

Thus the map $\tau_{z,w}(f)$ is the expansion of f in $|z| > |w|$ and $\tau_{w,z}(f)$ is the expansion of f in $|w| > |z|$.

Lemma 2.1 *For any $\mathbb{C}$-algebra R and any $f \in R[z, z^{-1}]$, where $R[z, z^{-1}]$ is the set of all Laurent polynomials in the variable z with coefficients in R, we have*

$$f(z)\delta(z-w) = f(w)\delta(z-w). \tag{2.3}$$

Proof Note that $f(z) - f(w)$ is divisible by $z - w$. We have

$$\begin{aligned}(z-w)\delta(z-w) &= (z-w)\left(\tau_{z,w}\left(\frac{1}{(z-w)}\right) - \tau_{w,z}\left(\frac{1}{(z-w)}\right)\right)\\ &= \tau_{z,w}(1) - \tau_{w,z}(1) = 0,\end{aligned}$$

whence the assertion. □

Remark 2.1 In fact, for any formal series $f \in R[[z, z^{-1}]]$, the multiplication $f(z)\delta(z-w)$ makes sense and the equality (2.3) holds.

Both homomorphisms $\tau_{z,w}$ and $\tau_{w,z}$ commute with ∂_w and ∂_z. Therefore, it follows in the same way as above that

$$(z-w)^{n+1}\frac{1}{n!}\partial_w^n \delta(z-w) = 0 \tag{2.4}$$

for $n \geqslant 0$.

Lemma 2.2 *For $m, n \geqslant 0$ we have*

$$\mathrm{Res}_{z=0}\left((z-w)^m \frac{1}{n!}\partial_w^n \delta(z-w)\right) = \delta_{m,n}$$

Proof Observe that

$$(z-w)^m \frac{1}{n!}\partial_w^n \delta(z-w) = \tau_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right) - \tau_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right).$$

The meromorphic function $f(z) = \frac{1}{(z-w)^{n-m+1}}$, for fixed $w \in \mathbb{C}$, has poles contained in $\{w, 0, \infty\}$. It admits the following Laurent series expansions:

$$f(z) = \begin{cases} \tau_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right) = \sum_{m\in\mathbb{Z}} a_m(w) z^m \text{ if } |z| > |w| \\ \tau_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right) = \sum_{n\in\mathbb{Z}} b_n(w) z^n \text{ if } |w| > |z|, \end{cases}$$

with $a_m(w), b_n(w) \in \mathbb{C}[[w^{\pm}]]$,

Now we have

$$\mathrm{Res}_{z=0}\left(\tau_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right)\right) = a_{-1}(w) = \frac{1}{2\pi\sqrt{-1}}\int_{C_{1,w}} \frac{dz}{(z-w)^{n-m+1}}, \tag{2.5}$$

where $C_{1,w}$ is the contour described in Figure 2.1 (a circle centred at 0 with radius $r_1 > |w| = r$), while

$$\mathrm{Res}_{z=0}\left(\tau_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right)\right) = b_{-1}(w) = \frac{1}{2\pi\sqrt{-1}}\int_{C_{2,w}} \frac{dz}{(z-w)^{n-m+1}}, \tag{2.6}$$

where $C_{2,w}$ is the contour described in Figure 2.1 (a circle centred at 0 with radius $r_2 < |w| = r$).

Clearly, by the residue theorem applied to the meromorphic function $f(z)$ defined on the domain $\mathbb{C} \setminus \{w, 0\}$, we get

$$(2.5) - (2.6) = \frac{1}{2\pi\sqrt{-1}}\int_{C_w} \frac{dz}{(z-w)^{n-m+1}} = \delta_{m,n},$$

where C_w is the contour described in Figure 2.1 (a circle with center w and radius $< \min(r - r_2, r_1 - r)$). This concludes the proof.

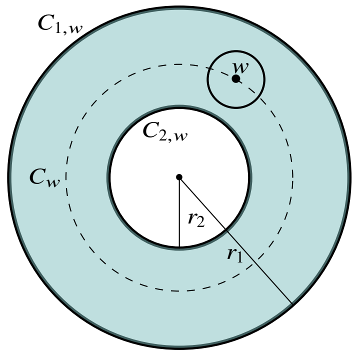


Fig. 2.1 The contour C_w

2.3 Locality and Operator product expansion (OPE)

Let V be a vector space over $\mathbb{C}$. We call elements $a(z)$ of $(\operatorname{End} V)[[z, z^{-1}]]$ *series* on V. For a series $a(z)$ on V, we set

$$a_{(n)} = \operatorname{Res}_{z=0} a(z) z^n$$

so that the expansion of $a(z)$ is

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}. \tag{2.7}$$

The coefficient $a_{(n)}$ is called a *Fourier mode* of $a(z)$. We write

$$a(z)b = \sum_{n \in \mathbb{Z}} a_{(n)} b z^{-n-1}$$

for $b \in V$.

Definition 2.1 A series $a(z) \in (\operatorname{End} V)[[z, z^{-1}]]$ is called a *field* on V if for any $b \in V$, $a(z)b \in V((z))$, that is, for any $b \in V$, $a_{(n)}b = 0$ for large enough n.

In the sequel, the space of all fields on V will be denoted by $\mathscr{F}(V)$.

For $a(z), b(z) \in \mathscr{F}(V)$, the product $a(z)b(z)$ does not make sense in general. However, the *normally ordered product*

$$:a(z)b(z): = a(z)_+b(z) + b(z)a(z)_-,$$

where

$$a(z)_+ = \sum_{n<0} a_{(n)}z^{-n-1}, \qquad a(z)_- = \sum_{n\geqslant 0} a_{(n)}z^{-n-1},$$

does make sense and belongs to $\mathscr{F}(V)$. However, the normally ordered product is neither commutative nor associative. By definition, $:a(z)b(z)c(z):$ stands for $:a(z):b(z)c(z)::$.

Although $a(z)b(w)$ makes sense, we also consider the following normally ordered product in $\mathrm{End}(V)[[z^{\pm}, w^{\pm}]]$:

$$:a(z)b(w): = a(z)_+b(w) + b(w)a(z)_-.$$

Note that $:a(z)b(w):v \in V[[z,w]][z^{-1}, w^{-1}]$, while $a(z)b(w)v \in V((z))((w))$, for $a(z), b(z) \in \mathscr{F}(V)$, $v \in V$.

Definition 2.2 We say two fields $a(z), b(z)$ on V are *mutually local* if

$$(z-w)^N [a(z), b(w)] = 0$$

in $(\mathrm{End}\, V)[[z^{\pm}, w^{\pm}]]$ for a sufficiently large N.

We note that a field $a(z)$ needs not be local to itself. The following proposition is stated in [160], see also [213].

Proposition 2.1 *Fix two fields $a(z), b(z)$ on a vector space V. The following assertions are equivalent:*

(i) *$a(z)$ and $b(z)$ are mutually local, that is, $(z-w)^N [a(z), b(w)] = 0$ for some $N \in \mathbb{Z}_{\geqslant 0}$ in $(\mathrm{End}\, V)[[z^{\pm}, w^{\pm}]]$;*

(ii) *There exist $c_0(w), c_1(w), \ldots, c_{N-1}(w) \in \mathscr{F}(V)$ such that*

$$[a(z), b(w)] = \sum_{n=0}^{N-1} c_n(w) \frac{1}{n!} \partial_w^n \delta(z-w).$$

in $(\mathrm{End}\, V)[[z^{\pm 1}, w^{\pm 1}]]$;

(iii) *There exist $c_0(w), c_1(w), \ldots, c_{N-1}(w) \in \mathscr{F}(V)$ such that*

$$a(z)b(w) = \sum_{n=0}^{N-1} c_n(w) \tau_{z,w}\left(\frac{1}{(z-w)^{n+1}}\right) + :a(z)b(w):$$

and

$$b(w)a(z) = \sum_{n=0}^{N-1} c_n(w) \tau_{w,z}\left(\frac{1}{(z-w)^{n+1}}\right) + :a(z)b(w):$$

in $(\mathrm{End}\, V)[[z^{\pm 1}, w^{\pm 1}]]$.

Proof The direction (iii) ⇒ (ii) is obvious and (ii) ⇒ (ii) follows from (2.4). We shall show that (i) ⇒ (iii). We have

$$
\begin{aligned}
a(z)b(w) - {:}a(z)b(w){:} &= [a(z)_-, b(w)],\\
b(w)a(z) - {:}a(z)b(w){:} &= [b(w), a(z)_+].
\end{aligned}
$$

By the locality assumption,

$$(z-w)^N[a(z)_-, b(w)] = (z-w)^N[b(w), a(z)_+]. \tag{2.8}$$

Observe that the left-hand-side of (2.8) does not have terms greater than $N-1$ in z whereas the right does not have terms of negative degree in z. Hence, they are polynomials of degree at most $N-1$ in z. It follows that there exists $c_j(w) \in (\operatorname{End} V)[[w, w^{-1}]]$, $j = 0, \dots, N-1$, such that

$$(z-w)^N[a(z)_-, b(w)] = \sum_{j=0}^{N-1} c_j(w)(z-w)^{N-j-1}.$$

For $v \in V$, the element $[a(z)_-, b(w)]v = (a(z)b(w) - {:}a(z)b(w){:})v$ belongs to $V((z))((w))$, which is a vector space over $\mathbb{C}((z))((w))$. We have

$$
\begin{aligned}
[a(z)_-, b(w)]v &= \tau_{z,w}\left(\frac{1}{(z-w)^N}\right)(z-w)^N[a(z)_-, b(w)]v\\
&= \tau_{z,w}\left(\frac{1}{(z-w)^N}\right)\sum_{j=0}^{N-1}(z-w)^{N-j-1}c_j(w)v\\
&= \sum_{j=0}^{N-1}\tau_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c_j(w)v.
\end{aligned}
$$

Since $v \in V$ is an arbitrary, we have obtained the first formula of (iii). The second formula is similarly shown. Finally, we need to show that each $c_j(w)$ is a field. Since we have shown that $[a(z), b(w)] = \sum_{j=0}^{N-1} c_j(w)\frac{1}{j!}\partial_w^j\delta(z-w)$, it follows from Lemma 2.2 that

$$c_j(w) = \operatorname{Res}_{z=0}((z-w)^j[a(z), b(w)]). \tag{2.9}$$

As both $a(z)$ and $b(w)$ are fields, $c_j(w)$ is a field as well. □

By abuse of notation we often just write

$$a(z)b(w) \sim \sum_{n=0}^{N-1}\frac{c_n(w)}{(z-w)^{n+1}} \tag{2.10}$$

for the relations of Proposition 2.1 (iii).

Definition 2.3 Formula (2.10) is called the *operator product expansion* (OPE) of $a(z)$ and $b(w)$.

Proposition 2.2 *The OPE* (2.10)*, or the relations of Proposition 2.1 (iii), is equivalent to the relation,*

$$[a_{(m)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} (c_j)_{(m+n-j)} \quad (m, n \in \mathbb{Z})$$

for $a, b \in V$*,* $m, n \in \mathbb{Z}$*, in* $\operatorname{End} V$.

In the above formulas, the notation $\binom{m}{j}$ for $j \geqslant 0$ and $m \in \mathbb{Z}$ means

$$\binom{m}{j} = \frac{m(m-1) \times \cdots \times (m-j+1)}{j(j-1) \times \cdots \times 1},$$

with the convention $\binom{m}{0} = 1$.

Proof We only show that (2.10) implies that the above relation. The other direction is easy to see. We have

$$[a_{(m)}, b_{(n)}] = \operatorname{Res}_{w=0} \left(w^n \operatorname{Res}_{z=0}(z^m [a(z), b(w)]) \right).$$

As in the same manner as in the proof of Lemma 2.2, we get

$$\operatorname{Res}_{z=0}(z^m [a(z), b(w)]) = \sum_{j=0}^{N-1} \operatorname{Res}_{z=w} \left(\frac{z^m}{(z-w)^{j+1}} \right) c_j(w) = \sum_{j=0}^{N-1} \binom{m}{j} c_j(w) w^{m-j-1}.$$

This completes the proof. □

Proposition 2.2 says that the right-hand-side of the OPE encodes all the brackets between all the coefficients of mutually local fields $a(z)$ and $b(z)$.

2.4 Example

Let $\mathcal{B}$ be the unital associative algebra generated by elements b_n, for $n \in \mathbb{Z}$, with relations

$$[b_m, b_n] = m\delta_{m+n,0}, \quad m, n \in \mathbb{Z}.$$

A $\mathcal{B}$-module M is called *smooth* if for each $m \in M$ there exits an integer N such that $b_n m = 0$ for $n > N$. If M is a smooth $\mathcal{B}$-module,

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}$$

is a field on M. We have

$$[b(z), b(w)] = \sum_{m,n \in \mathbb{Z}} [b_m, b_n] z^{-m-1} w^{-n-1} = \sum_{m \in \mathbb{Z}} m z^{-m-1} w^{m-1} = \partial_w \delta(z-w).$$

Hence, $b(z)$ is local to itself and

$$b(z)b(w) \sim \frac{1}{(z-w)^2}.$$

2.5 n-th product of fields

Let $a(z), b(z)$ be mutually local fields on V, so that $(z-w)^N [a(z), b(w)] = 0$ for some N. For $n \geqslant 0$, define the field $a(z)_{(n)} b(z)$ by

$$a(z)_{(n)} b(z) := \mathrm{Res}_{w=0}((w-z)^n [a(w), b(z)]).$$

Then, the OPE of $a(z)$ and $b(z)$ is expressed as

$$a(z)b(w) \sim \sum_{j \geqslant 0} \frac{a(w)_{(j)} b(w)}{(z-w)^{j+1}}, \tag{2.11}$$

see (2.9). (Note that $a(z)_{(j)} b(z) = 0$ for $j \geqslant N$.)

In fact, the field $a(z)_{(n)} b(z)$ makes sense for all $n \in \mathbb{Z}$, where we undestand $\mathrm{Res}_{w=0}((w-z)^n [a(w), b(z)])$ as

$$\mathrm{Res}_{w=0} \left(\tau_{w,z}((w-z)^n) a(w) b(z)\right) - \mathrm{Res}_{w=0} \left(\tau_{z,w}((w-z)^n) b(z) a(w)\right).$$

Explicitely, we have

$$a(z)_{(n)} b(z) = \sum_{k \in \mathbb{Z}} \left(\sum_{i \geqslant 0} \binom{n}{i} \left(a_{(n-i)} b_{(k+i)} - (-1)^n b_{(n+k-i)} a_{(i)} \right) \right) z^{-k-1}. \tag{2.12}$$

Definition 2.4 The field $a(z)_{(n)} b(z)$ is called the *n-th product* of $a(z)$ and $b(z)$.

Note that

$$a(z)_{(-1)} b(z) = {:}a(z)b(z){:}, \tag{2.13}$$

$$a(z)_{(-n)} \mathrm{id}_V = \begin{cases} \frac{1}{(n-1)!} \partial_z^{(n-1)} a(z) & \text{if } n > 0, \\ 0 & \text{if } n \leqslant 0. \end{cases} \tag{2.14}$$

The last formula follows from the fact that

$$a(z)_{(-n)}\,\mathrm{id}_V = \mathrm{Res}_{w=z}\left(\frac{a(w)}{(w-z)^n}\right)$$

(recall the proof of Lemma 2.2), where

$$\mathrm{Res}_{w=z}\left(\frac{a(w)}{(w-z)^n}\right) := \sum_{n\in\mathbb{Z}} a_{(n)}\,\mathrm{Res}_{w=z}\left(\frac{w^{-n-1}}{(w-z)^n}\right).$$

Similarly, we have

$$(\mathrm{id}_V)_{(n)}a(z) = \delta_{n,-1}a(z). \tag{2.15}$$

Also, we have

$$\partial_z(a(z)_{(n)}b(z)) = (\partial_z a(z))_{(n)}\,b(z) + a(z)_{(n)}\,(\partial_z b(z))\,. \tag{2.16}$$

This is clear from the fact that $\mathrm{Res}_{z=0}\,\partial_z(\ldots) = 0$.

The n-th product is not associative. By definition, $a(z)_{(m)}b(z)_{(n)}c(z)$ stands for $a(z)_{(m)}(b(z)_{(n)}c(z))$ as in the case of the normally ordered product.

Next lemma is usually referred to as *Dong's Lemma* ([193]).

Lemma 2.3 (Dong's Lemma) *If $a(z)$, $b(z)$, $c(z)$ are three mutually local fields on a vector space V, then the fields $a(z)_{(n)}b(z)$ and $c(z)$ are also mutually local for all $n \in \mathbb{Z}$.*

Proof By assumption there exists $N \geqslant 0$ such that

$$(z-w)^N a(z)b(w) = (z-w)^N b(w)a(z), \tag{2.17}$$

$$(z-u)^N a(z)c(u) = (z-u)^N c(u)a(z), \tag{2.18}$$

$$(w-u)^N b(w)c(u) = (w-u)^N c(u)b(w). \tag{2.19}$$

We may assume that $N + n \geqslant 0$. We claim that

$$\begin{aligned}&(w-u)^{4N}\left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right)c(u)\\ &= (w-u)^{4N}c(u)\left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right).\end{aligned} \tag{2.20}$$

Indeed, we have

$$(w-u)^{4N} = (w-u)^N\sum_{s=0}^{3N}\binom{3N}{s}(z-u)^s(w-z)^{3N-s}.$$

If $0 \leqslant s \leqslant N$, $(w-z)^{3N-s}\tau_{z,w}((z-w)^n) = (-1)^{3N-s}\tau_{z,w}((z-w)^{3N-s+n})$ and $3N-s+n \geqslant N$. Thus, $(w-z)^{3N-s}(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)) = 0$ by (2.17), and so the left-hand-side of (2.20) is equal to

$$\sum_{s=N+1}^{3N} (w-u)^N (z-u)^s (w-z)^{3N-s} \left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right) c(u).$$

Similarly, the right-hand-side of (2.20) is equal to

$$\sum_{s=N+1}^{3N} (w-u)^N (z-u)^s (w-z)^{3N-s} c(u) \left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right).$$

But these two are equal thanks to (2.18) and (2.19).

The assertion follows by taking $\mathrm{Res}_{z=0}$ of both sides of (2.20). □

The following assertion should be compared with Proposition 2.2.

Proposition 2.3 *Let $a(z)$, $b(z)$, $c(z)$ be three mutually local fields on a vector space V. Then,*

(2.21)
$$a(z)_{(m)}b(z)_{(n)}c(z) - b(z)_{(n)}a(z)_{(m)}c(z) = \sum_{j\geqslant 0} \binom{m}{j} (a(z)_{(j)}b(z))_{(m+n-j)}c(z)$$

for $m, n \in \mathbb{Z}$.

Proof The left-hand-side is equal to the sum of the following two terms:

$$\begin{aligned} &\mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{w,z}((w-z)^m)\tau_{u,z}((u-z)^n)a(w)b(u)c(z)\right) \\ &- \mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{w,z}((w-z)^m)\tau_{u,z}((u-z)^n)b(u)a(w)c(z)\right), \end{aligned} \tag{2.22}$$

$$\begin{aligned} &- \mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{z,w}((w-z)^m)\tau_{z,u}((u-z)^n)c(z)a(w)b(u)\right) \\ &+ \mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{z,w}((w-z)^m)\tau_{z,u}((u-z)^n)c(z)b(u)a(w)\right). \end{aligned} \tag{2.23}$$

By using the formula

$$(w-z)^m = \sum_{j\geqslant 0} \binom{m}{j} (w-u)^j (u-z)^{m-j},$$

and the fact that

$$\tau_{w,z}((w-u)^j(u-z)^{m-j}) = \begin{cases} \tau_{w,u}((w-u)^j)\tau_{u,z}((u-z)^{m-j}) & \text{in } \mathbb{C}((w))((u))((z)), \\ \tau_{u,w}((w-u)^j)\tau_{u,z}((u-z)^{m-j}) & \text{in } \mathbb{C}((u))((w))((z)), \end{cases}$$

we find that (2.22) is equal to

$$\sum_{j\geqslant 0} \binom{m}{j} \mathrm{Res}_{u=0} \left(\tau_{u,z}((u-z)^{m+n-j}) \left(a(u)_{(j)}b(u)\right) c(z)\right).$$

Similarly, we find (2.23) is equal to

$$- \sum_{j \geqslant 0} \binom{m}{j} \mathrm{Res}_{u=0} \left(\tau_{z,u} ((u-z)^{m+n-j}) \left(a(u)_{(j)} b(u) \right) c(z) \right).$$

This completes the proof. □

Exercise 2.1 Show that

$$a(z)_{(-n-1)} b(z) = \frac{1}{n!} {:}(\partial_z^n a(z)) b(z){:} \quad \text{for } n \geqslant 0.$$

2.6 Wick formula and λ-bracket notation

Wick's formulas that we shall present below are powerful tools to compute OPE's between mutually local fields.

Let $a(z)$, $b(z)$, $c(z)$ be three mutually local fields on a vector space V. Using Proposition 2.3 for $n = -1$ we first obtain the *noncommutative Wick formula*:

(2.24)

$$a(z)_{(m)} {:}b(z)c(z){:} = {:}\left(a(z)_{(m)} b(z)\right) c(z){:} + {:}b(z) \left(a(z)_{(m)} c(z)\right){:} + \sum_{j=0}^{m-1} \binom{m}{j} \left(a(z)_{(j)} b(z)\right)_{(m-1-j)} c(z).$$

We now introduce the λ-bracket notation, convenient to compute OPE's in practice. Assume that $a(z), b(z)$ are mutually local with $(z-w)^N [a(z), b(w)] = 0$. Write a in place of $a(z)$, $a_{(j)} b$ in place of $a(z)_{(j)} b(z)$, etc. Set

$$[a_\lambda b] := \sum_{j=0}^{N-1} a_{(j)} b \frac{\lambda^j}{j!}.$$

For example, $[b_\lambda b] = \lambda 1 = \lambda$ and $[b_\lambda 1] = [1_\lambda b] = [1_\lambda 1] = 0$, with $b(z)$ as in the previous paragraph. One can develop a calculus for the λ-bracket and the normally ordered product ${:}\;{:}$, and prove the following formula:

$$[(\partial a)_\lambda b] = -\lambda [a_\lambda b] \qquad [a_\lambda (\partial b)] = (\partial + \lambda)[a_\lambda b]$$
$$[b_\lambda a] = -[a_{-\lambda-\partial} b] \qquad \partial({:}ab{:}) = {:}(\partial a) b{:} + {:}a(\partial b){:}.$$

We leave the verifications as an exercise. From it we obtain a λ-bracket interpretation of (2.24):

(2.25) $$[a_\lambda {:}bc{:}] = {:}[a_\lambda b]c{:} + {:}b[a_\lambda c]{:} + \int_0^\lambda [[a_\lambda b]_\mu c] \mathrm{d}\mu.$$

In the case where $[a(z), b(w)] = 0$, that is, $[a_\lambda b] = 0$, we get the formula:

(2.26)
$$ {}^{\circ}_{\circ} a(z) \left({}^{\circ}_{\circ} b(z)c(z) {}^{\circ}_{\circ} \right) {}^{\circ}_{\circ} = {}^{\circ}_{\circ} \left({}^{\circ}_{\circ} a(z), b(z) {}^{\circ}_{\circ} \right) c(z) {}^{\circ}_{\circ} + {}^{\circ}_{\circ} b(z) \left({}^{\circ}_{\circ} a(z)c(z) {}^{\circ}_{\circ} \right) {}^{\circ}_{\circ} $$

equivalently,

$$ [a_\lambda {}^{\circ}_{\circ} bc {}^{\circ}_{\circ}] = {}^{\circ}_{\circ} [a_\lambda b]c {}^{\circ}_{\circ} + {}^{\circ}_{\circ} b[a_\lambda c] {}^{\circ}_{\circ} . \tag{2.27} $$

One can generalize the above *commutative Wick formulas* (2.26) and (2.27).

Recall that the normally ordered product of fields $a^1(z), \dots, a^k(z)$ over a vector space V is defined inductively from right to left:

$$ {}^{\circ}_{\circ} a^1(z) \dots a^k(z) {}^{\circ}_{\circ} = {}^{\circ}_{\circ} a^1(z) \dots {}^{\circ}_{\circ} a^{k-1}(z) a^k(z) {}^{\circ}_{\circ} \dots {}^{\circ}_{\circ} . $$

It is a sum of 2^k terms of the form

$$ a^{i_1}(z)_+ a^{i_2}(z)_+ \dots a^{j_1}(z)_- a^{j_2}(z)_- \dots , \tag{2.28} $$

where $i_1 < i_2 \cdots, j_1 > j_2 > \cdots$ is a permutation of the index set $\{1, \dots, k\}$.

Remark 2.2 It is clear from (2.28) that if $[a^i(z)_\pm, a^j(z)_\pm] = 0$ for all i, j, then ${}^{\circ}_{\circ} a^1(z) \dots a^k(z) {}^{\circ}_{\circ} = {}^{\circ}_{\circ} a^{i_1}(z) \dots a^{i_k}(z) {}^{\circ}_{\circ}$ for any permutation $\{i_1, \dots, i_k\}$. It follows that in this case the normally ordered products is commutative and associative.

We write $\langle a^i, a^j \rangle = [a^i(z)_-, a^j(w)]$ for the *contraction* of $a^i(z)$ and $a^j(w)$. As already observed in the proof of Proposition 2.1, we have

$$ \langle a^i, a^j \rangle = a^i(z) a^j(w) - {}^{\circ}_{\circ} a^i(z) a^j(w) {}^{\circ}_{\circ} $$

so that $\langle a^i, a^j \rangle$ represents the “singular part” of $a^i(z) a^j(w)$.

Theorem 2.1 (Wick's formula) *Let $a^1(z), \dots, a^m(z)$ and $b^1(z), \dots, b^n(z)$ be two collections of fields such that the following properties hold:*

(i) $[\langle a^i, b^j \rangle, a^k(z)_\pm] = 0$ *and* $[\langle a^i, b^j \rangle, b^k(z)_\pm] = 0$ *for all i, j, k;*
(ii) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all i and j.*

Then one has the following OPE:

(2.29)
$$ {}^{\circ}_{\circ} a^1(z) \dots a^m(z) {}^{\circ}_{\circ} {}^{\circ}_{\circ} b^1(w) \dots b^n(w) {}^{\circ}_{\circ} $$
$$ = \sum_{s=0}^{\min(m,n)} \sum_{\substack{i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \left(\langle a^{i_1}, b^{j_1} \rangle \cdots \langle a^{i_s}, b^{j_s} \rangle {}^{\circ}_{\circ} a^1(z) \dots a^m(z) b^1(w) \dots b^n(w) {}^{\circ}_{\circ (i_1, \dots, i_s; j_1, \dots, j_s)} \right), $$

where the subscript $(i_1, \dots, i_s; j_1, \dots, j_s)$ means that the fields $a^{i_1}(z) \dots a^{i_s}(z) b^{j_1}(w) \dots b^{j_s}(w)$ are removed.

The theorem can be deduced from (2.27), and so from (2.24), by induction but we can also prove it directly following [160, §3.3]. Since the proof is short, we present it.

Proof The typical term of the left-hand-side of (2.29) is

$$\left(a^{j_1}(z)_+ a^{j_2}(z)_+ \dots a^{i_1}(z)_- a^{i_2}(z)_- \dots\right)\left(b^{k_1}(w)_+ b^{k_2}(w)_+ \dots b^{l_1}(w)_- b^{l_2}(w)_- \dots.\right)$$

Then we have to move the $a^i(z)_-$ across the $b^j(w)_+$ in order to bring this product to the normally ordered product as in (2.28). Due to the condition (ii) of the theorem, we have

$$a^i(z)_- b^j(w)_+ = b^j(w)_+ a^i(z)_- + \langle a^i, b^j \rangle.$$

But due to condition (i), the contractions commute with all fields $b^k(w)_\pm$, hence can be moved to the left. This proves the theorem. □

Remark 2.3 There is a Mathematica package [246] which provides a computer program for these OPE calculations.

Remark 2.4 It is not difficult to adapt Wick's formulas (2.29) and (2.24) in the case where V is a superspace as in [160, Section 3.3].

Exercise 2.2 Keep the notation of Section 2.4. For $\alpha \in \mathbb{C}$, set

$$L(z) = \frac{1}{2} {:}b(z)^2{:} + \alpha \partial_z b(z).$$

By Lemma 2.3, $L(z)$ is local to $b(z)$ and itself. Using Wick's formulas, show that

$$\begin{aligned}
&{:}b(z)^2{:}\,{:}b(w)^2{:} \sim \frac{2}{(z-w)^4} + \frac{4}{(z-w)^2}{:}b(w)^2{:} + \frac{4}{(z-w)}{:}(\partial_w b(w))b(w){:},\\
&\partial_z b(z)({:}b(w)^2{:}) \sim -\frac{4}{(z-w)^3}b(w),\\
&({:}b(z)^2{:})\partial_w b(w) \sim \frac{4}{(z-w)^3}b(w) + \frac{4}{(z-w)^2}\partial_w b(w) + \frac{2}{(z-w)}\partial_w^2 b(w),\\
&\partial_z b(z)\partial_w b(w) \sim -\frac{6}{(z-w)^4},\\
&L(z)b(w) \sim -\frac{2\alpha}{(z-w)^3} + \frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)},
\end{aligned}$$

(2.30)

$$L(z)L(w) \sim \frac{(1-12\alpha^2)/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)},$$

or, equivalently, that

$$[{:}b^2{:}_\lambda {:}b^2{:}] = \frac{1}{3}\lambda^3 + 4b^2\lambda^3 + 4{:}(\partial b)b{:}, \qquad [(\partial b)_\lambda {:}b^2{:}] = -2b\lambda^2,$$
$$[{:}b^2{:}_\lambda(\partial b)] = 2b\lambda^2 + 4(\partial b)\lambda + 2\partial^2 b, \qquad [(\partial b)_\lambda(\partial b)] = -\lambda^3,$$
$$[L_\lambda b] = -\alpha\lambda^2 + b\lambda + \partial b, \qquad [L_\lambda L] = \frac{(1-12\alpha^2)}{12}\lambda^3 + 2\lambda L + \partial L.$$

2.7 Definition of vertex algebras

The following definition of a vertex algebra is due to De Sole and Kac [87].

Definition 2.5 A vector space V is a *vertex algebra* if it is equipped with a nonzero vector $|0\rangle$, an endomorphism $T\colon V \to V$ and a set of mutually local fields $\mathcal{S}$ on V such that the following conditions holds:

(i) $T|0\rangle = 0$ and $[T, a(z)] = \partial_z a(z)$ for all $a(z) \in \mathcal{S}$,
(ii) V is spanned by the vectors

$$a^{i_1}_{(n_1)} \dots a^{i_m}_{(n_m)}|0\rangle$$

for $a^{i_1}(z), \dots, a^{i_m}(z) \in \mathcal{S}$ written as $a^{i_j}(z) = \sum_{n\in\mathbb{Z}} a^{i_j}_{(n)} z^{-n-1}$.

If this is the case we say V is a *vertex algebra generated by* $\mathcal{S}$.

Note that the second condition of (i) is equivalent to

$$[T, a_{(n)}] = -na_{(n-1)} \tag{2.31}$$

for all $n \in \mathbb{Z}$.

Let V be a vertex algebra generated by a set of mutually local fields $\mathcal{S}$.

Lemma 2.4 *We have $a(z)|0\rangle \in V[[z]]$ for all $a \in \mathcal{S}$, that is, $a_{(n)}|0\rangle = 0$ for all $n \geqslant 0$.*

Proof Since $a(z)$ is a field, there exists $M \in \mathbb{Z}$ such that $a_{(n)}|0\rangle = 0$ for all $n \geqslant M$. Choose such minimal M. We wish to show that $M = 0$. Applying both sides of (2.31) for $n = M$ to $|0\rangle$, we obtain $Ta_{(M)}|0\rangle = -Ma_{(M-1)}|0\rangle$ by the first condition of (i) in Definition 2.5. Thus $M > 0$ implies that $a_{(M-1)} = 0$, a contradiction. □

Denote by $\langle\mathcal{S}\rangle_V$ the subspace of $\mathscr{F}(V)$ spanned by the fields constructed by successive application of the n-th products to the fields in $\mathcal{S}$ as well as the identify field id_V.

Lemma 2.5 *We have $[T, a(z)] = \partial_z a(z)$ for $a(z) \in \langle\mathcal{S}\rangle_V$.*

Proof By induction, it is enough to show that $[T, a(z)_{(n)}b(z)] = \partial_z(a(z)_{(n)}b(z))$ for all $a(z), b(z) \in \langle\mathcal{S}\rangle_V$ since the identity $[T, a(z)] = \partial_z a(z)$ obviously holds for the field $a(z) = \mathrm{id}_V$. But this can be checked by explicit computations using (2.16), (2.12) and the above identity (2.31). □

Lemma 2.6 *We have* $a(z)|0\rangle \in V[[z]]$ *for all* $a(z) \in \langle \mathcal{S} \rangle_V$.

Proof V can be viewed as a vertex algebra generated by $\langle \mathcal{S} \rangle_V$. Hence the assertion follows from Lemma 2.4. □

By Lemma 2.6, one can define linear map

$$s\colon \langle \mathcal{S} \rangle_V \longrightarrow V, \quad a(z) \longmapsto \lim_{z\to 0} a(z)|0\rangle.$$

Lemma 2.7 *For* $a(z), b(z) \in \langle \mathcal{S} \rangle_V$, *set* $a = s(a(z))$, $b = s(b(z))$.

(i) $s(a(z)_{(n)} b(z)) = a_{(n)} b$,
(ii) $s(\partial_z a(z)) = Ta$,
(iii) *For* $a(z) \in \langle \mathcal{S} \rangle_V$, $a(z)|0\rangle = e^{zT} a$.

Proof (i) We have $a(z)_{(n)} b(z)|0\rangle \in V[[z]]$ and

$$\lim_{z\to 0} a(z)_{(n)} b(z)|0\rangle = a_{(n)} b_{(-1)}|0\rangle = a_{(n)} b,$$

see (2.12), which gives (i).

(ii) Note that by (2.14), $\partial_z a(z) \in \langle \mathcal{S} \rangle_V$. By Lemma 2.5, we have $\partial_z a(z)|0\rangle = [T, a(z)]|0\rangle = Ta(z)|0\rangle$, whence $s(\partial_z a(z)) = \lim_{z\to 0} Ta(z)|0\rangle = Ta$ as expected.

(iii) A rapid induction from $\partial_z a(z)|0\rangle = Ta(z)|0\rangle$ gives $\partial_z^n a(z)|0\rangle = T^n a(z)|0\rangle$ since $(\partial_z a(z))|0\rangle = \partial_z (a(z)|0\rangle)$ which can be checked directly. In particular, we have

$$\lim_{z\to 0} \partial_z^n a(z)|0\rangle = T^n a.$$

Therefore,

$$a(z)|0\rangle = \sum_{n\geqslant 0} \frac{z^n}{n!} \lim_{z\to 0} \partial_z^n a(z)|0\rangle = \sum_{n\geqslant 0} \frac{z^n}{n!} T^n a = e^{zT} a, \tag{2.32}$$

whence (iii). □

By Lemma 2.7 and its proof, we have

$$(Ta)_{(n)} = -na_{(n-1)}$$

for all $a \in V, n \in \mathbb{Z}$, and

$$a_{(-n-1)}|0\rangle = \frac{1}{n!} T^n a, \quad \text{for all} \quad n > 0. \tag{2.33}$$

Theorem 2.2 *Let* V *be a vertex algebra generated by a set of mutually local fields* $\mathcal{S}$. *The map* $s\colon \langle \mathcal{S} \rangle_V \to V$ *is a linear isomorphism.*

Proof By Lemma 2.7 (i), s is surjective because $a = a_{(-1)}|0\rangle$. Let $a(z) \in \ker s$. Then $a(z)|0\rangle = 0$, by Lemma 2.7 (iii). Then for $b(z) \in \mathcal{S}$, we have

$$(z-w)^N a(z)b(w)|0\rangle = (z-w)^N b(w)a(z)|0\rangle = 0$$

by the locality, for a sufficiently large N. We deduce evaluating at $w = 0$ that $a(z)s(b(z)) = 0$. By the surjectivety of s, this implies that $a(z) = 0$ as required. □

By Theorem 2.2, we can define a linear map

$$Y\colon V \longrightarrow \mathscr{F}(V), \quad a \longmapsto Y(a,z) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

by setting

$$Y(a,z) = s^{-1}(a).$$

Then we have

- $Y(|0\rangle, z) = \mathrm{id}_V$. Furthermore, for all $a \in V$,

  $$Y(a,z)|0\rangle \in V[[z]]$$

  and $\lim_{z\to 0} Y(a,z)|0\rangle = a$. In other words, $a_{(n)}|0\rangle = 0$ for $n \geqslant 0$ and $a_{(-1)}|0\rangle = a$,
- $T|0\rangle = 0$ and for any $a \in V$,

  $$[T, Y(a,z)] = \partial_z Y(a,z), \quad \text{and} \quad Y(Ta,z) = \partial_z Y(a,z),$$

- for all $a, b \in V$, the fields $Y(a,z)$ and $Y(b,z)$ are mutually local, that is,

  $$(z-w)^N [Y(a,z), Y(b,w)] = 0 \tag{2.34}$$

  for some $N = N_{a,b} \in \mathbb{Z}_{\geqslant 0}$.
- for all $a, b \in V$, $n \in \mathbb{Z}$,

  $$Y(a_{(n)}b, z) = Y(a,z)_{(n)} Y(b,z)$$

  for all $n \in Z$, $a, b \in V$.

Let V, W be vertex algebras. The tensor product $V \otimes W$ is a vertex algebra with the vacuum vector $|0\rangle \otimes |0\rangle$, the translation operator $T \otimes 1 + 1 \otimes T$, and the vertex operator $Y(a \otimes b, z) = Y(a,z) \otimes Y(b,z)$.

A *vertex algebra homomorphism* from V to W is a linear map $\phi\colon V \to W$ such that $\phi(|0\rangle) = |0\rangle$, $\phi(Ta) = T\phi(a)$, and $\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b)$ for all $a, b \in V$, $n \in V$.

A collection $\{a^i : i \in I\}$ of elements of a vertex algebra V is called *strong generators* of V if V is spanned by

$$a^{i_1}_{(n_1)} \dots a^{i_s}_{(n_s)} |0\rangle$$

with $s \geqslant 0$, $n_r < 0$ and $i_r \in I$. The structure of V is completely determined by the OPEs among the fields $a^i(z) = Y(a^i, z)$, $i \in I$.

Remark 2.5 By Exercise 2.1, the vertex operator $Y(-,z)$ for V can be explicitly described as

$$Y(a^{i_1}_{(-n_1-1)}a^{i_2}_{(-n_2-1)}\dots a^{i_r}_{(-n_r-1)}|0\rangle, z) \\ = \frac{1}{n_1!n_2!\dots n_r!} {}^{\circ}_{\circ}(\partial_z^{n_1}a^{i_1}(z))(\partial_z^{n_2}a^{i_2}(z))\dots(\partial_z^{n_r}a^{i_r}(z)){}^{\circ}_{\circ}$$

for $n_i \geqslant 1$.

Exercise 2.3 (Goddard's uniqueness theorem) Let V be a vertex algebra, and $A(z)$ a field on V. Suppose there exists a vector $a \in V$ such that

$$A(z)|0\rangle = Y(a,z)|0\rangle$$

and $A(z)$ is local with $Y(b,z)$ for all $b \in V$. Show that $A(z) = Y(a,z)$.

2.8 Skew-symmetry and Borcherds identities

Proposition 2.4 (skew-symmetry) *Let V be a vertex algebra. Then for all $a, b \in V$ the identity*

$$Y(a,z)b = e^{zT}Y(b,-z)a$$

holds in $V((z))$.

Note that $e^{zT}Y(b,-z)a$ stands for

$$\sum_{n\in\mathbb{Z}} e^{zT}a_{(n)}b(-z)^{-n-1} = \sum_{n\in\mathbb{Z}}\sum_{j\geqslant 0}\frac{T^j(a_{(n)}b)}{j!}z^j(-z)^{-n-1}.$$

Proof By (2.32) and the locality,

$$(z-w)^N Y(a,z)e^{wT}b = (z-w)^N Y(a,z)Y(b,w)|0\rangle = (z-w)^N Y(b,w)Y(a,z)|0\rangle \\ = (z-w)^N Y(b,w)e^{zT}a$$

for a sufficiently large N. Now we have

$$e^{zT}Y(b,w)e^{-zT} = \sum_{n\geqslant 0}\frac{1}{n!}\operatorname{ad}(zT)^n(Y(b,w)) = \sum_{n\geqslant 0}\frac{z^n}{n!}\partial_w^n Y(b,w) = Y(b,z+w)$$

in $(\operatorname{End} V)[[z^{\pm}, w^{\pm}]]$, where by $(z+w)^{-1}$ we understand its expansion $\tau_{z,w}(1/(z+w))$. (The formal variable version of the Taylor formula.) Hence,

$$(z-w)^N Y(a,z)e^{wT}b = (z-w)^N e^{zT}Y(b,w-z)a,$$

where by $(w-z)^{-1}$ we understand its expansion $\tau_{z,w}(1/(w-z))$. Since there is no negative power of w on the left-hand-side, we can set $w=0$ on both sides to get the desired formula. □

Theorem 2.3 (Borcherds identities) *Let V be a vertex algebra, $a, b \in V$. We have*

$$[a_{(m)}, b_{(n)}] = \sum_{i\geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}, \tag{2.35}$$

$$(a_{(m)}b)_{(n)} = \sum_{j\geqslant 0} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}), \tag{2.36}$$

for $m, n \in \mathbb{Z}$.

Proof By (2.11) and lemma 2.7, (i), we have

$$Y(a,z)Y(b,w) \sim \sum_{i\geqslant 0} \frac{Y(a_{(i)}b, w)}{(z-w)^{i+1}}.$$

Hence, (2.35) follows from Proposition 2.1. As for (2.36), it is equivalent to the statement of Lemma 2.7, (i), and formula (2.12). □

The relations (2.35) and (2.36) are called *Borcherds identities*.

Remark 2.6 The two identifies (2.35) and (2.36) are equivalent to the following single identity, for $p, q, r \in \mathbb{Z}$:

(2.37)
$$\sum_{i\geqslant 0} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)} = \sum_{i\geqslant 0} (-1)^i \binom{r}{i} \left(a_{(p+r-i)}b_{(q+i)} - (-1)^r b_{(q+r-i)}a_{(p+i)}\right),$$

which is equivalent to the *Jacobi identity* in [190], see [213]. Note that (2.37) is also equivalent to the following identity:

(2.38)
$$\begin{aligned}&\operatorname{Res}_{z-w} Y(Y(a,z-w)b,w)\tau_{w,z-w}F(z,w)\\ &\quad = \operatorname{Res}_z Y(a,z)Y(b,w)\tau_{z,w}F(z,w) - \operatorname{Res}_z Y(b,w)Y(a,z)\tau_{w,z}F(z,w),\end{aligned}$$

where $F(z,w) = z^p w^q (z-w)^r$.

Remark 2.7 It is easy to adapt the definition of a vertex algebra to the supercase. To be more specific, if $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a superspace, then the data and axioms have be modified as follows: for each $a(z) \in \mathcal{S}$, all endomorphisms $a_{(n)}$ have the same parity, $|0\rangle$ is an element of $V_{\bar{0}}$, T has even parity and the locality axiom is:

$$(z-w)^N a(z)b(w) = (-1)^{|a||b|}(z-w)^N b(w)a(z)$$

for N sufficiently large, where $|a|$ denotes the common parity of all $a_{(n)} \in \operatorname{End} V$. The Borcherds identities have to be understood in the supercase as follows:

(2.39)

$$[a_{(m)}, b_{(n)}] = a_{(m)}b_{(n)} - (-1)^{|a||b|}b_{(n)}a_{(m)} = \sum_{i\geqslant 0}\binom{m}{i}(a_{(i)}b)_{(m+n-i)},$$

(2.40)

$$(a_{(m)}b)_{(n)} = \sum_{j\geqslant 0}(-1)^j\binom{m}{j}(a_{(m-j)}b_{(n+j)} - (-1)^{|a||b|}(-1)^m b_{(m+n-j)}a_{(j)}).$$

2.9 Modules over vertex algebras, vertex ideals and quotient vertex algebras

Let $\mathcal{S}$ be a set of pairwise mutually local fields on a vector space M. Denote by $\langle\mathcal{S}\rangle_M$ the subspace of $\mathscr{F}(M)$ spanned by the fields constructed by successive application of the n-th products to the fields in $\mathcal{S}$ as well as the identify field id_M.

Next theorem is proved in [193].

Theorem 2.4 *The space $\langle\mathcal{S}\rangle_M$ has the structure of a vertex algebra generated by $\mathcal{S}$ with the vacuum vector* id_M*, the translation operator ∂_z, and*

$$Y(a(z),\xi) = \sum_{n\in\mathbb{Z}} a(z)_{(n)}\xi^{-n-1},$$

where $a(z)_{(n)}$ denotes the linear map $b(z)\mapsto a(z)_{(n)}b(z)$ on $\mathcal{V}$.

Proof By (2.14) for $n=2$, $\langle\mathcal{S}\rangle_M$ is stable under the translation operator $T=\partial_z$. The vaccum axiom is satisfied by (2.14) and (2.15). By (2.16), we have $[\partial_z, Y(a(z),\xi)] = Y(\partial_z a(z),\xi)$. Since $\mathrm{Res}_{w=0}\,\partial_w((w-z)^n[a(w),b(z)]) = 0$, we get that $(\partial_z a(z))_{(n)} = -na(z)_{(n-1)}$, and hence the translation axiom (i) of Definition 2.5 holds. The locality axiom holds by Proposition 2.3, in view of Proposition 2.1. □

Let V be a vertex algebra generated by $\mathcal{S}$. In view of Theorem 2.4, the map

$$V \xrightarrow{\sim} \langle\mathcal{S}\rangle_V, \quad a \longmapsto Y(a,z),$$

is an isomorphism of vertex algebras, whose inverse is given by $s(a(z)) = \lim_{z\to 0} a(z)|0\rangle$.

Definition 2.6 A vector space M is called a *V-module*, or a *representation of V*, if there exists a vertex algebra homomorphism $V\to\langle\mathcal{S}'\rangle_M$ for some set $\mathcal{S}'$ of pairwise mutually local fields on M.

If M is a V-module, we denote by $Y^M(a,z) = a^M(z) = \sum_{n\in\mathbb{Z}} a^M_{(n)}z^{-n-1}$ the image of $a\in V$ in $\langle\mathcal{S}'\rangle_M\subset\mathscr{F}(M)$, or simply by $Y(a,z) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$ if no confusion should occur.

The following assertion is clear.

Lemma 2.8 *A vector space M is a module over a vertex algebra V if and only if there exists a linear map $V \to \mathscr{F}(M)$, $a \mapsto Y^M(a, z)$, such that*

$$Y^M(|0\rangle, z) = \mathrm{id}_M, \tag{2.41}$$

$$[Y^M(a, z), Y^M(b, w)] = \sum_{j \geqslant 0} Y^M(a_{(j)}b, w) \frac{1}{j!} \partial_w^j \delta(z - w), \tag{2.42}$$

$$Y^M(a_{(n)}b, z) = Y^M(a, z)_{(n)} Y^M(b, z) \tag{2.43}$$

for all $a, b \in V$, $n \in \mathbb{Z}$,

A vertex algebra is clearly a module over itself, which is called the *adjoint representation*.

By definition, a subspace N of a V-module M is a *submodule* if $a_{(n)}N \subset N$ for all $a \in V$, $n \in \mathbb{Z}$. It is clear that the category V-Mod of V-modules is an abelian category.

A T-stable proper submodule of the adjoint representation is called an *ideal* of V. If $f : V \to V'$ is a vertex algebra homomorphism, $\ker f$ is an ideal of V. For an ideal I of V, the quotient V/I inherits the vertex algebra structure from V. Indeed, there are two ways to see this. One is to apply directly Definition 2.5 since V/I is spanned by the images of $a_{(-1)}|0\rangle$, $a \in V$. The other one is to use the skew-symmetry (Proposition 2.4), as it shows that $Y(a, z)b = e^{zT} Y(b, -z)a \in I$ for $a \in I$, $b \in V$.

The category V/I-Mod is a full subcategory of V-Mod consisting of objects M such that $Y^M(a, z) = 0$ for all $a \in I$.

2.10 Commutative vertex algebras

A vertex algebra V is called *commutative* if all vertex operators $Y(a, z)$, $a \in V$, commute each other (i.e., we have $N_{a,b} = 0$ in the locality axiom (2.34)). This condition is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0 \quad \text{for all} \quad a, b \in V,\ m, n \in \mathbb{Z}.$$

This condition is also equivalent to that $a_{(n)}b = 0$ for all $n \geqslant 0$, $a, b \in V$, that is, $Y(a, z) \in \operatorname{End} V[[z]]$ for all $a \in V$. Indeed, if $Y(a, z) \in \operatorname{End} V[[z]]$ for all $a \in V$ then V is commutative by (2.35). Conversely, if V is commutative, then $a_{(n)}b = a_{(n)}b_{(-1)}|0\rangle = b_{(-1)}a_{(n)}|0\rangle = 0$ for $n \geqslant 0$.

Suppose that V is commutative. Then, the relation (2.36) for $m = n = -1$ simplifies to $(a_{(-1)}b)_{(-1)} = a_{(-1)}b_{(-1)}$, that is,

$$(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c).$$

for all $a, b, c \in V$. It follows that a commutative vertex algebra has a structure of a unital commutative algebra with the product:

$$a \cdot b = a_{(-1)}b,$$

where the unit is given by the vacuum vector $|0\rangle$. The translation operator T of V acts on V as a derivation with respect to this product:

$$T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb).$$

Therefore a commutative vertex algebra has the structure of a differential algebra, see Definition **??**.

The converse holds according to the following exercice.

Exercise 2.4 Show that a differential algebra R with a derivation ∂ carries a canonical commutative vertex algebra structure such that the vacuum vector is the unit, and

$$Y(a, z)b = \left(e^{z\partial}a\right) b = \sum_{n\geqslant 0} \frac{z^n}{n!}(\partial^n a)b \quad \text{for all} \quad a, b \in R.$$

This correspondence gives the following result ([59]).

Theorem 2.5 *The category of commutative vertex algebras is the same as that of differential algebras.*

Example 2.1 If $X = \operatorname{Spec} R$ is an affine scheme, then $(\mathcal{O}(\mathcal{J}_\infty R), T)$ is a differential algebra (see Section 1.1 and Section 1.2) hence a commutative vertex algebra by Theorem 2.5, where $T = \partial$ is the derivation defined by (1.2). More generally, $(\pi_\infty)_* \mathcal{O}_{\mathcal{J}_\infty X}$ is a sheaf of commutative vertex algebras on a scheme X.

2.11 Vertex subalgebra, commutant and center

Definition 2.7 A subspace W of a vertex algebra V is called a *vertex subalgebra* if $|0\rangle \in W$, $TW \subset W$, and $a_{(n)}b \in W$ for all $a, n \in W$, $n \in \mathbb{Z}$.

Let W be a vertex subalgebra of V. We set

$$\begin{aligned}\operatorname{Com}(W, V) :=& \{v \in V\colon [w_{(m)}, v_{(n)}] = 0 \text{ for all } w \in W,\ m, n \in \mathbb{Z}\},\\ =& \{v \in V\colon w_{(n)}v = 0 \text{ for all } w \in W, n \geqslant 0\},\\ =& \{v \in V\colon v_{(n)}w = 0 \text{ for all } w \in W, n \geqslant 0\}.\end{aligned}$$

Let us explain the second equality. If $w_{(n)}v = 0$ for all $w \in W$, $n \geqslant 0$, then $v \in \operatorname{Com}(W, V)$ by (2.35). Conversely, if $v \in \operatorname{Com}(W, V)$ then $w_{(n)}v = w_{(n)}v_{(-1)}|0\rangle = v_{(-1)}w_{(n)}|0\rangle = 0$ for $n \geqslant 0$. The same line of arguments shows the third equality. It is straightforward to see that $\operatorname{Com}(W, V)$ is a vertex subalgebra of V, called the *commutant* of W in V, or the *coset* of V by W.

Vertex subalgebras W_1, W_2 of V are said to form a *dual pair* if $W_1 = \operatorname{Com}(W_2, V)$ and $W_2 = \operatorname{Com}(W_1, V)$. The commutant $\operatorname{Com}(V, V)$ of V in V is called the *center* of V, and it is denoted also by $Z(V)$.

2.12 Example (continued from Section 2.6)

Let

$$\pi = \mathbb{C}[b_{-1}, b_{-2}, \ldots,].$$

Then π is a smooth $\mathcal{B}$-module on which b_n, $n \geqslant 0$, acts as $n\frac{\partial}{\partial b_{-n}}$, and b_{-n}, $n > 0$, acts as multiplication by b_{-n}. Define

$$T = \sum_{n>0} n b_{-n-1} \frac{\partial}{\partial b_{-n}} \in \operatorname{End} \pi.$$

Then $[T, b(z)] = \partial_z b(z)$ on π. It follows from Definition 2.5 that there is a unique vertex algebra structure on π such that 1 is the vacuum vector and $Y(b_{-1}, z) = b(z)$.

Exercise 2.5 Let M be a smooth $\mathcal{B}$-module.

(i) Show that the following correspondence gives the vertex algebra $\langle b(z)\rangle_M$ a $\mathcal{B}$-module structure:

$$\mathcal{B} \to \operatorname{End}(\langle b(z)\rangle_M) \quad b_n \mapsto b(z)_{(n)}$$

(ii) Show that there is a surjective homomorphism $\pi \to \langle b(z)\rangle_M$ of vertex algerbas.

Exercise 2.6 Set $\omega = \frac{1}{2}b_{-1}^2 + \alpha b_{-2} \in \pi$, so that $L(z) = Y(\omega, z)$.

(i) Verify that the OPE (2.30) is equivalent to the following relations:

$$b_0\omega = 0, \quad b_1\omega = b_{-1}, \quad b_2\omega = 2\alpha.$$

(ii) Show that $L_{-1} = T$ on π.

Exercise 2.7 Show that the vertex algebra π is simple, that is, there is no non-trivial ideal of π. This implies that the vertex algebra $\langle b(z)\rangle_M$ of local fields on *any* non-trivial smooth $\mathcal{B}$-module M is isomorphic to π.

2.13 Loop spaces and the vertex algebra of jet algebra

Let V be a commutative vertex algebra, and let M be a V-module. Then,

$$[Y^M(a, z), Y^M(b, w)] = 0$$

for all $a, b \in V$ by (2.42). However, $Y^M(a, z)$ needs not be in $(\operatorname{End} M)[[z]]$. This implies that a V-module as a vertex algebra is *not* the same as a V-module as a differential algebra. In fact, we have the following assertion.

Theorem 2.6 *Let X be an affine scheme. Then the category of vertex $\mathcal{O}(\mathcal{J}_\infty X)$-modules is the same as the category of smooth $\mathcal{O}(\mathscr{L}X)$-modules.*

Here, by *smooth* $\mathcal{O}(\mathscr{L}X)$*-module* we mean an $\mathcal{O}(\mathscr{L}X)$-module M such that, in the notation of §1.6, $f_{(n)}.m = 0$ for sufficiently large n.

Proof First, let $X = \mathbb{A}^N = \operatorname{Spec}\mathbb{C}[x^1, \ldots, x^N]$. Recall that

$$\mathcal{O}(\mathscr{L}X) = \varprojlim_r \mathbb{C}[x^i_{(n)} : i = 1, \ldots, N,\ n \leqslant r]$$

see (1.13). Let M be a smooth $\mathcal{O}(\mathscr{L}X)$-module. Then

$$x^i(z) := \sum_{n\in\mathbb{Z}} x^i_{(n)} z^{-n-1}$$

is a field on M, since $x^i_{(n)}$ acts as zero for a sufficiently large n because M is smooth. Moreover, $x^i(z)$ and $x^j(z)$ are mutually local as they commute each other. Therefore, we have a well-defined vertex algebra homomorphism

$$\mathcal{O}(\mathcal{J}_\infty X) \to \langle x^i(z) : i = 1, \ldots N\rangle_M \subset (\operatorname{End} M)[[z, z^{-1}]]$$

that sends $x^i \in \mathcal{O}(X) \subset \mathcal{O}(\mathcal{J}_\infty X)$ to $x^i(z)$. Conversely, let M be a vertex $\mathcal{O}(\mathcal{J}_\infty X)$-module. Then the correspondence

$$\mathcal{O}(\mathscr{L}X) \to \operatorname{End}(M), \quad x^i_{(n)} \mapsto \operatorname{Res}_{z=0} z^n Y^M(x^i, z),$$

defines a smooth $\mathcal{O}(\mathscr{L}X)$-module structure on M. It is clear that this correspondence is compatible with the morphisms.

Next, let $X = \operatorname{Spec} R$ with

$$R = \mathbb{C}[x^1, x^2, \cdots, x^N]/(f_1, f_2, \cdots, f_r).$$

Then $\mathcal{O}(\mathcal{J}_\infty X) = \mathcal{O}(\mathcal{J}_\infty \mathbb{A}^N)/I$, where $I = \langle T^j f_i : i = 1, \ldots, r, j \geqslant 0\rangle$. Hence, $\mathcal{O}(\mathcal{J}_\infty X)$ -Mod is the full subcategory of $\mathcal{O}(\mathcal{J}_\infty \mathbb{A}^N)$ -Mod consisting of modules M such that

$$Y^M(f_i, z) = 0$$

for all $i = 1, \ldots, r$. (Here we have used the fact that $Y^M(Ta, z) = \partial_z Y^M(a, z)$.) But under the above identification of $\mathcal{O}(\mathcal{J}_\infty \mathbb{A}^N)$ -Mod with the category of smooth $\mathcal{O}(\mathscr{L}\mathbb{A}^N)$-modules, this is nothing but the category of smooth $\mathcal{O}(\mathscr{L}X)$-modules. □

One of the advantages of vertex algebras to loop spaces is that one can avoid the use of completions, which can be sometimes tedious.

2.14 Conical vertex algebras

A *Hamiltonian* of a vertex algebra V is a semisimple operator H on V satisfying

$$[H, a_{(n)}] = -(n+1)a_{(n)} + (Ha)_{(n)} \tag{2.44}$$

for all $a \in V$, $n \in \mathbb{Z}$.

Definition 2.8 A vertex algebra equipped with a Hamiltonian H is called *graded*. In that case, set $V_\Delta = \{a \in V\colon Ha = \Delta a\}$ for $\Delta \in \mathbb{C}$, so that $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$. For $a \in V_\Delta$, Δ is called the *conformal weight* of a and it is denoted by Δ_a. We have

$$a_{(n)}b \in V_{\Delta_a + \Delta_b - n - 1} \tag{2.45}$$

for homogeneous elements $a, b \in V$. A graded vertex algebra is called *conical* if there exists a positive integer m such that $V = \bigoplus_{\Delta \in \frac{1}{m}\mathbb{Z}_{\geqslant 0}} V_\Delta$ and $V_0 = \mathbb{C}$.

We set

$$a_n = a_{(n+\Delta_a - 1)}$$

for $n \in -\Delta_a + \mathbb{Z}$, so that $a_n V_\Delta \subset V_{\Delta - n}$. Then we have

$$a(z) = \sum_{n \in -\Delta_a + \mathbb{Z}} a_n z^{-n-\Delta_a}, \tag{2.46}$$

which is more standard notation in physics than (2.7).

Any (proper) graded ideal of a conical vertex algebra V does not contain the vacuum vector $|0\rangle$, and hence, there is a unique simple graded quotient of V.

Let X be a conical affine scheme, that is, $X = \operatorname{Spec} R$ with a graded ring $R = \bigoplus_{\Delta \in \frac{1}{m}\mathbb{Z}_{\geqslant 0}} R_\Delta$ such that $R_0 = \mathbb{C}$, where m is some positive integer. Then the commutative vertex algebra $\mathcal{O}(\mathcal{J}_\infty X) = \mathcal{J}_\infty R$ is conical, where the Hamiltonian is defined by

$$[H, f_{(-n)}] = (\Delta + n - 1) f_{(-n)}, \quad f \in R_\Delta.$$

In particular the scheme $\mathcal{J}_\infty X$ is conical, and we have a contracting $\mathbb{C}^*$-action on $\mathcal{J}_\infty X$ corresponding to the comorphism $\mathcal{J}_\infty R \to \mathbb{C}[t, t^{-1}] \otimes \mathcal{J}_\infty R$, $f_{(-n)} \mapsto t^{\Delta + n - 1} \otimes f_{(-n)}$ ($f \in R_\Delta$).

Chapter 3
Examples of noncommutative vertex algebras

We present in this chapter the first important examples of noncommutative vertex algebras: the universal affine vertex algebras (cf. Section 3.1), that includes the Heisenberg vertex algebras (see Example 3.1), the Virasoro vertex algebras (cf. Section 3.2), and the chiral differential operators on an algebraic group (cf. Section 3.4) These examples are all constructed from infinite-dimensional Lie algebras. We will see Part III more sophisticated examples, based on the quantized–Drinfeld reduction, that generalize some of these examples.

It was observed that many vertex algebras that appear in the nature carry an action of the Virasoro algebra, and satisfy a bounded-below property with respect to an energy, or Hamiltonian, operator. This motivates the notion of conformal vertex algebras and vertex operator algebras introduced in this chapter as well (cf. Section 3.3).

By considering quotients vertex algebras of these examples, that is, quotient by vertex ideals, we construct many other interesting families of vertex algebras.

3.1 Universal affine vertex algebras

Let $\mathfrak{a}$ be a Lie algebra endowed with a symmetric invariant bilinear form κ. Here, a bilinear form κ on $\mathfrak{a}$ is called invariant if $\kappa([x, y], z) = \kappa(x, [y, z]) = 0$ for $x, y, z \in \mathfrak{a}$. Let

$$\hat{\mathfrak{a}}_\kappa = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$$

be the *Kac-Moody affinization* of $\mathfrak{a}$. It is a Lie algebra with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}\kappa(x, y)\mathbf{1}, \qquad [\mathbf{1}, \hat{\mathfrak{a}}_\kappa] = 0,$$

for all $x, y \in \mathfrak{a}$ and all $m, n \in \mathbb{Z}$, where $\delta_{i,j}$ is the Kronecker symbol.

An $\hat{\mathfrak{a}}_\kappa$-module M is called *smooth* if for any $m \in M$ there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $xt^n m = 0$ for all $x \in \mathfrak{g}$, $n \geqslant N$ or, equivalently,

$$x(z) = \sum_{n\in\mathbb{Z}} (xt^n) z^{-n-1}$$

is a field on M for all $x \in \mathfrak{a}$.

Lemma 3.1 *For any smooth $\hat{\mathfrak{a}}_\kappa$-module M the fields $x(z)$, $y(z)$, $x, y \in \mathfrak{a}$, are mutually local, and we have*

$$x(z)y(w) \sim \frac{1}{z-w}[x,y](w) + \frac{\kappa(x,y)}{(z-w)^2}.$$

Proof The assertion is equivalent to the fact that

$$[x(z), y(w)] = [x,y](w)\delta(z-w) + \kappa(x,y)\partial_w \delta(z-w),$$

which can be checked directly. □

Let M be a smooth $\hat{\mathfrak{a}}_\kappa$-module on which the central element $\mathbf{1}$ acts as the identity. By Lemma 3.1, $\langle x(z) \colon x \in \mathfrak{a}\rangle_M$ has a structure of vertex algebras. Moreover, the correspondence

$$\hat{\mathfrak{a}}_\kappa \ni x \otimes t^n \mapsto x(z)_{(n)} \in \mathrm{End}(\langle x(z) \colon x \in \mathfrak{a}\rangle_M)$$

gives an $\hat{\mathfrak{a}}_\kappa$-module structure on the vertex algebra $\langle x(z) \colon x \in \mathfrak{a}\rangle_M$, see Proposition 2.3. By Proposition 2.3, we find that the $\hat{\mathfrak{a}}_\kappa$-module $\langle x(z) \colon x \in \mathfrak{a}\rangle_M$ is generated by the vector id_M, which satisfies the condition

$$\mathfrak{a}[t]\, \mathrm{id}_M = 0.$$

Hence, by the Frobenius reciprocity, there is an $\hat{\mathfrak{a}}_\kappa$-module homomorphism from the $\hat{\mathfrak{a}}_\kappa$-module

$$V^\kappa(\mathfrak{a}) := U(\hat{\mathfrak{a}}_\kappa) \otimes_{U(\mathfrak{a}[t]\oplus\mathbb{C}\mathbf{1})} \mathbb{C}, \tag{3.1}$$

where $\mathbb{C}$ is a one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{a}[t]$ acts trivially and $\mathbf{1}$ acts as the identity, to the $\hat{\mathfrak{a}}_\kappa$-module $\langle x(z) \colon x \in \mathfrak{a}\rangle_M$.

By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition (as a vector space)

$$\hat{\mathfrak{a}}_\kappa = (\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1})$$

gives us the isomorphism of vector spaces

$$U(\hat{\mathfrak{a}}_\kappa) \cong U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes U(\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}),$$

whence

$$V^\kappa(\mathfrak{a}) \cong U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}])$$

as $\mathbb{C}$-vector spaces.

The space $V^\kappa(\mathfrak{a})$ is naturally graded,

$$V^\kappa(\mathfrak{a}) = \bigoplus_{\Delta\in\mathbb{Z}_{\geqslant 0}} V^\kappa(\mathfrak{a})_\Delta, \tag{3.2}$$

where the grading is defined by

$$\deg(x^{i_1}t^{-n_1})\dots(x^{i_m}t^{-n_m})|0\rangle = \sum_{i=1}^m n_i,$$

where $|0\rangle = 1\otimes 1$. We have $V^\kappa(\mathfrak{a})_0 = \mathbb{C}|0\rangle$, and we identify $\mathfrak{a}$ with $V^\kappa(\mathfrak{a})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

From the works of Frenkel and Zhu [121], the space $V^\kappa(\mathfrak{a})$ has naturally a vertex algebra structure (see also [193]).

Proposition 3.1 *There is a unique vertex algebra structure on $V^\kappa(\mathfrak{a})$ such that $|0\rangle = 1\otimes 1$ is the vacuum vector and $\mathcal{S} = \{x(z) \colon x \in \mathfrak{a}\}$ is a set of generating fields. Moreover, there is a surjective homomorphism $V^\kappa(\mathfrak{a}) \to \langle x(z)\colon x\in\mathfrak{a}\rangle_M$ of vertex algebras for any smooth $\hat{\mathfrak{a}}_\kappa$-module M on which $\mathbf{1}$ acts as the identity.*

Proof For the first assertion we just need to show that there is a unique T-action on $V^k(\mathfrak{g})$ such that $T|0\rangle = 0$ and $[T, x(x)] = \partial_z x(z)$, because the property (ii) in Definition 2.5 holds obviously. The uniqueness is clear, so let us show the existence.

Let $\hat{\mathfrak{g}}\rtimes\mathbb{C}T$ be the vector space $\hat{\mathfrak{g}}\oplus\mathbb{C}T$ viewed as a Lie algebra such that the natural map $\hat{\mathfrak{g}} \hookrightarrow \hat{\mathfrak{g}}\rtimes\mathbb{C}T$ is a Lie algebra homomorphism and

$$[T, xt^n] = -nxt^{n-1} \tag{3.3}$$

for $x\in\mathfrak{g}$, $n\in\mathbb{Z}$. Then $(\mathfrak{g}[t]\oplus\mathbb{C}K)\rtimes\mathbb{C}T := (\mathfrak{g}[t]\oplus\mathbb{C}K)\oplus\mathbb{C}T$ is a Lie subalgebra and

$$U(\hat{\mathfrak{g}}\rtimes\mathbb{C}T)\otimes_{U((\mathfrak{g}[t]\oplus\mathbb{C}K)\rtimes\mathbb{C}T)}\mathbb{C}_k \cong V^k(\mathfrak{g}) \tag{3.4}$$

as $\hat{\mathfrak{g}}$-modules, where on the left-hand-side $\mathbb{C}_k$ is the one-dimensional representation of $(\mathfrak{g}[t]\oplus\mathbb{C}K)\rtimes\mathbb{C}T$ on which $\mathfrak{g}[t]\oplus\mathbb{C}T$ acts trivially and K acts as multiplication by k. It is clear that (3.4) gives the required $\mathbb{C}T$-module structure on $V^k(\mathfrak{g})$.

For the second one, first recall that there is a homomorphism of $\hat{\mathfrak{a}}_\kappa$-modules $V^\kappa(\mathfrak{a}) \to \langle x(z)\colon x\in\mathfrak{a}\rangle_M$. It is clearly a homomorphism of vertex algebras by construction. Since both $V^\kappa(\mathfrak{a})$ and $\langle x(z)\colon x\in\mathfrak{a}\rangle_M$ are generated by $|0\rangle$ as $\hat{\mathfrak{g}}$-modules, the surjectivity follows. □

The vertex algebra $V^\kappa(\mathfrak{a})$ is called the *universal affine vertex algebra* associated with $\mathfrak{a}$ and κ. It is a conical vertex algebra by the grading (3.2). The unique simple graded quotient $L_\kappa(\mathfrak{a})$ of $V^\kappa(\mathfrak{a})$ is called the *simple affine vertex algebra* associated with $\mathfrak{a}$ and κ.

Proposition 3.2 *The category $V^\kappa(\mathfrak{a})$ -Mod of $V^\kappa(\mathfrak{a})$-modules is the same as that of smooth representations of $\hat{\mathfrak{a}}_\kappa$ on which $\mathbf{1}$ acts as the identity.*

Proof Any $V^\kappa(\mathfrak{a})$-modules is a smooth $\hat{\mathfrak{a}}_\kappa$-module by the correspondence $xt^n \mapsto \operatorname{Res}_{z=0}(z^n x(z))$. Conversely, we have a vertex algebra homomorphism $V^\kappa(\mathfrak{a}) \to \langle x(z)\rangle_M$ for any smooth $\hat{\mathfrak{a}}_\kappa$-module M on which $\mathbf{1}$ acts as the identity, and hence, M is a $V^\kappa(\mathfrak{a})$-module. It is clear that this correspondence is compatible with morphisms.□

By Proposition 3.2, the category $L_\kappa(\mathfrak{a})$-Mod of $L_\kappa(\mathfrak{a})$-modules is a full-subcategory of the category of smooth $\hat{\mathfrak{a}}_\kappa$-module consisting of objects M on which $Y_M(v,z) = 0$ for any element v in the kernel of the natural surjection $V^\kappa(\mathfrak{a}) \to L_\kappa(\mathfrak{a})$.

Example 3.1 Let $\mathfrak{h}$ be a vector space viewed as a commutative Lie algebra, and κ be any bilinear form on $\mathfrak{h}$. Then $V^\kappa(\mathfrak{h})$ is the *Heisenberg vertex algebra associated with* $\mathfrak{h}$ *and* κ. In the case that $\mathfrak{h}$ is one-dimensional and κ is a nonzero bilinear form, then $V^\kappa(\mathfrak{h})$ is isomorphic to the vertex algebra π in Section 2.12.

Example 3.2 Let us consider another important example. Assume that $\mathfrak{a}$ is a simple Lie algebra $\mathfrak{g}$, and that $\kappa = k(-|-)$, where

$$(-|-) := \frac{k}{2h^\vee} \times \text{Killing form of } \mathfrak{g}, \qquad \text{for } k \in \mathbb{C},$$

where $h^\vee$ its dual Coxeter number of $\mathfrak{g}$. The reader is referred to Appendix A for main notations and standard facts about simple Lie algebras (Section A.1), and the corresponding affine Kac-Moody Lie algebras (Section A.2).

In this case, $V^\kappa(\mathfrak{a})$ is identical to the $\hat{\mathfrak{g}}$-module

$$V^k(\mathfrak{g}) := U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k,$$

where $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$ is the affine Kac-Moody algebra associated with $\mathfrak{g}$ as in Appendix A, and $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as multiplication by k. We will preferably use the notation $V^k(\mathfrak{g})$ in this case.

The representation $V^k(\mathfrak{g})$ is a highest weight representation of $\hat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, where Λ_0 is the highest weight of the *basic* representation (it corresponds to $k = 1$)[1], and highest weight vector v_k, where v_k denotes the image of $1 \otimes 1$ in $V^k(\mathfrak{g})$. According to the well-known Schur Lemma, any central element of a Lie algebra acts as a scalar on a simple finite dimensional representation. As the Schur Lemma extends to a representation with countable dimension[2], the result holds for highest weight $\hat{\mathfrak{g}}$-modules.

A representation M is said to be *of level* k if K acts as $k\mathrm{Id}$ on M (see §A.5.2). Then $V^k(\mathfrak{g})$ is by construction of level k.

The vertex algebra $V^k(\mathfrak{g})$ is also called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at level* k. The simple quotient $L_\kappa(\mathfrak{g})$ is denoted also by $L_k(\mathfrak{g})$ and is called the *simple affine vertex algebra associated with* $\mathfrak{g}$ *at level* k.

[1] that is, the dual of K in $\hat{\mathfrak{h}}^*$ with respect to a basis of $\hat{\mathfrak{h}}$ adapted to the decomposition $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$.

[2] i.e., it admits a countable set of generators.

Exercise 3.1 Let V be a vertex algebra, and suppose that there exists a vertex algebra homomorphism $\phi\colon V^{\kappa}(\mathfrak{g}) \to V$, so that V is a $\hat{\mathfrak{g}}_{\kappa}$-module. Show that

$$\mathrm{Com}(\phi(V^{\kappa}(\mathfrak{g})), V) = V^{\mathfrak{g}[[t]]},$$

where $V^{\mathfrak{g}[[t]]} = \{v \in V : \mathfrak{g}[[t]]v = 0\}$.

3.2 The Virasoro vertex algebra

Let $Vir = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$ be the *Virasoro Lie algebra*, with the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0}C,$$
$$[C, Vir] = 0.$$

A Vir-module M is called *smooth* if

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

is a field on M. For any smooth Vir-module M the fields $L(z)$ is local to itself, and we have

$$L(z)L(w) \sim \frac{1}{z-w}\partial_w L(w) + \frac{2}{(z-w)^2}L(w) + \frac{C/2}{(z-w)^4}.$$

A Vir-module M is said to be of *central charge* $c \in \mathbb{C}$ if the central element C acts as multiplication by c.

Let M be a smooth Vir-module of central charge c. Then $\langle L(z)\rangle_M$ is a smooth Vir-module of central charge c by the action $L_n \mapsto L(z)_{(n+1)}$. It is generated by id_M and we have $L(z)_{(n)}\,\mathrm{id}_M = 0$ for $n \geqslant 0$. Similarly to the case of $V^{\kappa}(\mathfrak{a})$ we obtain that $\langle L(z)\rangle_M$ is a quotient of the induced representation

$$\mathrm{Vir}^c := U(Vir) \otimes_{U(\bigoplus_{n\geqslant -1}\mathbb{C}L_n \oplus \mathbb{C}C)} \mathbb{C}_c,$$

where C acts as multiplication by c and L_n, $n \geqslant -1$, acts by 0 on the one-dimensional module $\mathbb{C}_c$.

By the PBW Theorem, Vir^c has a basis of the form

$$L_{j_1} \dots L_{j_m}|0\rangle, \qquad j_1 \leqslant \cdots \leqslant j_m \leqslant -2,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in Vir^c. Next result is proved in [121, 193].

Proposition 3.3 *There is a unique vertex algebra structure on* Vir^c *such that* $|0\rangle = 1 \otimes 1$ *is the vacuum vector and* $L(z)$ *is a generating field. Moreover, there is a* *surjective homomorphism* $\mathrm{Vir}^c \to \langle L(z)\rangle_M$ *of vertex algebras for any smooth Vir-module M of central charge c.*

The vertex algebra Vir^c is called the *universal Virasoro vertex algebra with central charge c*.

Note that $T = L_{-1}$ on Vir^c since $L(z)_{(0)}L(z) = \partial_z L(z)$ (or equivalently, $L_{-1}L_{-2}|0\rangle = L_{-3}|0\rangle$). Also, Vir^c is conical by the Hamiltonian $H = L_0$:

$$\mathrm{Vir}^c = \bigoplus_{\Delta\in\mathbb{Z}_{\geqslant 0}} \mathrm{Vir}^c_\Delta, \quad \mathrm{Vir}^c_0 = \mathbb{C}|0\rangle,\ \mathrm{Vir}^c_1 = 0,\ \mathrm{Vir}^c_2 = \mathbb{C}\omega, \tag{3.5}$$

where $\omega = L_{-2}|0\rangle$. The unique simple quotient of Vir^c is called the *simple Virasoro vertex algebra with central charge c* and is denoted by Vir_c.

Proposition 3.4 *The category* Vir^c -Mod *of* Vir^c*-modules is the same as that of smooth representations of Vir of central charge c.*

3.3 Conformal vertex algebras

Definition 3.1 A vertex algebra V is called *conformal* if there exists a vector ω, called the *stress tensor*, or the *conformal vector*, such that the corresponding field

$$Y(\omega, z) = T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfies the following conditions:

(1) $[L_n, L_m] = (n-m)L_{n+m} + \dfrac{n^3-n}{12}\delta_{n+m,0}c$, where c is a constant called the *central charge* of V,
(2) $\omega_{(0)} = L_{-1} = T$,
(3) $\omega_{(1)} = L_0 = H$ is a Hamiltonian so that V is graded by L_0: $V = \bigoplus_{\Delta\in\mathbb{C}} V_\Delta$, with $L_0|_{V_\Delta} = \Delta \mathrm{Id}_{V_\Delta}$ for all $\Delta\in\mathbb{C}$.

A $\mathbb{Z}$-graded conformal vertex algebra such that $\dim V_\Delta < \infty$ for all $\Delta\in\mathbb{Z}$ and $V_\Delta = 0$ for sufficiently small Δ is also called a *vertex operator algebra*.

For a conformal vertex algebra of central charge c, we have a homomorphism $\mathrm{Vir}^c \to V$, $\omega \mapsto \omega$, of vertex algebras. Let M be a module over a conformal vertex algebra V of central charge c. Then the Virasoro algebra acts on M via the vertex algebra homomorphism $\mathrm{Vir}^c \to V$.

Definition 3.2 The module M is called a *positive energy representation* if there is $\lambda\in\mathbb{C}$ such that $M = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} M_{\lambda+n}$, where

$$M_d = \{m \in M : L_0^M m = dm\}.$$

A positive energy representation M is called an *ordinary representation* if each M_d is finite-dimensional.

For an ordinary representation M the *normalized character*

$$\chi_M(q) = \mathrm{tr}_M(q^{L_0 - c/24}) = q^{-c/24}\sum_d (\dim M_d) q^d \tag{3.6}$$

is well-defined.

Example 3.3 The Virasoro vertex algebra Vir^c is clearly conformal with central charge c and conformal vector $\omega = L_{-2}|0\rangle$.

Example 3.4 The Heisenberg vertex algebra $V^\kappa(\mathfrak{h})$ associated with a nondegenerate bilinear form κ is conformal. Indeed, $T(z) = \frac{1}{2}\sum_{i=1}^{r} {:}x_i(z)x^i(z){:}$ is a conformal field with central charge r, where $\{x_i\}_{1\leqslant i\leqslant r}$ and $\{x^i\}_{1\leqslant i\leqslant r}$ are dual basis of $\mathfrak{h}$ with respect to κ (see Exercise 2.6 for the case $r = 1$).

Example 3.5 The universal affine vertex algebra $V^k(\mathfrak{g})$, with $\mathfrak{g}$ simple of dimension d, is conformal by *Sugawara construction* provided with $k \neq -h^\vee$ (here $h^\vee$ is the dual Coxeter number): Set

$$S = \frac{1}{2}\sum_{i=1}^{d} x_{i,(-1)}x^i_{(-1)}|0\rangle,$$

where $\{x_i\}_{1\leqslant i\leqslant d}$ and $\{x^i\}_{1\leqslant i\leqslant d}$ are dual basis with respect to $(-|-)$. Then for $k \neq -h^\vee$, $L = \dfrac{S}{k+h^\vee}$ is a stress tensor of $V^k(\mathfrak{g})$ with central charge

$$c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee}.$$

We refer to [113, §3.4.8] or to [112, 3.1.1] for a proof of this nontrivial statement; see also [160, Theorem 5.7] and its proof. We have

$$[L_m, x_{(n)}] = -n x_{(m+n)} \quad x \in \mathfrak{g},\ m, n \in \mathbb{Z}. \tag{3.7}$$

It follows from Exercise 3.1 that $Z(V^k(\mathfrak{g})) = V^k(\mathfrak{g})^{\mathfrak{g}[[t]]}$, where

$$V^k(\mathfrak{g})^{\mathfrak{g}[[t]]} := \{a \in V^k(\mathfrak{g}) : x_{(m)}a = 0 \text{ for all } x \in \mathfrak{g},\ m \in \mathbb{Z}_{\geqslant 0}\}.$$

The following exercise gives a description of the vertex center of $V^k(\mathfrak{g})$ which has a priori nothing to do the vertex algebra structure.

Exercise 3.2 Show that we have the following isomorphism of commutative $\mathbb{C}$-algebras (the product on the commutative vertex algebra $Z(V^k(\mathfrak{g}))$ is the normally ordered product):

$$Z(V^k(\mathfrak{g})) \cong \mathrm{End}_{\hat{\mathfrak{g}}}(V^k(\mathfrak{g})).$$

Remark 3.1 It is easily seen that $Z(V^k(\mathfrak{g})) = \mathbb{C}|0\rangle$ for $k \neq -h^\vee$ using the stress tensor L. For $k = -h^\vee$, the center

$$Z(V^{-h^\vee}(\mathfrak{g})) =: \mathfrak{z}(\hat{\mathfrak{g}})$$

is huge, and it is usually referred as the *Feigin–Frenkel center* [109][3]: we have $\operatorname{gr}\mathfrak{z}(\hat{\mathfrak{g}}) \cong \mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}/\!\!/G))$, with $\mathfrak{g}/\!\!/G = \operatorname{Spec}\mathcal{O}(\mathfrak{g})^G$.

3.4 Chiral differential operators on an algebraic group

Let G be a affine algebraic group, $\mathfrak{g} = \operatorname{Lie}(G)$, κ an invariant bilinear form on $\mathfrak{g}$, and set

$$\mathcal{A}_G = U(\hat{\mathfrak{g}}_\kappa) \otimes \mathcal{O}(\mathscr{L}G),$$

where $\mathscr{L}G$ is the loop space of G (see Section 1.6), and consider $\mathcal{A}_G$ as an algebra such that the natural embeddings $U(\hat{\mathfrak{g}}_\kappa) \hookrightarrow \mathcal{A}_G$, $\mathcal{O}(\mathscr{L}G) \hookrightarrow \mathcal{A}_G$, are embeddings of algebras and

(3.8) $$[xt^m, f_{(n)}] = (x_L f)_{(m+n)} \quad \text{for } x \in \mathfrak{g},\ f \in \mathcal{O}(G),\ m, n \in \mathbb{Z}.$$

We regard $\mathcal{O}(\mathcal{J}_\infty G)$ as a module over the subalgebra

$$\mathcal{A}_{G,+} = U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1}) \otimes \mathcal{O}(\mathscr{L}G) \subset \mathcal{A}_G$$

on which $\mathcal{O}(\mathscr{L}G)$ acts via the natural surjection $\mathcal{O}(\mathscr{L}G) \to \mathcal{O}(\mathcal{J}_\infty G)$, an element of $\mathfrak{g}[t] \subset \mathfrak{g}[[t]]$ acts as a left invariant vector field on $\mathcal{O}(\mathcal{J}_\infty G)$ (see Example 1.2), and $\mathbf{1}$ acts as the identity. Define

(3.9) $$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} = \mathcal{A}_G \otimes_{\mathcal{A}_{G,+}} \mathcal{O}(\mathcal{J}_\infty G).$$

Note that

(3.10) $$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1})} \mathcal{O}(\mathcal{J}_\infty G)$$

as $\hat{\mathfrak{g}}$-modules. We have the mutually local fields

$$x(z) = \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1} \quad (x \in \mathfrak{g}), \qquad f(z) = \sum_{n \in \mathbb{Z}} f_{(n)} z^{-n-1} \quad (f \in \mathcal{O}(G))$$

on $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ satisfying the OPEs

[3] See Example 1.3 for more details about the scheme $\mathcal{J}_\infty(\mathfrak{g}/\!\!/G)$.

$$x(z)y(w) \sim \frac{1}{z-w}[x,y](w) + \frac{\kappa(x,y)}{(z-w)^2}, \quad f(z)g(w) \sim 0, \tag{3.11}$$

$$x(z)f(w) \sim \frac{1}{z-w}(x_L f)(w) \tag{3.12}$$

for $x, y \in \mathfrak{g}$, $f, g \in \mathcal{O}(G)$.

The following assertion, first observed in [137, 41], is clear from Definition 2.5.

Theorem 3.1 *There is a unique vertex algebra structure on $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ such that the embeddings*

$$\pi_L : V^\kappa(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{G,\kappa}, \quad u|0\rangle \mapsto u \otimes 1,$$
$$j : \mathcal{O}(\mathcal{J}_\infty G) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{G,\kappa}, \quad f \mapsto 1 \otimes f,$$

are homomorphisms of vertex algebras, and

$$x(z)f(w) \sim \frac{1}{z-w}(x_L f)(w) \tag{3.13}$$

for $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$.

The vertex algebra $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ is called the vertex algebra of (global) *chiral differential operators* (cdo) on G. It is naturally $\mathbb{Z}_{\geqslant 0}$-graded by the following conditions:

- elements of $\mathfrak{g}$, embedded in $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ through π_L, have weight 1,
- elements of $\mathcal{O}(G)$, embedded in $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ through j, have weight 0.

Let Ω be the subspace of $(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})_1$ spanned by vectors $f\partial g$, with $f, g \in \mathcal{O}(G)$, where $\partial = T$ is the translation operator on $\mathcal{O}(\mathcal{J}_\infty G)$. Recall that the embedding $\mathfrak{g} \hookrightarrow \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$, $x \mapsto x_L$, induces an isomorphism of left $\mathcal{O}(G)$-modules

$$\mathcal{O}(G) \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)). \tag{3.14}$$

It is easy to see that

$$(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})_1 = \Omega \oplus \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)) \tag{3.15}$$

as vector spaces. Let $\Omega^1(G)$ be the space of global differential forms on G as in Appendix C. Recall that $\Omega^1(G)$ is generated as a $\mathcal{O}(G)$-module by the elements df, for $f \in \mathcal{O}(G)$, where d is the de Rham differential (see Appendix C), and that

$$\Omega^1(G) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G))$$

through the map $df \mapsto (x \mapsto x_L f)$ (see Lemma C.2).

Lemma 3.2 *The $\mathbb{C}$-linear map sending $f\partial g \in \Omega$ to the element $h \otimes x \mapsto (hx)_{(1)}(f\partial g)$ of $\mathrm{Hom}_{\mathcal{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)), \mathcal{O}(G))$, with $h \in \mathcal{O}(G)$ and $x \in \mathfrak{g}$, is an isomorphism of vector spaces. Therefore $\Omega \cong \Omega^1(G)$ as vector spaces.*

Proof First of all, note that for $x \in \mathfrak{g}$ and $f, g, h \in \mathcal{O}(G)$,

$$(hx)_{(1)}(f\partial g) = hf(x_L g) \in \mathcal{O}(G)$$

and, clearly, the map sending $h \otimes x \in \mathcal{O}(G) \otimes \mathfrak{g}$ to $(hx)_{(1)}(f\partial g) = hf(x_L g)$ is a morphism of $\mathcal{O}(G)$-modules. Hence the map of the lemma is well-defined. Let is denote it by Γ.

By the Frobenius reciprocity,

$$\mathrm{Hom}_{\mathcal{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)), \mathcal{O}(G)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G)),$$

and through this isomorphism, the map $\Gamma(f\partial g)$ sends $x \in \mathfrak{g}$ to $f(x_L g)$. Hence it suffices to show that the $\mathcal{O}(G)$-linear map sending $\partial g \in \Omega$ to the element $(x \mapsto x_L g)$ of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{O}(G))$ is an isomorphism of $\mathcal{O}(G)$-modules. By Lemma C.2, this is equivalent to showing that the $\mathcal{O}(G)$-linear map sending $\partial g \in \Omega$ to $dg \in \Omega^1(G)$ is an isomorphism.

But $\partial g = g_{(-2)}|0\rangle$ is by construction a regular function on $\mathcal{J}_1(G) \cong TG$, where TG is the tangent bundle of G, and through this identification, ∂g is nothing but dg, so the statement is obvious. □

We denote by $\langle -, - \rangle \colon \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G)) \times \Omega \to \mathcal{O}(G)$ the canonical $\mathcal{O}(G)$-bilinear pairing. The Lie algebra $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$ acts on Ω by the Lie derivative given by (C.6).

Lemma 3.3 *Let $x \in \mathfrak{g}$ and $\omega \in \Omega$. Then $x_{(1)}\omega = \langle x, \omega \rangle$ and $x_{(0)}\omega = (\mathrm{Lie}\, x).\omega$*

Proof The identity $x_{(1)}\omega = \langle x, \omega \rangle$ is clear by Lemma 3.2. Let us prove the second one using it. The Lie derivative action can be written as:

$$y_{(1)}((\mathrm{Lie}\, x).\omega) = x_L(y_{(1)}\omega) - [x, y]_{(1)}\omega = x_{(0)}y_{(1)}\omega - [x, y]_{(1)}\omega,$$

for all $y \in \mathfrak{g}$. But

$$y_{(1)}(x_{(0)}\omega) = x_{(0)}y_{(1)}\omega - [x, y]_{(1)}\omega$$

for all $y \in \mathfrak{g}$, whence $x_{(0)}\omega = (\mathrm{Lie}\, x).\omega$. □

Let

$$\kappa^* = -\kappa - \kappa_{\mathfrak{g}}, \tag{3.16}$$

where $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$.

Theorem 3.2 *Let κ and κ^* be as in (3.16).*

(i) *There is a vertex algebra embedding*

$$\pi_R \colon V^{\kappa^*}(\mathfrak{g}) \hookrightarrow \mathrm{Com}(V^{\kappa}(\mathfrak{g}), \mathcal{D}^{\mathrm{ch}}_{G,\kappa}) \subset \mathcal{D}^{\mathrm{ch}}_{G,\kappa}$$

such that

$$[\pi_R(x)_{(m)}, f_{(n)}] = (x_R f)_{(m+n)} \quad \textit{for } x \in \mathfrak{g},\ f \in \mathcal{O}(G),\ m, n \in \mathbb{Z},$$

where x_R denotes the right invariant vector field corresponding to $x \in \mathfrak{g}$.

(ii) *There is a vertex algebra isomorphism*

$$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong \mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}$$

that sends $f \in \mathcal{O}(G)$ to $S(f) \in \mathcal{O}(G)$, where $S\colon \mathcal{O}(G) \to \mathcal{O}(G)$ is the antipode.

Proof (i) From now, we identify $x \in \mathfrak{g}$ with its image in $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ through π_L.

Let $\{x^1, \dots, x^d\}$ be a basis of $\mathfrak{g}$, and $\{\omega^1, \dots, \omega^d\}$ the dual $\mathcal{O}(G)$-basis of $\Omega \cong \Omega^1(G)$. The isomorphism (C.2) tells that $\{x^1, \dots, x^d\}$ forms an $\mathcal{O}(G)$-basis of $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}(G))$. In particular,

$$x^i_R = \sum_p f^{i,p} x^p, \quad i = 1, \dots, d,$$

for some invertible matrix $(f^{i,p})_{1 \leqslant i,p \leqslant d}$ over $\mathcal{O}(G)$. We will repeatidily make use of the identities of Lemma C.2 and Lemma C.3.

We set for all $i \in \{1, \dots, d\}$,

$$\pi_R(x^i) = x^i_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{i,p} \omega^q. \tag{3.17}$$

We first verify that for all i, j and $n \geqslant 0$,

$$(x^i)_{(n)} \pi_R(x^j) = 0. \tag{3.18}$$

By (3.11), (3.12), the condition (3.18) is clearly satisfied for $n \geqslant 2$.

Fix i, j. We first verify that $(x^i)_{(1)} \pi_R(x^j) = 0$. First, by (C.3), (3.11), (3.12) and Borcherds identity (2.36), we have

$$\begin{aligned}(x^i_R)_{(1)} x^j &= \sum_p (f^{i,p}_{(-1)} x^p)_{(1)} x^j = \sum_j (f^{i,p}_{(-1)} x^p_{(1)} x^j + x^p_{(0)} f^{i,p}_{(0)} x^j) \\ &= \sum_p (f^{i,p} \kappa(x^p, x^j) - x^p_L(x^j_L f^{i,p}))\end{aligned} \tag{3.19}$$

Using Lemma C.2 (i) twice, we get

$$-x^p_L(x^j_L f^{i,p}) = \sum_s x^p_L(c^{j,s}_p f^{i,s}) = -\sum_{s,u} c^{p,u}_s c^{j,s}_p f^{i,u} = \sum_{s,u} c^{u,p}_s c^{j,s}_p f^{i,u}.$$

Since

$$\kappa_{\mathfrak{g}}(x^i, x^j) = \sum_{p,q} c^{i,q}_p c^{j,p}_q, \quad i, j = 1, \dots, d,$$

we deduce that

$$-\sum_p x_L^p(x_L^j f^{i,p}) = \sum_u \kappa_{\mathfrak{g}}(x^u, x^j) f^{i,u}. \tag{3.20}$$

Combining (3.19), (3.20) and (3.16), we obtain that for $i, j = 1, \dots, d$,

$$(x_R^i)_{(1)} x^j = -\sum_p \kappa^*(x^p, x^j) f^{i,p}.$$

On the other hand, by Lemma 3.3, we have for any $p, q \in \{1, \dots, d\}$,

$$(\kappa^*(x^p, x^q) f^{i,p} \omega^q)_{(1)} x^j = \kappa^*(x^p, x^q) f^{i,p} \langle \omega^q, x^j \rangle = \kappa^*(x^p, x^j) f^{i,p},$$

whence $(x^i)_{(1)} \pi_R(x^j) = 0$.

We now wish to prove that $(x^i)_{(0)} \pi_R(x^j) = 0$. We have

$$(x^i)_{(0)} \pi_R(x^j) = (x^i)_{(0)} (x_R^j + \sum_{p,q} \kappa^*(x^p, x^q) f^{j,p} \omega^q).$$

One one hand, using Lemma C.2 (i),

$$x_{(0)}^i x_R^j = \sum_q x_{(0)}^i (f^{j,q} x^q) = \sum_q x_{(0)}^i f_{(-1)}^{j,q} x^q = \sum_q ((x_L^i f^{j,q})_{(-1)} x^q + f_{(-1)}^{j,q} [x^i, x^q]) = 0.$$

On the other hand, using Lemma C.2 (i) and Lemma 3.3, we get

$$\begin{aligned} x_{(0)}^i (\kappa^*(x^p, x^q) f^{j,p} \omega^q) &= \kappa^*(x^p, x^q) \left((x_L^i f^{j,p}) \omega^q + f^{j,p} (\operatorname{Lie} x^j).\omega^q\right) \\ &= -\sum_{s,r} \left(\kappa^*(x^p, x^r) c_p^{i,s} f^{j,s} \omega^r + \kappa^*(x^p, x^q) c_q^{i,r} f^{j,p} \omega^r\right) \\ &= -\sum_{s,r} \left(\kappa^*([x^i, x^s], x^r) f^{j,s} \omega^r - \kappa^*(x^p, [x^i, x^r]) f^{j,p} \omega^r\right) = 0 \end{aligned}$$

due to the invariance of κ^*. This proves that $(x^i)_{(0)} \pi_R(x^j) = 0$.

In conclusion, (3.18) holds for any $i, j = 1, \dots, d$ and $n \geqslant 0$ as desired. It remains to verify that π_R defines a vertex algebra homomorphism which is injective. Due to the decomposition (3.15), we see that the map π_R defined by (3.17) is injective since the map $\mathfrak{g} \to \operatorname{Der}_{\mathbb{C}}(\mathcal{O}(G))$, $x \mapsto x_L$, is.

For the vertex algebra homomorphism part, we have to show that

$$(\pi_R(x))\,(z)\,(\pi_R(y))\,(w) \sim \frac{1}{z-w} \pi_R([x, y])(w) + \frac{\kappa^*(x, y)}{(z-w)^2} \tag{3.21}$$

for all $x, y \in V^{\kappa^*}(\mathfrak{g})$, that is,

$$\pi_R(x)_{(1)} \pi_R(y) = \kappa^*(x, y), \tag{3.22}$$

$$\pi_R(x)_{(0)} \pi_R(y) = \pi_R([x, y]). \tag{3.23}$$

for all $x, y \in V^{\kappa^*}(\mathfrak{g})$.

To compute $\pi_R(x)_{(1)}\pi_R(y)$ we notice that for $i, j \in \{1, \dots, d\}$,

$$\pi_R(x^i)_{(1)}\pi_R(x^j) = \left(x^i_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{i,p}\omega^q\right)_{(1)} \left(x^j_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{j,p}\omega^q\right)$$

is a sum of four terms. We have

$$\begin{aligned}(x^i_R)_{(1)}x^j_R &= \sum_{p,s}(f^{i,p}x^p)_{(1)}(f^{j,s}x^s)\\ &= \sum_{p,s}(f^{i,p}f^{j,s}\kappa(x^p, x^s) - f^{i,p}x^s_L(x^p_L f^{j,s}) - f^{j,s}x^p_L(x^s_L f^{i,p}) - (x^p_L f^{j,s})(x^s_L f^{i,p}))\end{aligned}$$

Using (3.16) and Lemma C.2 we see that

$$-f^{i,p}x^s_L(x^p_L f^{j,s}) = -f^{j,s}x^p_L(x^s_L f^{i,p}) = (x^p_L f^{j,s})(x^s_L f^{i,p}) = \kappa_{\mathfrak{g}}(x^p, x^s) f^{i,p} f^{j,s},$$

for all i, j, s, p, whence

$$(x^i_R)_{(1)}x^j_R = -\sum_{p,s} \kappa^*(x^p, x^s) f^{i,p} f^{j,s}.$$

Next, using Lemma 3.3, we get for all i, j, p, q,

$$\left(\kappa^*(x^p, x^q) f^{i,p}\omega^q\right)_{(1)} x^j_R = \kappa^*(x^p, x^q) f^{i,p} f^{j,q}.$$

Similarly,

$$(x^i_R)_{(1)} \left(\kappa^*(x^s, x^u) f^{j,s}\omega^u\right)_{(1)} x^j_R = \kappa^*(x^p, x^q) f^{i,p} f^{j,q},$$

and evidently,

$$\left(\kappa^*(x^p, x^q) f^{i,p}\omega^q\right)_{(1)} \left(\kappa^*(x^s, x^u) f^{j,s}\omega^u\right) = 0$$

Adding up, we get

$$\pi_R(x^i)_{(1)}\pi_R(x^j) = \sum_{p,q} \kappa^*(x^p, x^q) f^{i,p} f^{j,q}. \tag{3.24}$$

We differentiate the above relation. By Lemma C.2, we have for i, j, p, q, s,

$$
\begin{aligned}
x^s_L\left(\sum_{p,q}\kappa^*(x^p,x^q)f^{i,p}f^{j,q}\right) &= \sum_{p,q}\kappa^*(x^p,x^q)\left((x^s_L f^{i,p})f^{j,q}+f^{i,p}(x^s_L f^{j,q})\right)\\
&= \sum_{p,q,u,v}\kappa^*(x^p,x^q)\left(-c^{s,u}_p f^{i,u}f^{j,q}-c^{s,v}_q f^{i,p}f^{j,v}\right)\\
&= \sum_{p,q,u,v}\left(-\kappa^*([x^s,x^u],x^q)f^{i,u}f^{j,q}-\kappa^*(x^p,[x^s],x^v])f^{i,p}f^{j,v}\right)\\
&= 0.
\end{aligned}
$$

Therefore, (3.24) is constant. This constant can be computed by observing that the matrix $(f^{i,j})$, considered as a function on the group G, is equal to the identity at the neutral element of G by the identity (C.3). Hence, (3.24) is equal to $\kappa^*(x^i,x^j)$, which proves (3.22).

We now compute $\pi_R(x)_{(0)}\pi_R(y)$. For $i,j\in\{1,\dots,d\}$ we have

$$
\pi_R(x^i)_{(0)}\pi_R(x^j)=\pi_R(x^i)_{(0)}\left(\sum_q f^{j,q}x^q+\sum_{s,u}\kappa^*(x^s,x^u)f^{j,s}\omega^u\right).
$$

Using (3.18) with $n=0$, we have

$$
\begin{aligned}
\pi_R(x^i)_{(0)}(f^{j,q}x^q) &= (\pi_R(x^i)_{(0)}f^{j,q})x^q\\
&= \sum_p f^{i,p}(x^p_L f^{j,q})x^q = [x^i_R,x^j_R]=[x^i,x^j]_R,
\end{aligned}
\tag{3.25}
$$

by Lemma 3.3, Lemma C.2 (ii), Lemma C.3.

On the other hand,

$$
\begin{aligned}
\pi_R(x^i)_{(0)}(\kappa^*(x^s,x^u)f^{j,s}\omega^u) &= (x^i_R)_{(0)}(\kappa^*(x^s,x^u f^{j,s}\omega^u)\\
&= \kappa^*(x^s,x^u)x^i_R(f^{j,s}\omega^u)\\
&= \kappa^*(x^s,x^u)f^{j,p}x^p_L(f^{j,s})\omega^u)=\kappa^*(x^s,x^u)c^{i,j}_q f^{q,s}\omega^u)
\end{aligned}
\tag{3.26}
$$

by Lemma 3.3, Lemma C.2 (ii), and Lemma C.3. Adding up (3.25) and (3.26) we obtain that

$$
\pi_R(x^i)_{(0)}\pi_R(x^j)=\pi_R([x^i,x^j])
$$

for all $i,j=1,\dots,d$, which proves (3.23). To sum up, we have proven that π_R is an injective vertex algebra homomorphism. We leave to the reader the verification of the identity $[\pi_R(x)_{(m)},f_{(n)}]=(x_R f)_{(m+n)}$ for all $x\in\mathfrak{g}$, $f\in\mathcal{O}(G)$, $m,n\geqslant 0$.

(ii) Consider the unique vertex algebra homomorphism

$$
\Phi\colon\mathcal{D}^{\mathrm{ch}}_{G,\kappa}\longrightarrow\mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}
$$

whose restriction to $\mathcal{O}(G)$ is given by the antipode S, and restriction to $V^\kappa(\mathfrak{g})$ is the map $\pi_L(x)\mapsto\pi_R(x)$. It is easy to verify that Φ is indeed a vertex algebra homomorphism by (3.21), since

$$(\Phi(x))(z)(\Phi(f))(w) \sim \frac{1}{z-w}\left(\Phi(x)_L\Phi(f)\right)(w)$$

for $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$ which holds by

$$x_R S(f) = S(x_L f)$$

for $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$.

It remains to show that Φ is an isomorphism. Consider the vertex algebra homomorphism from $\mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}$ to $\mathcal{D}^{\mathrm{ch}}_{G,(\kappa^*)^*}$ whose restriction to $\mathcal{O}(G)$ is given by the antipode, and restriction to $V^{\kappa^*}(\mathfrak{g})$ is the map $\pi_R(x) \to \pi_L(x)$. Note that $(\kappa^*)^* = \kappa$. Similarly to Φ, we verify that Ψ is indeed a vertex algebra homomorphism. Moreover, we have $\Psi \circ \Phi = \mathrm{id}_{\mathcal{D}^{\mathrm{ch}}_{G,\kappa}}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}}$. This concludes the proof of (ii). □

Theorem 3.3 *Suppose that G is connected. The vertex algebras $V^\kappa(\mathfrak{g})$ and $V^{\kappa^*}(\mathfrak{g})$ form a dual pair in $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$.*

Proof We have to show that

$$V^\kappa(\mathfrak{g}) = (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])} \quad \text{and} \quad V^{\kappa^*}(\mathfrak{g}) = (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_L(\mathfrak{g}[[t]])}.$$

By Theorem 3.2, we have already established the inclusions $V^\kappa(\mathfrak{g}) \subset (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])}$ and $V^{\kappa^*}(\mathfrak{g}) \subset (\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_L(\mathfrak{g}[[t]])}$. To show the other inclusions, observe that

$$\begin{aligned}(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])} &= \left(U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}\mathbf{1})} \mathcal{O}(\mathcal{J}_\infty G)\right)^{\pi_R(\mathfrak{g}[[t]])} \\ &= U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}\mathbf{1})} \mathcal{O}(\mathcal{J}_\infty G)^{\pi_R(\mathfrak{g}[[t]])}\end{aligned}$$

since the image by π_R of $V^{\kappa^*}(\mathfrak{g})$ commutes with $U(\hat{\mathfrak{g}}_\kappa)$. But since G is connected, we get that

$$\mathbb{C} \cong \mathcal{O}(\mathcal{J}_\infty G)^{\mathcal{J}_\infty(G)} = \mathcal{O}(\mathcal{J}_\infty G)^{\mathfrak{g}[[t]]} = \mathcal{O}(\mathcal{J}_\infty G)^{\pi_R(\mathfrak{g}[[t]])}.$$

As a result,

$$(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])} \cong V^\kappa(\mathfrak{g}).$$

Using the isomorphism $\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong \mathcal{D}^{\mathrm{ch}}_{G,\kappa^*}$ of Theorem 3.2, we obtain that

$$(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_L(\mathfrak{g}[[t]])} \cong V^{\kappa^*}(\mathfrak{g}).$$

This concludes the proof of the theorem. □

Suppose that G is reductive. The *algebraic Peter-Weyl theorem* states that

$$\mathcal{O}(G) \cong \bigoplus_{\chi \in \hat{G}} V_\lambda \otimes V_{\lambda^*} \tag{3.27}$$

as $G \times G$-modules, where $\hat{G}$ is the set of isomorphism classes of finite-dimensional simple rational G-modules, V_λ denotes the finite-dimensional representation of $\lambda \in \hat{G}$ and λ^* is an element of $\hat{G}$ such that V_{λ^*} is the dual G-module to V_λ (see e.g. [245, Theorem 27.3.9]). The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is reductive, so that

$$[\mathfrak{g}, \mathfrak{g}] = \bigoplus_{i=1}^{r} \mathfrak{g}_i,$$

where each $\mathfrak{g}_i$ is a simple Lie subalgebra of $\mathfrak{g}$. Then $\kappa|_{\mathfrak{g}_i}$ is a constant multiplication of the Killing form $\kappa_{\mathfrak{g}_i}$ of $\mathfrak{g}_i$. We say κ is *irrational* if and $\kappa|_{\mathfrak{g}_i}/\kappa_{\mathfrak{g}_i} \notin \mathbb{Q}$ for all i.

Remark 3.2 If $\mathfrak{g} = \mathrm{Lie}(G)$ is simple, then $\lambda^* = -w_0(\lambda)$, that is, $E_{\lambda^*} = (E_\lambda)^*$, where w_0 is the longest element of the Weyl group.

Proposition 3.5 *Let G be reductive, and suppose that $\kappa|_{[\mathfrak{g},\mathfrak{g}]}$ is irrational and that $\kappa|_{\mathfrak{z}(\mathfrak{g})}$ is non-degenerate, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Then we have*

$$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong \bigoplus_{\lambda \in \hat{G}} \mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda^*,\kappa^*},$$

where $\mathbb{V}_{\lambda,\kappa} = U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1})} V_\lambda$ and V_λ is considered to be a $\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1}$-module on which $\mathfrak{g}[t]$ acts by the projection $\mathfrak{g}[t] \to \mathfrak{g}$ and $\mathbf{1}$ as the identity.

Proof We follow the arguments of [41]. By the assumption on κ, $\mathbb{V}_{\lambda,\kappa}$ and $\mathbb{V}_{\lambda^*,\kappa^*}$ for $\lambda \in \hat{G}$ are irreducible $\hat{\mathfrak{g}}_\kappa$-module and $\hat{\mathfrak{g}}_{\kappa^*}$-module, respectively ([172]). Moreover, $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ is completely reducible as $\hat{\mathfrak{g}}_\kappa \oplus \hat{\mathfrak{g}}_{\kappa^*}$-modules and a direct sum of $\mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\mu^*,\kappa^*}$ with $\lambda, \mu \in \hat{G}$. Because κ is generic (and so is κ^*), the category of integrable $\hat{\mathfrak{g}}_{\kappa^*}$-modules is equivalent to the category of integrable $\mathfrak{g}$-modules and the equivalence is given by $M \to M^{t\mathfrak{g}[[t]]}$, see [172] and [173, Section 30]. Since $\mathbb{V}_{\mu^*,\kappa^*}^{t\mathfrak{g}[[t]]} = V_{\mu^*}$, it is sufficient to show that $(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(t\mathfrak{g}[[t]])} \cong \bigoplus_{\lambda \in \hat{G}} \mathbb{V}_{\lambda,\kappa} \otimes V_{\lambda^*}$ as $\hat{\mathfrak{g}}_\kappa \times \mathfrak{g}$-modules. But we have $(\mathcal{D}^{\mathrm{ch}}_{G,\kappa})^{\pi_R(t\mathfrak{g}[[t]])} = U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1})} \left(\mathcal{O}(\mathcal{J}_\infty G)^{\pi_R(t\mathfrak{g}[[t]])}\right) = U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1})} \mathcal{O}(G)$. Hence the assertion follows from the algebraic Peter-Weyl theorem. □

Example 3.6 Let G be a torus T, $\{h_1, \dots, h_r\}$ a basis of the abelian Lie algebra $\mathfrak{g} = \mathrm{Lie}(T)$. Then $\mathcal{O}(G) = \mathbb{C}[P]$, where $P = \bigoplus_{i=1}^r \mathbb{Z}\varpi_i$ is the weight lattice of $\mathfrak{g}$. The subalgebra $V^\kappa(\mathfrak{g}) \subset \mathcal{D}^{\mathrm{ch}}_{T,\kappa}$ is the Heisenberg vertex algebra generated by the field $h_L(z)$, $h \in \mathfrak{g}$, satisfying the OPE

$$h_L(z) h'_L(w) \sim \frac{\kappa(h, h')}{(z-w)^2}.$$

The OPE (3.13) reads as

$$h_L(z) e^\alpha(w) \sim \frac{\alpha(h)}{z-w} e^\alpha(w) \quad (h \in \mathfrak{g},\ \alpha \in P).$$

Since the Killing form of $\mathfrak{g}$ is zero, we have $\kappa^* = -\kappa$. The subalgebra $\pi_R(V^{\kappa^*}(\mathfrak{g})) \subset \mathcal{D}^{\mathrm{ch}}_{T,\kappa}$ is generated the fields $h_R(z)$, $h \in \mathfrak{g}$, defined by

$$h_R(z) := h_L(z) - \sum_{i=1}^{r} \kappa(h, h_i) e^{-\varpi_i}(z) \partial e^{\varpi_i}(z) \tag{3.28}$$

(Note that $e^{-\varpi_i}(z)\partial e^{\varpi_i}(z) = {:}e^{-\varpi_i}(z)\partial e^{\varpi_i}(z){:} = (e^{-\varpi_i}\partial e^{\varpi_i})(z)$ since $\mathcal{O}(\mathcal{J}_\infty G)$ is commutative.) We have

$$h_L(z) h'_R(w) \sim 0, \quad h_R(z) h'_R(w) \sim \frac{\kappa^*(h, h')}{(z-w)^2}.$$

The stress tensor vector of $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$ is given by

$$T(z) = \sum_{i=1}^{r} {:}h_i(z)(e^{-\varpi_i}\partial e^{\varpi_i})(z){:} - \frac{1}{2}\sum_{i,j=1}^{r} \kappa(h_i, h_j) {:}(e^{-\varpi_i}\partial e^{\varpi_i})(z)(e^{-\varpi_j}\partial e^{\varpi_j})(z){:},$$

which has central charge $2r$.

Now suppose that κ is non-degenerate. Then we have the embedding of vertex algebras

$$V^{\kappa}(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{G,\kappa},$$

and we have

$$T(z) = T_L(z) + T_R(z),$$

where $T_L(z) = \frac{1}{2}\sum_{i=1}^{r} {:}h_{i,L}(z) h^i_L(z){:}$ and $T_R(z) = -\frac{1}{2}\sum_{i=1}^{r} {:}h_{i,R}(z) h^i_R(z){:}$ are the stress tensors of the vertex subalgebra $V^{\kappa}(\mathfrak{g})$ and $V^{\kappa^*}(\mathfrak{g})$, respectively. As a $V^{\kappa}(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$-module we have

$$\mathcal{D}^{\mathrm{ch}}_{G,\kappa} \cong \bigoplus_{\lambda \in P_+} \mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda,\kappa^*}, \tag{3.29}$$

Here $\mathbb{V}_{\lambda,\kappa}$ is the highest weight representation of the Heisenberg algebra $\hat{\mathfrak{g}}_\kappa$ with highest weight λ.

In the case that κ is non-degenerate it is possible to give the vertex algebra structure using the decomposition (3.29) as explained in [120]. Note that the vector $|\lambda\rangle = v_\lambda \otimes v_\lambda \in \mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda,\kappa^*}$, where v_λ is the highest weight vector of $\mathbb{V}_{\lambda,\kappa}$ or $\mathbb{V}_{\lambda,\kappa^*}$, corresponds to the vector $e^\lambda \in \mathcal{O}(G)$ on the left-hand-side. Observe from (3.28) that

$$\partial_z Y(|\lambda\rangle, z) = {:}(\lambda_L(z) - \lambda_R(z)) Y(|\lambda\rangle, z){:}, \tag{3.30}$$

where we identified P_+ as a subspace of $\mathfrak{g}$ via the form κ. In view of Theorem **??**, (3.30) together with the relation $Y(|\lambda\rangle, z)|0\rangle|_{z=0} = |\lambda\rangle$ completely determines the field $Y(|\lambda\rangle, z)$. As a result, we find that

$$Y(|\lambda\rangle, z) = e^{\lambda} \exp\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(\lambda_L)_{(n)} - (\lambda_R)_{(n)}}{-n} z^{-n}\Big) \quad \text{for } \lambda \in P_+,$$

where e^{λ} is the operator on $\bigoplus_{\lambda\in\mathbb{Z}} \mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda,\kappa^*}$ defined by $e^{\lambda}|\mu\rangle = |\lambda+\mu\rangle$, $[(h_L)_{(n)}, e^{\lambda}] = [(h_R)_{(n)}, e^{\lambda}] = 0$ for $n \neq 0$ and $[(h_L)_{(0)}, e^{\lambda}] = [(h_R)_{(0)}, e^{\lambda}] = \lambda(h)e^{\lambda}$. Here note that $\lambda_L(z) - \lambda_R(z)$ generates a commutative vertex subalgebra and $(\lambda_L)_{(0)} - (\lambda_R)_{(0)}$ acts as zero on the whole space (compare with (3.33) below). It is straightforward to check that

$$Y(|\lambda\rangle, z)Y(|\mu\rangle, w) \sim 0, \quad Y(|\lambda\rangle, z)Y(|\mu\rangle, z) = Y(|\lambda+\mu\rangle, z), \tag{3.31}$$

$$h_L(z)Y(|\lambda\rangle, w) \sim \frac{\lambda(h)}{z-w} Y(|\lambda\rangle, w), \quad h_R(z)Y(|\lambda\rangle, w) \sim \frac{\lambda(h)}{z-w} Y(|\lambda\rangle, w). \tag{3.32}$$

This construction is useful to construct $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$-modules for a non-degenerate κ. For $\lambda \in P_+ = \hat{G}$, set

$$M_{\lambda,\kappa} = \bigoplus_{\mu\in P_+} \mathbb{V}_{\lambda+\mu,\kappa} \otimes \mathbb{V}_{\mu,\kappa^*},$$

which is naturally a $V^{\kappa}(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$-modules. The $V^{\kappa}(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$-module structure extends to the $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$-module structure by setting

$$Y_{M_{\lambda,\kappa}}(|\alpha\rangle, z) = e^{\alpha} \exp\Big(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(\alpha_L)_{(n)} - (\alpha_R)_{(n)}}{-n} z^{-n}\Big) z^{(\alpha_L)_{(0)} - (\alpha_R)_{(0)}} \tag{3.33}$$

for $\alpha \in P_+$, where e^{α} is defined by $e^{\alpha}(v_{\lambda+\mu} \otimes v_{\mu}) = v_{\lambda+\mu+\alpha} \otimes v_{\mu+\alpha}$, $[(h_L)_{(n)}, e^{\alpha}] = [(h_R)_{(n)}, e^{\alpha}] = 0$. (Note that $z^{(\alpha_L)_{(0)} - (\alpha_R)_{(0)}} = z^{\kappa(\lambda,\alpha)}$ on $M_{\lambda,\kappa}$.)

Exercise 3.3 Show that $M_{\lambda,\kappa}$ is a simple $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$-module for all $\lambda \in P_+$.

Remark 3.3 One can show that the modules $M_{\lambda,\kappa}$, for $\lambda \in P_+$ are the only simple $\mathcal{D}^{\mathrm{ch}}_{G,\kappa}$-modules which are $\mathfrak{g}[[t]]$-integrable with respect to the left action. We refer to [237] for more details about this topic.

Example 3.7 Assume that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple and that $V^{\kappa}(\mathfrak{g}) = V^{k}(\mathfrak{g})$ for $k \in \mathbb{C}$ as in Example 3.2. It follows from [137, Corollary 9.3] that for any $k \in \mathbb{C}$, $\mathcal{D}^{\mathrm{ch}}_{G,k}$ is conformal with central charge $2\dim\mathfrak{g}$. Note that $c(k) + c(k^{\vee}) = 2\dim\mathfrak{g}$ if $k \neq -h^{\vee}$, where $c(k) = \dfrac{k\dim\mathfrak{g}}{k+h^{\vee}}$ is the central charge of $V^{k}(\mathfrak{g})$. The conformal vector of $\mathcal{D}^{\mathrm{ch}}_{G,k}$ is the sum of the Sugawara conformal vector of $V^{k}(\mathfrak{g})$ and $V^{k^{\vee}}(\mathfrak{g})$ provided that $k \neq -h^{\vee}$.

Remark 3.4 Assume that G is a connected reductive algebraic group, with Lie algebra $\mathfrak{g}$. We reformulate in this remark one of the main results of [114]. A module over $\mathcal{D}^{\mathrm{ch}}_{G,k}$ is the same as a smooth $\hat{\mathfrak{g}}$-module M at level k equipped with an action of the commutative vertex algebra $\mathcal{O}(\mathcal{J}_\infty G)$ such that $[x_{(m)}, f_{(n)}] = (x_L f)_{(m+n)}$ on M for $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$, $m, n \in \mathbb{Z}$. From this, one obtains the equivalence of the categories

$$\mathcal{D}^{\mathrm{ch}}_{G,k}\text{-Mod} \cong \mathcal{D}_{G((t)),k}\text{-Mod},$$

where $\mathcal{D}_{G((t)),k}$ -Mod is the category of the (properly defined) k-twisted $\mathcal{D}$-modules on $G((t))$. In particular,

$$\mathcal{D}^{\mathrm{ch}}_{G,k}\text{-Mod}^{G[[t]]} \cong \mathcal{D}_{\mathbf{Gr},k}\text{-Mod},$$

where $\mathcal{D}^{\mathrm{ch}}_{G,k}$ -Mod$^{G[[t]]}$ is the full subcategory of $\mathcal{D}^{\mathrm{ch}}_{G,k}$ -Mod consisting of integrable[4] $G[[t]]$-modules (with respect to the action π_R) $\mathcal{D}^{\mathrm{ch}}_{G,k}$-modules, and **Gr** is the affine Grassmannian $G((t))/G[[t]]$.

We refer to §9.1.6 for additional properties of the vertex algebra $\mathcal{D}^{\mathrm{ch}}_{G,k}$.

[4] Here, by *integrable* $G[[t]]$*-module* we mean a $\mathcal{D}^{\mathrm{ch}}_{G,k}$-module M which is integrable as a module over the pro-Lie algebra $\mathfrak{g}[[t]]$, and this is equivalent to the following conditions:

- $t\mathfrak{g}[[t]]$ acts locally nilpotently,
- the action of $\mathfrak{g}$ on M is integrable, that is, exponentiates to an action of G on N for any finite-dimensional submodule N of M.

Part II
Poisson vertex algebras, Li filtration and associated varieties

This part is about important objects attached to vertex algebras and the connections between them.

Chapter 4 gives a concise presentation of Poisson vertex algebras (a particular class of commutative vertex algebras). It will be observed that any vertex algebra is naturally filtered and that the corresponding graded space has a structure of a Poisson vertex algebra. The Zhu C_2-functor $V \mapsto R_V$ associates with any vertex algebra V a certain quotient that has a Poisson algebra structure and generates the Poisson vertex algebra $\operatorname{gr} V$ as a differential algebra. The spectrum of the Zhu C_2-algebra R_V is called the associated scheme. Its maximal spectrum is called the associated variety. Associated schemes and associated varieties, as well as their spaces of arcs, are introduced in this chapter. They will occupy a central place in the rest of the book. Not surprisingly, the associated variety is usually easier to describe than the associated scheme as it will observed in many examples in the following parts. It already contains meaningful information. One can, for example, detect from it the lisse condition.

Zhu's functor $V \mapsto \mathrm{Zhu}(V)$ is introduced in Chapter 5. It gives a correspondence between the theory of modules over a vertex algebra and the representation theory of its Zhu's algebra. This correspondence is particularly well-understood in the case of the universal affine vertex algebras, where Zhu's algebras are enveloping algebras of the corresponding finite-dimensional simple Lie algebras.

In Chapter 6, we develop the theory of Poisson vertex modules and Frenkel–Zhu's bimodules. These notions generalize all above constructions to the setting of modules over a vertex algebras.

Recall that Figure 0.1 in the introduction summarizes the main objects of this part.

We do not claim that this diagram commutes. In general, only the upper right triangle does. We will see, however, many interesting examples where all arrows in Figure 0.1 commute, so that $\operatorname{gr}\, \mathrm{Zhu}(V) \cong R_V$. We will also discuss relations between the Poisson vertex algebra $\operatorname{gr} V$ and the coordinate ring over the arc space of the associated scheme $\operatorname{Spec} R_V$.

Chapter 4
Poisson vertex algebras

Recall that a Poisson algebra structure is a combination of a commutative associative algebra structure and a Lie algebra structure with certain compatibility conditions. We refer the reader to Appendix B for basics on Poisson algebras and Poisson varieties. In the same spirit, a Poisson vertex algebra structure is a combination of a commutative vertex algebra structure (or, equivalently, a differential algebra structure) and a Lie vertex algebra structure (that is, a conformal Lie structure in Kac's terminology[1]) with natural compatibility conditions.

In this book, we follow the approach of [44] using the λ-bracket for the definition of a Poisson vertex algebra. That point of view is, for example, particularly suited to the study of integrable PDEs. We also present the n-th product approach.

Just as the graded space of an almost-commutative filtered associative algebra with unit have naturally a structure of a Poisson algebra, any vertex algebra is naturally filtered and the corresponding graded space is naturally a Poisson vertex algebra. The corresponding scheme, referred to as the singular support, is of great interest. In parallel, the C_2-cofiniteness condition introduced by Zhu [257] is based on a certain quotient, the Zhu's C_2 quotient R_V, of a vertex algebras. This condition was interpreted geometrically by the first author in [12] using the variety X_V associated with R_V. This variety turns out to be an extremely powerful invariant. To explore the properties of the associated variety and the singular support is the main objective of this part.

The chapter is structured as follows.

Section 4.1 is about the definition of Poisson vertex algebras. Section 4.2 is devoted to the Poisson vertex structure on arc spaces of Poisson schemes. The structure of invariant arc spaces is then discussed in Section 4.3. Section 4.4 is about the Li filtration and the corresponding graded vertex algebras. Section 4.5 contains important properties of the associated variety, the associated scheme and the singular support. Various examples of such geometrical invariants are then computed in Section 4.6. Section 4.7 aims to compare the Li filtration and the weight filtration.

[1] None of these notions will be discussed in this book.

The lisse condition is discussed in Section 4.8. Finally, the chapter concludes with Section 4.9 on a remark on the Poisson center of the Zhu C_2-algebra.

Our basic references for this chapter are [113, 53, 91, 194, 12].

4.1 Definition

Recall that a vertex algebra V is a *commutative vertex algebra* (cf. Section 2.10), that is, a unital commutative algebra equipped with a derivation, if and only if $a_{(n)} = 0$ in $\mathrm{End}(V)$ for all $n \geqslant 0$.

Definition 4.1 A commutative vertex algebra V is called a *Poisson vertex algebra* if there exists a linear map

(4.1) $$V \otimes V \to V[\lambda], \quad a \otimes b \mapsto \{a_\lambda b\} = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} a_{(n)} b,$$

called the *λ-bracket*, such that

(4.2) $$\{(\partial a)_\lambda b\} = -\lambda\{a_\lambda b\}, \quad \{a_\lambda \partial b\} = (\lambda + \partial)\{a_\lambda b\}, \quad \text{(sesquilinearity)}$$

(4.3) $$\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}, \quad \text{(skewsymmetry)}$$

(4.4) $$\{a_\lambda\{b_\mu c\}\} - \{b_\mu\{a_\lambda c\}\} = \{\{a_\lambda b\}_\mu c\}, \quad \text{(the Jacobi identity)}$$

(4.5) $$\{a_\lambda(bc)\} = \{a_\lambda b\}c + b\{a_\lambda c\}, \quad \text{(left Leibniz rule).}$$

Here, in (4.1), $a_{(n)}$, for $n \geqslant 0$, are "new" operators. (The "old" ones given by the field $a(z)$ being zero for $n \geqslant 0$ since V is commutative.)

In terms of operators $a_{(n)}$, "sesquilinearity", "skewsymmetry", the "Jacobi identity" and the "left Leibniz rule" are equivalent to the following properties, respectively:

(4.6) $$(\partial a)_{(n)} = [\partial, a_{(n)}] = -n a_{(n-1)},$$

(4.7) $$a_{(n)} b = \sum_{j \geqslant 0} (-1)^{n+j+1} \frac{1}{j!} \partial^j (b_{(n+j)} a),$$

(4.8) $$[a_{(m)}, b_{(n)}] = \sum_{j \geqslant 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)},$$

(4.9) $$a_{(n)}(b \cdot c) = (a_{(n)} b) \cdot c + b \cdot (a_{(n)} c)$$

for all $a, b, c \in V$ and all $n, m \geqslant 0$.

The equation (4.9) says that $a_{(n)}$, for $n \geqslant 0$, is a derivation of the ring V. (Do not confuse $a_{(n)} \in \mathrm{Der}(V)$, for $n \geqslant 0$, with the multiplication $a_{(n)}$ as a vertex algebra, which should be zero for a commutative vertex algebra.)

It follows from the definition that we also have the "right Leibniz rule" ([161, Exercise 4.2])

$$\{(ab)_\lambda c\} = \{b_{\lambda+\partial} c\}_{\to} a + \{a_{\lambda+\partial} c\}_{\to} b, \tag{4.10}$$

where $\{b_{\lambda+\partial} c\}_{\to}$ means that ∂ is moved to the right, that is,

$$\{b_{\lambda+\partial} c\}_{\to} a = \sum_{n\geqslant 0} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \lambda^j (b_{(n)} c)(\partial^{n-j} a).$$

One finds that (4.10) is equivalent to

$$(a \cdot b)_{(n)} c = \sum_{i\geqslant 0} (a_{(-i-1)} b_{(n+i)} c + b_{(-i-1)} a_{(n+i)} c),$$

for all $a, b, c \in V$, and $n \in \mathbb{Z}_{\geqslant 0}$ (compare with (2.36)), cf. Section 4.4).

4.2 Poisson vertex structure on arc spaces

Arc spaces over an affine Poisson scheme naturally give rise to a Poisson vertex algebras, as shows the following result ([12, Proposition 2.3.1]).

Theorem 4.1 *Given an affine Poisson scheme X, that is, X = Spec R for some Poisson algebra R, there is a unique Poisson vertex algebra structure on $\mathcal{J}_\infty(R) = \mathcal{O}(\mathcal{J}_\infty(X))$ such that*

$$\{a_\lambda b\} = \{a, b\}$$

for all $a, b \in R$, that is,

$$a_{(n)} b = \begin{cases} \{a, b\} & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

for all $a, b \in R$.

Proof The bilinear map

$$R \otimes R \to R[\lambda], \quad a \otimes b \mapsto \{a_\lambda b\} = \{a, b\}, \tag{4.11}$$

clearly satisfies $\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}$, $\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_\mu c\}$. This extends uniquely to the linear map

$$\mathcal{J}_\infty R \otimes \mathcal{J}_\infty(R) \to \mathcal{J}_\infty(R)\{\lambda\}, \quad a \otimes b \mapsto \{a_\lambda b\}, \tag{4.12}$$

satisfying (4.2), (4.3) and (4.5). Here, the well-defindness of this map follows from the fact that the relations in $\mathcal{J}_\infty R$ is spanned by the relations of the form $\partial^n a$, where a is a relation in R, and that (4.11) is well-defined.

Finally we need to show that the Jacobi identify (4.4) is satisfied. By the Leibniz rule it is sufficient to show this for the generators $\partial^n a$, for $a \in R$, $n \in \mathbb{Z}$, but this is easily done. □

Remark 4.1 More generally, given a Poisson scheme X, not necessarily affine, the structure sheaf $\mathscr{O}_{\mathcal{J}_\infty(X)}$ carries a unique Poisson vertex algebra structure such that

$$f_{(n)}g = \delta_{n,0}\{f, g\}$$

for all $f, g \in \mathscr{O}_X \subset \mathscr{O}_{\mathcal{J}_\infty(X)}$, see [30, Lemma 2.1.3.1].

Example 4.1 Recall that $\mathbb{C}[\mathfrak{g}^*]$ has naturally a Poisson structure induced from the Kirillov-Kostant-Souriau Poisson structure on $\mathfrak{g}^*$ (see Example B.2). Namely, for all $f, g \in \mathcal{O}(\mathfrak{g}^*)$ and all $x \in \mathfrak{g}^*$,

$$\{f, g\}(x) = \langle x, [d_x f, d_x g]\rangle,$$

where $d_x f, d_x g$ are the differentials of f, g, respectively, at $x \in \mathfrak{g}^*$ viewed as elements of $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. In particular, for $f, g \in \mathfrak{g} = (\mathfrak{g}^*)^* \subset \mathcal{O}(\mathfrak{g}^*)$,

$$\{f, g\} = [f, g].$$

Since

$$\mathcal{J}_\infty(\mathfrak{g}^*) = \operatorname{Spec} \mathbb{C}[x^i_{(-n)} \; ; \; i = 1, \ldots, d, \; n \geqslant 1],$$

where $\{x^1, \ldots, x^d\}$ is a basis of $\mathfrak{g}$, it follows from Theorem 4.1 that $\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}^*))$ inherits a Poisson vertex algebra from that of $\mathcal{O}(\mathfrak{g}^*)$.

We may identify $\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}^*))$ with the symmetric algebra $S(t^{-1}\mathfrak{g}[t^{-1}])$ via

$$x_{(-n)} \longmapsto xt^{-n}, \qquad x \in \mathfrak{g}, \; n \geqslant 1.$$

For $x \in \mathfrak{g}$, identify x with $x_{(-1)}|0\rangle = (xt^{-1})|0\rangle$, where $|0\rangle$ stands for the unit element in $S(t^{-1}\mathfrak{g}[t^{-1}])$. Then (4.8) gives that

$$[x_{(m)}, y_{(n)}] = (x_{(0)}y)_{m+n} = \{x, y\}_{(m+n)} = [x, y]_{(m+n)},$$

for all $x, y \in \mathfrak{g}$ and all $m, n \in \mathbb{Z}_{\geqslant 0}$. So the Lie algebra $\mathcal{J}_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$ acts on $\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}^*))$ by:

$$\mathfrak{g}[[t]] \to \operatorname{End}(\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}^*))), \quad xt^n \mapsto x_{(n)}, \quad n \geqslant 0,$$

where $x_{(n)}$, for $n \geqslant 0$, is the endomorphism of $\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}^*))$ given by the Poisson vertex structure on $\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}^*))$. This action coincides with that obtained by differentiating the action of $\mathcal{J}_\infty(G) = G[[t]]$ on $\mathcal{J}_\infty(\mathfrak{g}^*)$ induced by the coadjoint action of G (see Example 1.2). In other words, the Poisson vertex algebra structure of $\mathcal{O}(\mathcal{J}_\infty(\mathfrak{g}^*))$ comes from the $\mathcal{J}_\infty(G)$-action on $\mathcal{J}_\infty(\mathfrak{g}^*)$.

Example 4.2 Consider the cotangent bundle T^*G to an affine algebraic group G, which is a smooth affine symplectic variety. In particular, $\mathscr{O}(T^*G)$ is a Poisson algebra. Since $T^*G = G \times \mathfrak{g}^*$, we have

$$\mathscr{O}(T^*G) = \mathscr{O}(\mathfrak{g}^*) \otimes \mathscr{O}(G).$$

The Poisson algebra structure of $\mathscr{O}(T^*G)$ is described as follows. The natural embeddings

$$\mathscr{O}(\mathfrak{g}^*) \hookrightarrow \mathscr{O}(T^*G), \quad \mathscr{O}(G) \hookrightarrow \mathscr{O}(T^*G),$$

are homomorphisms of Poisson algebras, where $\mathscr{O}(\mathfrak{g}^*)$ is equipped with the Kirillov-Kostant-Souriau Poisson structure and $\mathscr{O}(G)$ is equipped with the trivial Poisson structure. Finally, the Poisson bracket between $\mathscr{O}(\mathfrak{g}^*)$ and $\mathscr{O}(G)$ is described by the following formula:

$$\{x, f\} = x_L f,$$

for $x \in \mathfrak{g} \subset \mathscr{O}(\mathfrak{g}^*)$, $f \in \mathscr{O}(G)$.

By Theorem 4.1, $\mathscr{O}(\mathcal{J}_\infty T^*G)$ is naturally a Poisson vertex algebra. Since $\mathcal{J}_\infty T^*G = \mathcal{J}_\infty G \times \mathcal{J}_\infty \mathfrak{g}^*$ by Lemma 1.4, we have

$$\mathscr{O}(\mathcal{J}_\infty T^*G) = \mathscr{O}(\mathcal{J}_\infty \mathfrak{g}^*) \otimes \mathscr{O}(\mathcal{J}_\infty G),$$

and the Poisson vertex algebra structure is given by the following formulas:

$$\begin{aligned}
&\{x_\lambda y\} = [x, y], && x, y \in \mathfrak{g} \subset \mathscr{O}(\mathfrak{g}^*) \subset \mathscr{O}(\mathcal{J}_\infty \mathfrak{g}^*),\\
&\{f_\lambda g\} = 0, && f, g \in \mathscr{O}(G) \subset \mathscr{O}(\mathcal{J}_\infty G),\\
&\{x_\lambda f\} = x_L f && x \in \mathfrak{g},\ f \in \mathscr{O}(G).
\end{aligned}$$

4.3 Arc spaces and invariants

Let A be a unital commutative algebra of finite type over $\mathbb{C}$, and write $Y = \operatorname{Spec}(A)$. Let G be an affine algebraic group with a left action $G \times Y \to Y$. By Lemma 1.9 this induces the action $\mathcal{J}_\infty(G) \times \mathcal{J}_\infty Y \to J_\infty Y$ of the (ind-)group scheme $\mathcal{J}_\infty(G)$ on $\mathcal{J}_\infty(Y)$. In other words the morphism of algebras

$$\rho\colon A \longrightarrow \mathcal{O}(G) \otimes A \tag{4.13}$$

uniquely gives rise to a morphism of commutative vertex algebras

$$\rho_\infty\colon \mathcal{J}_\infty(A) \longrightarrow \mathcal{J}_\infty(\mathcal{O}(G)) \otimes \mathcal{J}_\infty(A). \tag{4.14}$$

Let $A^G \subset A$ and $\mathcal{J}_\infty(A)^{\mathcal{J}_\infty(G)} \subset \mathcal{J}_\infty(A)$ be the subalgebras of G-invariant and $\mathcal{J}_\infty(G)$-invariant elements, respectively. By definition, $A^G = \ker(\rho - \iota)$ and $\mathcal{J}_\infty(A)^{\mathcal{J}_\infty(G)} = \ker(\rho_\infty - \iota_\infty)$ where $\iota \colon A \xrightarrow{\sim} \mathbb{C} \otimes A \hookrightarrow \mathcal{O}(G) \otimes A$.

Since both ρ_∞ and ι_∞ are morphisms of vertex algebras, $\mathcal{J}_\infty(A)^{\mathcal{J}_\infty(G)}$ is a vertex subalgebra of $\mathcal{J}_\infty(A)$, and the morphism $\mathcal{J}_\infty(A^G) \to \mathcal{J}_\infty(A)$ induced by the inclusion $A^G \hookrightarrow A$ factors through the following natural morphism of vertex algebras

$$j_A \colon \mathcal{J}_\infty(A^G) \longrightarrow (\mathcal{J}_\infty(A))^{\mathcal{J}_\infty(G)}. \tag{4.15}$$

In general, the morphism j_A is neither injective, nor surjective. For example, if G is finite nontrivial and A is a G-module, then j_A is not surjective ([198, Theorem 3.13.]); if $G = \mathrm{SL}_3$ and $Y = (\mathbb{C}^3)^{\oplus 6}$ is the direct sum of six copies of the standard representation, then $j_{\mathcal{O}(Y]}$ is not injective ([198, Example 6.6])[2]. See [151] for other counter-examples.

There are, however, interesting examples where j_A is an isomorphism.

Example 4.3 If $G = \mathrm{GL}_N$ and $Y = (\mathbb{C}^N)^{\oplus p} \oplus ((\mathbb{C}^N)^*)^{\oplus q}$ is a sum of p copies of the standard module and q copies of its dual, then $j_{\mathcal{O}(Y)}$ is an isomorphism due to the main result of [198].

We refer to [198, 197] for other examples where the morphism (4.15) is an isomorphism in the case where $G = \mathrm{SL}_N, \mathrm{Gl}_N, \mathrm{SO}_N$ or Sp_{2N}, and Y is a finite-dimensional G-module such that $Y /\!\!/ G$ is smooth or a complete intersection.

Example 4.4 Let G be a connected reductive group with Lie algebra $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ and its dual by the adjoint and the coadjoint action, respectively. As noticed in Example 1.3, we have

$$(\mathcal{J}_\infty \mathcal{O}(\mathfrak{g}^*))^{\mathcal{J}_\infty G} \cong \mathcal{J}_\infty \mathcal{O}(\mathfrak{g}^*)^G.$$

Assume from now on that A is equipped with a Poisson bracket and that G acts by Poisson automorphisms. Note that the jet algebra $\mathcal{J}_\infty(A^G)$ is a Poisson vertex subalgebra of $\mathcal{J}_\infty(A)$. Indeed, we compute that $\{\partial^r f_\lambda \partial^s g\} \in \mathcal{J}_\infty(A^G)[\lambda]$ for any $f, g \in A^G$ and $r, s \geqslant 0$ due to the Leibniz rules and the fact that $\{f_\lambda g\} = \{f, g\}$ is G-invariant. The following theorem ([63]) guarantees that $(\mathcal{J}_\infty A)^{\mathcal{J}_\infty G}$ is also a Poisson vertex subalgebra of $\mathcal{J}_\infty(A)$.

Theorem 4.2 *The invariant algebra $(\mathcal{J}_\infty A)^{\mathcal{J}_\infty G}$ is a Poisson vertex subalgebra of $\mathcal{J}_\infty A$. Moreover, the morphism $j_A \colon \mathcal{J}_\infty(A^G) \to (\mathcal{J}_\infty A)^{\mathcal{J}_\infty G}$ is a Poisson vertex algebra morphism.*

Proof Assume first that G is connected, and let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathcal{J}_\infty \mathfrak{g} = \mathfrak{g}[[t]]$ is the Lie algebra of $\mathcal{J}_\infty G$, and connectedness implies that

$$(\mathcal{J}_\infty A)^{\mathcal{J}_\infty G} = (\mathcal{J}_\infty A)^{\mathcal{J}_\infty \mathfrak{g}},$$

[2] In this example, the kernel of $j_{\mathcal{O}(Y)}$ consists of nilpotent elements, and $\mathcal{J}_\infty(Y /\!\!/ G)$ is not reduced.

where

$$(\mathcal{J}_\infty A)^{\mathcal{J}_\infty \mathfrak{g}} = (\mathcal{J}_\infty A)^{\mathfrak{g}[[t]]} = \{a \in \mathcal{J}_\infty A : x_{(k)} a = 0 \text{ for all } x \in \mathfrak{g},\ k \in \mathbb{Z}_{\geqslant 0}\},$$

with $x_{(k)} a := (xt^k).a$ given by the infinitesimal action of $xt^k \in \mathfrak{g}[[t]]$ on $\mathcal{J}_\infty A$. Since G acts as Poisson automorphisms on A, the Lie algebra $\mathfrak{g}$ acts on A by derivations for both the commutative and the Poisson algebra structures on A. Let $Y := \operatorname{Spec} A$. The left action $G \times Y \to Y$ induces an algebra morphism

$$\mathcal{J}_\infty(A) \longrightarrow \mathcal{J}_\infty(\mathbb{C}[G]) \otimes \mathcal{J}_\infty(A).$$

Therefore, the Lie algebra $\mathfrak{g}[[t]]$ acts on $\mathcal{J}_\infty A$ by derivations for the commutative algebra structure. The action of $\mathfrak{g}[[t]]$ on $\mathcal{J}_\infty A$ is entirely determined by the action of $\mathfrak{g}$ on A as follows:

$$x_{(k)}(\partial^l a) = \begin{cases} \dfrac{l!}{(l-k)!}\partial^{l-k}(x.a) & \text{if } k \leqslant l, \\ 0 & \text{otherwise,} \end{cases} \tag{4.16}$$

for $k, l \geqslant 0$ and $a \in A$. In particular, $x_{(0)} a = x.a$.

The following identity, for $a \in \mathcal{J}_\infty A$, $x \in \mathfrak{g}$ and $k \in \mathbb{Z}_{\geqslant 0}$ is a direct induction from the identity (4.6).

$$x_{(k)}(\partial a) = \partial(x_{(k)} a) + k x_{(k-1)} a. \tag{4.17}$$

We are now in a position to prove Theorem 4.2 in the case where G is connected. Assume that $a \in (\mathcal{J}_\infty A)^{\mathfrak{g}[[t]]}$. Then $x_{(k)} a = 0$ for all $k \geqslant 0$, and so, by (4.17), we see that $x_{(k)}(\partial a) = 0$ for any $k \geqslant 0$ as well, that is ∂a is in $(\mathcal{J}_\infty A)^{\mathfrak{g}[[t]]}$. Next, we have to show that $a_{(n)} b$ is in $(\mathcal{J}_\infty A)^{\mathfrak{g}[[t]]}$ for any $n \geqslant 0$ if it is the case for both a and b. One can assume that a, b are of the form

$$a = \partial^{i_1} a_1 \ldots \partial^{i_r} a_r \quad \text{and} \quad b = \partial^{j_1} b_1 \ldots \partial^{j_s} b_s. \tag{4.18}$$

The strategy is to establish the following formula for any $a, b \in \mathcal{J}_\infty A$ as in (4.18) (not necessarily $\mathfrak{g}[[t]]$-invariant):

$$x_{(k)}(a_{(n)} b) = a_{(n)}(x_{(k)} b) + \sum_{\ell=0}^{k} \binom{k}{\ell} (x_{(k-\ell)} a)_{(n+\ell)} b \tag{4.19}$$

If moreover a, b are $\mathfrak{g}[[t]]$-invariant, then so is $a_{(n)} b$ according to (4.19). The formula (4.19) is shown by induction on r in [63]. We omit the details here.

Next, if $G = \Gamma$ is finite, then $\mathcal{J}_\infty \Gamma = \Gamma$ and, hence, $\mathcal{J}_\infty \Gamma$ acts by Poisson vertex algebra automorphisms on $\mathcal{J}_\infty A$.

For the general case, since G is an affine algebraic group, the component groups $\Gamma := G/G^0$ is finite. Let $\{g_1, \ldots, g_\ell\}$ be a finite set of representatives of Γ in G so that any element g of G is uniquely written as $g = g^0 g_i$, with $g^0 \in G^0$ and

$i \in \{1, \dots, \ell\}$. Because $\mathcal{J}_\infty G \cong G \times \mathfrak{g}[[t]]$ as topological space, any element g of $\mathcal{J}_\infty G$ is also uniquely written as $g = g^0 g_i$, with $g^0 \in \mathcal{J}_\infty G^0$ and $i \in \{1, \dots, \ell\}$. Let now $a, b \in (\mathcal{J}_\infty A)^{\mathcal{J}_\infty G}$, $n \in \mathbb{Z}_{\geqslant 0}$. Write $g = g^0 g_i$ as above. Then

$$g(a_{(n)}b) = g^0((g_i a)_{(n)}(g_i b)) = g^0(a_{(n)}b) = a_{(n)}b.$$

The first equality holds because g_i acts as a Poisson vertex algebra automorphism; the last because $(\mathcal{J}_\infty A)^{\mathcal{J}_\infty G^0}$ is a Poisson vertex subalgebra by the connected case. Similarly, we get

$$g(\partial a) = g^0 g_i(\partial a) = g^0(\partial a) = \partial a.$$

This concludes the proof of the theorem. □

Remark 4.2 Recall from Example 4.4 that for the algebra $A = \mathcal{O}(\mathfrak{g}^*)$ equipped with the induced coadjoint action of G, the morphism (4.15) is an isomorphism. The Poisson vertex algebra structure on $\mathcal{J}_\infty\mathcal{O}(\mathfrak{g}^*)$ can be understood in two ways: from the Kirillov-Kostant-Souriau Poisson structure on $A = \mathcal{O}(\mathfrak{g}^*)$, or from the $\mathcal{J}_\infty(G)$-action on its Lie algebra $\mathcal{J}_\infty(\mathfrak{g})$ (see Example 4.1).

4.4 The Li filtration and the corresponding Poisson vertex structure

Our second basic example of Poisson vertex algebras comes from the graded vertex algebra associated with the canonical filtration, that is, the *Li filtration*.

Definition 4.2 Let V be a vertex algebra. A set $\{a^i : i \in I\}$ of vectors in V is called *a set of strong generators* if V is spanned by $|0\rangle$ and the elements of the form

$$a^{i_1}_{(-n_1-1)}, \dots, a^{i_r}_{(-n_r-1)}|0\rangle$$

with $r \geqslant 0$, $i_j \in I$, $n_j \geqslant 0$. A vertex algebra V is called *finitely strongly generated* if there exist a finite set of strong generators.

Note that $\{a : a \in V\}$ is a set of strong generators.

The universal affine vertex algebra $V^\kappa(\mathfrak{a})$, the vertex algebra of cdo $\mathcal{D}^{ch}_{G,\kappa}$ on an affine algebraic group G, the Virasoro vertex algebra Vir^c and their quotient vertex algebra are strongly finitely generated.

Haisheng Li [195] has shown that every vertex algebra is canonically filtered. For a vertex algebra V, choose a set $\{a^i : i \in I\}$ of strong generators of V. Let $F^p V$ be the subspace of V spanned by the elements

$$a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}|0\rangle, \tag{4.20}$$

with $i_j \in I$, $n_j \geqslant 0$, $n_1 + n_2 + \cdots + n_r \geqslant p$. Then

$$V = F^0 V \supset F^1 V \supset \dots .$$

It is clear from the definition that $TF^pV \subset F^{p+1}V$, where T is the translation operator of V.

Definition 4.3 The decreasing filtration $F^\bullet V$ is called the *Li filtration*.

Set for $n \in \mathbb{Z}$,

$$(F^pV)_{(n)}F^qV := \mathrm{span}_{\mathbb{C}}\{a_{(n)}b\ ;\ a \in F^pV,\ b \in F^qV\}.$$

Lemma 4.1 *For $p \geqslant 1$ we have*

$$F^pV = \sum_{j=1}^{p}(F^0V)_{(-j-1)}F^{p-j}V.$$

In particular, the Li filtration $F^\bullet V$ is independent of the choice of the strong generators of V.

Proof Let $v \in (F^0V)_{(-j-1)}F^{p-j}V$, for $j \in \{1, \dots, p\}$. Since $\{a^i : i \in I\}$ is a set of strong generators of F^0V, one can write

$$v = (a^{i_1}_{(-n_1-1)}a^{i_2}_{(-n_2-1)}\cdots a^{i_r}_{(-n_r-1)}|0\rangle)_{(-j-1)}b,$$

with $i_j \in I$, $n_i \geqslant 0$, $b \in F^{p-j}V$. Then it follows from Borcherds identity (2.36) and induction that v is a linear combination of elements of the form

$$a^{i_1}_{(-m_1-1)}a^{i_2}_{(-m_2-1)}\cdots a^{i_r}_{(-m_r-1)}b, \tag{4.21}$$

with $m_j \geqslant 0$ such that $m_1 + \cdots + m_r = \sum\limits_{j=1}^{r} n_j + j \geqslant j$. From this, it is now easy to see that $v \in F^pV$ because b is in $F^{p-j}V$. This shows the inclusion

$$\sum_{j=1}^{p}(F^0V)_{(-j-1)}F^{p-j}V \subset F^pV.$$

To show the other inclusion, set

$$\tilde{F}_pV = \sum_{j=1}^{p}(F^0V)_{(-j-1)}F^{p-j}V.$$

It is enough to prove that any monomial of F^pV of the form (4.20) is contained in $\tilde{F}_pV$. We argue by induction on r, the length of a monomial (4.20). Let $v \in F^pV$ be a monomial as in (4.20). Then $v = a^{i_1}_{(-n_1-1)}b$, with $b \in F^{p-n_1}V$. If $n_1 \geqslant 1$, we clearly get $v \in \tilde{F}_pV$ since $F^0V = V$.

Assume that $n_1 = 0$. Then $b \in F^pV$ is a monomial of length $r-1$. By the induction hypothesis, it is a sum of elements of the form $w_{(-j-1)}c$, with $w \in F^0V$, $j \in \{1, \dots, p\}$, $c \in F^{p-j}V$. By Borcherds identity (2.35), we have

$$a^{i_1}_{(-1)} w_{(-j-1)} c = w_{(-j-1)} a^{i_1}_{(-1)} c + \sum_{i\geqslant 0} \binom{-1}{i} (a_{(i)} w)_{(-j-i-2)} c.$$

Since $a^{i_1}_{(-1)} c \in F^{p-j} V$ and $w \in F^0 V$, we see that $w_{(-j-1)} a^{i_1}_{(-1)} c \in \tilde{F}^p V$. Next, $a_{(i)} w \in F^0 V$ and $c \in F^{p-j} V \subset F^{p-j-i-1} V$. Therefore $(a_{(i)} w)_{(-j-i-2)} c \in \tilde{F}_p V$. This shows that $v \in \tilde{F}_p V$, whence the expected inclusion. ◻

Proposition 4.1 *Let $p, q \in \mathbb{Z}$. We have $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n-1} V$ for all $n \in \mathbb{Z}$. Moreover, if $n \geqslant 0$, then $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n} V$. Here we have set $F^p V = V$ for $p < 0$.*

Proof $*$ First case. $n \leqslant -1$, that is, $n = -j-1$ with $j \geqslant 0$. Any element of $(F^p V)_{(n)} (F^q V)$ is a linear combination of elements of the form

$$(a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)} |0\rangle)_{(-j-1)} b,$$

with $i_j \in I$, $n_i \geqslant 0$, $n_1 + \cdots + n_r \geqslant p$, $b \in F^q V$. So, arguing as in the proof of Lemma 4.1 by using Borcherds identity (2.36) and induction, we easily obtain that any element of $(F^p V)_{(n)} (F^q V)$ is a linear combination of elements of the form

$$a^{i_1}_{(-m_1-1)} a^{i_2}_{(-m_2-1)} \cdots a^{i_r}_{(-m_r-1)} b,$$

with $m_j \geqslant 0$, $m_1 + \cdots + m_r = \sum\limits_{j=1}^{r} n_j + j \geqslant p + j = p - n - 1$, $b \in F^q V$, whence the inclusion $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n-1} V$.

$*$ Second case. $n \geqslant 0$. Since $F^{p+q-n} V \subset F^{p+q-n-1} V$, it suffices to show that $(F^p V)_{(n)} (F^q V) \subset F^{p+q-n} V$. We prove the statement by induction on q, observing that for $q \leqslant n - p$, the inclusion is clear because $F^{p+q-n} V = V$.

Assume $q > n - p$. The space $(F^p V)_{(n)} (F^q V)$ is generated by vectors $a_{(n)} b$, with $a \in F^p V$, $b \in F^q V$. By Lemma 4.1, a vector $b \in F^q V$ is a sum of vectors $u_{(-j-1)} c$, with $u \in V$, $j \in \{1, \ldots, q\}$, $c \in F^{q-j} V$. By Borcherds identity (2.35), we have

$$a_{(n)} u_{(-j-1)} c = u_{(-j-1)} a_{(n)} c + \sum_{i\geqslant 0} \binom{n}{i} (a_{(i)} u)_{(n-i-j-1)} c.$$

By the induction hypothesis, $a_{(n)} c \in F^{p+q-j-n} V$ since $q - j < q$ and, hence, $u_{(-j-1)} a_{(n)} c \in F^{p+q-n} V$. Next, assume for awhile that $a_{(i)} u \in F^{p-i} V$. Then by the first case, $(a_{(i)} u)_{(n-i-j-1)} c \in F^{p-i+q-j-(n-i-j-1)-1} V = F^{p+q-n} V$ and, therefore, $a_{(n)} u_{(-j-1)} c \in F^{p+q-n} V$, which shows the expected conclusion.

So, it remains to show that for $p \in \mathbb{Z}$ and $n \geqslant 0$, we have $(F^p V)_{(n)} V \subset F^{p-n} V$. We prove this fact by induction on p, observing that the statement is obvious for $p \leqslant 0$ since then $p - n \leqslant 0$ and $F^{p-n} V = V$. Assume $p > 0$. By Lemma 4.1, a vector $a \in F^p V$ is a sum of vectors $u_{(-j-1)} b$, with $u \in V$, $j \in \{1, \ldots, p\}$, $b \in F^{p-j} V$. By Borcherds identity (2.36), we have for $c \in V$,

$$(u_{(-j-1)}b)_{(n)}c = \sum_{i\geqslant 0}(-1)^i \binom{-j-1}{i} (u_{(-j-i-1)}b_{(n+i)}c - (-1)^{j+1}b_{(-j-1+n-i)}u_{(i)}c).$$

By the induction hypothesis, $b_{(n+i)}c \in F^{p-j-n-i}V$ because $p-j<p$. Hence $u_{(-j-i-1)}b_{(n+i)}c \in F^{p-n}V$. On the other hand, by the first case or the induction hypothesis if $-j-1+n-i \geqslant 0$, $b_{(-j-1+n-i)}u_{(i)}c \in F^{p-j-(-j-1+n-i)-1}V = F^{p-n+i}V \subset F^{p-n}V$ because $i \geqslant 0$. In conclusion, we have shown the inclusion $(F^pV)_{(n)}V \subset F^{p-n}V$ for $n \geqslant 0$, as desired.

This concludes the proof of the lemma. □

Definition 4.4 A vertex algebra V is called *good* if the filtration $F^\bullet V$ is *separated*, that is, $\bigcap_{p\geqslant 0} F^pV = \{0\}$.

In Corollary 4.2 below we show that any positively graded vertex algebra is good.

Set

$$\mathrm{gr}^F V = \bigoplus_{p\geqslant 0} F^pV/F^{p+1}V.$$

We denote by $\sigma_p\colon F^pV \mapsto F^pV/F^{p+1}V$, for $p \geqslant 0$, the canonical quotient map. When the filtration F is obvious, we often briefly write $\mathrm{gr}\, V$ for the space $\mathrm{gr}^F V$.

We have $V \cong \mathrm{gr}\, V$ as vector space for a good vertex algebra V. Next proposition is due to Li ([195]).

Proposition 4.2 *The space* $\mathrm{gr}^F V$ *is a Poisson vertex algebra by*

(4.22) $$\sigma_p(a)\cdot\sigma_q(b) := \sigma_{p+q}(a_{(-1)}b),$$

(4.23) $$\partial\sigma_p(a) := \sigma_{p+1}(Ta),$$

(4.24) $$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n}(a_{(n)}b),$$

for all $a \in F^pV \setminus F^{p+1}V$, $b \in F^qV$, $n \geqslant 0$.

Proof First of all, the space $\mathrm{gr}^F V$ naturally inherits a graded vertex algebra structure from the vertex algebra structure on V. The vertex operator is given by

$$Y(\sigma_p(a), z)b := \sum_{n\in\mathbb{Z}} \sigma_{p+q-n-1}(a_{(n)}b)z^{-n-1},$$

for $a \in F^pV \setminus F^{p+1}V$, $b \in F^qV$, $n \in \mathbb{Z}$, the vacuum is $|0\rangle = \sigma_0(|0\rangle)$ and the translation operator is the linear map sending $a \in F^pV \setminus F^{p+1}V$ to $\sigma_p(a)_{(-2)}|0\rangle = \sigma_{p+1}(Ta)$, since $Ta = a_{(-2)}|0\rangle$. The axioms are easy to check. The verifications are left to the reader.

Furthermore, by Proposition 4.1, $Y(\sigma_p(a), z)_{(n)}Y(\sigma_q(b), z) = 0$ for $a \in F^pV \setminus F^{p+1}V$, $b \in F^qV$, $n \geqslant 0$ and, hence, $\mathrm{gr}^F V$ is a commutative vertex algebra whose product is given by (4.22), and derivation is given by (4.23).

It remains to show that (4.24) defines a Poisson vertex algebra on $\mathrm{gr}^F V$. It is easy to check that the axioms (4.6), (4.7) and (4.8) are satisfied. We prove only (4.9). Let $a \in F^pV \setminus F^{p-1}V$, $b \in F^qV$, $c \in F^rV$, $n \geqslant 0$. By Borcherds identity (2.36), we have

$$a_{(n)}(b_{(-1)}c) = b_{(-1)}a_{(n)}c + \sum_{i\geqslant 0}\binom{n}{i}(a_{(i)}b)_{(n-1-i)}c$$
$$= b_{(-1)}a_{(n)}c + (a_{(n)}b)_{(-1)}c + \sum_{i=0}^{n-1}\binom{m}{i}(a_{(i)}b)_{(n-1-i)}c.$$

For $i \in \{0,\dots,n-1\}$, $(a_{(i)}b)_{(n-1-i)}c \in F^{p+q+r-n+1}V$ since $n-1-i \geqslant 0$, while $a_{(n)}(b_{(-1)}c)$, $b_{(-1)}a_{(n)}c$ and $(a_{(n)}b)_{(-1)}c$ are in $F^{p+q+r-n}V$. Hence,

(4.25)
$$\sigma_p(a)_{(n)}(\sigma_q(b).\sigma_r(c)) = \sigma_q(b).(\sigma_p(a)_{(n)}\sigma_r(c)) + (\sigma_p(a)_{(n)}\sigma_q(b)).\sigma_r(c),$$

whence the expected statement. □

Define

$$R_V := V/F^1V = F^0V/F^1V \subset \mathrm{gr}^F V. \tag{4.26}$$

Note that

$$F^1V = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}b\colon a,b \in V\} \tag{4.27}$$

by Lemma 4.1. We also write $C_2(V)$ for the space F^1V by historical reason. Zhu observed that the quotient R_V has a natural structure of a Poisson algebra ([257]).

Proposition 4.3 *The quotient R_V is a Poisson algebra by*

$$\bar{a}\cdot\bar{b} = \overline{a_{(-1)}b}, \quad \{\bar{a},\bar{b}\} = \overline{a_{(0)}b}$$

for $a,b \in V$, where $\bar{a} = \sigma_0(a)$.

Proof First, by (4.22) with $p = q = 0$, the product $\bar{a}\cdot\bar{b} = \overline{a_{(-1)}b}$, for $a,b \in V$, gives to R_V a ommutative associative algebra structure, with unit $\overline{|0\rangle}$.

Let us prove that the bracket $\{\bar{a},\bar{b}\} = \overline{a_{(0)}b}$, for $a,b \in V$, is Poisson for the commutative algebra R_V. It verifies the skew-symmetry property by (4.7) with $n = 0$ and $a,b \in F^0V$ so that $\partial^j(\bar{b}_{(j)}\bar{a}) \in F^1V$ for $j > 0$, and the left Leibniz rule by (4.25) with $p = q = 0$ and $n = 0$. Then it also verifies the right Leibniz rule by the skew-symmetry. As for the Jacobi identity, it follows from (4.8) with $m = n = 0$. □

Definition 4.5 The Poisson algebra R_V is called the *Zhu C_2-algebra* of V.

Li proved the following result ([195]).

Proposition 4.4 *As a differential algebra, $\mathrm{gr}^F V$ is generated by R_V.*

Proof Set $A = \bigoplus_{p\geqslant 0} A_p = \mathrm{gr}^F V$, $A_p = F^pV/F^{p+1}V$. We wish to show that the graded differential algebra A is generated by $A_0 = R_V$ as a differential algebra.

First, note that we have

$$A_+ := \bigoplus_{p>0} A_p = A\partial A. \tag{4.28}$$

Indeed, it is clear that $A\partial A \subset A_+$.

Conversely, let us show that $A_+ \subset A\partial A$. Let $v \in F^p V$. By Lemma 4.1, v is a sum of terms of the form $a_{(-j-1)}b$, with $j \in \{1, \ldots, p\}$, $a \in F^0 V$ and $b \in F^{p-j}V$. By Borcherds identity (2.35) and (2.33), we have

$$\begin{aligned} a_{(-j-1)}b = a_{(-j-1)}b_{(-1)}|0\rangle &= b_{(-1)}a_{(-j-1)}|0\rangle + \sum_{l\geqslant 0}\binom{-j-1}{l}(a_{(l)}b)_{(-j-2-l)}|0\rangle \\ &= b_{(-1)}\left(\frac{T^j a}{j!}\right) + \sum_{l\geqslant 0}\binom{-j-1}{l}\frac{T^{j+l+1}(a_{(l)}b)}{(j+l+1)!}. \end{aligned}$$

Hence,

$$\sigma(a_{(-j-1)}b) = \sigma_{p-j}(b).\partial^j\left(\frac{\sigma_0(a)}{j!}\right) + \sum_{l\geqslant 0}\binom{-j-1}{l}\partial^{j+l+1}\left(\frac{\sigma_{p-j-l-1}(a_{(l)}b)}{(j+l+1)!}\right).$$

This shows that A_p is contained in $A\partial A$ for all $p > 0$, whence $A_+ \subset A\partial A$.

Let A' be the differential subalgebra of A generated by A_0. We will show by induction on p that $A_p \subset A'$.

Clearly, $A_0 \subset A'$. So let $p > 0$. By (4.28), $A_p = \sum_{i=0}^{p-1} A_i\partial A_{p-i-1}$, which is contained in A' by the induction hypothesis. □

Corollary 4.1 *Let $\{a^i : i \in I\}$ be a set of vectors of a good vertex algebra V. The following are equivalent:*

(i) *$\{a^i : i \in I\}$ are strong generators of V,*
(ii) *the image of $\{a^i : i \in I\}$ generates R_V.*

In particular, a vertex algebra V is finitely strongly generated if and only if R_V is finitely generated.

In this book we will always assume that a vertex algebra V is finitely strongly generated.

Definition 4.6 Let $\phi\colon V \to W$ be a map between two Poisson vertex algebras. We say that ϕ is a *Poisson vertex algebra homomorphism* if ϕ is a homomorphism of differential algebras such that

$$\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b),$$

for all $a, b \in V$, $n \geqslant 0$.

The following assertion is clear.

Lemma 4.2 *If $\phi\colon V \to W$ is a homomorphism of vertex algebras, then ϕ respects the canonical filtration, that is, $\phi(F^p V) \subset F^p W$. Hence it induces a homomorphism $\mathrm{gr}^F V \to \mathrm{gr}^F W$ of Poisson vertex algebra homomorphism which we denote by $\mathrm{gr}^F \phi$. The map $\mathrm{gr}^F \phi$ restricts to a Poisson algebra homomorphism $R_V \to R_W$, which we denote by $\bar{\phi}$. If in addition ϕ is surjective, then $\phi(F^p V) = F^p W$. In particular, $\mathrm{gr}\,\phi : \mathrm{gr}^F V \to \mathrm{gr}^F W$ and $\bar{\phi} : R_V \to R_W$ are surjective homomorphisms of Poisson vertex algebras and Poisson algebras, respectively.*

4.5 Associated variety and singular support

We now focus on geometrical objects associated with R_V and $\mathrm{gr}^F V$.

Definition 4.7 Define the *associated scheme* $\tilde{X}_V$ and the *associated variety* X_V of a vertex algebra V as

$$\tilde{X}_V := \mathrm{Spec}\, R_V, \qquad X_V := \mathrm{Specm}\, R_V = (\tilde{X}_V)_{\mathrm{red}}.$$

Let X be an affine Poisson variety. A vertex algebra V is called a *chiral quantization* of X if $\tilde{X}_V \cong X$ as Poisson varieties.

By Proposition 4.4, $\mathrm{gr}^F V$ is generated by the subring R_V as a differential algebra. Thus, we have a surjection $\mathcal{J}_\infty(R_V) \to \mathrm{gr}^F V$ of differential algebras by Lemma 1.1 since R_V generates $\mathcal{J}_\infty(R_V)$ as a differential algebra, too.

This is in fact a homomorphism of Poisson vertex algebras ([12, Proposition 2.5.1]).

Proposition 4.5 *The identity map $R_V \to R_V$ induces a surjective Poisson vertex algebra homomorphism*

$$\mathcal{J}_\infty(R_V) = \mathcal{O}(\mathcal{J}_\infty(\tilde{X}_V)) \twoheadrightarrow \mathrm{gr}^F V.$$

Proof As noticed just above, the identity map $R_V \to R_V$ induces a surjective homomorphism of differential algebras $f\colon \mathcal{J}_\infty(R_V) \to \mathrm{gr}^F V$. Let us show that f is a Poisson vertex algebra homomorphism. It suffices to verify that $f(a_{(n)}b) = f(a)_{(n)} f(b)$, for all $a, b \in \mathcal{J}_\infty(R_V)$ and all $n \geqslant 0$.

By construction, this is true for all $a, b \in R_V$ and $n \geqslant 0$, since the restriction of f to R_V is the identity map, and $a_{(n)}b = \delta_{n,0}\{a, b\}$ for $a, b \in R_V$. The statement is then a direct consequence of Lemma 4.3 below. ◻

Remark 4.3 Suppose that the Poisson structure of R_V is trivial. Then the Poisson vertex algebra structure of $\mathcal{J}_\infty(R_V)$ is trivial, and so is that of $\mathrm{gr}^F V$ by Proposition 4.5. This happens if and only if

$$(F^p V)_{(n)}(F^q V) \subset F^{p+q-n+1} V \quad \text{for all} \quad n \geqslant 0.$$

If this is the case, one can give $\mathrm{gr}^F V$ yet another Poisson vertex algebra structure by setting

$$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n+1}(a_{(n)}b) \quad \text{for all} \quad n \geqslant 0.$$

(We can repeat this procedure if this Poisson vertex algebra structure is again trivial.)

Lemma 4.3 *Let V, W be two Poisson vertex algebras, and $\phi\colon V \to W$ an algebra homomorphism such that $\phi\partial = \partial\phi$. Suppose that*

$$\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b) \ \text{ for all } \ a, b \in R \text{ and } n \geqslant 0,$$

where R is a generating subset of V as a differential algebra. Then ϕ is a vertex Poisson algebra homomorphism.

Proof We follow this argument of [194, Lemma 3.3]. Let $a, b \in V$ be such that

$$\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b) \ \text{ for all } \ n \geqslant 0. \tag{4.29}$$

Using (4.6) for both V and W, the assumption $\phi\partial = \partial\phi$ and (4.29), we obtain for all $n \geqslant 0$:

$$\begin{aligned}\phi(a_{(n)}\partial b) &= \phi(\partial a_{(n)}b) + n\phi(a_{(n-1)}b)\\ &= \partial\phi(a)_{(n)}\phi(b) + n\phi(a)_{(n-1)}\phi(b)\\ &= \phi(a)_{(n)}(\partial\phi(b)).\end{aligned}$$

By the left Leibniz rule (4.9) and induction we deduce that (4.29) holds for all $a \in R$ and $b \in V$.

Next, using the skew-symmetry (4.7) and $\phi\partial = \partial\phi$ we get that $\phi(\partial a_{(n)}b) = \phi(\partial a)_{(n)}\phi(b)$ for all $a \in R$, $b \in V$ and $n \geqslant 0$. Again by the left Leibniz rule (4.9) and induction, we deduce that (4.29) holds for all $a, b \in V$.

This concludes the proof of the lemma. □

Definition 4.8 Define the *singular support* $SS(V)$ of a vertex algebra V as

$$SS(V) = \mathrm{Spec}(gr^F V) \subset \mathcal{J}_\infty \tilde{X}_V.$$

Definition 4.9 We say that a vertex algebra V admits a *PBW basis* if there exists a collection $\{a^i : i \in I\}$ of vectors of V, with I a finite set, such that the set of monomials

$$a^{i_1}_{(-n_1)}a^{i_2}_{(-n_2)}\dots a^{i_r}_{(-n_r)}|0\rangle, \quad i_1 \leqslant i_2 \leqslant \dots \leqslant i_r,\ n_s \leqslant n_{s+1} \text{ if } i_s = i_{s+1}, \tag{4.30}$$

form a basis of V.

Lemma 4.4 *The following conditions are equivalent:*

(i) *V admits a PBW basis,*

(ii) *R_V is isomorphic to a polynomial ring and $SS(V) = \mathcal{J}_\infty \tilde{X}_V$.*

Proof (i) $\Rightarrow$ (ii) In this case, R_V is generated by the images of the elements $a^{i_1}_{(-1)} a^{i_2}_{(-1)} \dots a^{i_r}_{(-1)}|0\rangle$, for $i \in I$, and they are linearly independent by the hypothesis. Hence, R_V is isomorphic to the polynomial algebra in the variables $\bar{a}^i$, $i \in I$, with $\bar{a}^i$ the image of $a^i_{(-1)}|0\rangle$ in R_V and, so, $\mathcal{J}_\infty R_V \cong \mathbb{C}[\partial^n \bar{a}^i : i \in I, n \geqslant 0]$.

On the other hand, it is clear that $F^p V$ is generated as a vector space by the monomials (4.30) such that $n_1 + \dots + n_r \geqslant p + r$. As a result, $\mathrm{gr}^F V$ is generated by the monomials (4.30) such that $n_1 + \dots + n_r = p + r$. Then we get an isomorphism $\mathrm{gr}^F V \xrightarrow{\sim} \mathcal{J}_\infty R_V$ given by $a^i_{(-p-1)} \mapsto \partial^p \bar{a}^i$ for $p \geqslant 0$, $i \in I$, whence $SS(V) = \mathcal{J}_\infty \tilde{X}_V$.

(ii) $\Rightarrow$ (i) Conversely, assume that $R_V = V/F^1 V \cong \mathbb{C}[x^i : i \in I]$ and that $\mathrm{gr}^F V \cong \mathcal{J}_\infty R_V$. For each i, let $a^i = a^i_{(-1)}|0\rangle$ be a representative of x^i in V. From the isomorphisms

$$\mathcal{J}_\infty R_V \cong \mathbb{C}[\partial^n x^i : i \in I, n \geqslant 0] \cong \mathrm{gr}^F V,$$

we deduce that $\mathrm{gr}^F V$ is generated as a vector space by the monomials (4.30), whence (i). □

4.6 Examples

The following assertion is useful to compute $F^1 V$ in practice.

Lemma 4.5 *Let $\{a^i : i \in I\}$ be a set of strong generators of a vertex algebra V such that for all $i_1, i_2 \in I$ and all $n \geqslant 0$, $a^{i_1}_{(n)} a^{i_2}$ is a linear combination of $|0\rangle$ and the a^i's, $i \in I$. Then*

$$F^1 V = \mathrm{span}_{\mathbb{C}} \{a^i_{(-n_i-2)} v : i \in I,\ n_i \geqslant 0, v \in V\}.$$

Proof Writing v as a linear combination of elements of the form (4.20), we see that $a^i_{(-n_i-2)} v$, $i \in I$, $n_i \geqslant 0$, $v \in V$ belongs to $F^1 V$, whence one inclusion.

Conversely, show by induction on the length r of monomials v of the form (4.20) that $F^1 V$ is contained in the right-hand-side. Let $v = a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}|0\rangle \in F^1 V$. At least one of the n_j's is greater than 1. If $n_1 \geqslant 1$, then the statement is clear. In particular, the statement is clear if $r = 1$. Assume $n_1 = 0$. Then $v' = a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}|0\rangle \in F^1 V$ and by the induction hypothesis, there is $j \in I$, $m \geqslant 0$, $w \in V$ such that

$$v = a^{i_1}_{(-1)} a^j_{(-m-2)} w = a^j_{(-m-2)} a^{i_1}_{(-1)} w + \sum_{l \geqslant 0} \binom{-1}{l} (a^{i_1}_{(l)} a^j)_{(-m-3-l)} w.$$

The element $a^j_{(-m-2)} a^{i_1}_{(-1)} w$ lies in the right-hand-side set of the lemma. Moreover, by the hypothesis of the lemma, $(a^{i_1}_{(l)} a^j)_{(-m-3-l)} w$ is a linear combination of elements

$a^i_{(-m-3-l)}w$, $i \in I$. Note that $|0\rangle_{(-m-3-l)}w = 0$ because $-m-3-l$ cannot be equal to -1. Since $m, l \geqslant 0$, we get that $v \in \operatorname{span}_{\mathbb{C}}\{a^i_{(-n_i-2)}v \colon i \in I,\ n_i \geqslant 0, v \in V\}$, as desired. □

Example 4.5 Consider the universal affine vertex algebra $V^\kappa(\mathfrak{a})$ as defined in Section 3.1. Since $V^\kappa(\mathfrak{a})$ is strongly generated by $x \in \mathfrak{a} \subset V^\kappa(\mathfrak{a})$, we have

$$F^1 V^\kappa(\mathfrak{a}) = t^{-2}\mathfrak{a}[t^{-1}]V^\kappa(\mathfrak{a})$$

by Lemma 4.5. Therefore,

$$R_{V^\kappa(\mathfrak{a})} = V^\kappa(\mathfrak{a})/t^{-2}\mathfrak{a}[t^{-1}]V^\kappa(\mathfrak{a}).$$

By the PBW theorem we have an isomorphism linear map

$$\mathscr{O}(\mathfrak{a}^*) = S(\mathfrak{a}) \xrightarrow{\sim} R_{V^\kappa(\mathfrak{a})} \tag{4.31}$$

that sends the monomial $x^1x^2\ldots x^r \in S(\mathfrak{a})$, for $x^i \in \mathfrak{a}$, to the image of $x^1_{(-1)}\ldots x^r_{(-1)}|0\rangle$ in $R_{V^\kappa(\mathfrak{a})}$. This is in fact an homomorphism of Poisson algebras. Therefore,

$$\tilde{X}_{V^\kappa(\mathfrak{a})} = X_{V^\kappa(\mathfrak{a})} = \mathfrak{a}^*.$$

In particular, $V^\kappa(\mathfrak{a})$ is a chiral quantization of $\mathfrak{a}^*$. Moreover, the surjection

$$\mathcal{O}(\mathcal{J}_\infty\mathfrak{a}^*) \to \operatorname{gr}^F V^\kappa(\mathfrak{a})$$

is an isomorphism since both sides have the same graded dimension with respect to

$$\deg xt^{-n} = n, \qquad n \in \mathbb{Z}_{>0}.$$

(Here we have used the fact that $V^\kappa(\mathfrak{a})$ is good, see also Example 4.6 below.) Hence

$$SS(V^\kappa(\mathfrak{a})) = \mathcal{J}_\infty\mathfrak{a}^*.$$

For the simple quotient $L_\kappa(\mathfrak{a})$ of $V^\kappa(\mathfrak{a})$, the surjection $V^\kappa(\mathfrak{a}) \twoheadrightarrow L_\kappa(\mathfrak{a})$ induces a surjection $\mathscr{O}(\mathfrak{a}^*) = R_{V^\kappa(\mathfrak{a})} \twoheadrightarrow R_{L_\kappa(\mathfrak{a})}$. Thus,

$$R_{L_\kappa(\mathfrak{a})} \cong \mathscr{O}(\mathfrak{a}^*)/I$$

for some graded Poisson ideal I of $\mathscr{O}(\mathfrak{a}^*)$, and $X_{L_\kappa(\mathfrak{a})}$ is the zero locus of I in $\mathfrak{a}^*$, which is a conic Poisson subvariety. Similarly, $SS(L_\kappa(\mathfrak{a}))$ is a $\mathbb{C}^*$-invariant closed subscheme of $\mathcal{J}_\infty\mathfrak{a}^*$.

Exercise 4.1 Let Vir^c be the universal Virasoro vertex algebra of central charge $c \in \mathbb{C}$.

(i) Show that $\operatorname{gr}^F \mathrm{Vir}^c \cong \mathbb{C}[L_{-2}, L_{-3}, \ldots]$.

(ii) Deduce from (i) that $R_{\mathrm{Vir}^c} \cong \mathbb{C}[x]$, where x is the image of $L := L_{-2}|0\rangle$ in R_{Vir^c}, with the trivial Poisson structure.
(iii) Show that one can endow $\mathrm{gr}^F \mathrm{Vir}^c$ with a nontrivial Poisson vertex algebra structure such that
$$\{L_\lambda L\} = TL + 2\lambda L.$$

4.7 The conformal weight filtration and comparison with the Li filtration

Suppose that V is positively graded:

$$V = \bigoplus_{\Delta \in \frac{1}{r_0}\mathbb{Z}_{\geqslant 0}} V_\Delta,$$

where r_0 is some positive integer. (In most cases we assume that $r_0 = 1$ or 2.) There is another natural filtration of V defined as follows [194].

Choose a set $\{a^i : i \in I\}$ of homogeneous strong generators of V. Let $G_p V$, $p \in \frac{1}{r_0}\mathbb{Z}_{\geqslant 0}$, be the subspace of V spanned by the vectors

$$a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}|0\rangle \tag{4.32}$$

with $i_j \in I$, $n_j \geqslant 0$, $\Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_r}} \leqslant p$. Then $G_\bullet V$ defines an increasing filtration of V:

$$0 = G_{-1}V \subset G_0 V \subset \ldots G_1 V \subset \ldots, \quad V = \bigcup_p G_p V.$$

Definition 4.10 The increasing filtration $G_\bullet V$ is called the *conformal weight filtration*.

Lemma 4.6 *We have*

$$T G_p V \subset G_p V, \tag{4.33}$$
$$(G_p V)_{(n)} G_q V \subset G_{p+q} V \quad \textit{for } n \in \mathbb{Z}, \tag{4.34}$$
$$(G_p V)_{(n)} G_q V \subset G_{p+q-1} V \quad \textit{for } n \in \mathbb{Z}_{\geqslant 0}. \tag{4.35}$$

Proof Since $[T, a^i_{(-n)}] = n a^i_{(-n-1)}$, for any $i \in I$, $n \geqslant 0$, and $T|0\rangle = 0$, (4.33) is easily seen.

For $n < 0$, we establish (4.34) exactly as for the proof of Proposition 4.1, using Borcherds identity (2.36).

Assume $n \geqslant 0$. Since $G_{p+q-1}V \subset G_{p+q}V$ it suffices to establish (4.35). In addition, it suffices to prove that $a_{(n)}b \subset G_{p+q-1}V$ for all $a \in G_p V$, $b \in G_q V$ that are homogeneous.

Recall that by (2.44), we have for $n \geqslant 0$,

$$(V_\Delta)_{(n)}V_{\Delta'} \subset V_{\Delta+\Delta'-n-1}. \tag{4.36}$$

Therefore, (4.35) will be a consequence of the equality (4.37) in Lemma 4.7 below. Indeed, setting $F^iV_\Delta := F^iV \cap V_\Delta$, $G_iV_\Delta := G_iV \cap V_\Delta$ for $i \geqslant 0$, we obtain by (4.37) and Proposition 4.1,

$$\begin{aligned} a_{(n)}b \in (F^{\Delta_a-p}V_{\Delta_a})_{(n)}F^{\Delta_b-q}V_{\Delta_b} &\subset F^{\Delta_a-p+\Delta_b-q-n}V_{\Delta_a+\Delta_b-n-1} \\ &= G_{p+q-1}V_{\Delta_a+\Delta_b-n-1} \subset G_{p+q-1}V \end{aligned}$$

for homogenous elements $a \in G_pV$, $b \in G_qV$ and $n \geqslant 0$.

Notice that the proof of Lemma 4.7 uses (4.34) for $n < 0$, but does not use (4.35) or (4.34) for $n \geqslant 0$. ◻

It follows that $\mathrm{gr}_G V = \bigoplus_p G_pV/G_{p-1}V$ is naturally a Poisson vertex algebras. Moreover, we have the following remarkable result ([12, Proposition 2.6.1]).

Lemma 4.7 *We have*

$$F^pV_\Delta = G_{\Delta-p}V_\Delta, \tag{4.37}$$

where $F^pV_\Delta = V_\Delta \cap F^pV$, $G_pV_\Delta = V_\Delta \cap G_pV$. Therefore

$$\mathrm{gr}^F V \cong \mathrm{gr}_G V$$

as Poisson vertex algebras.

Proof The second assertion easily deduces from the first one. Let us prove the first assertion. Clearly, $V_\Delta \subset G_\Delta V_\Delta$ since for $a \in V_\Delta$, one can write $a = a_{(-1)}|0\rangle \in G_\Delta V$. The other inclusion is obvious and, hence, $V_\Delta = G_\Delta V_\Delta$, that is,

$$F^0V_\Delta = G_\Delta V_\Delta.$$

We now show the inclusion $F^pV_\Delta \subset G_{\Delta-p}V_\Delta$ by induction on $p \geqslant 0$. Let $p > 0$. By Lemma 4.1, F^pV_Δ is generated by elements $v = a_{(-i-1)}b$, with $a \in V_{\Delta_a}$, $b \in F^{p-i}V_{\Delta_b}$, $i \geqslant 1$, $\Delta_a + \Delta_b + i = \Delta$. Hence it suffices to show that for such elements, $v \in G_{\Delta-p}V_\Delta$. By the induction hypothesis, $F^{p-i}V_{\Delta_b} \subset G_{\Delta_b-p+i}V_{\Delta_b}$. Because $a \in V_{\Delta_a} \subset G_{\Delta_a}V$, we have by (4.34) with $n = -i-1 < 0$,

$$v = a_{(-i-1)}b \in (G_{\Delta_a}V_{\Delta_a})_{(-i-1)}G_{\Delta_b-p+i}V_{\Delta_b} \subset G_{\Delta_a+\Delta_b-p+i}V_\Delta = G_{\Delta-p}V_\Delta.$$

Hence $F^pV_\Delta \subset G_{\Delta-p}V_\Delta$.

It remains to show the opposite inclusion $G_{\Delta-p}V_\Delta \subset F^pV_\Delta$. We prove that any element v of the form (4.32) belongs to F^pV_Δ by induction on $r \geqslant 0$. For $r = 0$, the statement is obvious. Assume $r > 0$. Then $v = a^{i_1}_{(-n_1-1)}w$, with $w = a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}|0\rangle$, $n_j \geqslant 0$, $\sum\limits_j \Delta_{a^{i_j}} \leqslant p$, $\Delta_{a^{i_1}} + \Delta_w + n_1 = \Delta$, where each a^{i_j}

is homogeneous. Because $w \in G_{p-\Delta_{a^{i_1}}} V_{\Delta_w}$, the induction hypothesis gives that $w \in F^{\Delta_{a^{i_1}}+\Delta_w-p} V_{\Delta_w}$. Hence

$$v = a^{i_1}_{(-n_1-1)} w \in F^{\Delta_{a^{i_1}}+\Delta_w-p+n_1} V_{\Delta_{a^{i_1}}+\Delta_w+n_1} = F^{\Delta-p} V_\Delta$$

since $a \in F^0 V_{\Delta_{a^{i_1}}}$. □

By Lemma 4.7, it follows in particular that the conformal weight filtration is independent of the choice of the set of strong generators.

Corollary 4.2 *A vertex algebra is good if it is positively graded.*

Proof This is clear from Lemma 4.7 since $F^p V_\Delta = G_{\Delta-p} V_\Delta = 0$ if $p > \Delta$ for each Δ. □

Example 4.6 Consider the universal affine vertex algebra $V^\kappa(\mathfrak{a})$. Since $V^\kappa(\mathfrak{a})$ is strongly generated by $x \in \mathfrak{a} \subset V^\kappa(\mathfrak{a})$, which has conformal weight one, it follows that

$$G_p V^\kappa(\mathfrak{a}) = U_p(\mathfrak{a}[t^{-1}]t^{-1})|0\rangle,$$

where $U_\bullet(\mathfrak{a}[t^{-1}]t^{-1})$ is the PBW filtration of $U(\mathfrak{a}[t^{-1}]t^{-1})$ (see Example B.2). On the other hand, we have the isomorphisms (cf. Example 4.1)

$$\operatorname{gr} U(\mathfrak{a}[t^{-1}]t^{-1}) \cong S(\mathfrak{a}[t^{-1}]t^{-1}) \cong \mathcal{O}(\mathcal{J}_\infty(\mathfrak{a}^*)).$$

Hence, as a consequence of Lemma 4.7, we reconfirm the fact that

$$\operatorname{gr}^F V^\kappa(\mathfrak{a}) \cong \operatorname{gr}_G V^\kappa(\mathfrak{a}) \cong \mathcal{O}(\mathcal{J}_\infty(\mathfrak{a}^*))$$

as Poisson vertex algebras.

Example 4.7 Consider the vertex algebra $\mathcal{D}^{ch}_{G,\kappa}$ of chiral differential operators on G at level κ (see Section 3.4). We have

$$G_p \mathcal{D}^{ch}_{G,\kappa} = U_p(\mathfrak{g}[t^{-1}]t^{-1}) \otimes \mathcal{O}(J_\infty G).$$

Thus

$$\operatorname{gr}^F \mathcal{D}^{ch}_{G,\kappa} \cong \operatorname{gr}_G \mathcal{D}^{ch}_{G,\kappa} \cong \mathbb{C}[\mathcal{J}_\infty(\mathfrak{g}^*)] \otimes \mathcal{O}(J_\infty G) = \mathcal{O}(\mathcal{J}_\infty(T^*G)),$$

which restricts to the isomorphism

$$R_{\mathcal{D}^{ch}_{G,\kappa}} \cong \mathcal{O}(T^*G). \tag{4.38}$$

In parcticular, we have

$$\tilde{X}_{\mathcal{D}^{ch}_{G,\kappa}} \cong T^*G, \quad SS(\mathcal{D}^{ch}_{G,\kappa}) \cong \mathcal{J}_\infty T^*G. \tag{4.39}$$

4.8 The lisse condition

Geometrical properties of the associated variety X_V should reflect important information about the vertex algebra V. It is natural to first consider the simplest case where X_V has dimension 0.

Recall that we are assuming that a vertex algebra V is finitely strongly generated so that $\tilde{X}_V$ is a scheme of finite type. The following statement is mentioned in [42, Exercise 8.3].

Lemma 4.8 *Let $X = \operatorname{Spec} R$ be an affine scheme of finite type over a field K. Then the following assertion are equivalent:*

(i) $\dim X = 0$,
(ii) *R is a finite dimensional K-algebra.*

If so, then X is a finite discrete topological space.

Proof To prove the equivalence (i) $\iff$ (ii), recall that a Noetherian ring has dimension zero if and only if it is Artinian [208, Theorem 3.2 and Example 2 in §5]. So the converse implication (ii) $\Rightarrow$ (i) is clear because a finite dimensional algebra is an Artinian ring. Indeed, if R is a finite dimensional K-vector space, then it is Artinian as K-vector space. But every ideal of R is a K-vector space and thus they satisfy the descending chain condition, which proves that R is Artinian as ring.

Conversely, assume that $\dim X = 0$, that is, R is Artinian. Then R is a finite product of Artinian local rings (cf. [42, Theorems 8.7]). So one may assume that R is an Artinian local ring, with maximal ideal $\mathfrak{m}$. Then $R/\mathfrak{m}$ if a finite extension of K by Zariski lemma. Since R is Artinian, $\mathfrak{m}$ is the radical of A ([208, proof of Theorem 3.2]) and thus $\mathfrak{m}^n = 0$ for some n. Thus we have a chain

$$R \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \supseteq \mathfrak{m}^n = 0.$$

Since R is Noetherian, $\mathfrak{m}$ is finitely generated and each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite dimensional $R/\mathfrak{m}$-vector space, hence $\mathfrak{m}$ is a finite dimensional vector space. This completes the proof. □

Definition 4.11 A vertex algebra V is called *lisse* (or *C_2-cofinite*) if $\dim X_V = 0$ or, equivalently, if $R_V = V/F^1(V)$ is finite dimensional.

As a consequence of Proposition 4.5, we have the following result.

Theorem 4.3 *We have* $\dim SS(V) = 0$ *if and only if* $\dim X_V = 0$.

Proof The "only if" part is obvious since $\pi_\infty(SS(V)) = \tilde{X}_V$. The "if" part follows from Corollary 1.2 and Proposition 4.5. □

By Theorem 4.3 we get:

Lemma 4.9 *The vertex algebra V is lisse if and only if* $\dim SS(V) = 0$.

The C_2-cofiniteness condition, $\dim V/C_2(V) < \infty$, was introduced by Zhu [257], while the term *lisse* has been borrowed from Beilinson, Feigin and Mazur who considered the finiteness condition $\dim SS(V) = 0$ in the case of Virasoro vertex algebras. The equivalence between these two notions was established in [12]. In this book, we will be rather using the name *lisse*.

Lemma 4.10 *Suppose that* V *is conical, so that* $V = \bigoplus_{\Delta\geqslant 0} V_\Delta$ *and* $V_0 = \mathbb{C}|0\rangle$. *The algebras* $\mathrm{gr}^F V$ *and* R_V *are equipped with the induced grading:*

$$\mathrm{gr}^F V = \bigoplus_{\Delta\geqslant 0}(\mathrm{gr}^F V)_\Delta, \quad (\mathrm{gr}^F V)_0 = \mathbb{C},$$

$$R_V = \bigoplus_{\Delta\geqslant 0}(R_V)_\Delta, \quad (R_V)_0 = \mathbb{C}.$$

Then the following conditions are equivalent:

(i) *V is lisse,*
(ii) $X_V = \{\text{point}\}$,
(iii) *the image of any vector* $a \in V_\Delta$ *for* $\Delta > 0$ *in* R_V *is nilpotent,*
(iv) *the image of any vector* $a \in V_\Delta$ *for* $\Delta > 0$ *in* $\mathrm{gr}^F V$ *is nilpotent.*

Thus, lisse vertex algebras can be regarded as a generalization of finite-dimensional algebras. Examples of lisse vertex algebras will be given in Chapter 11.

Proof The equivalence (i) $\iff$ (ii) follows from Lemma 4.8. Let us prove the equivalence (i) $\iff$ (iii). One can assume that $R_V = \mathbb{C}[x^1, \ldots, x^N]/I$ for some ideal I. If $X_V = \{\text{point}\}$, then $R_V/\sqrt{0} = \mathbb{C}$. So $\sqrt{I}$ is the argumentation ideal of $\mathbb{C}[x^1, \ldots, x^N]$ or, equivalently, each x^i is nilpotent. Conversely, if each x^i is nilpotent, $\sqrt{I}$ is the argumentation ideal of $\mathbb{C}[x^1, \ldots, x^N]$, that is, $R_V/\sqrt{0} = \mathbb{C}$ and so $\dim X_V = 0$.

To prove the equivalence (i) $\iff$ (iv), write

$$\mathrm{gr}^F V = \mathbb{C}[x^1_{(-j)}, \ldots, x^N_{(-j)},\ j \geqslant 1]/J$$

for some ideal J, which is possible by Proposition 4.5. Part (i) is equivalent to that $\dim SS(V) = 0$ by Lemma 4.9. Hence one can argue as for the equivalence (i) $\iff$ (iii). Namely, if $SS(V) = \{\text{point}\}$, then $\mathrm{gr}^F V/\sqrt{0} = \mathbb{C}$. So $\sqrt{J}$ is the argumentation ideal of $\mathbb{C}[x^1_{(-j)}, \ldots, x^N_{(-j)},\ j \geqslant 1]$ or, equivalently, each $x^i_{(-j)}$ is nilpotent. Conversely, if each $x^i_{(-j)}$ is nilpotent, $\sqrt{J}$ is the argumentation ideal of $\mathbb{C}[x^1_{(-j)}, \ldots, x^N_{(-j)},\ j \geqslant 1]$, that is, $\mathrm{gr}^F V/\sqrt{0} = \mathbb{C}$ and so $\dim SS(V) = 0$. □

Lemma 4.11 *Let* V *be a conical vertex algebra,* $\{a^i : i \in I\}$ *a set of homogenous strong generators, so that* $\mathcal{O}(\mathscr{L}\tilde{X}_V)$ *is a topological ring generated by the image* $\bar{a}^i_{(n)}$, $i \in I$, *where* $\bar{a}^i$ *is the the image of* a^i *in* R_V. *If* V *is lisse, then each* $\bar{a}^i_{(n)}$ *is nilpotent in* $\mathcal{O}(\mathscr{L}\tilde{X}_V)$.

Proof Recall that the ind-schemes $\mathscr{L}\tilde{X}_V$ is the direct limit of schemes $\mathscr{L}_n\tilde{X}_V$, with $\mathscr{L}_0\tilde{X}_V = \mathscr{J}_\infty\tilde{X}_V$. The canonical morphism $(\tilde{X}_V)_{\mathrm{red}} = X_V \to \tilde{X}_V$ induces morphisms $\mathscr{L}_n X_V \to \mathscr{L}_n\tilde{X}_V$ for each n and, hence, a morphism of ind-schemes $\mathscr{L}X_V \to \mathscr{L}\tilde{X}_V$. Since $\mathbb{C}((z))$ is a field, similarly to Lemma 1.7 we establish that

$$\mathscr{L}X_V \xrightarrow{\sim} \mathscr{L}\tilde{X}_V,$$

as topological spaces, whence $\mathscr{L}_n X_V \xrightarrow{\sim} \mathscr{L}_n\tilde{X}_V$ as topological spaces for each n as well.

Moreover, if X_V is a point as topological space, then $\mathscr{L}X_V$ is also a point since $\mathrm{Hom}_{Alg}(\mathbb{C}, \mathbb{C}((z))) \cong \mathrm{Hom}_{Sch}(\mathrm{Spec}\,\mathbb{C}((z)), X_V) \cong \mathrm{Hom}_{Sch}(\mathrm{Spec}\,\mathbb{C}, \mathscr{L}X_V)) \cong \mathrm{Hom}_{Alg}(\mathcal{O}(\mathscr{L}X_V), \mathbb{C})$ consists of only one point. It follows that if $\tilde{X}_V$ is zero-dimensional, then each $\mathscr{L}_n\tilde{X}_V$ is zero-dimensional too.

Hence, if V is lisse, then $\mathcal{O}(\mathscr{L}_n\tilde{X}_V)/\sqrt{0} = \mathbb{C}$, that is, $\mathbb{C}[\bar{a}^i_{(-j-1)} : i \in I]_{j \geqslant -n}/\sqrt{0} = \mathbb{C}$. So the augmentation ideal of $\mathbb{C}[\bar{a}^i_{(-j-1)} : i \in I]_{j \geqslant -n}$ is generated by the $\bar{a}^i_{(-j-1)}$'s. In particular each $\bar{a}^i_{(-j-1)}$, for $i \in I$ and $j \geqslant -n$, is nilpotent in $\mathcal{O}(\mathscr{L}_n\tilde{X}_V)$. Since this is true for each n we get the statement. □

Lemma 4.11 will be used in the proof of Theorem 5.4.

4.9 Remarks on the Poisson center of the Zhu C_2-algebra.

Let $Z(R_V)$ be the Poisson center of R_V defined by

$$Z(R_V) = \{\bar{a} \in R_V : \{\bar{a}, \bar{b}\} = 0 \text{ for all } b \in V\}.$$

If V is conformal with conformal vector ω, then $\bar{\omega}$ belongs to $Z(R_V)$. Indeed, for any $a \in V$, $\omega_{(0)}a = L_1^V a = Ta = a_{(-2)}|0\rangle$, whence $\{\bar{\omega}, \bar{a}\} = 0$ in R_V.

Lemma 4.12 *Let $a \in V$ such that $\bar{a} \in Z(R_V)$. Then*

$$D_a : R_V \to R_V, \quad \bar{b} \mapsto \overline{a_{(1)}b},$$

defines a derivation of R_V.

Proof Since $\bar{a} \in Z(R_V)$, we have $a_{(n)}F^pV \subset F^{p-n+1}V$. In particular, $a_{(1)}F^1V \subset F^1V$. Thus, the linear map $D_a : R_V \to R_V$ is well-defined. We have

$$a_{(1)}(b_{(-1)}c) = [a_{(1)}, b_{(-1)}]c + b_{(-1)}a_{(1)}c = (a_{(0)}b)_{(0)}c + (a_{(1)}b)_{(1)}c + b_{(-1)}a_{(1)}c.$$

Since $a \in Z(R_V)$, $a_{(0)}b \in F^1V$. Hence $(a_{(0)}b)_{(0)}c \in F^1V$, Therefore D_a is a derivation as required. □

Note that $D_\omega(\bar{a}) = \Delta_a\bar{a}$ for a homogenous element a of V of conformal weight Δ_a.

Theorem 4.4 *Let V be a conical conformal vertex algebra. The following conditions are equivalent:*

(i) *$\bar{\omega}$ is nilpotent in R_V,*
(ii) *the augmentation ideal of $Z(R_V)$ is contained in the radical of R_V.*

Proof The direction (ii) ⇒ (i) is obvious. So let us show that (i) ⇒ (ii). Let a be a homogenous element a of V of conformal weight $\Delta_a > 0$ such that $\bar{a} \in Z(R_V)$. Since $\bar{\omega} \in \sqrt{(0)}$, we have $D_a(\bar{\omega}) \in \sqrt{(0)}$. However,

$$a_{(1)}\omega = [a_{(1)}, \omega_{(-1)}]|0\rangle = -[\omega_{(-1)}, a_{(1)}]|0\rangle = -\sum_{j\geqslant 0}(-1)^j(\omega_{(j)}a)_{(-j)}|0\rangle$$
$$= -\sum_{j\geqslant 1}(-1)^j(\omega_{(j)}a)_{(-j)}|0\rangle \equiv -(\omega_{(1)}a)_{(-1)}|0\rangle = -\Delta_a a \pmod{F^1 V}.$$

Therefore, $a \in \sqrt{(0)}$. □

Corollary 4.3 *Let V be a conical conformal vertex algebra such that R_V is Poisson commutative. Then the following conditions are equivalent.*

(i) *V is lisse.*
(ii) *$\bar{\omega}$ is nilpotent in R_V.*

We will see an analogue statement for *quasi-lisse* affine vertex algebras in the next part (cf. Theorem 13.2).

Chapter 5
Modules over vertex algebras and Zhu's functor

We introduce in this chapter the Zhu functor $V \mapsto \mathrm{Zhu}(V)$ assigning to a vertex operator algebra the associative algebra $\mathrm{Zhu}(V)$. Zhu established that the equivalence classes of the irreducible representations of V are in one-to-one correspondence with the equivalence classes of the irreducible representations of $\mathrm{Zhu}(V)$. The associative algebra $\mathrm{Zhu}(V)$ has a much simpler structure than V, for example, the one-to-one correspondence theorem implies that if V is *rational* then $\mathrm{Zhu}(V)$ is semisimple. The algebra $\mathrm{Zhu}(V)$ also plays a crucial role in the proof of the modular invariance [257, 216].

Section 5.1 is about Zhu's functor. To prove the main theorem of the chapter (Theorem 5.2), we need an alternative description of the Zhu's algebra given in Section 5.2. Section 5.3 is devoted to the proof of Theorem 5.2. In Section 5.4, we discuss the connexion between the Zhu's algebra and the Zhu C_2-algebra. The second crucial result of this chapter (Theorem 5.4) is proved in Section 5.5. To finish, using the technics of Section 5.4, we explicitly compute in Section 5.6 the Zhu's algebra in some examples.

We continue to assume that a vertex algebra V is finitely strongly generated.

5.1 Zhu's algebra and Zhu's functor

We assume in this chapter that V be a $\mathbb{Z}$-graded vertex algebra (see Definition 2.8).

Definition 5.1 For homogeneous elements a, b of V, set

$$a \circ b := \mathrm{Res}_z \left(Y(a,z) b \frac{(z+1)^{\Delta_a}}{z^2} \right) = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} b,$$

and extend the products $\circ$ linearly. The expression $(z+1)^k$, for $k \in \mathbb{Z}$, means $\sum\limits_{j \geqslant 0} \binom{k}{j} z^j$. We set

$$\mathrm{Zhu}(V) := V/V \circ V,$$

where $V \circ V := \mathrm{span}\{a \circ b \colon a, b \in V\}$.

Next theorem is due to Frenkel–Zhu and Zhu ([121, 257]).

Theorem 5.1 (Frenkel–Zhu) *The quotient* Zhu(V) *is an associative algebra, called the* Zhu algebra *of* V*, with multiplication defined as*

$$a * b := \mathrm{Res}_z \left(Y(a,z)b \frac{(z+1)^{\Delta_a}}{z} \right) = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-1)} b$$

for homogeneous elements $a, b \in V$*. Its unit is the image of the vacuum* $|0\rangle$ *in the quotient* Zhu(V).

A vertex algebra V is called a *chiralization* of an algebra A if $\mathrm{Zhu}(V) \cong A$.
Before proving the theorem, we need some lemmas.

Lemma 5.1 *For a homogenous element a and* $n \in \mathbb{Z}_{\geqslant 0}$*, we have*

$$T^n a = n! \binom{-\Delta_a}{n} a \quad (\mathrm{mod}\ V \circ V).$$

Proof From $a \circ |0\rangle = a_{(-2)}|0\rangle + \Delta_a a = Ta + \Delta_a a$, we deduce that $Ta = -\Delta_a a$ $(\mathrm{mod}\ V \circ V)$. Using this relation and $\Delta_{Ta} = \Delta_a + 1$, the identities follows from an easy induction on n. □

Lemma 5.2 *For* a, b *homogeneous elements,*

$$b * a = \mathrm{Res}_z \left(Y(a,z)b \frac{(z+1)^{\Delta_a - 1}}{z} \right) = \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i-1)} b \quad (\mathrm{mod}\ V \circ V).$$

Proof By skew-symmetry (Proposition 2.4) and Lemma 5.1, we have

$$\begin{aligned} Y(b,z)a = e^{zT} Y(a,-z) b &= \sum_{n \in \mathbb{Z}} e^{zT} a_{(n)} b (-z)^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} \sum_{j \geqslant 0} \frac{T^j (a_{(n)} b)}{j!} z^j (-z)^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} \sum_{j \geqslant 0} \binom{-\Delta_a - \Delta_b + n + 1}{j} z^j a_{(n)} b (-z)^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} (-z)^{-n-1} (z+1)^{-\Delta_a - \Delta_b + n + 1} a_{(n)} b \\ &= (z+1)^{-\Delta_a - \Delta_b} Y(a, -\frac{z}{z+1}) b. \end{aligned}$$

Therefore, we get

$$b * a = \mathrm{Res}_z \left(Y(b,z) a \frac{(z+1)^{\Delta_b}}{z} \right) = \mathrm{Res}_z \left(Y(a, -\frac{z}{z+1}) b \frac{(z+1)^{\Delta_b}}{z} (z+1)^{-\Delta_a - \Delta_b} \right).$$

Recall the formula for change of variable for residue. For $g(w) = \sum_{m \geqslant M} v_m w^m \in V((w))$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n \in z\mathbb{C}[[z]]$ with $a_1 \neq 0$, the power series $g(f(z)) \in V((z))$ is defined as

$$g(f(z)) = \sum_{m \geqslant M} v_m f(z)^m = \sum_{m \geqslant M} \sum_{j=1}^{\infty} v_m (a_1 z)^m \binom{m}{j} \bar{f}^j,$$

where $\bar{f} = \sum_{i=2}^{\infty} \frac{a_i}{a_1} z^{i-1}$. Then we have the following formula:

$$\text{(5.1)} \qquad \mathrm{Res}_w\, g(w) = \mathrm{Res}_z (g(f(z))) \frac{d}{dz} f(z).$$

Using the formula of change of variable (5.1) with $w = -\frac{z}{z+1}$ we deduce that

$$b * a = \mathrm{Res}_w \left(Y(a,w) b \frac{(w+1)^{\Delta_a - 1}}{w} \right),$$

whence the expected result. □

Lemma 5.3 *For homogeneous elements a, b, we have*

$$a * b - b * a = \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i)} b \pmod{V \circ V}$$

In particular, the image of the conformal vector belongs to the center of Zhu(V).

Proof The first assertion is an easy consequence of Lemma 5.2. The last assertion follows from the fact that $\omega * a - a * \omega \equiv \sum_{i \geqslant 0} \binom{1}{i} \omega_{(i)} a \equiv Ta + Ha = a \circ |0\rangle$. □

Lemma 5.4 *For every homogeneous element $a \in V$, and $m \geqslant n \geqslant 0$,*

$$\mathrm{Res}_z \left(Y(a,z) \frac{(z+1)^{\Delta_a + n}}{z^{2+m}} b \right) \in V \circ V.$$

Proof Since

$$\frac{(z+1)^{\Delta_a + n}}{z^{2+m}} = \sum_{i=0}^{n} \binom{n}{i} \frac{(z+1)^{\Delta_a}}{z^{2+m-i}},$$

we only need to prove the lemma for the case $n = 0$ and $m \geqslant 0$. We prove the statement by induction on m, the case $m = 0$ being clear from the definition of $V \circ V$. Assume the statement true for any $m \leqslant k$, and prove it for $m = k + 1$. By induction, we have

$$\mathrm{Res}_z \left(Y(Ta,z) \frac{(z+1)^{\Delta_a + 1}}{z^{2+k}} b \right) \in V \circ V.$$

On the other hand,

$$\begin{aligned}\mathrm{Res}_z\left(Y(Ta,z)\frac{(z+1)^{\Delta_a+1}}{z^{2+k}}b\right) &= \mathrm{Res}_z\left(\frac{\partial}{\partial z}Y(a,z)\frac{(z+1)^{\Delta_a+1}}{z^{2+k}}b\right)\\ &= -\mathrm{Res}_z\left(Y(a,z)\frac{\partial}{\partial z}\frac{(z+1)^{\Delta_a+1}}{z^{2+k}}b\right)\\ &= -(\Delta_a+1)\,\mathrm{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z^{2+k}}b\right)\\ &\quad + (2+k)\,\mathrm{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z^{2+k+1}}b\right),\end{aligned}$$

whence the statement for $m = k+1$ since the first term of the right-hand-side is in $V \circ V$ by induction. □

Proof (of Theorem 5.1) First, for homogenous a,

(5.2) $$a * |0\rangle = a_{(-1)}|0\rangle = a \quad \text{and} \quad |0\rangle * a = |0\rangle_{(-1)}a = a.$$

To prove the theorem, we have to show that $V \circ V$ is a two-sided ideal of $\mathrm{Zhu}(V)$, so that $*$ is well-defined on $\mathrm{Zhu}(V)$, and that $\mathrm{Zhu}(V)$ is an associative algebra for $*$. It suffices to show that the following relations hold for homogeneous elements a, b, c:

(5.3) $$a * (V \circ V) \subset V \circ V,$$

(5.4) $$(V \circ V) * a \subset V \circ V,$$

(5.5) $$(a * b) * c - a * (b * c) \in V \circ V,$$

since (5.2) will ensure that the image of $|0\rangle$ is a unit for the multiplication $*$ on $\mathrm{Zhu}(V)$.

We only detail the proof of (5.3), the other identities are proven using the same technics. The idea is to show that for homogeneous elements a, b, c of V, we have $a * (b \circ c) - b \circ (a * c) \in V \circ V$ so that $a * (b \circ c) \in V \circ V$. Using (2.38), we get

$$\begin{aligned}a * (b \circ c) - b \circ (a * c) &= \mathrm{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z}\right)\mathrm{Res}_w\left(Y(b,w)\frac{(w+1)^{\Delta_b}}{w^2}c\right)\\ &\quad - \mathrm{Res}_w\left(Y(b,w)\frac{(w+1)^{\Delta_b}}{w^2}\right)\mathrm{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z}c\right)\\ &= \mathrm{Res}_w\left(\mathrm{Res}_{z-w}\left(Y(Y(a,z-w)b,w)\frac{(z+1)^{\Delta_a}}{z}\frac{(w+1)^{\Delta_b}}{w^2}c\right)\right)\\ &= \sum_{i=0}^{\Delta_a}\sum_{j\geqslant 0}\binom{\Delta_a}{i}\mathrm{Res}_w\left(Y(a_{(i+j)}b,w)(-1)^j\frac{(w+1)^{\Delta_a+\Delta_b-i}}{w^{j+3}}c\right).\end{aligned}$$

Since $\Delta_{a_{(i+j)}b} = \Delta_a + \Delta_b - i - j - 1$, the right hand side of the last equality belongs to $V \circ V$ in virtue of Lemma 5.4. □

For a simple positive energy representation (cf. Definition 3.2) $M = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} M_{\lambda+n}$, $M_\lambda \neq 0$, of V, let M_{top} be the *top degree component* M_λ of M. Using (2.45), we see that for any d,

$$a^M_{(n)} M_d \subset M_{d+\Delta_a-n-1}. \tag{5.6}$$

For a homogeneous vector $a \in V$, let $o(a) = a_{(\Delta_a-1)} = a^M_{(\Delta_a-1)}$, so that $o(a)$ preserves the homogeneous components of any graded representation of V by (5.6).

Remark 5.1 There is also an alternative construction of the Zhu's algebra by van Ekeren and Heluani that uses elliptic functions and does not use the conformal structure [94]. As a matter of fact, the Zhu's algebra structure is independent of the conformal structure.

The importance of Zhu's algebra in vertex algebra theory comes from the following fact that was established by Zhu [257].

Theorem 5.2 (Zhu) *Assume that V is conformal. For any positive energy representation M of V, $[a] \mapsto o(a)$ gives a well-defined representation of* Zhu(V) *on M_{top}, where $[a]$ is the image of a in* Zhu(V)*. Moreover, the correspondence $M \mapsto M_{\text{top}}$ gives a bijection between the set of isomorphism classes of irreducible positive energy representations of V and that of simple* Zhu(V)*-modules.*

The proof of this theorem will be given after Theorem 5.3.

5.2 Current algebra and Zhu algebra

The next two lemmas are due to Borcherds ([59]).

Lemma 5.5 *For a vertex algebra V, V/TV is a Lie algebra by*

$$[a + TV, b + TV] = a_{(0)}b + TV, \quad a, b \in V$$

Proof The skew symmetry property follows from the skew symmetry property of vertex algebra, which is equivalent to (4.7). The Jacobi identity $[a+TV, [b+TV, c+TV]] = [[a+TV, b+TV], c+TV] + [b+TV, [a+TV, c+TV]]$ follows from the Borcherds identity (2.35) for $m = n = 0$. ◻

Lemma 5.6 *Let V be a vertex algebra, (R, ∂) a differential algebra. Then*

$$\mathrm{Lie}(V, R) := (V \otimes R)/(T \otimes 1 + 1 \otimes \partial)(V \otimes R)$$

is a Lie algebra by

$$[a \otimes r, b \otimes s] = \sum_{j\geqslant 0} a_{(j)}b \otimes \left(\frac{1}{j!}\partial^j r\right) s. \tag{5.7}$$

Proof Since R is a commutative vertex algebra, $V \otimes R$ is a vertex algebra with the translation operator $T \otimes 1 + 1 \otimes \partial$. The assertion follows by applying Lemma 5.5 to the vertex algebra $V \otimes R$. □

The *Borcherds Lie algebra* associated with a vertex algebra V is by definition the Lie algebra

$$\mathrm{Lie}(V) := \mathrm{Lie}(V, \mathbb{C}[t,t^{-1}]) = V \otimes \mathbb{C}[t,t^{-1}]/(T \otimes 1 + 1 \otimes \partial_t)(V \otimes \mathbb{C}[t,t^{-1}]),$$

where $\mathbb{C}[t,t^{-1}]$ is viewed as a differential algebra with the derivation ∂_t. We have

$$[a_{\{m\}}, b_{\{n\}}] = \sum_{j \geqslant 0} \binom{m}{j} (a_{(j)}b)_{\{m+n-j\}}, \tag{5.8}$$

where $a_{\{n\}}$ is the image of $a \otimes t^n \in V \otimes \mathbb{C}[t,t^{-1}]$ in $\mathrm{Lie}(V)$. By definition, we have $(Ta)_{\{n\}} = -na_{\{n-1\}}$.

The following is clear from (2.42).

Lemma 5.7 *Any V-module M is a* $\mathrm{Lie}(V)$*-module by $a_{\{n\}} \mapsto a_{(n)} = a^M_{(n)}$ for $a \in V$, $n \in \mathbb{Z}$.*

Note that a $\mathrm{Lie}(V)$-module needs not to be a V-module since the identities (2.42) and (2.43) may not be satisfied.

Recall that (2.43) is equivalent to the identity (2.36), which contains an infinite sum. In order to make sense of (2.36), we shall introduce a completion $\widetilde{U(\mathrm{Lie}(V))}$ of the the universal enveloping algebra $U(\mathrm{Lie}(V))$ of $\mathrm{Lie}(V)$ as follows.

Assume that V is $\mathbb{Z}$-graded by a Hamiltonian H. Then $\mathrm{Lie}(V)$ is a graded Lie algebra, by defining the action $\operatorname{ad} H$ of H on $\mathrm{Lie}(V)$ by

$$\operatorname{ad} H(a_{\{n\}}) = -(n+1)a_{\{n\}} + (Ha)_{\{n\}}.$$

We have

$$\mathrm{Lie}(V) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Lie}(V)_d, \quad \mathrm{Lie}(V)_d = \{x \in \mathrm{Lie}(V) \colon (\operatorname{ad} H)x = dx\}.$$

Let $U(\mathrm{Lie}(V)) = \bigoplus_{d \in \mathbb{Z}} U(\mathrm{Lie}(V))_d$ be the induced $\mathbb{Z}$-grading on $U(\mathrm{Lie}(V))$.

Define

$$\widetilde{U(\mathrm{Lie}(V))} = \bigoplus_{d \in \mathbb{Z}} \widetilde{U(\mathrm{Lie}(V))}_d,$$
$$\widetilde{U(\mathrm{Lie}(V))}_d = \varprojlim_r U(\mathrm{Lie}(V))_d \Big/ \sum_{p \leqslant r} U(\mathrm{Lie}(V))_{d-p} U(\mathrm{Lie}(V))_p.$$

The space $\widetilde{U(\mathrm{Lie}(V))}$ is a $\mathbb{Z}$-graded topological ring with each component $\widetilde{U(\mathrm{Lie}(V))}_d$ being complete. Now the identity

$$(a_{(m)}b)_{\{n\}} = \sum_{j\geqslant 0}(-1)^j \binom{m}{j} (a_{\{m-j\}}b_{\{n+j\}} - (-1)^m b_{\{m+n-j\}}a_{\{j\}}) \tag{5.9}$$

makes sense as an element of $U(\widetilde{\mathrm{Lie}(V)})$. Let $I = \bigoplus_{d\in\mathbb{Z}} I_d$ be the graded ideal of $U(\widetilde{\mathrm{Lie}(V)})$ generated by (5.9) and $(|0\rangle)_{\{n\}} = \delta_{n,-1}$. Let

$$\mathscr{U}(V) := \bigoplus_{d\in\mathbb{Z}} \mathscr{U}(V)_d, \quad \mathscr{U}(V)_d = U(\widetilde{\mathrm{Lie}(V)})_d/\overline{I}_d,$$

where $\overline{I}_d$ is the closure of I_d in $U(\widetilde{\mathrm{Lie}(V)})_d$. Then $\mathscr{U}(V)$ is again a $\mathbb{Z}$-graded topological ring with each component $\mathscr{U}(V)_d$ being complete, which is called the *universal enveloping algebra* [121] , or the *current algebra* [209] of V.

A $\mathscr{U}(V)$-module M is called *smooth* if the action $\mathscr{U}(V)\times M \to M$ is continuous, where M is equipped with the discrete topology.

Lemma 5.8 *A V-module is the same as a smooth $\mathscr{U}(V)$-module.*

Clearly, $\mathscr{U}(V)_0$ is a subalgebra of $\mathscr{U}(V)$. Define the algebra $A(V)$ by

$$A(V) := \mathscr{U}(V)_0/\overline{\sum_{r>0} \mathscr{U}(V)_r \mathscr{U}(V)_{-r}},$$

where – denotes the closure.

Theorem 5.3 *We have the isomorphism of algebras*

$$\mathrm{Zhu}(V) \cong A(V).$$

Let $\mathscr{U}(V)_{\leqslant 0} = \bigoplus_{d\leqslant 0} \mathscr{U}(V)_d \subset \mathscr{U}(V)$. For an $A(V)$-module E, define the positive energy representation $\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E)$ of V by

$$\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E) := \mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leqslant 0}} E, \tag{5.10}$$

where $\mathscr{U}(V)_{\leqslant 0}$ acts on E by the projection $\mathscr{U}(V)_{\leqslant 0} \to A(V)$.

Assume for awhile that Theorem 5.3 is proven.

Proof (of Theorem 5.2) Let M be a simple positive energy representation V. Then M_{top} is a simple $\mathscr{U}(V)_0$-module on which $\mathscr{U}(V)_{-r}$ acts trivially for $r > 0$. Hence M_{top} is a simple module over $A(V) = \mathrm{Zhu}(V)$. Conversely, let E be a simple $\mathrm{Zhu}(V)$-module. Since $\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E)$ is a positive energy representation of such that $\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E)_{\mathrm{top}} = E$, any proper graded submodule of $\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E)$ intersects E trivially. Indeed, if N is any such proper graded submodule such that $N_{\mathrm{top}} = N\cap E \neq \{0\}$, then for any nonzero element v in the intersection, we have $E = A(V).v \subset N$ since E a simple $\mathrm{Zhu}(V)$-module. But E generates $\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E)$ as V-modules, because $\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E) \cong \mathscr{U}(V)_{>0}.E$, whence $N = \mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E)$. Hence there exists a unique simple graded quotient $L(E)$ of $\mathrm{Ind}_{A(V)}^{\mathscr{U}(V)}(E)$. Clearly, the maps $E \mapsto L(E)$ and $M \mapsto M_{\mathrm{top}}$ are inverse to each other. □

5.3 Proof of Theorem 5.3

This section is devoted to the proof of Theorem 5.3, following [146]. We use the following notation, which is defined for any homogeneous element $a \in V$ and extend linearly to V:

$$J_n(a) := a_{\{\Delta_a - 1 + n\}}.$$

The advantage of this notation is that $J_n(a)$ has always degree $-n$. The proof of the following combinatorial lemma essentially follows from (5.9), see [146, Corollary A.2].

Lemma 5.9 *For all integers s, t, N satisfying $N + s \geqslant 0$, the following identity holds in the universal enveloping algebra of V:*

$$\begin{aligned} J_{-s}(a)J_t(b) &= \sum_{j=0}^{N}\sum_{i\geqslant 0}(-1)^i\binom{N+\Delta_a}{i}\binom{-N-s-1}{j}J_{t-s}(a_{(-N-s-i-j-1)}b) \\ &\quad - \sum_{k\geqslant N+1}\sum_{j=0}^{N}(-1)^j\binom{N+s+j}{j}\binom{N+s-k}{k-j}J_{-k-s}(a)J_{k+t}(b) \\ &\quad + \sum_{j=0}^{N}\sum_{i\geqslant 0}(-1)^{N+s+1}\binom{N+s+j}{j}\binom{N+s+j+i}{i}J_{t-N-s-1-i}(b)J_{N+1+i}(a). \end{aligned}$$

Lemma 5.10 *Every element $a = J_{n_1}(a_1)\dots J_{n_m}(a_m)$ can be expressed in the quotient $A(V)$ as $J_0(v)$ for some $v = v(a)$ in V depending on a.*

Proof We prove the statement by induction on the length m. If $m = 1$, there is nothing to do. Let $m \geqslant 2$, and assume the statement true for every monomial of length $< m$. Apply Lemma 5.9 to $J_{n_{m-1}}(a_{m-1})J_{n_m}(a_m)$, where

$$-s = n_{m-1}, \quad t = n_m, \quad a = a_{m-1}, \quad b = a_m.$$

In Lemma 5.9, choose N big enough so that $\min\{N + n_m, N\} > 0$. Then $J_{k+n_m}(a_m)$ and $J_{N+1+i}(a_{m-1})$ are both contained in $\bigoplus_{j<0}\mathscr{U}(V)_j$ for $k \geqslant N+1$, and so a is congruent to a linear combination of the following terms with length $< m$:

$$J_{n_1}(a_1)\dots J_{n_{m-2}}(a_{m-2})J_{n_{m-1}+n_m}((a_{m-1})_{(-N+n_{m-1}-i-j-1)}a_m).$$

By induction, these terms are congruent to monomials of the form $J_0(v')$, $v' \in V$. So a is itself congruent to some monomial $J_0(v)$. Here, notice that for any $n \in \mathbb{Z}$, $a, b \in V$, we have $J_n(a) + J_n(b) = J_n(a+b)$. □

We are now in a position to prove Theorem 5.3. Let φ be the composition map of the linear map from V to $\mathscr{U}(V)_0$ sending homogeneous element a to $a_{\{\Delta_a-1\}}$ with the canonical quotient map from $\mathscr{U}(V)_0$ to $A(V)$. Lemma 5.10 ensures that this map is surjective.

Let us show now that φ factors through $\mathrm{Zhu}(V)$, that is,

$$\varphi(V\circ V)\subset\overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}}.$$

Let a, b be homogeneous elements $a, b\in V$. We have $\Delta_{a_{(i-2)}b}=\Delta_a+\Delta_b-i+1$. Using the identity (5.9), we get

$$\begin{aligned}\varphi(a\circ b)&=\sum_{i\geqslant 0}\binom{\Delta_a}{i}(a_{(i-2)}b)_{\{\Delta_a+\Delta_b-i\}}\\&=\sum_{i\geqslant 0}(-1)^i\binom{-2}{i}(a_{\{\Delta_a-2-i\}}b_{\{\Delta_b+i\}}-b_{\{\Delta_b-2-i\}}a_{\{\Delta_a+i\}})\\&=\sum_{i\geqslant 0}(-1)^i\binom{-2}{i}(J_{-i-1}(a)J_{i+1}(b)-J_{-i-1}(b)J_{i+1}(a)).\end{aligned}$$

Since $\deg J_{i+1}(b)=\deg J_{i+1}(a)=-i-1<0$, we get that

$$\varphi(a\circ b)\in\overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}},$$

whence the statement. As a result, we get a well-defined map, still denoted by φ, from $\mathrm{Zhu}(V)$ to $A(V)$ which is surjective.

Next, we prove that φ is an algebra homomorphism. It is enough to show that $\varphi(a*b)=\varphi(a)\varphi(b)$ for homogeneous elements $a, b\in V$. Again using the identity (5.9), we get

$$\begin{aligned}\varphi(a*b)&=\sum_{i\geqslant 0}\binom{\Delta_a}{i}(a_{(i-1)}b)_{\{\Delta_a+\Delta_b-i-1\}}\\&=\sum_{i\geqslant 0}(-1)^i\binom{-1}{i}(a_{\{\Delta_a-1-i\}}b_{\{\Delta_b-1+i\}}+b_{\{\Delta_b-2-i\}}a_{\{\Delta_a+i\}})\\&=\sum_{i\geqslant 0}(-1)^i\binom{-2}{i}(J_{-i}(a)J_i(b)+J_{-i-1}(b)J_{i+1}(a))\\&=J_0(a)J_0(b)\quad(\mathrm{mod}\ \overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}}).\end{aligned}$$

On the other hand, by letting $s=t=N=0$ in Lemma 5.10, we have

$$\begin{aligned} J_0(a)J_0(b) &= \sum_{i\geqslant 0}(-1)^i\binom{\Delta_a}{i}J_0(a_{(-i-1)}b) \pmod{\overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}}} \\ &= \sum_{i\geqslant 0}(-1)^i\binom{\Delta_a}{i}(a_{(-i-1)}b)_{\{\Delta_a+\Delta_b+i-1\}} \pmod{\overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}}}, \end{aligned}$$

whence $\varphi(a*b) = \varphi(a)\varphi(b)$ in $A(V)$.

It remains to construct an inverse map for φ. By Lemma 5.10 every element of $A(V)$ can be expressed as $J_0(a) + \pmod{\overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}}}$. We want to define a map ψ from $A(V)$ to $\mathrm{Zhu}(V)$ sending $J_0(a) + \pmod{\overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}}}$ to $a + V\circ V$. Once we can show this, it is clear that ψ and φ are inverse to each other. The well-definedness requires that whenever $J_0(a) \in \pmod{\overline{\sum_{r>0}\mathscr{U}(V)_r\mathscr{U}(V)_{-r}}}$, then $a \in V\circ V$. This can be shown using the functor, denoted by L^0, constructed by Dong, Li and Mason [90].

5.4 Relations between the Zhu algebra and the Zhu C_2-algebra

We define an increasing filtration of the Zhu algebra. For this, we assume that V is $\mathbb{Z}_{\geqslant 0}$-graded, that is, $V = \bigoplus_{\Delta\geqslant 0} V_\Delta$. Then $V_{\leqslant p} := \bigoplus_{\Delta=0}^{p} V_\Delta$ gives an increasing filtration of V. Define

$$\mathrm{Zhu}_p(V) := \mathrm{im}(V_{\leqslant p} \to \mathrm{Zhu}(V)).$$

Obviously, we have

$$0 = \mathrm{Zhu}_{-1}(V) \subset \mathrm{Zhu}_0(V) \subset \mathrm{Zhu}_1(V) \subset \cdots, \quad \text{and} \quad \mathrm{Zhu}(V) = \bigcup_{p\geqslant -1}\mathrm{Zhu}_p(V).$$

Also, since $a_{(n)}b \in V_{\Delta_a+\Delta_b-n-1}$ for $a \in V_{\Delta_a}$, $b \in V_{\Delta_b}$, we have

$$\mathrm{Zhu}_p(V) * \mathrm{Zhu}_q(V) \subset \mathrm{Zhu}_{p+q}(V). \tag{5.11}$$

The following assertion follows from the skew symmetry.

By Lemma 5.3, we have

$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(V)] \subset \mathrm{Zhu}_{p+q-1}(V). \tag{5.12}$$

This means that the filtered associative algebra $\mathrm{Zhu}(V)$ is *almost-commutative* (see Section B.3). By (5.11) and (5.12) the associated graded space,

$$\mathrm{gr}\,\mathrm{Zhu}(V) = \bigoplus_{p\geqslant 0}\mathrm{Zhu}_p(V)/\mathrm{Zhu}_{p-1}(V),$$

is so naturally a graded Poisson algebra (see Section B.3).

Our next focus is to explore the connections between the Zhu algebra and the Zhu C_2-algebra or, equivalently, between the Poisson schemes $\tilde{X}_V$ and $\operatorname{Spec} \operatorname{gr} \operatorname{Zhu}(V)$. First, note that $a \circ b \equiv a_{(-2)} b \pmod{\bigoplus_{\Delta \leqslant \Delta_a + \Delta_b} V_\Delta}$ for homogeneous elements a, b in V.

We can now establish a link between the Zhu's and the Zhu C_2-algebras ([87, Proposition 2.17(c)] and [31, Proposition 3.3]).

Lemma 5.11 *The following map defines a well-defined surjective Poisson algebra homomorphism:*

$$\begin{aligned} \eta_V : R_V &\longrightarrow \operatorname{gr} \operatorname{Zhu}(V) \\ \bar{a} &\longmapsto a \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta}. \end{aligned}$$

Proof We have $a \circ b = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} b = a_{(-2)} b + \sum_{i \geqslant 1} \binom{\Delta_a}{i} a_{(i-2)} b$. Since the degree of $a_{(i-2)} b$ is $\Delta_a + \Delta_b + 1 - i < \deg a_{(-2)} b$ if $i \geqslant 1$, we get that

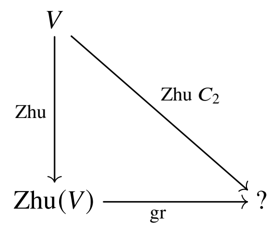
$$a_{(-2)} b = a \circ b \pmod{\bigoplus_{\Delta < \Delta_{a_{(-2)} b}} V_\Delta}.$$

This shows that $C_2(V)$ is contained in $V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta$ and, hence, η_V is well-defined. Clearly, it is surjective. It remains to show that η_V is an algebra homomorphism. But $\eta_V(\bar{a}.\bar{b}) = \eta_V(\overline{a_{(-1)} b}) = a_{(-1)} b \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a + \Delta_b} V_\Delta}$ while the image of $a * b$ in $\operatorname{gr} \operatorname{Zhu}(V)$ is $a_{(-1)} b \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a + \Delta_b} V_\Delta}$ since the degree of $a_{(i-1)} b$ is $\Delta_a + \Delta_b - i < \Delta_a + \Delta_b$ for $i \geqslant 1$. $\square$

Although the map η_V is an isomorphism in several examples, e.g., the universal affine vertex algebra $V^k(\mathfrak{g})$ (cf. §5.6.2), the fermion Fock space (cf. §5.6.3), the *universal W-algebra* $\mathscr{W}^k(\mathfrak{g}, f)$ (cf. 9.2.1), etc., it is not true an isomorphism in general.

Example 5.1 Let $\mathfrak{g}$ be the simple Lie algebra of type E_8 and $V = L_1(\mathfrak{g})$. Then $\dim R_V > \dim \operatorname{Zhu}(V) = 1$. This counter-example was discovered by Gaberdiel and Gannon[124] [1]. In other words, the diagram

[1] The equality $\dim \operatorname{Zhu}(V) = 1$ follows from the fact that $L_1(\mathfrak{g})$ is *holomorphic*, that is, the only simple module is itself, because it is the only integrable affine $\mathfrak{g}$-module of level 1. Because Zhu's algebra of any holomorphic vertex operator algebra is one-dimensional, we get that $\operatorname{gr} \operatorname{Zhu}(V) \cong \mathbb{C}$. On the other hand, it is easy to check that $\dim R_V > 1$ since the unique proper maximal submodule of $V^1(\mathfrak{g})$ is generated by $(e_\theta t^{-1} |0\rangle)^2$: see Section 11.1.



is not always commutative.

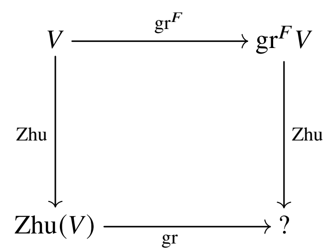
Remark 5.2 It was shown in [107] that $R_V \cong \operatorname{gr}\operatorname{Zhu}(V)$ for $V = L_k(\mathfrak{g})$ for any nonnegative $k \in \mathbb{Z}_{\geqslant 0}$, if $\mathfrak{g}$ is the simple Lie algebra $\mathfrak{sl}_n$.

By Lemma 5.11, we have $\operatorname{Specm}(\operatorname{gr}\operatorname{Zhu}(V)) \subset X_V$. The stronger fact in conjectured in [16].

Conjecture 5.1 If V is a simple $\mathbb{Z}_{\geqslant 0}$-graded conformal vertex algebra, then

$$X_V \cong \operatorname{Specm}(\operatorname{gr}\operatorname{Zhu}(V)).$$

Remark 5.3 One may also ask when the following diagram is commutative.



In other words, one may ask when one has $\operatorname{Zhu}(\operatorname{gr}^F V) \cong \operatorname{gr}\operatorname{Zhu}(V)$. Note that R_V is not a priori isomorphic to $\operatorname{Zhu}(\operatorname{gr}^F V)$ since $C_2(V) \neq V \circ V$ even for commutative vertex algebras.

Exercise 5.1 Verify that the example $V = L_1(\mathfrak{g})$, with $\mathfrak{g}$ simple of type E_8 as in Example 5.1, furnishes an example of vertex algebra V such that $\operatorname{Zhu}(\operatorname{gr}^F V) \ncong \operatorname{gr}\operatorname{Zhu}(V)$.

? Open problem

Is there an example of a vertex algebra for which $R_V \ncong \operatorname{Zhu}(\operatorname{gr}^F V)$?

Corollary 5.1 *If V is lisse then* $\operatorname{Zhu}(V)$ *is finite dimensional. Hence the number of isomorphic classes of simple positive energy representations of V is finite.*

5.5 Filtration of current algebra

We continue to assume that V is $\mathbb{Z}_{\geqslant 0}$-graded.

Recall the increasing, conformal weight filtration $G_\bullet V$ (Section 4.7). This induces the increasing filtration $G_\bullet \operatorname{Lie}(V)$ of $\operatorname{Lie}(V)$ such that

$$[G_p \operatorname{Lie}(V), G_q \operatorname{Lie}(V)] \subset G_{p+q-1} \operatorname{Lie}(V),$$

where $G_p \operatorname{Lie}(V)$ is the image of $G_p V \otimes \mathbb{C}[t, t^{-1}]$ in $\operatorname{Lie}(V)$. Hence, $\operatorname{gr}_G \operatorname{Lie}(V) = \bigoplus_p G_p \operatorname{Lie}(V)/G_{p-1} \operatorname{Lie}(V)$ is naturally a commutative Lie algebra. On the other hand, $\operatorname{Lie}(\operatorname{gr}_G V)$ is also a commutative Lie algebra since $\operatorname{gr} V$ is commutative.

Lemma 5.12 *There is a surjective Lie algebra homomorphism*

$$\operatorname{Lie}(\operatorname{gr}_G V) \longrightarrow \operatorname{gr}_G \operatorname{Lie}(V)$$

that sends $\sigma_p(a)_{\{n\}}$ *to* $\sigma_p(a_{\{n\}})$.

Proof The map obtained by composing the quotient map $V \otimes \mathbb{C}[t, t^{-1}] \to \operatorname{Lie}(V)$ with the quotient map $\operatorname{Lie}(V) \to \operatorname{gr}_G \operatorname{Lie}(V)$ is clearly surjective, and it factorizes through the composition map $V \otimes \mathbb{C}[t, t^{-1}] \to \operatorname{gr}_G V \otimes \mathbb{C}[t, t^{-1}] \to \operatorname{Lie}(\operatorname{gr}_G V)$ since any element $T\sigma_p(a)_{\{n\}} + n\sigma_p(a)_{\{n-1\}}$ of $(T \otimes 1 + 1 \otimes \partial_t)\operatorname{gr}_G V$ is mapped to the element $\sigma_{p+1}(Ta_{\{n\}}) + n\sigma_p(a_{\{n-1\}})$ of $(T \otimes 1 + 1 \otimes \partial_t)G_p V$ for $a \in G_p V \setminus G_{p-1} V$, $n \in \mathbb{Z}$. It remains to verify that the resulting surjective map is a Lie algebra homomorphism, but this is clear from (5.8). (Note that both Lie algebras are commutative.) □

The filtration $G_\bullet \operatorname{Lie}(V)$ induces a filtration $G_\bullet U(\operatorname{Lie}(V))$ of the universal enveloping algebra $U(\operatorname{Lie}(V))$. This in turn induces a filtration $G_\bullet \mathscr{U}(V)$ of the current algebra $\mathscr{U}(V)$, where $G_p \mathscr{U}(V)$ is the closure of the image of $G_p U(\operatorname{Lie}(V))$ in $\mathscr{U}(V)$. Let $\operatorname{gr}_G \mathscr{U}(V) = \bigoplus_p G_p \mathscr{U}(V)/G_{p-1}\mathscr{U}(V)$ be the associated graded topological algebra.

Lemma 5.12 immediately gives the following result:

Lemma 5.13 *There is a surjective algebra homomorphism*

$$\mathscr{U}(\operatorname{gr}_G V) \longrightarrow \operatorname{gr}_G \mathscr{U}(V).$$

The surjection in Proposition 4.5 induces a surjection $\mathscr{O}(\mathscr{L}\tilde{X}_V) \to \mathscr{U}(\operatorname{gr}_G V)$. Thus by Lemma 5.13 we have a surjection

$$\mathscr{O}(\mathscr{L}\tilde{X}_V) \twoheadrightarrow \operatorname{gr}_G \mathscr{U}(V). \tag{5.13}$$

The following stronger fact is known ([2]).

Theorem 5.4 (Abe–Buhl–Dong) *Let V be a strongly finitely generated conformal lisse vertex algebra. Then any simple V-module is an ordinary positive energy representation. Therefore the number of isomorphic classes of simple V-modules is finite.*

Proof Let $m \in M \setminus \{0\}$. Then $M = \mathscr{U}(V)m$ since M is simple. Define an increasing filtration $G_p M$ by setting $G_p M = G_p \mathscr{U}(V)m$. Then $\mathrm{gr}_G M = \bigoplus_p G_p M / G_{p-1} M$ is naturally a module over $\mathrm{gr}_G \mathscr{U}(V)$, and hence over $\mathscr{O}(\mathscr{L}\tilde{X}_V)$. By construction, we have $\mathrm{gr}_G M = \mathscr{O}(\mathscr{L}\tilde{X}_V)\bar{m}$, where $\bar{m}$ is the image of m in $\mathrm{gr}_G M$.

Let us first now that the L_0-module $\mathrm{span}_{\mathbb{C}}\{L_0^n m \colon n \in \mathbb{Z}_{\geqslant 0}\}$ generated by m is finite-dimensional. Let $\{a^i : i \in I\}$ be a finite set of strong generators of V.

Since $\mathrm{gr}_G M = \mathscr{O}(\mathscr{L}\tilde{X}_V)\bar{m}$, there is $A_0 \in \mathscr{O}(\mathscr{L}\tilde{X}_V)$ such that $\overline{L_0^n m} = A_0^n \bar{m}$. By Lemma 4.11, the images of the $a^i_{(n)}$'s in R_V are nilpotent in $\mathscr{O}(\mathscr{L}\tilde{X}_V)$, whence $A_0^n \bar{m} = 0$ for sufficiently large n. As a result, $L_0^n \bar{m} = 0$ for sufficiently large n too, and so $\mathrm{span}_{\mathbb{C}}\{L_0^n m \colon n \in \mathbb{Z}_{\geqslant 0}\}$ is finite-dimensional. This proves that the action of L_0 on M is locally finite. Therefore, M is a direct sum of generalized eigenspaces,

$$M = \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda\,\mathrm{Id})^{n_\lambda} \supset \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda\,\mathrm{Id}) =: M'.$$

Using (2.44) for $H = L_0$, we easily verify that M' is a vertex submodule of M which is nonzero since the action of L_0 is locally finite. Hence, $M = M'$ which proves that M is L_0-graded.

Let us now show that M is positively graded. We may assume that m is an L_0-eigenvector of weight $\lambda \in \mathbb{C}$. Notice that the L_0-weight of $a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)} m_0$ is

$$\lambda + \Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_r}} - n_1 - \cdots - n_r - r. \tag{5.14}$$

Since M is smooth and I is finite, there is $N > 0$ such that for all $n \geqslant N$ and all $i \in I$, $a^i_{(n)} m = 0$. Furthermore using again Lemma 4.11, we deduce that $(\bar{a}^{i_1}_{(n_1)})^{l_1} \dots (\bar{a}^{i_r}_{(n_r)})^{l_r}.m = 0$ in $\mathrm{gr}_G M$ if $n_j \geqslant N$ and l_j large enough for $j = 1, \dots, r$, whence the statement by (5.14).

It remains to prove that each graded component $M_{\lambda+n}$ is finite-dimensional. Since $M_\lambda \neq 0$, we may assume that $m \in M_\lambda$. A L_0-weight space in $\mathrm{gr}_G M$ is generated by some $(\bar{a}^{i_1}_{(n_1)})^{t_1} \dots (\bar{a}^{i_r}_{(n_r)})^{t_r} \bar{m}$, with $(\bar{a}^{i_1}_{(n_1)})^{t_1} \dots (\bar{a}^{i_r}_{(n_r)})^{t_r} \in \mathscr{O}(\mathscr{L}\tilde{X}_V)$. Since each $\bar{a}^i_{(n)}$ is nilpotent in $\mathscr{O}(\mathscr{L}\tilde{X}_V)$ and I is finite, each L_0-weight space is finite-dimensional. ◻

Moreover, we have the following statement ([89, 209]).

Theorem 5.5 *Let V be lisse. Then the abelian category of V-modules is equivalent to the module category of a finite-dimensional associative algebra.*

5.6 Computation of Zhu's algebras

This section describes some technics to compute the Zhu algebra, and contains some explicit examples.

5.6.1 PBW basis

Recall that a vertex algebra V admits a PBW basis if R_V is a polynomial algebra and if the map $\mathbb{C}[\mathcal{J}_\infty(X_V)] \twoheadrightarrow \mathrm{gr}^F V$ is an isomorphism (cf. Definition 4.9). Vertex algebras satisfying the second condition, $\mathcal{J}_\infty R_V \cong \mathrm{gr}^F V$, are called *classically free* in [95].

Theorem 5.6 *If V admits a PBW basis, then $\eta_V : R_V \twoheadrightarrow \mathrm{gr}(\mathrm{Zhu} V)$ is an isomorphism.*

Proof We have $\mathrm{gr}\,\mathrm{Zhu}(V) = V/\mathrm{gr}(V \circ V)$, where $\mathrm{gr}(V \circ V)$ is the associated graded space of $V \circ V$ with respect to the filtration induced by the filtration $V_{\leqslant p}$. We wish to show that $\mathrm{gr}(V \circ V) = F^1 V$. Since $a \circ b \equiv a_{(-2)} b \pmod{V_{\leqslant \Delta_a + \Delta_b}}$ for homogeneous $a, b \in V$, it is sufficient to show that $a \circ b \neq 0$ implies that $a_{(-2)} b \neq 0$.

Suppose that $a_{(-2)} b = (Ta)_{(-1)} b = 0$ for homogeneous $a, b \in V$. Since V admits a PBW basis, $\mathrm{gr}^F V$ has no zero divisors, whence $Ta = 0$. Also, from the PBW property we find that $Ta = 0$ implies that $a = c|0\rangle$ for some constant $c \in \mathbb{C}$. Thus, a is a constant multiple of $|0\rangle$, in which case $a \circ b = 0$. ◻

5.6.2 Universal affine vertex algebras

The universal affine vertex algebra $V^k(\mathfrak{g})$ admits a PBW basis. Therefore

$$\eta_{V^k(\mathfrak{g})} : R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*] \xrightarrow{\sim} \mathrm{gr}\,\mathrm{Zhu}(V^k(\mathfrak{g})).$$

On the other hand, from Lemma 5.3 one finds that

$$\begin{aligned} U(\mathfrak{g}) &\longrightarrow \mathrm{Zhu}(V^k(\mathfrak{g})) \\ \mathfrak{g} \ni x &\longmapsto [x_{(-1)}|0\rangle] \end{aligned} \tag{5.15}$$

gives a well-defined algebra homomorphism. This map respects the filtration on both sides, where the filtration in the left side is the PBW filtration. Hence it induces a map between their associated graded algebras, which is identical to $\eta_{V^k(\mathfrak{g})}$. Therefore (5.15) is an isomorphism, that is to say, $V^k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})$.

Exercise 5.2 Extend Theorem 5.6 to the case where $\mathfrak{g}$ is a Lie superalgebra.

Theorem 5.2 gives the following in this example. The top degree component of the irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight λ is $L_{\mathfrak{g}}(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of λ to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Let $N_k = N_k(\mathfrak{g})$ be the maximal ideal of $V^k(\mathfrak{g})$ as in Example 3.2 so that

$$L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k,$$

where $L_k(\mathfrak{g})$ is the unique graded quotient of $V^k(\mathfrak{g})$. We have the exact sequence $J_k \to U(\mathfrak{g}) \to \mathrm{Zhu}(L_k(\mathfrak{g})) \to 0$, where J_k is the image of N_k in $\mathrm{Zhu}(V) = U(\mathfrak{g})$

through the compound map $N_k \hookrightarrow V \twoheadrightarrow \mathrm{Zhu}(V)$, and thus

$$\mathrm{Zhu}(L_k(\mathfrak{g})) = U(\mathfrak{g})/J_k.$$

Hence when the homomorphism $\eta_{L_k(\mathfrak{g})}$ of Lemma 5.11 is an isomorphism, the associated variety $X_{L_k(\mathfrak{g})}$ can be viewed as an analog of associated varieties of primitive ideals (see Section D.5). However, there are substantial differences (see, for instance, Example 12.1). In general, it is a hard problem to compute N_k and I_k.

5.6.3 Free fermions

Let $\mathfrak{n}$ be a finite-dimensional vector space. We refer to Appendix E for basics on superalgebras and Clifford algebras.

Consider the *Clifford algebra* Cl associated with the vector space $\mathfrak{n} \oplus \mathfrak{n}^*$ and the non-degenerate bilinear forms $\langle - | - \rangle$ defined by $\langle \phi + x | \psi + y \rangle = \phi(y) + \psi(x)$ for $\phi, \psi \in \mathfrak{n}^*$, $x, y \in \mathfrak{n}$. Specifically, Cl is the unital $\mathbb{C}$-superalgebra that is isomorphic to $\bigwedge(\mathfrak{n}) \otimes \bigwedge(\mathfrak{n}^*)$ as $\mathbb{C}$-vector spaces, and

$$[x, \phi] = \phi(x), \qquad x \in \mathfrak{n} \subset \textstyle\bigwedge(\mathfrak{n}),\ \phi \in \mathfrak{n}^* \subset \bigwedge(\mathfrak{n}^*).$$

(Note that $[x, \phi] = x\phi + \phi x$ since x, ϕ are odd.) Define an increasing filtration on Cl by setting $Cl_p := \bigwedge^{\leqslant p}(\mathfrak{n}) \otimes \bigwedge(\mathfrak{n}^*)$. We have

$$0 = Cl_{-1} \subset Cl_0 \subset Cl_1 \subset \cdots \subset Cl_N = Cl,$$

where $N = \dim \mathfrak{n}$, and

$$Cl_p.Cl_q \subset Cl_{p+q}, \qquad [Cl_p, Cl_q] \subset Cl_{p+q-1}.$$

As a consequence, the associated graded algebra,

$$\overline{Cl} := \mathrm{gr}\, Cl = \bigoplus_{p \geqslant 0} \frac{Cl_p}{Cl_{p+1}},$$

is naturally a graded Poisson superalgebra. We have $\overline{Cl} = \bigwedge(\mathfrak{n}) \otimes \bigwedge(\mathfrak{n}^*)$ as a commutative superalgebra, and its Poisson (super)bracket is given by:

$$\{x, \phi\} = \phi(x), \quad \{x, y\} = 0, \quad \{\phi, \psi\} = 0, \quad x, y \in \mathfrak{n} \subset \textstyle\bigwedge(\mathfrak{n}),\ \phi, \psi \in \mathfrak{n}^* \subset \bigwedge(\mathfrak{n}^*).$$

The charged fermion Fock space

The *Clifford affinization* $\hat{Cl}$ of $\mathfrak{n}$ is the Clifford algebra associated with $\mathfrak{n}[t,t^{-1}] \oplus \mathfrak{n}^*[t,t^{-1}]$ and its symmetric bilinear form defined by

$$\langle xt^m | ft^n\rangle = \delta_{m+n,0} f(x), \quad \langle xt^m | yt^n\rangle = 0 = \langle ft^m | gt^n\rangle$$

for $x, y \in \mathfrak{n}$, $f, g \in \mathfrak{n}^*$, $m, n \in \mathbb{Z}$.

Let $\{x_i\}_{1\leqslant i\leqslant s}$ be a basis of $\mathfrak{n}$, and $\{x_i^*\}_{1\leqslant i\leqslant s}$ its dual basis. We write $\psi_{i,m}$ for $x_i t^m \in \hat{Cl}$ and $\psi^*_{i,m}$ for $x_i^* t^m \in \hat{Cl}$, so that $\hat{Cl}$ is the associative superalgebra with

- odd generators: $\psi_{i,m}, \psi^*_{i,m}$, $m \in \mathbb{Z}$, $i = \{1, \dots, s\}$,
- relations: $[\psi_{i,m}, \psi_{j,n}] = [\psi^*_{i,m}, \psi^*_{j,n}] = 0$, $[\psi_{i,m}, \psi^*_{j,n}] = \delta_{i,j}\delta_{m+n,0}$.

Define the *charged fermion Fock space* associated with $\mathfrak{n}$ as

$$\mathcal{F}(\mathfrak{n}) := \frac{\hat{Cl}}{\sum\limits_{\substack{m\geqslant 0\\ 1\leqslant i\leqslant s}} \hat{Cl}\psi_{i,m} + \sum\limits_{\substack{k\geqslant 1\\ 1\leqslant j\leqslant s}} \hat{Cl}\psi^*_{j,k}} \cong \bigwedge\left(\psi_{i,n}\right)_{\substack{n<0\\ 1\leqslant i\leqslant s}} \otimes \bigwedge\left(\psi^*_{j,m}\right)_{\substack{m\leqslant 0\\ 1\leqslant j\leqslant s}},$$

where $\bigwedge(a_i)_{i\in I}$ denotes the exterior algebra with generators a_i, $i \in I$. It is an irreducible $\hat{Cl}$-module, and as $\mathbb{C}$-vector spaces we have

$$\mathcal{F}(\mathfrak{n}) \cong \bigwedge(\mathfrak{n}^*[t^{-1}]) \otimes \bigwedge(\mathfrak{n}[t^{-1}]t^{-1}).$$

There is a unique vertex (super)algebra structure on $\mathcal{F}(\mathfrak{n})$ such that the image of 1 is the vacuum $|0\rangle$ and

$$Y(\psi_{i,-1}|0\rangle, z) = \psi_i(z) := \sum_{n\in\mathbb{Z}} \psi_{i,n} z^{-n-1}, \quad i = 1, \dots, s,$$
$$Y(\psi^*_{i,0}|0\rangle, z) = \psi^*_i(z) := \sum_{n\in\mathbb{Z}} \psi^*_{i,n} z^{-n}, \quad i = 1, \dots, s.$$

We have $F^1\mathcal{F}(\mathfrak{n}) = \mathfrak{n}^*[t^{-1}]t^{-1}\mathcal{F}(\mathfrak{n}) + \mathfrak{n}[t^{-1}]t^{-2}\mathcal{F}(\mathfrak{n})$, and it follows that there is an isomorphism

$$\begin{aligned} \overline{Cl} &\xrightarrow{\sim} R_{\mathcal{F}(\mathfrak{n})}, \\ x_i &\longmapsto \overline{\psi_{i,-1}|0\rangle}, \\ x_i^* &\longmapsto \overline{\psi^*_{i,0}|0\rangle} \end{aligned}$$

as Poisson superalgebras. Thus,

$$X_{\mathcal{F}(\mathfrak{n})} = T^*(\Pi\mathfrak{n}),$$

where $\Pi\mathfrak{n}$ is the space $\mathfrak{n}$ considered as a purely odd affine space. Its arc space $\mathcal{J}_\infty(T^*(\Pi\mathfrak{n}))$ is also regarded as a purely odd affine space, such that

$$\mathbb{C}[\mathcal{J}_\infty(T^*(\Pi\mathfrak{n}))] = \wedge(\mathfrak{n}^*[t^{-1}]) \otimes \wedge(\mathfrak{n}[t^{-1}]t^{-1}).$$

The map $\mathbb{C}[\mathcal{J}_\infty(X_{\mathcal{F}(\mathfrak{n})})] \to \operatorname{gr}\mathcal{F}(\mathfrak{n})$ is an isomorphism and, hence, $\mathcal{F}(\mathfrak{n})$ admits a PBW basis. Therefore we have the isomorphism

$$\eta_{\mathcal{F}(\mathfrak{n})}\colon R_{\mathcal{F}(\mathfrak{n})} = \overline{Cl} \xrightarrow{\sim} \operatorname{gr}\operatorname{Zhu}(\mathcal{F}(\mathfrak{n}))$$

by Exercise 5.2. On the other hand the map

$$\begin{aligned} Cl &\longrightarrow \operatorname{Zhu}(\mathcal{F}(\mathfrak{n})) \\ x_i &\longmapsto \overline{\psi_{i,-1}|0\rangle}, \\ x_i^* &\longmapsto \overline{\psi_{i,0}^*|0\rangle} \end{aligned}$$

gives an algebra homomorphism that respects the filtration. Hence we have

$$\operatorname{Zhu}(\mathcal{F}(\mathfrak{n})) \cong Cl.$$

That is, $\mathcal{F}(\mathfrak{n})$ is a chiralization of Cl.

Chapter 6
Poisson vertex modules and their associated variety

In this chapter we give the definition of a Poisson vertex modules over a Poisson vertex algebra and we study some their properties. This notion will be useful to construct new Poisson vertex algebras in Section 9.1 by applying the BRST reduction.

6.1 Poisson vertex modules

Definition 6.1 A *Poisson vertex module* over a Poisson vertex algebra V is a V-module M in the usual sense of vertex V-module, equipped with a linear map

$$V \mapsto (\operatorname{End} M)[[z^{-1}]]z^{-1}, \quad a \mapsto Y_-^M(a,z) = \sum_{n \geqslant 0} a_{(n)}^M z^{-n-1},$$

satisfying

$$a_{(n)}^M m = 0 \quad \text{for} \quad n \gg 0, \tag{6.1}$$

$$(Ta)_{(n)}^M = -n a_{(n-1)}^M, \tag{6.2}$$

$$a_{(n)}^M (bv) = (a_{(n)}^M b)v + b(a_{(n)}^M v), \tag{6.3}$$

$$[a_{(m)}^M, b_{(n)}^M] = \sum_{i \geqslant 0} \binom{m}{i} (a_{(i)} b)_{(m+n-i)}^M, \tag{6.4}$$

$$(ab)_{(n)}^M = \sum_{i=0}^{\infty} (a_{(-i-1)} b_{(n+i)}^M + b_{(-i-1)} a_{(n+i)}^M) \tag{6.5}$$

for all $a, b \in V$, $m, n \geqslant 0$, $v \in M$.

A Poisson vertex algebra V is naturally a Poisson vertex module over itself.

Example 6.1 Let M be a Poisson vertex module over $\mathscr{O}(\mathcal{J}_\infty(\mathfrak{g}^*))$. Then by (6.4), the assignment

$$xt^n \mapsto x^M_{(n)}, \qquad x \in \mathfrak{g} \subset \mathscr{O}(\mathfrak{g}^*) \subset \mathscr{O}(J_\infty(\mathfrak{g}^*)),\ n \geqslant 0,$$

defines a $J_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$-module structure on M. In fact, a Poisson vertex module over $\mathscr{O}(J_\infty(\mathfrak{g}^*))$ is the same as a $\mathscr{O}(J_\infty(\mathfrak{g}^*))$-module M in the usual associative sense equipped with an action of the Lie algebra $J_\infty(\mathfrak{g})$ such that $(xt^n)m = 0$ for $n \gg 0$, $x \in \mathfrak{g}$, $m \in M$, and

$$(xt^n) \cdot (am) = (x_{(n)}a) \cdot m + a(xt^n) \cdot m$$

for $x \in \mathfrak{g}$, $n \geqslant 0$, $a \in \mathscr{O}(J_\infty(\mathfrak{g}^*))$, $m \in M$.

Below we often write $a_{(n)}$ for $a^M_{(n)}$.

The proofs of the following assertions are straightforward. (We refer to §B.6 for the definition of *Poisson modules*.)

Lemma 6.1 *Let R be a Poisson algebra, E a Poisson module over R. There is a unique Poisson vertex $J_\infty(R)$-module structure on $J_\infty(R) \otimes_R E$ such that*

$$a_{(n)}(b \otimes m) = (a_{(n)}b) \otimes m + \delta_{n,0} b \otimes \{a, m\}$$

for $n \geqslant 0$, $a \in R \subset J_\infty(R)$, $b \in J_\infty(R)$, $m \in E$.

Lemma 6.2 *Let R be a Poisson algebra, M a Poisson vertex module over $J_\infty(R)$. Suppose that there exists a R-submodule E of M (in the usual commutative sense) such that $a_{(n)}E = 0$ for $n > 0$, $a \in R$, and that M is generated by E (in the usual commutative sense). Then there exists a surjective homomorphism*

$$J_\infty(R) \otimes_R E \twoheadrightarrow M$$

of Poisson vertex modules.

6.2 Canonical filtration of modules over vertex algebras

Let V be a vertex algebra graded by a Hamiltonian H. A *compatible filtration* of a V-module M is a decreasing filtration

$$M = \Gamma^0 M \supset \Gamma^1 M \supset \cdots$$

such that

$$\begin{aligned}
&a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n-1} M \quad \text{for } a \in F^p V,\ \forall n \in \mathbb{Z},\\
&a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n} M \quad \text{for } a \in F^p V,\ n \geqslant 0,\\
&H.\Gamma^p M \subset \Gamma^p M \quad \text{for all } p \geqslant 0,\\
&\bigcap_p \Gamma^p M = 0.
\end{aligned}$$

For a compatible filtration $\Gamma^\bullet M$, the associated graded space

$$\mathrm{gr}^\Gamma M = \bigoplus_{p\geqslant 0} \Gamma^p M/\Gamma^{p+1} M$$

is naturally a graded Poison vertex module over the graded Poisson vertex algebra $\mathrm{gr}^F V$, and hence, it is a graded Poison vertex module over $\mathcal{J}_\infty(R_V)$ by Theorem 4.1 and Proposition 4.5.

The Poisson vertex $\mathcal{J}_\infty(R_V)$-module structure of $\mathrm{gr}^\Gamma M$ restricts to a Poisson R_V-module structure of $M/\Gamma^1 M = \Gamma^0 M/\Gamma^1 M$, and $a_{(n)}(M/\Gamma^1 M) = 0$ for $a \in R_V \subset \mathcal{J}_\infty(R_V)$, $n > 0$. It follows that there is a homomorphism

$$\mathcal{J}_\infty(R_V) \otimes_{R_V} (M/\Gamma^1 M) \to \mathrm{gr}^\Gamma M, \quad a \otimes \bar{m} \mapsto a\bar{m},$$

of Poisson vertex modules by Lemma 6.2.

Let $\{a^i : i \in I\}$ be a set of strong generators of V. Set

$$F^p M = \mathrm{span}_{\mathbb{C}}\{a^1_{(-n_1-1)} \dots a^r_{(-n_r-1)} m \colon a^i \in V,\ m \in M,\ n_1 + \dots + n_r \geqslant p\}.$$

Proposition 6.1 *Let M be a vertex V-module. The filtration $F^\bullet M$ is a compatible filtration of M. In fact, it is the finest compatible filtration of M, that is, $F^p M \subset \Gamma^p M$ for all p for any compatible filtration $\Gamma^\bullet M$ of M. In particular, $F^\bullet M$ is independent of the choice of strong generators.*

The filtration $F^\bullet M$ is called the *Li filtration* [195] of M.

The subspace $F^1 M$ is spanned by the vectors $a_{(-2)}m$ with $a \in V$, $m \in M$, which is often denoted by $C_2(M)$ in the literature. Set

$$\overline{M} = M/F^1 M (= M/C_2(M)),$$

which is a Poisson module over $R_V = \overline{V}$.

We refer to [195, Proposition 4.12] for a proof of the next proposition.

Proposition 6.2 *The Poisson vertex module homomorphism*

$$\mathcal{J}_\infty(R_V) \otimes_{R_V} \overline{M} \to \mathrm{gr}^F M$$

is surjective.

Let $\{a^i : i \in I\}$ be elements of V such that their images generate R_V in the usual commutative sense, and let U be a subspace of M such that $M = U + F^1 M$. The surjectivity of the above map is equivalent to that

$$F^p M = \mathrm{span}_{\mathbb{C}}\{a^{i_1}_{(-n_1-1)} \dots a^{i_r}_{(-n_r-1)} m \colon m \in U,\ n_i \geqslant 0, n_1 + \dots + n_r \geqslant p,\ i_1, \dots, i_r \in I\}. \tag{6.6}$$

Lemma 6.3 *Let V be a vertex algebra, M a V-module. The Poisson vertex algebra module structure of $\mathrm{gr}^F M$ restricts to the Poisson module structure of $\overline{M} := M/F^1 M$*

over R_V, that is, $\overline{M}$ is a Poisson R_V-module by

$$\bar{a}\cdot\bar{m}=\overline{a_{(-1)}m},\quad \operatorname{ad}(\bar{a})(\bar{m})=\overline{a_{(0)}m},\qquad \bar{a}\in R_V,\ m\in M.$$

A V-module M is called *finitely strongly generated* if $\overline{M}$ is finitely generated as a R_V-module in the usual associative sense.

Definition 6.2 For a finitely strongly generated V-module M, define its *associated variety X_M* by

$$\begin{aligned} X_M &= \operatorname{supp}_{R_V}(\overline{M}) \\ &= \{\mathfrak{p}\in \operatorname{Spec} R_V : \mathfrak{p}\supset \operatorname{Ann}_{R_V}(\overline{M})\}\subset X_V, \end{aligned}$$

equipped with the reduced scheme structure.

A finitely strongly generated V-module M is called *lisse*, or *C_2-cofinite* if $\dim X_M=0$. Next lemma is stated in [12, Lemma 3.2.2].

Lemma 6.4 *Let M be a finitely strongly generated V-module. Then the following are equivalent:*

(i) *M is lisse.*
(ii) *$\bar{M}$ is finite-dimensional.*

6.3 Example: Associated varieties of modules over affine vertex algebras

For a $V=V^{\kappa}(\mathfrak{a})$-module M, or equivalently, a smooth $\widehat{\mathfrak{a}}_{\kappa}$-module (see Proposition 3.2), we have

$$\overline{M}=M/\mathfrak{a}[t^{-1}]t^{-2}M,$$

and the Poisson $\mathbb{C}[\mathfrak{a}^*]$-module structure is given by

$$x\cdot\bar{m}=\overline{(xt^{-1})m},\quad \operatorname{ad}(x)\overline{m}=\overline{xm},\qquad x\in\mathfrak{a},\ m\in M.$$

Now suppose that G is a connected semisimple group, $\mathfrak{g}=\operatorname{Lie}(G)$.

Let $\mathbf{KL}(\hat{\mathfrak{g}}_{\kappa})$ be the full subcateogory of the category of $\hat{\mathfrak{g}}_{\kappa}$-modules consisting of modules on which $t\mathfrak{g}[t]$ acts locally nilpotently and $\mathfrak{g}$ acts semisimply. Clearly, $\mathbf{KL}(\hat{\mathfrak{g}}_{\kappa})$ is an abelian category, which can be regarded as a full subcategory of the category of $V^{\kappa}(\mathfrak{g})$-modules.

For a $\mathfrak{g}$-module E, let

$$V^{\kappa}(E):=U(\hat{\mathfrak{g}}_{\kappa})\otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}\mathbf{1})}E,$$

where E is considered as a $\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1}$-module on which $\mathfrak{g}[t]$ acts trivially and $\mathbf{1}$ acts as the identity. Then $V^\kappa(E)$ is an object of $\mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$ for a finite dimensional representation E of $\mathfrak{g}$. Note that $V^\kappa(\mathfrak{g}) = V^\kappa(\mathbb{C})$ and its simple quotient $L_\kappa(\mathfrak{g})$ are also objects of $\mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$.

Lemma 6.5 *For $M \in \mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$ the following conditions are equivalent:*

(i) *M is finitely strongly generated as a $V^\kappa(\mathfrak{g})$-module,*
(ii) *M is finitely generated as a $\mathfrak{g}[t^{-1}]t^{-1}$-module,*
(iii) *M is finitely generated as a $\hat{\mathfrak{g}}_\kappa$-module.*

Exercise 6.1 We have $\overline{V^\kappa(E)} \cong \mathscr{O}(\mathfrak{g}^*) \otimes E$ and $X_{V^\kappa(E)} = \mathfrak{g}^*$ for a finite dimensional representation E of $\mathfrak{g}$.

6.4 Frenkel–Zhu's bimodules

Recall that for a graded vertex algebra V, its Zhu's algebra is defined by $\mathrm{Zhu}(V) = V/V \circ V$. There is a similar construction for modules due to Frenkel and Zhu [121]. For a V-module M, set

$$\mathrm{Zhu}(M) = M/V \circ M,$$

where $V \circ M$ is the subspace of M spanned by the vectors

$$a \circ m = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} m$$

for $a \in V_{\Delta_a}$, $\Delta_a \in \mathbb{Z}$, and $m \in M$.

Next statement is proved in [121, Theorem 1.5.1].

Proposition 6.3 $\mathrm{Zhu}(M)$ *is a bimodule over* $\mathrm{Zhu}(V)$ *by the multiplications*

$$a * m = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-1)} m, \quad m * a = \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i-1)} m$$

for $a \in V_{\Delta_a}$, $\Delta_a \in \mathbb{Z}$, and $m \in M$. Moreover, the two actions commute.

Proof The statements are equivalent to proving the following relations for all homogeneous elements $a, b \in V$, $m \in M$:

$$(V \circ V) * m \subset V \circ M, \qquad m * (V \circ V) \subset V \circ M,$$
$$a * (V \circ M) \subset V \circ M, \qquad (V \circ M) * a \subset V \circ M,$$

$$(a * b) * m - a * (b * m) \in V \circ M,$$
$$(m * a) * b - m * (a * b) \in V \circ M,$$
$$(a * m) * b - a * (m * b) \in V \circ M.$$

We omit the details. □

Thus, we have a right exact functor

$$V\text{-Mod} \to \mathrm{Zhu}(V)\text{-biMod}, \quad M \mapsto \mathrm{Zhu}(M),$$

where $\mathrm{Zhu}(V)$ -biMod is the category of bimodules over $\mathrm{Zhu}(V)$.

Lemma 6.6 *Let $M = \bigoplus_{d \in h+\mathbb{Z}_{\geqslant 0}} M_d$ be a positive energy representation of a $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra V. Define an increasing filtration $(\mathrm{Zhu}_p(M))_p$ on $\mathrm{Zhu}(V)$ by*

$$\mathrm{Zhu}_p(M) = \mathrm{im}\left(\bigoplus_{d=h}^{h+p} M_p \to \mathrm{Zhu}(M)\right).$$

(i) *We have*

$$\mathrm{Zhu}_p(V) \cdot \mathrm{Zhu}_q(M) \cdot \mathrm{Zhu}_r(V) \subset \mathrm{Zhu}_{p+q+r}(M),$$
$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(M)] \subset \mathrm{Zhu}_{p+q-1}(M).$$

Therefore $\mathrm{gr}\,\mathrm{Zhu}(M) = \bigoplus_p \mathrm{Zhu}_p(M)/\mathrm{Zhu}_{p-1}(M)$ is a Poisson $\mathrm{gr}\,\mathrm{Zhu}(V)$-module, and hence is a Poisson R_V-module through the homomorphism $\eta_V : R_V \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}(V)$.

(ii) *There is a natural surjective homomorphism*

$$\eta_M : \bar{M}(= M/F^1 M) \to \mathrm{gr}\,\mathrm{Zhu}(M)$$

of Poisson R_V-modules. This is an isomorphism if V admits a PBW basis and $\mathrm{gr}\,M$ is free over $\mathrm{gr}\,V$.

Example 6.2 Let $M = V^\kappa(E)$. Since $\mathrm{gr}\,V^\kappa(E)$ is free over $\mathscr{O}(\mathfrak{J}_\infty(\mathfrak{g}^*))$, we have the isomorphism

$$\eta_{V^\kappa(E)} : \overline{V^\kappa(E)} = E \otimes \mathbb{C}[\mathfrak{g}^*] \xrightarrow{\sim} \mathrm{gr}\,\mathrm{Zhu}(V^\kappa(E)).$$

On the other hand, there is a $U(\mathfrak{g})$-bimodule homomorphism

$$\begin{aligned} E \otimes U(\mathfrak{g}) &\to \mathrm{Zhu}(V^\kappa(E)), \\ v \otimes x_1 \dots x_r &\mapsto (1 \otimes v) * (x_1 t^{-1}) * \dots * (x_r t^{-1}) + V^\kappa(\mathfrak{g}) \circ V^\kappa(E) \end{aligned} \tag{6.7}$$

which respects the filtration. Here the $U(\mathfrak{g})$-bimodule structure of $U(\mathfrak{g}) \otimes E$ is given by

$$x(v \otimes u) = (xv) \otimes u + v \otimes xu, \quad (v \otimes u)x = v \otimes (ux),$$

and the filtration of $U(\mathfrak{g}) \otimes E$ is given by $\{U_i(\mathfrak{g}) \otimes E\}$. Since the induced homomorphism between associated graded spaces (6.7) coincides with $\eta_{V^\kappa(E)}$, (6.7) is an isomorphism.

Let $\mathcal{HC} = \mathcal{HC}(\mathfrak{g})$ be the category of *Harish-Chandra bimodules*, that is, the full subcategory of the category of $U(\mathfrak{g})$-bimodules consisting of objects M on which the adjoint action of $\mathfrak{g}$ is integrable, that is, locally finite.

Lemma 6.7 *For* $M \in \mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$*, we have* $\mathrm{Zhu}(M) \in \mathcal{HC}$*. If* M *is finitely generated, then so is* $\mathrm{Zhu}(M)$.

Part III
BRST cohomology and quantum Drinfeld–Sokolov reduction

Kostant and Sternberg [183], inspired by Feigin [106]. suggested a BRST cohomological interpretation of the Hamiltonian reduction.

This part studies the BRST cohomology in various contexts. Chapter 7 is within a framework of Poisson algebras. Applying the BRST reduction in a special case, we obtain a BRST interpretation of the Slodowy slices and equivariant Slodowy slices that also appear in the study of singularities of nilpotent orbit closures (see Section §D.6 in appendix about this topic).

Next, in Chapter 8 we consider the quantized BRST cohomology. In this setting, we construct the finite $\mathcal{W}$-algebras using BRST complexes. This follows the works of Kostant [182], further generalized by Lynch [206] and developed by Premet [227] and Gan–Ginzburg [128]. Chapter 8 also covers some aspects of the representation theory of $\mathcal{W}$-algebras, following the works of Premet and Losev for this topic.

Chapter 9 is the heart of the part and is about the chiral quantized BRST cohomology. This allows to defines the affine $\mathcal{W}$-algebras, simply often referred to as $\mathcal{W}$-algebras and the equivariant $\mathcal{W}$-algebras. In Chapter **??** we present the Wakimoto realization of $\mathcal{W}$-algebras. Chapter 10 presents an alternative description of the BRST reduction which allows to give the definition of $\mathcal{W}$-algebras following the Kac–Roan–Wakimoto approach [163]. We prove that the two definitions are equivalent.

Chapter 7
BRST cohomology and Slodowy slices

In this chapter we shall introduce the BRST cohomology (where BRST refers to the physicists Becchi, Rouet, Stora and Tyutin) for later purpose.

In more details, Section 7.1 is about the general BRST cohomology. Applying this construction to the BRST cohomology associated with nilpotent elements of a simple Lie algebra, we obtain in Section 7.2 the BRST realization of Slodowy slices (the Drinfeld–Sokolov reduction). Section 7.3 is about equivariant Slodowy slices. While Slodowy slices are merely (non symplectic) Poisson varieties, equivariant Slodowy slices are always symplectic varieties. In fact, we will see the former are obtained from $\mathscr{O}(\mathfrak{g}^*)$, for simple $\mathfrak{g} = \mathrm{Lie}(G)$, by Hamiltonian reduction while the later are obtained from $\mathscr{O}(T^*G) \cong \mathscr{O}(G) \otimes \mathscr{O}(\mathfrak{g}^*)$, where T^*G is the cotangent bundle G which is clearly symplectic. The Drinfeld–Sokolov reduction allows us to define in Section 7.4 the Moore–Tachikawa operation on the set of Poisson varieties acted by a simple Lie group with Hamiltonian actions. Lastly, we extend in Section 7.5 the BRST reduction to the setting of Poisson modules over $\mathscr{O}(\mathfrak{g}^*)$ (here, $\mathfrak{g}$ is an arbitrary algebraic Lie algebra). Then we apply this to the special case of the Drinfeld–Sokolov reduction.

We refer the reader to Appendix B about basics on Poissons algebras and to Appendix E for backgrounds on superspaces, superalgebras, Lie superalgebras, etc. and Clifford algebras.

7.1 BRST cohomology and BRST reduction

7.1.1 BRST cohomology

Let G be any connected affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$ (we do not assume G is semisimple). Let $\{x_i\}_{1\leqslant i\leqslant d}$ a basis of $\mathfrak{g}$, and let $\{x_i^*\}_{1\leqslant i\leqslant d}$ be the corresponding dual basis of $\mathfrak{g}^*$. Denote by $c_{i,j}^k$ the structure constants of $\mathfrak{g}$, that is, $[x_i, x_j] = \sum_{k=1}^{d} c_{i,j}^k x_k$ for $i, j = 1, \ldots, d$.

Let $Cl(\mathfrak{g})$ be the *Clifford algebra* associated with the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ and the nondegenerate bilinear form $\langle - | - \rangle$ defined by $\langle f + x | g + y \rangle = f(y) + g(x)$ for $f, g \in \mathfrak{g}^*$, $x, y \in \mathfrak{g}$. Namely, $Cl(\mathfrak{g})$ is the unital superalgebra isomorphic to $\bigwedge(\mathfrak{g} \oplus \mathfrak{g}^*) \cong \bigwedge(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}^*)$ as $\mathbb{C}$-vector spaces, the natural embeddings $\bigwedge(\mathfrak{g}) \hookrightarrow Cl(\mathfrak{g})$, $\bigwedge(\mathfrak{g}^*) \hookrightarrow Cl(\mathfrak{g})$ are homogeneous homomorphism of superalgebras, and

$$[x, f] = f(x) \quad x \in \mathfrak{g} \subset \bigwedge(\mathfrak{g}),\ f \in \mathfrak{g}^* \subset \bigwedge(\mathfrak{g}^*).$$

(Note that $[x, f] = xf + fx$ since x, f are odd.)

Lemma 7.1 *The following map gives a Lie algebra homomorphism:*

$$\begin{aligned} \rho \colon \mathfrak{g} &\longrightarrow Cl(\mathfrak{g}) \\ x_i &\longmapsto \sum_{1 \leqslant j,k \leqslant d} c_{i,j}^k x_k x_j^*. \end{aligned}$$

We have

$$[\rho(x), y] = [x, y] \in \mathfrak{g} \subset Cl(\mathfrak{g}),$$

for $x, y \in \mathfrak{g}$ where the first bracket is in $Cl(\mathfrak{g})$ while the second is in $\mathfrak{g}$.

Define an increasing filtration on $Cl(\mathfrak{g})$ by setting $Cl_p(\mathfrak{g}) := \bigwedge^{\leqslant p}(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}^*)$ where $\bigwedge(\mathfrak{g}) = \bigoplus_{i \geqslant 0} \bigwedge^i(\mathfrak{g})$ is the natural grading. We have

$$0 = Cl_{-1}(\mathfrak{g}) \subset Cl_0(\mathfrak{g}) \subset Cl_1(\mathfrak{g}) \cdots \subset Cl_d(\mathfrak{g}) = Cl(\mathfrak{g}),$$

and

$$Cl_p(\mathfrak{g}) \cdot Cl_q(\mathfrak{g}) \subset Cl_{p+q}(\mathfrak{g}), \quad [Cl_p(\mathfrak{g}), Cl_q(\mathfrak{g})] \subset Cl_{p+q-1}(\mathfrak{g}). \tag{7.1}$$

Let $\bar{C}l(\mathfrak{g})$ be its associated graded algebra:

$$\bar{C}l(\mathfrak{g}) := \operatorname{gr} Cl(\mathfrak{g}) = \bigoplus_{p \geqslant 0} \frac{Cl_p(\mathfrak{g})}{Cl_{p-1}(\mathfrak{g})}.$$

By (7.1), $\bar{C}l(\mathfrak{g})$ is naturally a graded Poisson superalgebra, called the *classical Clifford algebra* associated with $\mathfrak{g} \oplus \mathfrak{g}^*$.

We have $\bar{C}l(\mathfrak{g}) = \bigwedge(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}^*)$ as a commutative superalgebra. Its Poisson (super)bracket is given by

$$\begin{aligned} &\{x, f\} = f(x), \quad x \in \mathfrak{g} \subset \bigwedge(\mathfrak{g}),\ f \in \mathfrak{g}^* \subset \bigwedge(\mathfrak{g}^*), \\ &\{x, y\} = 0, \quad x, y \in \mathfrak{g} \subset \bigwedge(\mathfrak{g}), \quad \{f, g\} = 0,\ f, g \in \mathfrak{g}^* \subset \bigwedge(\mathfrak{g}^*). \end{aligned}$$

Lemma 7.2 *We have* $\bar{C}l(\mathfrak{g})^{\mathfrak{g}} = \bigwedge(\mathfrak{g})$*, where*

$$\bar{C}l(\mathfrak{g})^{\mathfrak{g}} := \{w \in \bar{C}l\colon \{x, w\} = 0 \textit{ for all } x \in \mathfrak{g}\}.$$

The Lie algebra homomorphism $\rho\colon \mathfrak{g} \to Cl_1(\mathfrak{g}) \subset Cl(\mathfrak{g})$ induces a Lie algebra homomorphism

$$\bar{\rho} := \sigma_1 \circ \rho\colon \mathfrak{g} \longrightarrow \bar{C}l(\mathfrak{g}), \tag{7.2}$$

where σ_1 is the projection $Cl_1(\mathfrak{g}) \to Cl_1(\mathfrak{g})/Cl_0(\mathfrak{g}) \subset \operatorname{gr} Cl(\mathfrak{g})$. We have for $x, y \in \mathfrak{g}$,

$$\{\bar{\rho}(x), y\} = [x, y].$$

Set

$$\bar{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \bar{C}l(\mathfrak{g}).$$

Since it is a tensor product of Poisson superalgebras, $\bar{C}(\mathfrak{g})$ is naturally a Poisson superalgebra.

Lemma 7.3 *For any character* χ *of* $\mathfrak{g}$*, the following map gives a Lie algebra homomorphism:*

$$\begin{aligned} \bar{\theta}_\chi\colon \mathfrak{g} &\longrightarrow \bar{C}(\mathfrak{g}) \\ x &\longmapsto (x - \chi(x)) \otimes 1 + 1 \otimes \bar{\rho}(x), \end{aligned}$$

that is, $\{\bar{\theta}_\chi(x), \bar{\theta}_\chi(y)\} = \bar{\theta}_\chi([x, y])$ *for* $x, y \in \mathfrak{g}$.

Let $\bar{C}(\mathfrak{g}) = \bigoplus_{n\in\mathbb{Z}} \bar{C}^n(\mathfrak{g})$ be the $\mathbb{Z}$-grading defined by $\deg \phi\otimes 1 = 0$ for $\phi \in \mathbb{C}[\mathfrak{g}^*]$, $\deg 1 \otimes f = 1$ for $f \in \mathfrak{g}^*$, and $\deg 1 \otimes x = -1$ for $x \in \mathfrak{g}$. We have

$$\bar{C}^n(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \left(\bigoplus_{j-i=n} \bigwedge\nolimits^i(\mathfrak{g}) \otimes \bigwedge\nolimits^j(\mathfrak{g}^*)\right).$$

The following result is due to Beilinson and Drinfeld ([54, Lemma 7.13.3]).

Lemma 7.4 *There exists a unique element* $\bar{Q} \in \bar{C}^1(\mathfrak{g})$ *such that*

$$\{\bar{Q}, 1 \otimes x\} = \bar{\theta}_\chi(x) \quad \textit{for} \quad x \in \mathfrak{g}.$$

We have $\{\bar{Q}, \bar{Q}\} = 0$.

Proof For the existence, it is straightforward to see that the element

$$\bar{Q} = \sum_i (x_i - \chi(x_i)) \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k x_i^* x_j^* x_k$$

satisfies the condition.

For the uniqueness, suppose that $\bar{Q}_1, \bar{Q}_2 \in \bar{C}^1(\mathfrak{g})$ satisfy the condition. Set $R = Q_1 - Q_2 \in \bar{C}^1(\mathfrak{g})$. Then $\{R, 1 \otimes x\} = 0$, and so, $R \in \mathbb{C}[\mathfrak{g}^*] \otimes \bar{C}l(\mathfrak{g})^{\mathfrak{g}}$. But by Lemma 7.2, $\mathbb{C}[\mathfrak{g}^*] \otimes \bar{C}l(\mathfrak{g})^{\mathfrak{g}} \cap \bar{C}^1(\mathfrak{g}) = 0$. Thus $R = 0$ as required.

To show that $\{\bar{Q}, \bar{Q}\} = 0$, observe that

$$\{1 \otimes x, \{1 \otimes y, \{\bar{Q}, \bar{Q}\}\}\} = 0 \quad \text{for all } x, y \in \mathfrak{g}$$

(note that $\bar{Q}$ is odd). Applying Lemma 7.2 twice, we get that $\{\bar{Q}, \bar{Q}\} = 0$. □

Since $\bar{Q}$ is odd, Lemma 7.4 implies that

$$\{\bar{Q}, \{\bar{Q}, a\}\} = \frac{1}{2}\{\{\bar{Q}, \bar{Q}\}, a\} = 0$$

for any $a \in \bar{C}(\mathfrak{g})$. That is, $\operatorname{ad} \bar{Q} := \{\bar{Q}, \cdot\}$ satisfies that

$$(\operatorname{ad} \bar{Q})^2 = 0.$$

Thus, $(\bar{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$ is a *differential graded Poisson superalgebra*. Its cohomology

$$H^{\bullet}_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[\mathfrak{g}^*]) := H^{\bullet}(\bar{C}(\mathfrak{g}), \operatorname{ad} \bar{Q}) = \bigoplus_{i \in \mathbb{Z}} H^i(\bar{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$$

inherits a graded Poisson superalgebra structure from $\bar{C}(\mathfrak{g})$.

More generally, let R be a Poisson algebra equipped with a Poisson algebra homomorphism $\mu^* \colon \mathbb{C}[\mathfrak{g}^*] \to R$. View $\bar{Q}$ as an element of $R \otimes \overline{Cl}(\mathfrak{g})$ through the map $\mu^* \otimes \mathrm{id}$. Then $\mu^* \otimes \mathrm{id} \colon \bar{C}(\mathfrak{g}^*) \to R \otimes \overline{Cl}(\mathfrak{g})$ is a Poisson superalgebra homomorphism, and $(R \otimes \overline{Cl}(\mathfrak{g}), \operatorname{ad} \bar{Q})$ is a differential graded Poisson superalgebra, where the image of $\bar{Q}$ is also denoted by $\bar{Q}$. Therefore, its cohomology

$$H^{\bullet}_{BRST,\chi}(\mathfrak{g}, R) := H^{\bullet}(R \otimes \overline{Cl}(\mathfrak{g}), \operatorname{ad} \bar{Q}) \tag{7.3}$$

inherits a graded Poisson superalgebra structure from $R \otimes \bar{C}l(\mathfrak{g})$.

7.1.2 BRST reduction

Let X be any affine Poisson scheme[1] equipped with a Hamiltonian G-action, $\chi \in \mathfrak{g}^*$ a one-point G-orbit, that is, χ is a character of $\mathfrak{g}$. The above construction gives the Poisson algebra $H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]) \subset H^{\bullet}_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])$. (Note that the degree zero part is purely even.) The affine Poisson scheme

$$X /\!\!/_{BRST,\chi} G := \operatorname{Spec}(H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]))$$

---

[1] It is assumed that all Poisson schemes are of finite type unless otherwise stated.

is called the *BRST reduction* of X. We write $X /\!\!/_{BRST} G$ for $X /\!\!/_{BRST,\chi} G$ if $\chi = 0$.

The BRST reduction coincides with the geometric Hamiltonian reduction in (cf. §B.5) in some "nice" cases.

Theorem 7.1 *Let $X = \operatorname{Spec} R$ be an affine Poisson scheme equipped with a Hamiltonian G-action, $\chi \in \mathfrak{g}^*$ a one-point G-orbit. Suppose that*

(i) *the moment map $\mu \colon X \to \mathfrak{g}^*$ is flat,*
(ii) *there exists a subscheme $\mathscr{S}$ of $\mu^{-1}(\chi)$ such that the action map gives the isomorphism $G \times \mathscr{S} \xrightarrow{\sim} \mu^{-1}(\chi)$.*

Then

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, R) \cong \mathbb{C}[\mathscr{S}] \otimes H^\bullet_{DR}(G).$$

Here $H^\bullet_{DR}(G)$ denotes the de Rham cohomology of G equipped with the trivial Poisson structure. In particular, $H^0_{DR}(G) = \mathbb{C}$ since G is connected.

Lemma 7.5 *Let A be a commutative $\mathbb{C}$-algebra with a regular sequence $x_1, x_2, \dots$, that is, $A/(x_1, \dots, x_r) \neq 0$ and x_{r+1} is not a zero devisor of $A/(x_1, \dots, x_r)$ for all r. If M is a flat A-module, that is, $M \otimes_A -$ is an exact functor, then*

$$H_i^{Kos}(A, M) = \delta_{i,0} M/(x_1, x_2, \dots)M.$$

Here $H_i^{Kos}(A, M)$ denotes the homology of the Koszul complex with respect to the sequence $x_1, x_2, \dots$.

Proof (of Theorem 7.1) Give a bigrading on $\bar{C} := R \otimes \overline{Cl}$ by setting

$$\bar{C}^{i,j} = R \otimes \textstyle\bigwedge^i(\mathfrak{g}^*) \otimes \bigwedge^{-j}(\mathfrak{g}),$$

so that $\bar{C} = \bigoplus_{i \geqslant 0, j \leqslant 0} \bar{C}^{i,j}$.

Observe that $\operatorname{ad} \bar{Q}$ decomposes as $\operatorname{ad} \bar{Q} = d_+ + d_-$ such that

$$d_-(\bar{C}^{i,j}) \subset \bar{C}^{i+1,j}, \qquad d_+(\bar{C}^{i,j}) \subset \bar{C}^{i,j+1}. \tag{7.4}$$

Explicitly, we have

$$d_- = \sum_i (x_i - \chi(x_i)) \otimes \operatorname{ad} x_i^*,$$

$$d_+ = \sum_i \operatorname{ad} x_i \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k x_i^* x_j^* \operatorname{ad} x_k + \sum_i 1 \otimes \bar{\rho}(x_i) \operatorname{ad} x_i^*.$$

Since $(\operatorname{ad} \bar{Q})^2 = 0$, (7.4) implies that

$$d_-^2 = d_+^2 = [d_-, d_+] = 0.$$

It follows that there exists a spectral sequence

$$E_r \Longrightarrow H^{\bullet}_{BRST,\chi}(\mathfrak{g}, R) = H^{\bullet}(\bar{C}, \operatorname{ad}\bar{Q})$$

such that

$$E_1^{\bullet,q} = H^q(\bar{C}, d_-) = H^q(R \otimes \textstyle\bigwedge(\mathfrak{g}), d_-) \otimes \textstyle\bigwedge^{\bullet}(\mathfrak{g}^*),$$
$$E_2^{p,q} = H^p(H^q(\bar{C}, d_-), d_+).$$

Observe that $(\bar{C}, d_-)$ is identical to the Koszul complex $\mathbb{C}[\mathfrak{g}^*]$ associated with the sequence $x_1 - \chi(x_1), x_2 - \chi(x_2), \dots, x_d - \chi(x_d)$ tensored with $\bigwedge(\mathfrak{g}^*)$. Hence by Lemma 7.5 we get that

$$H^i(\bar{C}, d_-) = \begin{cases} \mathbb{C}[\mu^{-1}(\chi)] \otimes \bigwedge(\mathfrak{g}^*), & \text{if } \quad i = 0, \\ 0, & \text{if } \quad i \neq 0. \end{cases}$$

Next, notice that $(H^{\bullet}(\bar{C}, d_-), d_+)$ is identical to the Chevalley complex for the Lie algebra cohomology $H^{\bullet}(\mathfrak{g}, \mathbb{C}[\mu^{-1}(\chi)])$. Since $\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathscr{S}] \otimes \mathbb{C}[G]$ by the assumption as G-modules, where G acts only on $\mathbb{C}[G]$ on the right-hand-side, we get that

$$H^i(H^j(\bar{C}, d_-), d_+) = \begin{cases} \mathbb{C}[\mathscr{S}] \otimes H^i(\mathfrak{g}, \mathbb{C}[G]), & \text{if } \quad j = 0, \\ 0, & \text{if } \quad j \neq 0. \end{cases}$$

Hence the spectral sequence collapses at $E_2 = E_\infty$. Thus there is an isomorphism

$$H^{\bullet}(\bar{C}, \operatorname{ad}\bar{Q}) \xrightarrow{\sim} H^{\bullet}(H^0(\bar{C}, d_-), d_+) = \mathbb{C}[\mathscr{S}] \otimes H^{\bullet}(\mathfrak{g}, \mathbb{C}[G]), \quad [c] \mapsto [c],$$

of Poisson algebras. The assertion follows noting that $H^i(\mathfrak{g}, \mathbb{C}[G]) = H^i_{DR}(G)$ as G is affine. □

We write $H^{\bullet}_{BRST}(\mathfrak{g}, \mathbb{C}[\mathfrak{g}^*])$ for $H^{\bullet}_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[\mathfrak{g}^*])$ if $\chi = 0$.

7.2 BRST realization of Slodowy slices

In this section, it is required that $\mathfrak{g} = \operatorname{Lie}(G)$ is a simple Lie algebra, and we keep the notations of Appendices A and D. Let $\kappa \colon \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphism induced from the nondegenerate bilinear form $(-|-) = \dfrac{1}{2h^\vee} \times$ Killing form of $\mathfrak{g}$.

7.2.1 Kirillov form

Fix (e, h, f) an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, and let $\chi = \kappa(f) = (f|-)$ be the linear form associated with f. Set for $j \in \mathbb{Z}$,

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \colon [h, x] = 2jx\}$$

as in §D.3. The restriction to $\mathfrak{g}_{\frac{1}{2}} \times \mathfrak{g}_{\frac{1}{2}}$ of the antisymmetric bilinear form,

$$\omega_\chi \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \qquad (x, y) \mapsto (f|[x, y]),$$

is nondegenerate. This results from the paring between $\mathfrak{g}_{\frac{1}{2}}$ and $\mathfrak{g}_{-\frac{1}{2}}$, and from the injectivity of the map $\operatorname{ad} f \colon \mathfrak{g}_{\frac{1}{2}} \to \mathfrak{g}_{-\frac{1}{2}}$. It is called the *Kirillov form associated with* f. Let ℓ be a Lagrangian subspace of $\mathfrak{g}_{\frac{1}{2}}$, that is, ℓ is maximal isotropic which means $\omega_\chi(\ell, \ell) = 0$ and $\dim \ell = \frac{1}{2} \dim \mathfrak{g}_{\frac{1}{2}}$. Set

$$\mathfrak{m} = \mathfrak{m}_{\chi,\ell} := \ell \oplus \bigoplus_{j > \frac{1}{2}} \mathfrak{g}_j. \tag{7.5}$$

Then $\mathfrak{m}$ is an ad-nilpotent[2] and ad h-graded subalgebra of $\mathfrak{g}$. Moreover, the algebra $\mathfrak{m}$ verifies the following properties:

(i) $\chi([\mathfrak{m}, \mathfrak{m}]) = (f|[\mathfrak{m}, \mathfrak{m}]) = 0$,
(ii) $\mathfrak{m} \cap \mathfrak{g}^f = \{0\}$,
(iii) $\dim \mathfrak{m} = \frac{1}{2} \dim G.f$.

Notice that if f is regular, then $\mathfrak{h} = \mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{m}$ is the nilpotent radical $\mathfrak{n}_+ = \bigoplus_{j>0} \mathfrak{g}_j$ of the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$.

7.2.2 Slodowy slices

Definition 7.1 The *Slodowy slice associated with* (e, h, f) is the affine space

$$\mathscr{S}_f := \kappa(f + \mathfrak{g}^e) = \chi + \kappa(\mathfrak{g}^e) \subset \mathfrak{g}^*$$

The affine space $\mathscr{S}_f$ is identified with $f + \mathfrak{g}^e$ through $(-|-)$ Note that $\kappa(\mathfrak{g}^e) \cong (\mathfrak{g}^f)^*$ by the theory of $\mathfrak{sl}_2$-triples. Consider the one-parameters $\rho \colon \mathbb{C}^* \to G$ and $\tilde{\rho} \colon \mathbb{C}^* \to G$ defined as in the proof of Lemma D.4 so that $\rho(t)x = t^{2j}x$ and $\tilde{\rho}(t)x = t^{2+2j}x$ for any $x \in \mathfrak{g}_j$. In particular, $\tilde{\rho}(t)f = f$ and the $\mathbb{C}^*$-action of $\tilde{\rho}$ stabilizes $\mathscr{S}_f$. Moreover, it is contracting to f on $\mathscr{S}_f$, that is,

$$\lim_{t \to 0} \tilde{\rho}(t)(f + x) = f$$

for any $x \in \mathfrak{g}^e$, because $\mathfrak{g}^e \subset \mathfrak{m}^\perp \subseteq \mathfrak{g}_{>-1}$. The same lines of arguments show that the action $\tilde{\rho}$ stabilizes $f + \mathfrak{m}^\perp$ and it is contracting to f on $f + \mathfrak{m}^\perp$.

Theorem 7.2 *The affine space* $\mathscr{S}_f$ *is transversal to the coadjoint orbits of* $\mathfrak{g}^*$. *More precisely, given any* $\xi \in \mathscr{S}_f$, *we have* $T_\xi(G.\xi) + T_\xi(\mathscr{S}_f) = \mathfrak{g}^*$. *An analogue statement holds for the affine variety* $\chi + \mathfrak{m}^\perp$.

[2] i.e., $\mathfrak{m}$ only consists of nilpotent elements of $\mathfrak{g}$.

Proof We have to show that $[\mathfrak{g}, x] + \mathfrak{g}^e = \mathfrak{g}$ for any $x \in f + \mathfrak{g}^e$ since $T_x(G.x) = [\mathfrak{g}, x]$ and $T_x(f + \mathfrak{g}^e) = \mathfrak{g}^e$. It suffices to verify that the map

$$\eta\colon G \times (f + \mathfrak{g}^e) \to \mathfrak{g}$$

is a submersion at any point (g, x) of $G \times (f + \mathfrak{g}^e)$, that is, the differential $d\eta_{(g,x)}$ of η at (g, x) is surjective for any point (g, x) of $G \times (f + \mathfrak{g}^e)$. The differential of η is the linear map $\mathfrak{g} \times \mathfrak{g}^e \to \mathfrak{g}$, $(v, w) \mapsto g([v, x]) + g(w)$. So $d\eta_{(\mathrm{Id}, f)}(v, w) = [v, f] + w$ and, hence, $d\eta_{(\mathrm{Id}, f)}$ is surjective, since $[\mathfrak{g}, f] + \mathfrak{g}^e = \mathfrak{g}$. Thus $d\eta_{(\mathrm{Id},x)}$ is surjective for any x in an open neighborhood Ω of f in $f + \mathfrak{g}^e$. Because the morphism η is G-equivariant for the action by left multiplication, we deduce that $d\eta_{(g,x)}$ is surjective for any $g \in G$ and any $x \in \Omega$. In particular, for any $x \in \Omega$, we get

$$\mathfrak{g} = [\mathfrak{g}, x] + \mathfrak{g}^e$$

We now use the contracting $\mathbb{C}^*$-action ρ on $f + \mathfrak{g}^e$ to show that η is actually a submersion at any point of $G \times (f + \mathfrak{g}^e)$. □

Consider the adjoint map

$$M \times (f + \mathfrak{m}^\perp) \to \mathfrak{g}, \quad (g, x) \mapsto g.x$$

Its image is contained in $f + \mathfrak{m}^\perp$. Indeed, for any $x \in \mathfrak{n}$ and any $y \in \mathfrak{m}^\perp$, $\exp(\operatorname{ad} x)(f + y) \in f + \mathfrak{m}^\perp$ since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ and $\chi([\mathfrak{m}, \mathfrak{m}]) = 0$. This is enough to conclude because, $\mathfrak{m}$ being ad-nilpotent, M is generated by the elements $\exp(\operatorname{ad} x)$ for x running through $\mathfrak{m}$. As a result, by restriction, we get a map

$$\alpha\colon M \times \mathscr{S}_f \to f + \mathfrak{m}^\perp.$$

The following result is stated in [128] (see also [133]).

Lemma 7.6 *Let $\alpha\colon X_1 \to X_2$ be a $\mathbb{C}^*$-equivariant morphism of smooth affine $\mathbb{C}^*$-varieties with contracting $\mathbb{C}^*$-actions which induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points. Then α is an isomorphism.*

Proof Let x_1, x_2 be the (only) fixed points of X_1, X_2, respectively, and write T_1, T_2 for the corresponding tangent spaces at x_1, x_2 of X_1, X_2, respectively. Since α induces an isomorphism $T_1 \xrightarrow{\sim} T_2$, α is an equivariant open embedding. Since X_2 is irreducible, $\alpha^*\colon \mathbb{C}[X_1] \to \mathbb{C}[X_2]$ is injective.

To show the surjectivity, observe that $\mathbb{C}[X_1]$ and $\mathbb{C}[X_2]$ are $\mathbb{Z}_{\geqslant 0}$-graded because the $\mathbb{C}^*$-action is contracting. The same goes for T_1 and T_2 since x_1 and x_2 are fixed points. Let $\chi_X \in \mathbb{C}[[t]]$ be the formal characters of a $\mathbb{C}^*$-variety with contracting $\mathbb{C}^*$-action. By [133, (7.7)], we have

$$\chi_{X_1} = \chi_{T_1} = \chi_{T_2} = \chi_{X_2},$$

whence the surjectivity of $\alpha^*\colon \mathbb{C}[X_1] \to \mathbb{C}[X_2]$. □

Theorem 7.3 (Gan–Ginzburg) *The map $\alpha\colon M\times\mathscr{S}_f \to f+\mathfrak{m}^\perp$ is an isomorphism of affine varieties.*

Proof We have a contracting $\mathbb{C}^*$-action on $M\times\mathscr{S}_f$ defined by:

$$t.(g,x) := (\rho(t^{-1})g\rho(t), \tilde{\rho}(t)x) \quad \text{for all} \quad t\in\mathbb{C}^*\ g\in M,\ x\in\mathscr{S}_f\,.$$

The morphism α is $\mathbb{C}^*$-equivariant with respect to this contracting $\mathbb{C}^*$-action, and the $\mathbb{C}^*$-action $\tilde{\rho}$ on $f+\mathfrak{m}^\perp$. This finishes the proof by Lemma 7.6. □

As a consequence of this result, we get the isomorphism:

$$\mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[f+\mathfrak{m}^\perp]^M.$$

Remark 7.1 More generally, if ℓ is any isotropic subspace of $\mathfrak{g}_{\frac{1}{2}}$, set $\mathfrak{m}_\ell := \ell\oplus\bigoplus_{j>\frac{1}{2}}\mathfrak{g}_j$ and $\mathfrak{n}_\ell := \ell^\perp\oplus\bigoplus_{j>\frac{1}{2}}\mathfrak{g}_j$, where $\ell^\perp$ is the orthogonal of ℓ in $\mathfrak{g}_{\frac{1}{2}}$ with respect to $(x,y)\mapsto\chi([x,y])$. Letting N_ℓ be the unipotent subgroup of G with Lie algebra $\mathfrak{n}_\ell$, we have ([128]):

$$N_\ell\times\mathscr{S}_f \xrightarrow{\sim} f+\mathfrak{m}^\perp.$$

We recover the previous cas when ℓ is Lagrangian.

7.2.3 Poisson structure on $\mathscr{S}_f$

We will see in this paragraph that $\mathscr{S}_f$ is naturally equipped with a Poisson structure. The arguments are adapted from [128].

The connected Lie group M acts on the Poisson variety $\mathfrak{g}^*$ by the coadjoint action. The action is Hamiltonian and the moment map,

$$\mu\colon\mathfrak{g}^*\to\mathfrak{m}^*, \tag{7.6}$$

is the restriction of functions from $\mathfrak{g}$ to $\mathfrak{m}$. Since $\chi|_\mathfrak{m}$ is a character on $\mathfrak{m}$, it is fixed by the coadjoint action of M. As a consequence, the set

$$\mu^{-1}(\chi|_\mathfrak{m}) = \{\xi\in\mathfrak{g}^*\colon \mu(\xi)=\chi|_\mathfrak{m}\}$$

is M-stable.

Lemma 7.7 *The restriction $\chi|_\mathfrak{m}$ is a regular value for the restriction of μ to each symplectic leaf of $\mathfrak{g}^*$.*

Proof Note that $\mu^{-1}(\chi|_\mathfrak{m}) = \chi+\mathfrak{m}^\perp$. Then we have to prove that for any $\xi\in\chi+\mathfrak{m}^\perp$, the map

$$d_\xi\mu\colon T_\xi(G.\xi)\to T_{\chi|_\mathfrak{m}}(\mathfrak{m}^*)$$

is surjective. But $T_\xi(G.\xi) \simeq [\mathfrak{g}, \xi]$ while $T_{\chi|_\mathfrak{m}}(\mathfrak{m}^*) = \mathfrak{m}^*$. Since $\chi + \mathfrak{m}^\perp$ is transversal to the coadjoint orbits in $\mathfrak{g}^*$ (cf. Theorem 7.2), we have

$$\mathfrak{g} = [\mathfrak{g}, \xi] + \mathfrak{m}^\perp.$$

Fix $\gamma \in \mathfrak{m}^*$ and write $\gamma = x + x'$, with $x \in [\mathfrak{g}, \xi]$ and $x' \in \mathfrak{m}^\perp$, according to the above decomposition of $\mathfrak{g}$. Then $\mu(x) = \gamma$. □

Since the map

$$M \times \mathscr{S}_f \longrightarrow \chi + \mathfrak{m}^\perp$$

is an isomorphism of affine varieties (cf. Theorem 7.3),

$$\mathscr{S}_f \cong (\chi + \mathfrak{m}^\perp)/M.$$

Therefore, by Theorem B.2 we get a symplectic structure on $\mathscr{S}_f$. In fact, thanks to Lemma 7.7, we have shown that the symplectic form on each leaf on $\mathscr{S}_f$ is obtained by symplectic reduction from the symplectic form of the corresponding leaf of $\mathfrak{g}^*$. The Poisson structure on $\mathscr{S}_f$ is described as follows. Let $\pi\colon \chi + \mathfrak{m}^\perp \twoheadrightarrow (\chi + \mathfrak{m}^\perp)/M \simeq \mathscr{S}_f$ be the natural projection map, and $\iota\colon \chi + \mathfrak{m}^\perp \hookrightarrow \mathfrak{g}^*$ be the natural inclusion. Then for any $f, g \in \mathbb{C}[\mathscr{S}_f]$,

$$\{f, g\}_{\mathscr{S}_f} \circ \pi = \{\tilde{f}, \tilde{g}\} \circ \iota$$

where $\tilde{f}, \tilde{g}$ are arbitrary extensions of $f \circ \pi, g \circ \pi$ to $\mathfrak{g}^*$.

Remark 7.2 On can show that the above Poisson structure on $\mathscr{S}_f$ is the same as the one we would have obtained by induction from the Poisson structure on $\mathfrak{g}^* \cong \mathfrak{g}$ (see Theorem B.1). Next exercise is about this.

Exercise 7.1 The aim of this exercise is to obtained the above Poisson structure on $\mathscr{S}_f$ in other way, by applying Theorem B.1.

(i) Show that for any $\xi \in \mathscr{S}_f$, $[\xi, [e, \mathfrak{g}]] \cap T_\xi(\mathscr{S}_f) = 0$, that is, $[\xi, [e, \mathfrak{g}]] \cap \mathfrak{g}^e = 0$.
(ii) Deduce from question 1 that for any coadjoint orbit $\mathbb{O}$ in $\mathfrak{g}^*$ and any $\xi \in \mathbb{O} \cap \mathscr{S}_f$ the restriction of the symplectic form on $T_\xi(\mathbb{O})$ to $T_\xi(\mathscr{S}_f) \cap T_\xi(\mathbb{O})$ is nondegenerate.
(iii) Conclude by applying Theorem B.1.

Alternatively, the Poisson structure of $\mathscr{S}_f$ can be described as follows. Let

$$I_\chi = \mathbb{C}[\mathfrak{g}^*] \sum_{x \in \mathfrak{m}} (x - \chi(x)),$$

so that

$$\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathfrak{g}^*]/I_\chi.$$

Then $\mathbb{C}[\mathscr{S}_f] = \mathbb{C}[\mu^{-1}(\chi)]^M$ is identified with the subspace of $\mathbb{C}[\mathfrak{g}^*]/I_\chi$ consisting of all cosets $\phi + I_\chi$ such that $\{x, \phi\} \in I_\chi$ for all $x \in \mathfrak{m}$. In this realization, the Poisson structure on $\mathbb{C}[\mathscr{S}_f]$ is defined by the formula

$$\{\phi + I_\chi, \phi' + I_\chi\} = \{\phi, \phi'\} + I_\chi$$

for ϕ, ϕ' such that $\{x, \phi\}, \{x, \phi'\} \in I_\chi$ for all $x \in \mathfrak{m}$.

7.2.4 BRST reduction of $\mathfrak{g}^*$

We now apply Theorem 7.1 to $X = \mathfrak{g}^*$, $G = M$, the moment map $\mu\colon X = \mathfrak{g}^* \to \mathfrak{m}^*$ and $\chi = (f|-)$. Observe that $\mu^{-1}(\chi) = f + \mathfrak{m}^\perp$. Clearly μ is flat, and the second assumption is satisfied by Theorem 7.3.

Theorem 7.4 (Gan–Ginzburg) *We have* $H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*]) = 0$ *for* $i \neq 0$ *and*

$$H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*]) \cong \mathbb{C}[\mathscr{S}_f]$$

as Poisson algebras.

Proof Since M is unipotent, we have $H^i_{DR}(M) = \delta_{i,0}\mathbb{C}$. Therefore the assertion follows immediately from Theorem 7.1. □

7.3 Equivariant Slodowy slices

Let x_L and x_R denote the action of x on $\mathbb{C}[G]$ as a left invariant vector field and a right invariant vector field respectively. Consider the cotangent bundle T^*G of G. We have $T^*G = G \times \mathfrak{g}^*$ and the Poisson structure of $\mathbb{C}[T^*G]$ is given by

$$\{x, y\} = [x, y], \qquad \{x, f\} = x_L(f), \quad x, y \in \mathfrak{g}, f \in \mathbb{C}[G].$$

There are the following two commuting Hamiltonian G action $g \mapsto g_L$ and $g \mapsto g_R$ on T^*G, where

$$g_L(a, x) = (ag^{-1}, g.x), \quad g_R(a, x) = (ga, x), \tag{7.7}$$

where $g.x = \mathrm{Ad}^* g(x)$ denotes the coadjoint action of g on x. The moment map corresponding to the former is just the projection

$$\mu_L\colon T^*G \ni (a, x) \longmapsto x \in \mathfrak{g}^*. \tag{7.8}$$

The moment map corresponding to the latter is given by

$$\mu_R\colon T^*G \ni (a, x) \longmapsto a.x \in \mathfrak{g}^*. \tag{7.9}$$

The action of $\mathfrak{g}$ on $\mathbb{C}[T^*G] = \mathbb{C}[G] \otimes \mathbb{C}[\mathfrak{g}^*]$ obtained by differentiating these actions are

$$\pi_L(x) = x_L + \operatorname{ad} x \quad , \quad \pi_R(x) = x_R,$$

where $\operatorname{ad} x$ denotes the action $f \mapsto \{x, f\}$ on $\mathbb{C}[\mathfrak{g}^*]$.

Now consider the composition

$$\mu \colon T^*G \xrightarrow{\mu_L} \mathfrak{g}^* \xrightarrow{\text{projection}} \mathfrak{m}^*.$$

Then μ is the moment map for the M-action by restriction to g_L. We have $\mu^{-1}(\chi) = G \times (\chi + \mathfrak{m}^\perp)$. Clearly the action of M on $\mu^{-1}(\chi)$ is free and χ is the regular value of μ. Thus,

$$\widetilde{\mathscr{S}_f} := \mu^{-1}(\chi)/M = G \times_M (\chi + \mathfrak{m}^\perp) \tag{7.10}$$

is a symplectic variety. We have

$$\widetilde{\mathscr{S}_f} \cong G \times \mathscr{S}_f$$

and $\widetilde{\mathscr{S}_f}$ is called the *equivariant Slodowy slice* [199].

As μ is clearly flat we can apply Theorem 7.1 to obtain the following.

Proposition 7.1 *We have* $H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[T^*G]) = 0$ *for* $i \neq 0$ *and*

$$H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[T^*G]) = \mathbb{C}[\widetilde{\mathscr{S}_f}]$$

as Poisson algebras.

Remark 7.3 The equivariant Slodowy slice $\widetilde{\mathscr{S}_f} = G \times_M (\chi + \mathfrak{m}^\perp)$ is naturally a vector bundle over G/M. As a bundle over G/M, $\widetilde{\mathscr{S}_f}$ is called a *twisted cotangent bundle* [81, 1.4.13].

The relation between the Slodowy slice $\mathscr{S}_f$ and the equivariant Slodowy slice $\widetilde{\mathscr{S}_f}$ is described as follows. There is an action of G on $\widetilde{\mathscr{S}_f}$ defined by

$$g(a, x) = (ga, x). \tag{7.11}$$

Proposition 7.2 *We have* $\mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[\widetilde{\mathscr{S}_f}]^G$ *as Poisson algebras.*

The G-action (7.11) is Hamiltonian and the corresponding moment map is given by

$$\mu \colon \widetilde{\mathscr{S}_f} = G \times_M (\chi + \mathfrak{m}^\perp) \ni (g, x) \longmapsto g.x \in \mathfrak{g}^*. \tag{7.12}$$

A proof of the following theorem can be found in [236, 227, 80].

Theorem 7.5 *The moment map $\mu \colon \widetilde{\mathscr{S}_f} \to \mathfrak{g}^*$ given by (7.12) is smooth onto a dense open subset of $\mathfrak{g}^*$ containing $G.\chi$. In particular, μ is flat.*

Proof Since the proof is short, we give the argument.

It suffices to prove that the morphism

$$\theta_f : G \times (f + \mathfrak{g}^e) \to \mathfrak{g}^*, \quad (g, x) \mapsto g.x$$

is smooth onto a dense open subset of $\mathfrak{g}^*$ containing $G.f$. Since $\mathfrak{g} = \mathfrak{g}^e + [f, \mathfrak{g}]$, θ_f is a submersion at $(1_G, f)$. Then θ_f is a submersion at all points of $G \times (f + \mathfrak{g}^e)$ since it is G-equivariant for the left multiplication in G and since

$$\lim_{t \to \infty} \rho(t).x = f$$

for all x in $f + \mathfrak{g}^e$. So, by [185, Chapter III, Proposition 10.4], θ_f is a smooth morphism onto a dense open subset of $\mathfrak{g}$, containing $G.f$. □

Theorem 7.6 *The natural map $\mathbb{C}[\mathfrak{g}^*]^G \to \mathbb{C}[\mathscr{S}_f] = H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{g}^*])$ defined by sending p to $p \otimes 1$ induces an isomorphism from $\mathbb{C}[\mathfrak{g}^*]^G$ to the Poisson center of $\mathbb{C}[\mathscr{S}_f]$.*

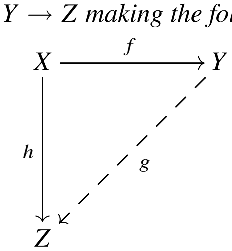
The fact that

$$Z(\mathbb{C}[\mathscr{S}_f]) \cong \mathbb{C}[\mathfrak{g}]^G,$$

where $Z(\mathbb{C}[\mathscr{S}_f])$ is the Poisson center of $\mathbb{C}[\mathscr{S}_f]$, was claimed by Ginzburg–Premet ([228, Question 5.1]).

Before proving the theorem we state two lemmas of independent interest. They are used also for the proof (omitted in this book) of Theorem 13.5. We refer to [136, Theorem A.2.9] for a proof of the next result.

Lemma 7.8 *Let X, Y, Z be irreducible affine varieties. Assume that $f \colon X \to Y$ and $h \colon X \to Z$ are dominant morphisms such that h is constant on the fibers of f. There there exists a rational map $g \colon Y \to Z$ making the following diagram commutative:*



Lemma 7.9 *Let X and Y be two normal irreducible affine varieties, and $f \colon X \to Y$ a flat morphism. Then $\mathbb{C}(Y) \cap \mathbb{C}[X] = \mathbb{C}[Y]$. Here, we view $\mathbb{C}[Y]$ as a subalgebra of $\mathbb{C}[X]$ using $f^* \colon \mathbb{C}[Y] \to \mathbb{C}[X]$.*

The result is probably is well-known. As we have not found any appropriate reference, we include a proof.

Proof Since X is normal and the fibers of f are all of dimension $\dim X - \dim Y$, the image of the set X' of smooth points of X is an open subset Y' of Y such that $Y \setminus Y'$ has codimension at least 2.

Let y in Y' and $x \in f^{-1}(y) \subset X'$. Then we have a flat extension of the local rings $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. Since $\mathscr{O}_{Y,y}$ and $\mathscr{O}_{X,x}$ are regular local rings, they are factorial. For $a \in \mathbb{C}(Y) \cap \mathbb{C}[X]$, write $a = p/q$ with p, q relatively prime elements of $\mathbb{C}[Y]$. Since p, q are relatively prime, the multiplication by p induces an injective homomorphism

$$\mathscr{O}_{Y,y}/q\mathscr{O}_{Y,y} \to \mathscr{O}_{Y,y}/q\mathscr{O}_{Y,y}.$$

Since $\mathscr{O}_{X,x}$ is flat over $\mathscr{O}_{Y,y}$, the base change $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} -$ yields an injective homomorphism

$$\mathscr{O}_{X,x}/q\mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/q\mathscr{O}_{X,x}.$$

Hence p and q are relatively prime in $\mathscr{O}_{X,x}$. In addition, the image of 1 is 0 because $a = p/q$ is regular in X. As a result, q is invertible in $\mathscr{O}_{X,x}$.

Since the maximal ideal of $\mathscr{O}_{Y,y}$ is the intersection of $\mathscr{O}_{Y,y}$ with the maximal ideal of $\mathscr{O}_{X,x}$, q is invertible in $\mathscr{O}_{Y,y}$, and so a is in $\mathscr{O}_{Y,y}$. As a result, a is regular on Y' and then extends to a regular function on Y since Y is normal. □

Proof (of Theorem 7.6) By Theorem 7.5, the moment map $\mu\colon \widetilde{\mathscr{S}_f} \to \mathfrak{g}^*$ induces an embedding $\mu^*\colon \mathbb{C}[\mathfrak{g}^*] \hookrightarrow \mathbb{C}[\widetilde{\mathscr{S}_f}]$ of Poisson algebras. By taking G-invariants, we get the embedding $\mathbb{C}[\mathfrak{g}^*]^G \hookrightarrow \mathbb{C}[\widetilde{\mathscr{S}_f}]^G = \mathbb{C}[\mathscr{S}_f]$.

Consider the morphism $\varphi\colon \mathscr{S}_f \to \mathfrak{g}^*/\!\!/G$. Each of the fibers is a finite union of symplectic leaves for $\mathscr{S}_f$. Remember that the symplectic leaves of $\mathscr{S}_f$ are the intersections $\mathscr{S}_f \cap G.\xi$ with $\xi \in \mathfrak{g}^*$. On the other hand, by [228, §§5.4 & 6.4], all scheme-theoretic fibers of φ are reduced and irreducible. Hence each fiber of φ is the closure of some symplectic leaf of $\mathscr{S}_f$. Let now z be in the Poisson center of $\mathbb{C}[\mathscr{S}_f]$. It is constant on each symplectic leaf by definition of the Hamiltonian flow: If σ_x is an integral curve of $\{H, -\}$, with $H \in \mathbb{C}[\mathscr{S}_f]$ and $\sigma_x(0) = x \in \mathscr{S}_f$, then $\dfrac{d}{dt}(z \circ \sigma_x) = \{H, z\} \circ \sigma_x = 0$, and so z is constant on all flows through x, that is, on the symplectic leaf of x. As a result, z is constant on all fibers of the morphism φ.

If z is constant, then clearly z lies in the Poisson center of $\mathbb{C}[\mathfrak{g}^*]$. In addition, one can assume that z is homogeneous for the *Slodowy grading* on $\mathbb{C}[\mathscr{S}_f]$ induced from the $\mathbb{C}^*$-action of $\tilde{\rho}$ on $\mathscr{S}_f$ since the Poisson center of $\mathbb{C}[\mathscr{S}_f]$ is Slodowy invariant. So for any $t \in \mathbb{C}^*$, $t.z = t^k z$ if k is the Slodowy degree of z. Hence one can assume that $z\colon \mathscr{S}_f \to \mathbb{C}$ is a dominant (and even surjective) morphism. So by Lemma 7.8, z induces a rational morphism on $\mathfrak{g}^*/\!\!/G$ since z is constant on the fibers of the dominant morphism φ. As a result, it remains to prove the following:

$$\mathbb{C}(\mathfrak{g}^*/\!\!/G) \cap \mathbb{C}[\mathscr{S}_f] = \mathbb{C}[\mathfrak{g}^*/\!\!/G], \tag{7.13}$$

since $\mathbb{C}[\mathfrak{g}^*/\!\!/G] \cong \mathbb{C}[\mathfrak{g}^*]^G$. But this follows from Lemma 7.9. □

7.4 Moore–Tachikawa operation and Drinfeld–Sokolov reduction

We continue to assume that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple, and that G is connected.

7.4.1 Moore–Tachikawa operation

Let X, Y be (any) affine Poisson schemes equipped with Hamiltonian G-action, $\mu_X : X \to \mathfrak{g}^*$, $\mu_Y : Y \to \mathfrak{g}^*$ the corresponding moment maps. Then the diagonal action of G on $X \times Y$ is Hamiltonian, with the moment map

$$\mu_{X\times Y} : X \times Y \ni (x, y) \longmapsto \mu_X(x) + \mu_Y(y) \in \mathfrak{g}^*.$$

Motivated by [219], we define the affine Poisson scheme $X \circ Y$ by

$$X \circ Y := (X \times Y) /\!\!/_{BRST} G = \mathrm{Spec}(H^0_{BRST}(\mathfrak{g}, \mathbb{C}[X] \otimes \mathbb{C}[Y])).$$

Clearly, $X \circ Y \cong Y \circ X$.

Proposition 7.3 *We have $T^*G \circ X \cong X$ for any affine Poisson scheme X equipped with a Hamiltonian G-action.*

Proof From Exercise 7.2 below, it is enough to show that $H^0_{BRST}(\mathfrak{g}, \mathbb{C}[T^*G]) = \mathbb{C}$. But this is easy to see. □

Exercise 7.2 Let X be an affine Poisson schemes equipped with a Hamiltonian G-action. There are the following four Hamiltonian G-actions on $T^*G \times X$:

$$\pi_{1,L}(g)(a, f, x) = (ag^{-1}, gf, gx), \quad \pi_{1,R}(g)(a, f, x) = (ga, f, x),$$
$$\pi_{2,L}(g)(a, f, x) = (ag^{-1}, gf, x), \quad \pi_{2,R}(g)(a, f, x) = (ga, f, gx).$$

Clearly the actions $\pi_{1,L}$ and $\pi_{1,R}$ (resp. $\pi_{2,L}$ and $\pi_{2,R}$) mutually commute. Consider the morphism

$$\Phi : T^*G \times X \to T^*G \times X$$

defined by $(g, f, x) \mapsto (g, f, gx)$ for $g \in G$, $f \in \mathfrak{g}^*$, $x \in X$. Check that Φ is an isomorphism of Poisson schemes such that

$$\Phi \circ \pi_{1,L} = \pi_{2,L} \circ \Phi, \quad \Phi \circ \pi_{1,R} = \pi_{2,R} \circ \Phi.$$

Theorem 7.7 *Let X be an affine Poisson scheme equipped with a Hamiltonian G-action, and $\mu_X : X \to \mathfrak{g}^*$ the corresponding moment map. Then $\widetilde{\mathscr{S}_f} \circ X$ is isomorphic to the scheme theoretic intersection $X \times_{\mathfrak{g}^*} \mathscr{S}_f = X \cap \mu_X^{-1}(\mathscr{S}_f)$, where $\mathscr{S}_f \to \mathfrak{g}^*$ is given by the inclusion $x \mapsto -x$.*

Proof Let

$$\mu\colon \widetilde{\mathscr{S}_f} \times X = G \times \mathscr{S}_f \times X \to \mathfrak{g}^*, \quad (g, s, x) \mapsto g.s + \mu_X(x),$$

be the moment map that is the sum of the moment maps. By Theorem 7.5, μ is flat. Further, the action map gives the isomorphism

$$G \times (\mathscr{S}_f \times_{\mathfrak{g}^*} X) \xrightarrow{\sim} \mu^{-1}(0).$$

Thus, Theorem 7.1 gives that

$$H^\bullet_{BRST}(\mathfrak{g}, \mathbb{C}[X \times \widetilde{\mathscr{S}_f}]) \cong \mathbb{C}[X \times_{\mathfrak{g}^*} \mathscr{S}_f] \otimes H^\bullet_{DR}(G).$$

Exercise 7.3 Show that $\mathfrak{g}^* \circ \mathfrak{g}^* \cong \mathfrak{g}^* /\!\!/ G$, that is, $\mathbb{C}[\mathfrak{g}^* \circ \mathfrak{g}^*] \cong \mathbb{C}[\mathfrak{g}^*]^G$.

7.4.2 Drinfeld–Sokolov reduction

Let X be an affine Poisson scheme equipped with a Hamiltonian G-action. The composition of the moment map μ_X with the projection $\mathfrak{g}^* \to \mathfrak{m}^*$ is the moment map for the M-action on X. We define the Poisson algebra $DS_f(\mathbb{C}[X])$ by

$$DS_f(\mathbb{C}[X]) := H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X]),$$

and the affine Poisson scheme $DS_f(X)$ by

$$DS_f(X) := X /\!\!/_{BRST,\chi} M = \operatorname{Spec}(H^0_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X])),$$

where $\chi = (f|-)$ is as before.

Note that Theorem 7.4 and Proposition 7.1 say that

$$DS_f(\mathfrak{g}^*) = \mathscr{S}_f, \quad DS_f(T^*G) = \widetilde{\mathscr{S}_f}.$$

The following theorem is proved in [128].

Theorem 7.8 *For any affine Poisson scheme X equipped with a Hamiltonian G-action, we have*

$$DS_f(X) \cong \widetilde{\mathscr{S}_f} \circ X \cong X \times_{\mathfrak{g}^*} \mathscr{S}_f.$$

Moreover, $H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X]) = 0$ for $i \neq 0$.

Proof Recall the isomorphism $\Phi\colon T^*G \times X \to T^*G \times X$ in Exercise 7.2. We will show that the following diagram commutes:

$$\begin{array}{ccc} T^*G \times X & \xrightarrow{/\!\!/_{BRST}G} & X \\ \Big\downarrow{\scriptstyle DS_f} & & \Big\downarrow{\scriptstyle DS_f} \\ \widetilde{\mathscr{S}_f} \times X & \xrightarrow[/\!\!/_{BRST}G]{} & DS_f(X). \end{array}$$

Set

$$C := \mathbb{C}[X] \otimes \mathbb{C}[T^*G] \otimes \bar{C}l(\mathfrak{m}) \otimes \bar{C}l(\mathfrak{g}).$$

Let $\bar{Q}_{\mathfrak{m},\chi} \in \bar{C}(\mathfrak{m})$ and $\bar{Q}_{\mathfrak{g}} \in \bar{C}(\mathfrak{g})$ be the elements that give the differentials of the BRST complex for $H^{\bullet}_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[\mathfrak{m}])$ and $H^{\bullet}_{BRST}(\mathfrak{g}, \mathbb{C}[\mathfrak{g}])$, respectively. Let $\bar{Q}_{1,\mathfrak{m}} \in C$ be the image of $\bar{Q}_{\mathfrak{m},\chi}$ by the embedding $\bar{C}(\mathfrak{m}) \hookrightarrow C$ given by the moment map with respect to the action $\pi_{1,L}$ of $\mathbf{M}$ (this corresponds to the right vertical arrow). Let $\bar{Q}_{1,\mathfrak{g}} \in C$ be the image of $\bar{Q}_{\mathfrak{g}}$, by the embedding $\bar{C}(\mathfrak{g}) \hookrightarrow C$ given by the moment map with respect to the action $\pi_{1,R}$ of G (this corresponds to the upper horizontal arrow). Let $\bar{Q}_{2,\mathfrak{m}} \in C$ be the image of $\bar{Q}_{\mathfrak{m},\chi}$ by the embedding $\bar{C}(\mathfrak{m}) \hookrightarrow C$ given by the moment map with respect to the action $\pi_{2,L}$ of $\mathbf{M}$ (this corresponds to the left vertical arrow). Let $\bar{Q}_{2,\mathfrak{g}} \in C$ be the image of $\bar{Q}_{\mathfrak{g}}$, by the embedding $\bar{C}(\mathfrak{g}) \hookrightarrow C$ given by the moment map with respect to the action $\pi_{2,R}$ of G (this corresponds to the lower horizontal arrow).

Define

$$Q_1 = \bar{Q}_{1,\mathfrak{m}} + \bar{Q}_{1,\mathfrak{g}}, \quad Q_2 = \bar{Q}_{2,\mathfrak{m}} + \bar{Q}_{2,\mathfrak{g}}.$$

Since $\operatorname{ad} \bar{Q}_{i,\mathfrak{m}}$ and $\operatorname{ad} \bar{Q}_{i,\mathfrak{g}}$ obviously commute each other, $(\operatorname{ad} Q_i)^2 = 0$ for $i = 1, 2$. Moreover Φ induces the isomorphism $(C, \operatorname{ad} Q_1) \xrightarrow{\sim} (C, \operatorname{ad} Q_2)$ of differential graded Poisson algebras. In particular

$$H^{\bullet}(C, \operatorname{ad} Q_1) \xrightarrow{\sim} H^{\bullet}(C, \operatorname{ad} Q_2). \tag{7.14}$$

To compute $H^{\bullet}(C, \operatorname{ad} Q_1)$ one can use the spectral sequence (E_r, d_r) whose d_0 is $\operatorname{ad} \bar{Q}_{1,\mathfrak{g}}$ and d_1 is $\operatorname{ad} \bar{Q}_{1,\mathfrak{m}}$. We have

$$\begin{aligned} E_1^{\bullet,q} = H^q(C, \operatorname{ad} \bar{Q}_{1,\mathfrak{g}}) &= \mathbb{C}[X] \otimes \overline{Cl}(\mathfrak{m}) \otimes H^q_{BRST}(\mathfrak{g}, \mathbb{C}[T^*G]) \\ &\cong \mathbb{C}[X] \otimes \overline{Cl}(\mathfrak{m}) \otimes H^q_{DR}(G). \end{aligned}$$

It follows that the complex $(E_1^{\bullet,q}, d_1)$ is the BRST complex for $H^{\bullet}_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X])$ tensored with $H^q_{DR}(G)$. Hence

$$E_2^{p,q} \cong H^p_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X]) \otimes H^q_{DR}(G). \tag{7.15}$$

We can therefore represent classes in $E_2^{p,q}$ as tensor products $\omega_1 \otimes \omega_2$ of a cocyle ω_1 in $\mathbb{C}[X] \otimes \overline{Cl}(\mathfrak{m})$ representing a class in $H^p_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X])$ and a cocycle ω_2 in

$\bigwedge^q \mathfrak{g}^* \subset \mathbb{C}[T^*G] \otimes \overline{Cl}(\mathfrak{g})$ representing a class in $H^q_{DR}(G) = H^q(\mathfrak{g}, \mathbb{C})$. Applying the differential $\operatorname{ad} Q_1$ to this class, we find that it is identically equal to zero. Therefore all the classes in E_2 survive. Moreover, all of the elements of E_2 in the decomposition (7.15) lifts canonically to the cohomology $H^\bullet(C, \operatorname{ad} Q_1)$, and thus, we get that

$$H^\bullet(C, \operatorname{ad} Q_1) \cong H^\bullet_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X]) \otimes H^\bullet_{DR}(G). \tag{7.16}$$

Similarly, to compute $H^\bullet(C, \operatorname{ad} Q_2)$ one can use the spectral sequence (E'_r, d'_r) such that $d'_0 = \operatorname{ad} \bar{Q}_{2,\mathfrak{m}}$ and $d'_1 = \operatorname{ad} \bar{Q}_{2,\mathfrak{g}}$. We have

$$\begin{aligned} E_1'^{\bullet,q} = H^q(C, \operatorname{ad} \bar{Q}_{2,\mathfrak{m}}) = \mathbb{C}[X] \otimes \overline{Cl}(\mathfrak{g}) \otimes H^p_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[T^*G]) \\ \cong \delta_{q,0}\mathbb{C}[X] \otimes \overline{Cl}(\mathfrak{g}) \otimes \mathbb{C}[\widetilde{\mathscr{S}_f}] \end{aligned}$$

It follows that the complex $(E_1'^{\bullet,q}, d_1)$ is the BRST complex for the operation $X \circ \widetilde{\mathscr{S}_f}$. Hence

$$E_2'^{p,q} \cong \delta_{q,0}\mathbb{C}[X \circ \widetilde{\mathscr{S}_f}] \otimes H^p_{DR}(G).$$

We conclude that the spectral sequence collapses at $E_2 = E_\infty$, and we get that

$$H^\bullet(C, \operatorname{ad} Q_2) \cong \mathbb{C}[X \circ \widetilde{\mathscr{S}_f}] \otimes H^\bullet_{DR}(G). \tag{7.17}$$

Finally (7.14), (7.16) and (7.17) give that

$$H^i_{BRST,\chi}(\mathfrak{m}, \mathbb{C}[X]) \cong \delta_{i,0}\mathbb{C}[X \circ \widetilde{\mathscr{S}_f}].$$

Let I be an $\operatorname{ad} \mathfrak{g}$-invariant graded ideal of $\mathbb{C}[\mathfrak{g}^*]$. Then I is a Poisson ideal, so that $\mathbb{C}[\mathfrak{g}^*]/I$ is a Poisson algebra. Set

$$\widetilde{\mathscr{V}}(I) := \operatorname{Spec}(\mathbb{C}[\mathfrak{g}^*]/I), \quad \mathscr{V}(I) := \operatorname{Specm}(\mathbb{C}[\mathfrak{g}^*]/I). \tag{7.18}$$

Thus, $\mathscr{V}(I)$ is the zero locus of I in $\mathfrak{g}^*$. The action of G on $\mathfrak{g}^*$ restricts to a Hamiltonian action on $\widetilde{\mathscr{V}}(I)$.

Corollary 7.1 *Let I be an* $\operatorname{ad} \mathfrak{g}$*-invariant graded ideal of* $\mathbb{C}[\mathfrak{g}^*]$.

(i) $DS_f(\widetilde{\mathscr{V}}(I)) \neq 0$ *if and only if* $\mathscr{V}(I) \supset \overline{G.f}$.
(ii) *The Poisson algebra* $\mathbb{C}[DS_f(\widetilde{\mathscr{V}}(I))]$ *is finite-dimensional if* $\mathscr{V}(I) = \overline{G.f}$.

Proof By applying Theorem 7.8, we get that

$$DS_f(\widetilde{\mathscr{V}}(I)) = \widetilde{\mathscr{V}}(I) \times_{\mathfrak{g}^*} \mathscr{S}_f,$$

which is isomorphic to $\mathscr{V}(I) \cap \mathscr{S}_f \subset \mathscr{S}_f$ as topological spaces.

(i) Since it is stable under the $\mathbb{C}^*$-action on $\mathscr{S}_f$, $DS_f(\widetilde{\mathscr{V}}(I))$ is nonzero if and only it contains the point $\{f\}$. As $\mathscr{V}(I)$ is G-invariant and closed, this is equivalent to that $\mathscr{V}(I) \supset \overline{G.f}$.

(ii) Clearly, $\mathbb{C}[DS_f(\widetilde{\mathscr{V}}(I))]$ is finite-dimensional if and only if $\dim DS_f(\widetilde{\mathscr{V}}(I)) = 0$, which is equivalent to that $\mathscr{V}(I) \cap \mathscr{S}_f = \{f\}$. On the other hand we have $\overline{G.f} \cap \mathscr{S}_f = \{f\}$ by the trasversality of $\mathscr{S}_f$ to G-orbits. □

7.5 BRST reduction of Poisson modules

The above results can be generalized to *Poisson modules* (see §B.6). For a Poisson algebra R, we write R -PMod for the category of Poisson modules over R.

We assume in this section that G is any connected affine algebraic group. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and χ a one-point orbit in $\mathfrak{g}^*$. Recall the differential graded algebra $(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})$ defined in §7.1.1.

For $N \in \mathbb{C}[\mathfrak{g}^*]$ -Pmod, $N \otimes \overline{Cl}(\mathfrak{g})$ is naturally a Poisson module over $\bar{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}(\mathfrak{g})$. (The notation of Poisson modules naturally extends to the Poisson superalgebras modules.) Thus, $(N \otimes \overline{Cl}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})$ is a differential graded Poisson module over the differential graded Poisson module $(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})$. Here $\mathrm{ad}\,\bar{Q}$ is defined using the Poisson module structure map,

$$\bar{C}(\mathfrak{g}) \times (N \otimes \overline{Cl}(\mathfrak{g})) \longrightarrow N \otimes \overline{Cl}(\mathfrak{g}).$$

In particular its cohomology

$$H^{\bullet}_{BRST,\chi}(\mathfrak{g}, N) := H^{\bullet}(N \otimes \overline{Cl}, \mathrm{ad}\,\bar{Q})$$

is a Poisson module over $H^{\bullet}(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})$, and thus over $H^0(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})$. So we get a functor

$$\mathbb{C}[\mathfrak{g}^*]\text{ -PMod} \longrightarrow H^0(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})\text{ -PMod}, \quad N \longmapsto H^0_{BRST,\chi}(\mathfrak{g}, N).$$

More generally, let R be a Poisson algebra equipped with a Poisson algebra homomorphism $\mu^* \colon \mathbb{C}[\mathfrak{g}^*] \to R$. Then for a Poisson R-module M, $H^0_{BRST,\chi}(\mathfrak{g}, M)$ is a Poisson module over $H^0_{BRST,\chi}(\mathfrak{g}, R)$. Thus we get a functor

$$R\text{ -Pmod} \longrightarrow H^0(\bar{C}(\mathfrak{g}), \mathrm{ad}\,\bar{Q})\text{ -PMod}, \quad N \longmapsto H^0_{BRST,\chi}(\mathfrak{g}, N).$$

7.5.1 Results for Poisson modules

Assume from now that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple. Let $\overline{\mathcal{HC}}(\mathfrak{g})$ be the full subcategory of the category of Poisson $\mathbb{C}[\mathfrak{g}^*]$-modules on which the Lie algebra $\mathfrak{g}$-action is *integrable*, that is, locally finite. If X is an affine Poisson scheme equipped with a Hamiltonian G-action then $\mathbb{C}[X]$ is an object of $\overline{\mathcal{HC}}(\mathfrak{g})$ (see Example B.5).

For $M, N \in \overline{\mathcal{HC}}(\mathfrak{g})$, define

$$M \circ N := H^0_{BRST}(\mathfrak{g}, M \otimes N),$$

where $\mathfrak{g}$ acts on $M \otimes N$ diagonally. Then $M \otimes N$ is a Poisson module over the trivial Poisson algebra $\mathbb{C}[\mathfrak{g}^* \circ \mathfrak{g}^*] = \mathbb{C}[\mathfrak{g}^*]^G$.

The proof of the following assertion is similar to that of Proposition 7.3.

Proposition 7.4 *For* $M \in \overline{\mathcal{HC}}(\mathfrak{g})$,

$$T^*G \circ M \cong M$$

as a Poisson module over $\mathbb{C}[T^*G \circ \mathfrak{g}^*] = \mathbb{C}[\mathfrak{g}^*]$.

For $M \in \overline{\mathcal{HC}}(\mathfrak{g})$, define

$$DS_f(M) = H^0_{BRST,\chi}(\mathfrak{m}, M),$$

which is a Poisson module over $\mathbb{C}[\mathscr{S}_f]$.

The following assertion can be proved in the same way as Theorem 7.8.

Theorem 7.9 *For* $M \in \overline{\mathcal{HC}}(\mathfrak{g})$, *we have* $H^i_{BRST,\chi}(\mathfrak{m}, M) = 0$ *for* $i \neq 0$. *Therefore, the functor*

$$\overline{\mathcal{HC}}(\mathfrak{g}) \longrightarrow \mathbb{C}[\mathscr{S}_f]\text{-PMod}, \quad M \longmapsto DS_f(M),$$

is exact. We have

$$DS_f(M) \cong \mathbb{C}[\widetilde{\mathscr{S}_f}] \circ M$$

as a Poisson module over $\mathbb{C}[\mathscr{S}_f] = \mathbb{C}[\widetilde{\mathscr{S}_f} \circ \mathfrak{g}^*]$.

Corollary 7.2 *Let* I *be an* $\operatorname{ad}\mathfrak{g}$*-invariant graded ideal of* $\mathbb{C}[\mathfrak{g}^*]$ *such that* $\mathscr{V}(I) = \overline{G.f}$ *Then*

$$\dim DS_f(\widetilde{\mathscr{V}}(I)) = \operatorname{mult}_{\overline{G.f}} \widetilde{\mathscr{V}}(I),$$

where the integer $\operatorname{mult}_{\overline{G.f}} \widetilde{\mathscr{V}}(I)$ *is defined in the below proof.*

Proof There is a filtration

$$\mathbb{C}[\mathfrak{g}^*]/I = M_0 \supset M_1 \supset \cdots \supset M_r = 0$$

Is it $\mathbb{C}[\mathfrak{g}^*]$-Poisson modules?

of $\mathbb{C}[\mathfrak{g}^*]$-modules such that $M_i/M_{i+1} = \mathbb{C}[\mathfrak{g}^*]/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal in $\mathbb{C}[\mathfrak{g}^*]$. The integer $\operatorname{mult}_{\overline{G.f}} \widetilde{\mathscr{V}}(I)$ is by definition the number of indexes i such that $\mathfrak{p}_i$ coincides with the prime ideal corresponding to $\overline{G.f}$. As $\mathscr{V}(\mathfrak{p}_i) \subset \mathscr{V}(I) = \overline{G.f}$,

$$DS_f(M_i) = \begin{cases} \mathbb{C} & \text{if } \mathscr{V}(\mathfrak{p}_i) = \overline{G.f}, \\ 0 & \text{otherwise.} \end{cases}$$

The assertion follows from the exactness of the functor $DS_f(-)$. □

Chapter 8
Quantized BRST cohomology and finite $\mathcal{W}$-algebras

The goal of this chapter is to quantize the BRST constructions made in the previous chapter. Then we will proceed with the study of finite $\mathcal{W}$-algebras. These algebras were introduced by Premet. They have caught attention for several reasons that are mostly related with classical problems of representation theory. For our purpose, they are important as Zhu's algebras of (affine) $\mathcal{W}$-algebras, as we will see in the next chapter. Finite $\mathcal{W}$-algebras are certain generalizations of the enveloping algebra of a simple Lie algebra. They can be defined through the BRST cohomology associated with nilpotent elements. So the definition and properties of finite $\mathcal{W}$-algebras are deeply related to the geometry of of nilpotent orbits. We recall that Appendix D gathers standard facts on nilpotent elements and nilpotent orbits in simple Lie algebras.

The chapter is structured as follows. Section 8.1, 8.2, 8.3 and 8.4 conern quantizations of the BRST constructions made in Section 7.1, 7.2, 7.3 and 7.4, respectively, essentially following [183]. In particular, we start in Sect. 8.1 with a general setting. After that, we focus from Section 8.2 on the special BRST reduction associated with nilpotent elements of a simple Lie algebra. In Section 8.5, we collect various important results on the representation theory of finite $\mathcal{W}$-algebras.

8.1 Quantized BRST cohomology and quantized BRST reduction

Let G be any connected affine algebraic group with Lie algebra $\mathfrak{g}$, and let $\chi\colon \mathfrak{g} \to \mathbb{C}$ be a character, that is, a one-point G-orbit.

8.1.1 Quantized BRST cohomology

Let $U_\bullet(\mathfrak{g})$ be the PBW filtration of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The PBW theorem gives isomorphisms of Poisson algebras (see Example B.2):

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i\geqslant 0} U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*].$$

Set

$$C(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cl(\mathfrak{g}).$$

It is naturally a $\mathbb{C}$-superalgebra, where $U(\mathfrak{g})$ is considered as a purely even subsuperalgebra. The filtrations of $U(\mathfrak{g})$ and $Cl(\mathfrak{g})$ induce a PBW filtration of $C(\mathfrak{g})$,

$$C_p(\mathfrak{g}) = \sum_{i+j\leqslant p} U_i(\mathfrak{g}) \otimes Cl_j(\mathfrak{g}),$$

and we have

$$\operatorname{gr} C(\mathfrak{g}) \cong \overline{C}(\mathfrak{g})$$

as Poisson superalgebras. Therefore, $C(\mathfrak{g})$ is a quantization of $\overline{C}(\mathfrak{g})$.

Define a $\mathbb{Z}$-grading $C(\mathfrak{g}) = \bigoplus_{n\in\mathbb{Z}} C^n(\mathfrak{g})$ by setting $\deg u \otimes 1 = 0$ for $u \in U(\mathfrak{g})$, $\deg 1 \otimes f = 1$ for $f \in \mathfrak{g}^*$, $\deg 1 \otimes x = -1$ for $x \in \mathfrak{g}$. Then

$$C^n(\mathfrak{g}) = U(\mathfrak{g}) \otimes (\bigoplus_{j-i=n} \textstyle\bigwedge^i(\mathfrak{g}) \otimes \bigwedge^j(\mathfrak{g}^*)).$$

Lemma 8.1 *The following map defines a Lie algebra homomorphism:*

$$\begin{aligned} \theta_\chi : \mathfrak{g} &\longrightarrow C(\mathfrak{g}) \\ x &\longmapsto (x - \chi(x)) \otimes 1 + 1 \otimes \rho(x). \end{aligned}$$

Next lemma was observed in [54, Lemma 7.13.7].

Lemma 8.2 *There exists a unique element $Q \in C^1(\mathfrak{g})$ such that*

$$[Q, 1 \otimes x] = \theta_\chi(x) \quad \textit{for all} \quad x \in \mathfrak{g}.$$

We have $Q^2 = 0$.

Proof The proof is similar to that of Lemma 7.4. In fact the element Q is explicitly given by the same formula as $\bar{Q}$:

$$Q = \sum_i (x_i - \chi(x_i)) \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k x_i^* x_j^* x_k.$$

Since Q is odd, Lemma 8.2 implies that

$$(\operatorname{ad} Q)^2 = 0.$$

Thus, $(C(\mathfrak{g}), \operatorname{ad} Q)$ is a *differential graded algebra*, and its cohomology

$$H^{\bullet}_{BRST,\chi}(\mathfrak{g}, U(\mathfrak{g})) := H^{\bullet}(C(\mathfrak{g}), \operatorname{ad} Q)$$

is a graded superalgebra.

More generally, let A be a $U(\mathfrak{g})$-algebra. Then $A \otimes Cl(\mathfrak{g})$ is naturally a $C(\mathfrak{g})$-algebra, and $(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q)$ is naturally a differential graded algebra, where the image of Q is also denoted by Q. Therefore, its cohomology

$$H^{\bullet}_{BRST,\chi}(\mathfrak{g}, A) := H^{\bullet}(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q) \tag{8.1}$$

inherits a graded Poisson superalgebra structure from $A \otimes Cl(\mathfrak{g})$. We write $H^{\bullet}_{BRST}(\mathfrak{g}, A)$ for $H^{\bullet}_{BRST,\chi}(\mathfrak{g}, A)$ if $\chi = 0$.

8.1.2 Kazhdan filtration

The operator on $\operatorname{gr} C(\mathfrak{g})$ induced by $\operatorname{ad} Q$ does not coincide with $\operatorname{ad} \bar{Q}$. To remedy this, we introduce the *Kazhdan filtration* $K_{\bullet}C(\mathfrak{g})$ of $C(\mathfrak{g})$ as follows.

We assume that there is a grading

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \tag{8.2}$$

of $\mathfrak{g}$ (compatible with the Lie algebra structure) such that $\chi(\mathfrak{g}_j) = 0$ unless $j = 1$. (If $\chi = 0$, we can choose the trivial grading.) The grading (8.2) extends to $C(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cl(\mathfrak{g})$ by setting $\deg x = j$, $\deg x^* = -j$ for $x, x^* \in Cl(\mathfrak{g})$ if $x \in \mathfrak{g}_j$. Here and after, we omit the tensor product sign. Let

$$C(\mathfrak{g}) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} C(\mathfrak{g})[j]$$

be the corresponding grading.

Put $C_i(\mathfrak{g})[j] = C_i(\mathfrak{g}) \cap C(\mathfrak{g})[j]$. Define

$$K_p C(\mathfrak{g}) = \sum_{i-j \leqslant p} C_i(\mathfrak{g})[j]$$

for $p \in \mathbb{Z}$ Then $K_{\bullet}C(\mathfrak{g})$ defines an increasing, exhaustive, separated filtration of $C(\mathfrak{g})$ such that

$$K_p C(\mathfrak{g}) \cdot K_q C(\mathfrak{g}) \subset K_{p+q} C(\mathfrak{g}), \qquad [K_p C(\mathfrak{g}), K_q C(\mathfrak{g})] \subset K_{p+q-1} C(\mathfrak{g}),$$

and $\operatorname{gr}_K C(\mathfrak{g}) = \bigoplus_p K_p C(\mathfrak{g}) / K_{p-1} C(\mathfrak{g})$ is isomorphic to $\overline{C}(\mathfrak{g})$ as Poisson superalgebras. Moreover,

$$(\operatorname{ad} Q) K_p C(\mathfrak{g}) \subset K_p C(\mathfrak{g}),$$

and the associated graded complex $(\operatorname{gr}_K C(\mathfrak{g}), \operatorname{ad} Q)$ is identical to $(\overline{C}(\mathfrak{g}), \operatorname{ad} \bar{Q})$.

Let $K_\bullet U(\mathfrak{g})$ and $K_\bullet Cl(\mathfrak{g})$ be the restrictions of $K_\bullet C(\mathfrak{g})$ to $U(\mathfrak{g})$ and $Cl(\mathfrak{g})$, respectively, so that $\operatorname{gr}_K U(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$, $\operatorname{gr}_K Cl(\mathfrak{g}) \cong \overline{Cl}[\mathfrak{g}^*]$.

8.1.3 Quantized BRST reduction

Let $X = \operatorname{Spec}(R)$ be an affine, Hamiltonian Poisson G-scheme, and $\mu_X : X \to \mathfrak{g}^*$ the moment map.

We wish to quantize the BRST reduction

$$X \rightsquigarrow X /\!\!/_{BRST,\chi} G = \operatorname{Spec}(H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])).$$

A *quantization* of the Hamiltonian G-scheme X is an almost commutative filtered $U(\mathfrak{g})$-algebra $(A, F_\bullet A)$ equipped with an action of G such that

(i) $\operatorname{gr}_F A \cong \mathbb{C}[X]$ as Poisson algebra,
(ii) $g(a.b) = (ga).(gb)$ for $g \in G$, $a, b \in A$,
(iii) the action of $\mathfrak{g}$ obtained by differentiating the action of G coincides with the adjoint action of $\mathfrak{g}$,
(iv) if we denote by $\tilde{\mu}^*$ the the natural algebra homomorphism $U(\mathfrak{g}) \to A$, $\tilde{\mu}^*(U_p(\mathfrak{g})) = F_p A \cap \tilde{\mu}^*(U(\mathfrak{g}))$, and the induced homomorphism $\mathbb{C}[\mathfrak{g}^*] = \operatorname{gr} U(\mathfrak{g}) \to \operatorname{gr}_F A = \mathbb{C}[X]$ coincides with μ_X^*.

Let $(A, F_\bullet A)$ be a quantization of the Hamiltonian G-scheme X, and $\chi \in \mathfrak{g}^*$. Let $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}} \mathfrak{g}_j$ be a grading of $\mathfrak{g}$ such that $\chi(\mathfrak{g}_j) = 0$ unless $j = 1$. A *compatible grading* on A is a grading $A = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} A[j]$ such that $\tilde{\mu}^*(\mathfrak{g}_j) \subset A[j]$ and $F_p A = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} F_p A[j]$, where $F_p A[j] = A[j] \cap F_p A$. With such a grading we can define the Kazhdan filtration $K_\bullet A$ on A by

$$K_p A = \sum_{i - j \leqslant p} F_i A(\mathfrak{g})[j]$$

We have $\operatorname{gr}_K A \cong \mathbb{C}[X]$.

Note that by definition the image of the left ideal $A \sum_{x \in \mathfrak{g}} (\tilde{\mu}^*(x) - \chi(x))$ of A in $\operatorname{gr}_K A = \mathbb{C}[X]$ coincides with the defining ideal $\sum_{x \in \mathfrak{g}} (\mu_X^*(x) - \chi(x))\mathbb{C}[X]$ of $\mu_X^{-1}(\chi)$ in X.

We have a natural algebra homomorphism $C(\mathfrak{g}) \to A \otimes Cl(\mathfrak{g})$, and so, $(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q)$ is a differential graded (super)algebra. Set

$$H^\bullet_{BRST,\chi}(\mathfrak{g}, A) := H^\bullet(A \otimes Cl(\mathfrak{g}), \operatorname{ad} Q).$$

Note that the filtrations $K_\bullet A$, $K_\bullet Cl$ induce a filtration $K_\bullet(A \otimes Cl(\mathfrak{g}))$ on $A \otimes Cl(\mathfrak{g})$ that is compatible with the action of $\operatorname{ad} Q$, and $(\operatorname{gr}_K(A \otimes Cl(\mathfrak{g})), \operatorname{ad} Q)$ is identical to the complex $(\mathbb{C}[X] \otimes \bar{C}l(\mathfrak{g}), \operatorname{ad} \bar{Q})$.

Let $K_\bullet H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ be the filtration of $H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ induced by $K_\bullet(A \otimes Cl(\mathfrak{g}))$, and $\operatorname{gr}_K H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ the associated graded.

Theorem 8.1 *Assume that the conditions (1) and (2) of Theorem 7.1 are verified, that is, there exists a subscheme $\mathscr{S}$ of $\mu_X^{-1}(\chi)$ such that the action map gives the isomorphism $G \times \mathscr{S} \xrightarrow{\sim} \mu_X^{-1}(\chi)$ and μ_X is flat.*

Under the above setting, there is an isomorphism of Poisson algebras

$$\mathrm{gr}_K\, H^\bullet_{BRST,\chi}(\mathfrak{g}, A) \cong H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]) = \mathbb{C}[X /\!\!/_{BRST,\chi} G] \otimes H^\bullet_{DR}(G).$$

In particular, $H^0_{BRST,\chi}(\mathfrak{g}, A)$ is a quantization of $\mathbb{C}[X /\!\!/_{BRST,\chi} G]$.

Proof Consider the spectral sequence $E_r \Rightarrow H^\bullet_{BRST,\chi}(\mathfrak{g}, A)$ such that $E_1 = H^\bullet(\mathrm{gr}_K(A \otimes Cl(\mathfrak{g})), \mathrm{ad}\, Q) = H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])$. By Theorem 7.1, $H^\bullet_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X]) = \mathbb{C}[\mathscr{S}] \otimes H^\bullet_{DR}(G)$. We can therefore represent classes in $E_1^{p,q}$ as tensor products $\omega_1 \otimes \omega_2$ of a cocyle ω_1 in $\mathbb{C}[X] \otimes \Lambda^p(\mathfrak{g})$ representing a class in $H^0_{BRST,\chi}(\mathfrak{g}, \mathbb{C}[X])$ and a cocycle $\omega_2 \in \Lambda^q \subset Cl(\mathfrak{g})$ representing a class in $H^q_{DR}(G) = H^q(\mathfrak{g}, \mathbb{C})$. Applying the differential $\mathrm{ad}\, Q$ to this class, we find that it is identically equal to zero. It follows that the spectral sequence collapses at $E_1 = E_\infty$. □

8.2 Finite $\mathcal{W}$-algebras

We will now apply the above construction to $(\mathfrak{g}^*, \mu, \chi)$, where $\mathfrak{g} = \mathrm{Lie}(G)$ is a simple Lie algebra, $\mu\colon \mathfrak{g}^* \to \mathfrak{m}^*$ is the moment map (7.6) and $\chi = (f|-) \in \mathfrak{m}^*$, with f a nilpotent element of $\mathfrak{g}$.

Clearly, $U(\mathfrak{g})$ with the PBW filtration is a quantization of the Hamiltonian G-scheme $\mathfrak{g}^*$. By restricting to the M-action, we may regard $U(\mathfrak{g})$ as a quantization of the Hamiltonian M-scheme $\mathfrak{g}^*$. The Dynkin grading (D.1) satisfies the condition of §8.1.2, as well as its restriction to $\mathfrak{m}$. So we have the corresponding Kazhdan filtrations $K_\bullet U(\mathfrak{m})$ and $K_\bullet U(\mathfrak{g})$.

By Theorem 8.1,

$$H^i_{BRST,\chi}(\mathfrak{m}, U(\mathfrak{g})) = 0 \quad \text{for} \quad i \neq 0, \tag{8.3}$$

and

$$U(\mathfrak{g}, f) := H^0_{BRST,\chi}(\mathfrak{m}, U(\mathfrak{g}))$$

is a quantization of $\mathbb{C}[\mathscr{S}_f]$.

Definition 8.1 The algebra $U(\mathfrak{g}, f)$ is is called the *finite $\mathcal{W}$-algebra* associated with $\mathfrak{g}$ and f.

8.2.1 Definition via Whittaker models

Let $\chi = (f|-) \in \mathfrak{m}^*$. It extends to a representation

$$\chi\colon U(\mathfrak{m}) \longrightarrow \mathbb{C}$$

and we denote by $\mathbb{C}_\chi$ the corresponding left $U(\mathfrak{m})$-module. The right multiplication by an element of $\mathfrak{m}$ induces a right $U(\mathfrak{m})$-module on $U(\mathfrak{g})$. Denote by I_χ the left ideal of $U(\mathfrak{g})$ generated by the elements $x - \chi(x)$, for $x \in \mathfrak{m}$,

$$I_\chi := \sum_{x\in\mathfrak{m}} U(\mathfrak{g})(x - \chi(x)),$$

and set

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi.$$

It is an $U(\mathfrak{g})$-module called a *generalized Gelfand-Graev module*.

The adjoint action of $\mathfrak{m}$ in $\mathfrak{g}$ uniquely extends to an action of $\mathfrak{m}$ in $U(\mathfrak{g})$ and the ideal I_χ is $\mathfrak{m}$-stable. Thus Q_χ is endowed with an $\mathfrak{m}$-module structure.

Definition 8.2 The algebra

$$Q_\chi^{\operatorname{ad}\mathfrak{m}} = \{\bar{u} \in Q_\chi\colon\ [y, u] \in I_\chi \text{ for any } y \in \mathfrak{m}\},$$

where $\bar{u}$ denotes the coset $u + I_\chi$ of $u \in U(\mathfrak{g})$, is called the *finite $\mathcal{W}$-algebra* associated with f.

We refer the above definition of $U(\mathfrak{g}, f)$ as the *Whittaker model realization* of $U(\mathfrak{g}, f)$.

Remark 8.1 The algebra $Q_\chi^{\operatorname{ad}\mathfrak{m}}$ is the space of *Whittaker vectors* of Q_χ,

$$Q_\chi^{\operatorname{ad}\mathfrak{m}} = \operatorname{Wh}(Q_\chi) := \{u \in Q_\chi : xu = \chi(x)u \text{ for any } x \in \mathfrak{m}\}.$$

The next result was obtained by Kostant ([182]) for the regular case. For the general case, see [87] or [10]. The proof of Arakawa [10] only concerns the regular case, but can be easily adapted to the general case. We follow here his proof.

Proposition 8.1 *We have*

$$U(\mathfrak{g}, f) \cong \operatorname{End}_{U(\mathfrak{g})}(Q_\chi)^{\mathrm{op}} \cong Q_\chi^{\operatorname{ad}\mathfrak{m}},$$

where the symbol "op" means that we consider the ring $\operatorname{End}_{U(\mathfrak{g})}(Q_\chi)$ *with reversed composition operation* $u.v := v \circ u$.

Proof As in the case of $\overline{C}(\mathfrak{g})$, $C(\mathfrak{g})$ is also bigraded, so we can also write $\operatorname{ad} Q = d_+ + d_-$ such that $d_+(C^{ij}) \subset C^{i+1,j}$, $d_-(C^{ij}) \subset C^{i,j+1}$ and get a spectral sequence

$$E_r \Longrightarrow H^\bullet(C(\mathfrak{g}), \operatorname{ad} Q)$$

such that

$$E_2^{p,q} = H^p(H^q(C(\mathfrak{g}), d_-), d_+) \cong \delta_{q,0} H^p(\mathfrak{n}, U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\chi)$$
$$\cong \delta_{p,0}\delta_{q,0} H^0(\mathfrak{m}, U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi) \cong \mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi)^{op}.$$

Thus we get the Whittaker model isomorphism

$$U(\mathfrak{g}, f) = H^0(C(\mathfrak{g}), \mathrm{ad}\, Q) \cong Q_\chi^{\mathrm{ad}\,\mathfrak{m}} \cong \mathrm{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi)^{op},$$

whence the statement. □

Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The restriction to $Z(\mathfrak{g})$ of the representation $U(\mathfrak{g}) \to \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)$ is injective. So we get an inclusion map,

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}, f).$$

By Theorem 7.6, the above map is surjective onto the center $Z(U(\mathfrak{g}, f))$ of $U(\mathfrak{g}, f)$ so that we get an algebra isomorphism

$$Z(\mathfrak{g}) \cong Z(U(\mathfrak{g}, f)).$$

According to a result of Kostant [182], if f is regular then $U(\mathfrak{g}, f)$ is isomorphic to $Z(\mathfrak{g})$, which is known to be a polynomial algebra in rank of $\mathfrak{g}$ variables. In particular, $U(\mathfrak{g}, f)$ is commutative in this case. The above result is a generalization of this fact.

Remark 8.2 The finite $\mathcal{W}$-algebra $Q_\chi^{\mathrm{ad}\,\mathfrak{m}} = U(\mathfrak{g}, f)$ a priori depends on the Lagrangian subspace $\ell \subset \mathfrak{g}_{\frac{1}{2}}$. By [128], the algebra $Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$ does not depend, up to isomorphism, on the choice of the Lagrangian subspace ℓ in $\mathfrak{g}_{\frac{1}{2}}$. Furthermore, according to the main result of [74], the algebra $U(\mathfrak{g}, f)$ does not depend, up to isomorphism, on the choice of the *good grading* adapted to f ([99]). Dynkin gradings are typical examples of good gradings. More generally, for (optimal) admissible gradings, the result is due to Sadaka ([232]).

8.3 Equivariant finite $\mathcal{W}$-algebras

We keep the notations of the previous section. In particular, $\mathfrak{g} = \mathrm{Lie}(G)$ is still assumed to be simple. We can also apply the above construction to quantize the equivariant Slodowy slices. Consider the ring $\mathcal{D}(G)$ of the (global) differential operators on G (see Appendix C). We have

$$\mathcal{D}(G) = U(\mathfrak{g}) \otimes \mathbb{C}[G]$$

as vector spaces, the natural maps

$$\mathbb{C}[G] \hookrightarrow \mathcal{D}(G), \quad \tilde{\mu}_L : U(\mathfrak{g}) \longrightarrow \mathcal{D}(G)$$

are embeddings of algebras (cf. Section C.4). The algebra $\mathcal{D}(G)$ is almost commutative by the standard (or, order) filtration $F_\bullet\mathcal{D}(G)$ defined by $F_p\mathcal{D}(G) = U_p(\mathfrak{g}) \otimes \mathbb{C}[G]$, and we have $\mathrm{gr}_F\mathcal{D}(G) \cong \mathbb{C}[T^*G]$ by (C.1) (see also Example B.3).

The embedding $\tilde{\mu}_L\colon U(\mathfrak{g}) \to \mathcal{D}(G)$ quantizes the comorphism $\mu_L^*\colon \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*G]$ of the moment map μ_L, see (7.8). This map is induced by the Lie algebra homomorphism $\mathfrak{g} \to \mathscr{X}(G)$, $x \mapsto x_L$, where $\mathscr{X}(G)$ is the Lie algebra of the vector fields on G and

$$(x_L f)(a) = \frac{d}{dt} f(a \exp(tx))|_{t=0}, \qquad x \in \mathfrak{g},\ f \in \mathbb{C}[G],\ a \in G.$$

Similarly, the Lie algebra homomorphism $\mathfrak{g} \to \mathscr{X}(G)$, $x \mapsto x_R$, where

$$(x_R f)(a) = \frac{d}{dt} f(\exp(tx)a)|_{t=0},$$

induces the algebra homomorphism

$$\tilde{\mu}_R\colon U(\mathfrak{g}) \longrightarrow \mathcal{D}(G),$$

which quantizes the moment map $\mu_R\colon T^*G \to \mathfrak{g}^*$, see §B.5 and Remark C.2 in Appendix. By definition, the two actions $\tilde{\mu}_L(x)$, $\tilde{\mu}_R(y)$ commute each other.

Thus, $\mathcal{D}(G)$ is a quantization of the Hamiltonian G-scheme T^*G (with respect to both actions).

The grading

$$\mathcal{D}(G) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathcal{D}(G)[j], \quad \mathcal{D}(G)[j] = \{\partial \in \mathcal{D}_G\colon [\tilde{\mu}_L(h), \partial] = 2j\partial\}$$

is compatible with the Dynkin grading (D.1). Thus, we have the corresponding Kazhdan filtration $K_\bullet\mathcal{D}(G)$.

By Theorem 8.1, we have $H^i_{BRST,\chi}(\mathfrak{m}, \mathcal{D}(G)) = 0$ for $i \neq 0$, and

$$\widetilde{U}(\mathfrak{g}, f) := H^0_{BRST,\chi}(\mathfrak{m}, \mathcal{D}(G))$$

is a quantization of the equivariant Slodowy slice $\mathscr{S}_f$, that is, the Kazhdan filtration on $\mathcal{D}(G)$ induces the filtration $K_\bullet\widetilde{U}(\mathfrak{g}, f)$ and we have $\mathrm{gr}_K \widetilde{U}(\mathfrak{g}, f) \cong \mathbb{C}[\widetilde{\mathscr{S}_f}]$. The algebra $\widetilde{U}(\mathfrak{g}, f)$ is called the *equivariant finite $\mathcal{W}$-algebra* ([199]).

In the definition of $\widetilde{U}(\mathfrak{g}, f)$, the BRST reduction is taken with respect to, say, the action $\tilde{\mu}_L$. So $\widetilde{U}(\mathfrak{g}, f)$ is a G-module with respect to the action $g \mapsto g_R$, and $\tilde{\mu}_R$ gives the algebra homomorphism

$$\tilde{\mu}_R\colon U(\mathfrak{g}) \longrightarrow \widetilde{U}(\mathfrak{g}, f), \quad u \mapsto \tilde{\mu}_R(u).$$

The adjoint action of $\mathfrak{g}$ on $\widetilde{U}(\mathfrak{g}, f)$ is the same as the action of $\mathfrak{g}$ obtained by differentiating G-action. Thus, $\widetilde{U}(\mathfrak{g}, f)$ is a quantization of the Hamiltonian G-scheme $\widetilde{\mathscr{S}_f}$.

Exercise 8.1 Show that $\widetilde{U}(\mathfrak{g}, f)$ is a simple algebra.

Proposition 8.2 *We have an algebra isomorphism*

$$U(\mathfrak{g}, f) \xrightarrow{\sim} \widetilde{U}(\mathfrak{g}, f)^G = \widetilde{U}(\mathfrak{g}, f)^{\operatorname{ad}\mathfrak{g}}.$$

Proof The map $\tilde{\mu}_L$ induces the algebra homomorphism $U(\mathfrak{g}, f) \to \widetilde{U}(\mathfrak{g}, f)$, and the image is contained in $\widetilde{U}(\mathfrak{g}, f)^G$. Moreover this is an isomorphism since it induces an isomorphism

$$\operatorname{gr}_K U(\mathfrak{g}, f) = \mathbb{C}[\mathscr{S}_f] \xrightarrow{\sim} \mathbb{C}[\tilde{\mathscr{S}}_f]^G \cong (\operatorname{gr}_K \widetilde{U}(\mathfrak{g}, f))^G = \operatorname{gr}_K \widetilde{U}(\mathfrak{g}, f)^{\mathfrak{g}}$$

by Proposition 7.2. In the above, the last equality is true since $\mathfrak{g}$ is simple and G connected. □

Remark 8.3 The algebra $\widetilde{U}(\mathfrak{g}, f)$ is the algebra of *twisted differential operators* (tdo) [186] that quantizes the twisted contangent bundle $G \times_{\mathbf{M}} (\chi + \mathfrak{m}^{\perp})$. Thus, the finite $\mathcal{W}$-algebra can be defined as the G-invariant subalgebra of this tdo.

8.4 Quantized Moore–Tachikawa operation and Drineld-Sokolov reduction

A *Harish–Chandra $U(\mathfrak{g})$-algebra* is a $U(\mathfrak{g})$-algebra A equipped with an action of G such that $(ga).(gb) = g(ab)$ for $g \in G$, $a, b \in A$, and the $\mathfrak{g}$-action on A obtained by differentiating the action of G coincides with the adjoint action of $\mathfrak{g}$. A *quantization A of a Hamiltonian G-scheme X* is a Harish–Chandra $U(\mathfrak{g})$-algebra such that $\operatorname{gr} A = \mathscr{O}(X)$, where $A_m := U_m(\mathfrak{g}).A$.

Let A, B be Harish–Chandra $U(\mathfrak{g})$-algebras. We define an algebra $A \circ B$ by

$$A \circ B := H^0_{BRST}(\mathfrak{g}, A \otimes B),$$

where $A \otimes B$ is considered as a diagonal $\mathfrak{g}$-module.

Exercise 8.2 Show that $U(\mathfrak{g}) \circ U(\mathfrak{g}) \cong U(\mathfrak{g})^G = Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

Proposition 8.3 *We have $\mathcal{D}(G) \circ A \cong A$ for any Harish–Chandra $U(\mathfrak{g})$-algebra A.*

Proof From Exercise 8.3 below, it is enough to show that $H^0_{BRST}(\mathfrak{g}, \mathcal{D}(G)) = \mathbb{C}$. But this is easy to see. □

Exercise 8.3 Let A be a Harish–Chandra $U(\mathfrak{g})$-algebra, and let $\rho^* \colon A \to \mathbb{C}[G] \otimes A$ be the comodule map, so that $\rho^*(a) = \sum_i f_i \otimes a_i$ if $ga = \sum_i f_i(g)a_i$, $f_i \in \mathbb{C}[G]$, $a_i \in A$, for all $g \in G$.

(i) Check that ρ^* is an algebra homomorphism.

(ii) Define the algebra homomorphism $\phi\colon \mathbb{C}[G] \otimes A \to \mathbb{C}[G] \otimes A$ as the composition

$$\mathbb{C}[G] \otimes A \xrightarrow{1\otimes\rho^*} \mathbb{C}[G] \otimes \mathbb{C}[G] \otimes A \xrightarrow{m^*\otimes 1} \mathbb{C}[G] \otimes A,$$

where $m\colon G\times G \to G$ is the multiplication map. Show that ϕ is an isomorphism such that

$$(g_L \otimes g) \circ \phi = \phi \circ (g_L \otimes 1), \quad (g_R \otimes 1) \circ \phi = \phi \circ (g_R \otimes g),$$

where $(g_L)(f)(g_1) = f(g^{-1}g_1)$, $(g_R)(f)(g_1) = f(g_1 g)$.

(iii) Define the algebra homomorphism

$$\Phi\colon \mathcal{D}(G) \otimes A \longrightarrow \mathcal{D}(G) \otimes A$$

by

$$\Phi((\tilde{\mu}_R(u)f) \otimes a) = (\tilde{\mu}_R(u) \otimes 1)\phi^{-1}(f \otimes a)$$

for $u \in U(\mathfrak{g})$, $f \in \mathbb{C}[G]$, $a \in A$. Show that Φ is an isomorphism such that

$$(g_L \otimes g) \circ \Phi = \Phi \circ (g_L \otimes 1), \quad (g_R \otimes 1) \circ \Phi = \Phi \circ (g_R \otimes g).$$

8.4.1 Drinfeld–Sokolov reduction in the algebra setting

For a Harish–Chandra $U(\mathfrak{g})$-algebra A, define the algebra $DS_f(A)$ by

$$DS_f(A) := H^0_{BRST,\chi}(\mathfrak{m}, A).$$

If A is a quantization of a Hamiltonian G-scheme X, one can define the Kazhdan filtration $K_\bullet A$ using the grading $A = \bigoplus_j A[j]$, $A[j] = \{a \in A\colon [h,a] = 2ja\}$. This induces a filtration $K_\bullet DS_f(A)$ of $DS_f(A)$.

Theorem **??** gives the following.

Theorem 8.2 *Let A be a quantization of a Hamiltonian G-scheme X. Then $H^i_{BRST,\chi}(\mathfrak{m}, A) = 0$ for $i \neq 0$, and*

$$DS_f(A) \cong \widetilde{U}(\mathfrak{g}, f) \circ A.$$

Moreover, we have the Poisson algebra isomorphism

$$\mathrm{gr}_K\, DS_f(A) \cong \mathbb{C}[DS_f(X)] = \mathbb{C}[X \times_{\mathfrak{g}^*} \mathscr{S}_f].$$

8.4.2 Drinfeld–Sokolov reduction for modules

Let $\mathcal{HC}(\mathfrak{g})$ be the category of Harish–Chandra bimodules, that is, the full subcategory of the cagegory of $U(\mathfrak{g})$-bimodules M consisting of objects M on which the adjoint action of $\mathfrak{g}$ is integrable, that is, locally finite.

A *good filtration* of $M \in \mathcal{HC}(\mathfrak{g})$ is an increasing, separated, exhaustive filtration $F_\bullet M$ of M such that $U_i(\mathfrak{g})F_p M U_j(\mathfrak{g}) \subset F_{p+i+j}M$, $[U_i(\mathfrak{g}), F_j M] \subset F_{i+j-1}M$, and the associated graded $\operatorname{gr}_F M = \bigoplus_p F_p M/F_{p-1}M$ is *finitely generated* as a $\mathbb{C}[\mathfrak{g}^*]$-module. Note that $\operatorname{gr}_F M \in \overline{\mathcal{HC}}(\mathfrak{g})$.

A good filtration exists if M is finitely generated.

For $M, N \in \mathcal{HC}(\mathfrak{g})$, $M \otimes N \otimes Cl(\mathfrak{g})$ is naturally a module over $C(\mathfrak{g}) \otimes Cl(\mathfrak{g})$, where $U(\mathfrak{g})$ acts on $M \otimes N$ diagonally. So we can define

$$H^\bullet_{BRST}(\mathfrak{g}, M \otimes N) = H^\bullet(M \otimes N \otimes Cl(\mathfrak{g}), \operatorname{ad} Q).$$

Let

$$M \circ N := H^0_{BRST}(\mathfrak{g}, M \otimes N).$$

Note that if M and N are bimodules over Harish–Chandra $U(\mathfrak{g})$-algebras A and B, respectively, then $M \circ N$ is naturally a bimodule over $A \circ B$. In particular it is a module over $U(\mathfrak{g}) \circ U(\mathfrak{g}) = Z(\mathfrak{g})$.

Proposition 8.4 *For any* $M \in \mathcal{HC}(\mathfrak{g})$ *we have*

$$\mathcal{D}(G) \circ M \cong M$$

as a bimodule over $\mathcal{D}(G) \circ U(\mathfrak{g}) = U(\mathfrak{g})$.

For $M \in \mathcal{HC}(\mathfrak{g})$, $M \otimes Cl(\mathfrak{m})$ is naturally a module over $C(\mathfrak{m}) = U(\mathfrak{m}) \otimes Cl(\mathfrak{m})$, and so we can define the cohomology $H^\bullet_{BRST,\chi}(\mathfrak{m}, M)$. Define

$$DS_f(M) := H^0_{BRST,\chi}(\mathfrak{m}, M),$$

which is a bimodule over $DS_f(U(\mathfrak{g}) = U(\mathfrak{g}, f)$.

Let $M \in \mathcal{HC}(\mathfrak{g})$ be finitely generated, and let $F_\bullet M$ a good filtration of M. Then we can define the corresponding Kazhdan filtration $K_\bullet M$ by

$$K_p M = \sum_{i-j \leqslant p} F_i M[j],$$

where $F_i M[j] = F_i M \cap M[j]$, with $M[j] = \{m \in M : [h, m] = 2jm\}$. Then $K_\bullet M$ is also good, and $\operatorname{gr}_K M \in \overline{HC}(\mathfrak{g})$. It induces a filtration $K_\bullet DS_f(M)$ of $DS_f(M)$, and $\operatorname{gr}_K DS_f(M)$ is naturally a Poisson module over $\operatorname{gr}_K U(\mathfrak{g}, f) = \mathbb{C}[\mathscr{S}_f]$.

Let $U(\mathfrak{g}, f)$ -biMod be the category of $U(\mathfrak{g}, f)$-bimodules. The following assertion follows from Theorem 7.9.

Theorem 8.3 *For $M \in \mathcal{HC}(\mathfrak{g})$, we have $H^i_{BRST,\chi}(\mathfrak{m}, M) = 0$ for $i \neq 0$. Therefore, the functor*

$$\mathcal{HC}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}, f)\text{-biMod}, \quad M \mapsto DS_f(M),$$

is exact. We have

$$DS_f(M) \cong \widetilde{U}(\mathfrak{g}, f) \circ M$$

as a bimodule over $\widetilde{U}(\mathfrak{g}, f) \circ U(\mathfrak{g}) = U(\mathfrak{g}, f)$. Moreover if $M \in \mathcal{HC}(\mathfrak{g})$ is finitely generated and $K_\bullet M$ is a good Kazhdan filtration then

$$\mathrm{gr}_K\, DS_f(M) \cong DS_f(\mathrm{gr}_K\, M)$$

as a Poisson module over $\mathbb{C}[\mathscr{S}_f]$.

8.5 Representation theory of finite $\mathcal{W}$-algebras

In this section, we will be concerned with representation theory of finite $\mathcal{W}$-algebras. More results presented in this section are due to Losev [199, 201, 202, 203, 204].

8.5.1 Primitive ideals and finite $\mathcal{W}$-algebras

Let I be a two-sided ideal of $U(\mathfrak{g})$. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on I, so that $\mathrm{gr}\, I$ becomes a graded Poisson ideal in $\mathbb{C}[\mathfrak{g}^*]$. Thus, $U(\mathfrak{g})/I$ is a quantization of the Hamiltonian G-scheme $\widetilde{\mathscr{V}}(\mathrm{gr}\, I) = \mathrm{Spec}\, \mathbb{C}[\mathfrak{g}^*]/\mathrm{gr}\, I$. Note that the variety $\mathscr{V}(\mathrm{gr}\, I) = \mathrm{Specm}\, \mathbb{C}[\mathfrak{g}^*]/\mathrm{gr}\, I = \widetilde{\mathscr{V}}(\mathrm{gr}\, I)_{\mathrm{red}} \subset \mathfrak{g}^*$ is nothing but the *associated variety* of I (see Section D.5).

Consider the exact sequence $0 \to I \to U(\mathfrak{g}) \to U(\mathfrak{g})/I \to 0$ in $\mathcal{HC}(\mathfrak{g})$. By Theorem 8.3, applying the exact functor $DS_f(-)$ we obtain the exact sequence

$$0 \longrightarrow DS_f(I) \longrightarrow U(\mathfrak{g}, f) \longrightarrow DS_f(U(\mathfrak{g})/I) \longrightarrow 0.$$

Following [202], we set

$$I_\dagger := DS_f(I),$$

which is a two-sided ideal of $U(\mathfrak{g}, f)$, so that

$$DS_f(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_\dagger.$$

By Theorem 8.3,

$$\mathrm{gr}_K\, DS_f\,(U(\mathfrak{g})/I) \cong DS_f\,(\mathrm{gr}_K\, U(\mathfrak{g})/I) = \mathbb{C}[\widetilde{\mathscr{V}}(\mathrm{gr}\, I) \times_{\mathfrak{g}^*} \mathscr{S}_f\,].$$

Recall that a proper two-sided ideal I of $U(\mathfrak{g})$ is called *primitive* if it is the annihilator of a simple left $U(\mathfrak{g})$-module (see Section D.5). Given a nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$, we denote by $\mathrm{Prim}_{\mathbb{O}} U(\mathfrak{g})$ the set of all primitive ideal of $U(\mathfrak{g})$ such that $\mathscr{V}(\mathrm{gr}\, I) = \overline{\mathbb{O}}$.

The following assertion immediately follows from Theorem 8.3 and Corollary 7.2 (see [199]).

Theorem 8.4 (Losev) *Let* $I \in \mathrm{Prim}_{\mathbb{O}} U(\mathfrak{g})$. *Then*

(i) $DS_f\,(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_\dagger$ *is nonzero if and only if* $G.f \subset \overline{\mathbb{O}}$.
(ii) $DS_f\,(U(\mathfrak{g})/I)$ *is finite-dimensional if and only if* $\overline{\mathbb{O}} = \overline{G.f}$. *Moreover, if this is the case,* $\dim DS_f\,(U(\mathfrak{g})/I)) = \mathrm{mult}_{\overline{G.f}}\, \widetilde{\mathscr{V}}(\mathrm{gr}\, I)$,

where the integer $\mathrm{mult}_{\overline{G.f}}\, \widetilde{\mathscr{V}}(\mathrm{gr}\, I)$ *is defined in the proof of Corollary 7.2.*

In fact, the following much stronger result is known ([202]).

Theorem 8.5 (Losev) *Let* $I \in \mathrm{Prim}_{G.f}\, U(\mathfrak{g})$. *Then* $DS_f\,(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_\dagger$ *is a (finite-dimensional) semisimple algebra.*

8.5.2 Skryabin equivalence

A $\mathfrak{g}$-module E is called a *Whittaker module* if for all $x \in \mathfrak{m}$, $x - \chi(x)$ acts on E locally nilpotently. A *Whittaker vector* in a Whittaker $\mathfrak{g}$-module E is a vector $v \in E$ which satisfies $(x - \chi(x)v = 0$ for any $x \in \mathfrak{m}$, i.e., $xv = \chi(x)v$ for any $x \in \mathfrak{m}$.

Let $\mathrm{Wh}(\mathfrak{g})$ be the category of finitely generated Whittaker $\mathfrak{g}$-modules and set for E an object of this category,

$$\mathrm{Wh}(E) := \{v \in E\colon\, (x - \chi(x))v = 0 \text{ for any } x \in \mathfrak{m}\}.$$

Observe that $\mathrm{Wh}(E) = 0$ implies that $E = 0$. Let $U(\mathfrak{g}, f)$ -Mod be the category of finitely generated $U(\mathfrak{g}, f)$-modules.

The following result is usually referred to as the *Skryabin equivalence*. See [227, Appendix] and [128, Theorem 6.1] for a proof.

Theorem 8.6 (Skryabin equivalence) *The functor*

$$Q_\chi \otimes_{U(\mathfrak{g},f)} -\colon U(\mathfrak{g}, f)\text{ -Mod} \longrightarrow \mathrm{Wh}(\mathfrak{g}), \quad V \longmapsto Q_\chi \otimes_{U(\mathfrak{g},f)} V$$

is an equivalence of categories, with

$$\mathrm{Wh}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}, f)\text{ -Mod}, \quad E \longmapsto \mathrm{Wh}(E),$$

as inverse.

Corollary 8.1 *Let I be a two-sided ideal of $U(\mathfrak{g})$. Then* $\mathrm{Wh}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}, f)$ -Mod *restricts to the equivalence*

$$\mathrm{Wh}(\mathfrak{g})^I \xrightarrow{\sim} U(\mathfrak{g}, f)/I_{\dagger}\text{ -Mod},$$

where $\mathrm{Wh}(\mathfrak{g})^I$ *is the full subcategory of* $\mathrm{Wh}(\mathfrak{g})$ *consisting of objects M that is annihilated by I.*

There is a ramification of the Skryabin's equivalence. It is an equivalence between the *category* $\mathcal{O}$ (see [75]) for a finite $\mathcal{W}$-algebra and the category of *generalized Whittaker* $U(\mathfrak{g})$*-modules*. This was conjectured in [75] and proved by Losev [203].

8.5.3 Classification of finite-dimensional representation of finite $\mathcal{W}$-algebras and primitive ideals

By Theorem 8.5, any $I \in \mathrm{Prim}_{G.f}\, U(\mathfrak{g})$ gives rise to an irreducible finite-dimensional representation of $U(\mathfrak{g}, f)$. Conversely, let E be a finite-dimensional irreducible representation of $U(\mathfrak{g}, f)$. Then, by Theorem 8.6, $Q_\chi \otimes_{U(\mathfrak{g},f)} E$ is simple, and thus, $I = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{U(\mathfrak{g},f)} E)$ is a primitive ideal of $U(\mathfrak{g})$. Moreover, $I \in \mathrm{Prim}_{G.f}\, U(\mathfrak{g})$ by [228]. In fact, E is a $U(\mathfrak{g}, f)/I_{\dagger}$-module ([201, 135]). In other words, the map

$$E \mapsto \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{U(\mathfrak{g},f)} E)$$

from the set of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, f)$-modules to the set $\mathrm{Prim}_{G.f}\, U(\mathfrak{g})$ is surjective. Moreover, any fiber of this map is a single $C(f)$-fibers, where $C(f) = G^\natural/(G^\natural)^\circ$ is the component group of the stabilizer $G^\natural := Z_G(e, h, f)$ of the $\mathfrak{sl}_2$-triple (e, h, f). This was partially proved by Losev in [203], and then completed by Losev and Ostrik [204].

8.5.4 Multiplicity free primitive ideals and one-dimensional representations of finite $\mathcal{W}$-algebras

A primitive ideal I of $U(\mathfrak{g})$ is called *multiplicity free* if $\mathrm{mult}_{\overline{\mathbb{O}}}\, \widetilde{\mathscr{V}}(\mathrm{gr}\, I) = 1$, where $\mathbb{O}$ is the nilpotent orbit such that $\mathscr{V}(\mathrm{gr}\, I) = \overline{\mathbb{O}}$. A multiplicity free primitive ideal I is *completely prime*, that is, $U(\mathfrak{g})/I$ is a domain.

By Theorem 8.5, if I is a multiplicity free primitive ideal such that $\mathscr{V}(\mathrm{gr}\, I) = \overline{G.f}$, then $DS_f(U(\mathfrak{g})/I) = U(\mathfrak{g}, f)/I_{\dagger}$ is one-dimensional. In particular, $U(\mathfrak{g}, f)$ admits a one-dimensional representation. Conversely, it is known that if E is a one-dimensional representation of $U(\mathfrak{g}, f)$ then $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{U(\mathfrak{g},f)} E)$ is multiplicity free.

Theorem 8.7 *Any finite $\mathcal{W}$-algebra admits a one-dimensional representation (equivalently, a two-sided ideal of codimension one).*

8.5.5 Classical Miura map

Assume that f is *even* (cf. Example D.3) or, equivalently, that the Dynkin grading $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ is even, so that $\ell = \mathfrak{g}_{\frac{1}{2}} = 0$. Then $\mathfrak{m} = \bigoplus_{j>0} \mathfrak{g}_j$.

Let $\mathfrak{m}_- = \bigoplus_{j<0} \mathfrak{g}_j$ be the opposed Lie subalgebra to $\mathfrak{m}$. We have

$$\mathfrak{g} = \mathfrak{m}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{m}. \tag{8.4}$$

Note that $\mathfrak{m}^\perp = \mathfrak{g}_0 \oplus \mathfrak{m}$ is a parabolic subalgebra of $\mathfrak{g}$ (containing the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$). Set $\mathfrak{m}_-^\perp = \mathfrak{m}_- \oplus \mathfrak{g}_0$.

Let $\{x_i\}_{1 \leqslant i \leqslant m}$ be a basis of $\mathfrak{m}$, and extend it to a basis $\{x_i\}_{1 \leqslant i \leqslant n}$ of $\mathfrak{g}$. Let $c_{i,j}^k$ be the structure constants with respect to this basis. Consider the linear map $\theta_0 \colon \mathfrak{m} \to C(\mathfrak{m})$ of Lemma 8.1 with respect to the Lie algebra $\mathfrak{m}$ (i.e., $\chi = 0$ and $\mathfrak{g} = \mathfrak{m}$ in this lemma). Extend it to a linear map $\theta_0 \colon \mathfrak{g} \to C(\mathfrak{g}, \mathfrak{m}) := U(\mathfrak{g}) \otimes Cl(\mathfrak{m})$ by setting

$$\theta_0(x_i) = x_i \otimes 1 + 1 \otimes \sum_{1 \leqslant j,k \leqslant m} c_{i,j}^k x_k x_j^*.$$

We already know that the restriction of θ_0 to $\mathfrak{m}$ is a Lie algebra homomorphism and

$$[\theta_0(x), 1 \otimes y] = 1 \otimes [x, y] \quad \text{for } x, y \in \mathfrak{m}.$$

Although θ_0 is not a Lie algebra homomorphism, we have the following.

Lemma 8.3 *The restriction of θ_0 to $\mathfrak{m}_-^\perp$ is a Lie algebra homomorphism. We have $[\theta_0(x), 1 \otimes y^*] = 1 \otimes \mathrm{ad}^*(x)(y)$ for $x \in \mathfrak{m}_-^\perp$, $y \in \mathfrak{m}^*$, where ad^* denote the coadjoint action and $\mathfrak{m}^*$ is identified with $(\mathfrak{g}/\mathfrak{m}_-^\perp)^*$.*

Recall that $U(\mathfrak{g}, f) = H^0(C(\mathfrak{g}, \mathfrak{m}), \mathrm{ad}\, Q)$, where $C(\mathfrak{g}, \mathfrak{m}) := U(\mathfrak{g}) \otimes Cl(\mathfrak{m})$. Let $C(\mathfrak{g}, \mathfrak{m})_+$ denote the subalgebra of $C(\mathfrak{g}, \mathfrak{m})$ generated by $\theta_0(\mathfrak{m})$ and $\bigwedge(\mathfrak{m}) \subset Cl(\mathfrak{m})$, and let $C(\mathfrak{g}, \mathfrak{m})_-$ denote the subalgebra generated by $\theta_0(\mathfrak{m}_-^\perp)$ and $\bigwedge(\mathfrak{m}^*) \subset Cl(\mathfrak{m})$.

Lemma 8.4 *The multiplication map gives a linear isomorphism*

$$C(\mathfrak{g}, \mathfrak{m})_- \otimes C(\mathfrak{g}, \mathfrak{m})_+ \xrightarrow{\sim} C(\mathfrak{g}, \mathfrak{m}).$$

Lemma 8.5 *The subspaces $C(\mathfrak{g}, \mathfrak{m}))_-$ and $C(\mathfrak{g}, \mathfrak{m})_+$ are subcomplexes of $(C(\mathfrak{g}, \mathfrak{m}), \mathrm{ad}\, Q)$. Hence $C(\mathfrak{g}, \mathfrak{m}) \cong C(\mathfrak{g}, \mathfrak{m})_- \otimes C(\mathfrak{g}, \mathfrak{m})_+$ as complexes.*

Proof The fact that $C(\mathfrak{g}, \mathfrak{m})_-$ is subcomplex is obvious (see Lemma 8.2). The fact that $C(\mathfrak{g}, \mathfrak{m})_+$ is a subcomplex follows from the following formulas.

$$[Q, \theta_0(x_i)] = \sum_{m+1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} c_{k,i}^j \theta_0(x_j)(1 \otimes x_k^*) - 1 \otimes \sum_{1 \leqslant j,k \leqslant m} c_{i,j}^k \chi(x_k) x_j^*$$

$$[Q, 1 \otimes x_i^*] = -1 \otimes \frac{1}{2} \sum_{1 \leqslant j,k \leqslant m} c_{j,k}^i x_j^* x_k^*.$$

Proposition 8.5 $H^\bullet(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q) \cong H^\bullet(C(\mathfrak{g},\mathfrak{m}), \operatorname{ad} Q)$.

Proof By Lemma 8.5 and Kunneth's Theorem,

$$H^p(C(\mathfrak{g},\mathfrak{m}), \operatorname{ad} Q) \cong \bigoplus_{i+j=p} H^i(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q) \otimes H^j(C(\mathfrak{g},\mathfrak{m})_+, \operatorname{ad} Q).$$

On the other hand, we have $\operatorname{ad}(Q)(1\otimes x_i) = \theta_\chi(x_i) = \theta_0(x_i) - \chi(x_i)$ for $i = 1, \dots, m$. Hence $C(\mathfrak{g},\mathfrak{m})_-$ is isomorphic to the tensor product of complexes of the form $\mathbb{C}[\theta_\chi(x_i)] \otimes \bigwedge(x_i)$ with the differential $\theta_\chi(x_i) \otimes x_i^*$, where x_i^* is the contraction with x_i. Each of these complexes has one-dimensional zeroth cohomology and zero first cohomology. Therefore $H^i(C(\mathfrak{g},\mathfrak{m})_+, \operatorname{ad} Q) = \delta_{i,0}\mathbb{C}$. This completes the proof. □

Note that the cohomological gradation takes only non-negative values on $C(\mathfrak{g},\mathfrak{m})_-$. Hence by Proposition 8.5 we may identify $U(\mathfrak{g}, f) = H^0(C(\mathfrak{g},\mathfrak{m}), \operatorname{ad} Q)$ with the subalgebra $H^0(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q) = \{c \in C(\mathfrak{g},\mathfrak{m})_-^0 \mid (\operatorname{ad} Q)c = 0\}$ of $C(\mathfrak{g},\mathfrak{m})_-$.

Consider the decomposition

$$C(\mathfrak{g},\mathfrak{m})_- = \bigoplus_{j\leqslant 0} C(\mathfrak{g},\mathfrak{m})_{-,j}, \quad C(\mathfrak{g},\mathfrak{m})_{-,j} = \{c \in C(\mathfrak{g},\mathfrak{m})_-^0 \mid [\theta_0(h), c] = 2jc\}.$$

Note that $C(\mathfrak{g},\mathfrak{m})_{-,0}$ is generated by $\theta_0(\mathfrak{g}_0)$ and is isomorphic to $U(\mathfrak{g}_0)$. The projection

$$C(\mathfrak{g},\mathfrak{m})_- \to C(\mathfrak{g},\mathfrak{m})_{-,0} \cong U(\mathfrak{g}_0)$$

is an algebra homomorphism, and hence, its restriction

$$\Upsilon\colon U(\mathfrak{g}, f) = H^0(C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} Q) \longrightarrow U(\mathfrak{g}_0)$$

is also an algebra homomorphism.

Proposition 8.6 *The map Υ is an embedding.*

Let $K_\bullet C(\mathfrak{g},\mathfrak{m})_\pm$ be the filtration of $C(\mathfrak{g},\mathfrak{m})_\pm$ induced by the Kazhdan filtration of $C(\mathfrak{g},\mathfrak{m})$. We have the isomorphism

$$\mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}(\mathfrak{m}) = \operatorname{gr}_K C(\mathfrak{g},\mathfrak{m}) \cong \operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_- \otimes \operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_+$$

as complexes. Similarly as above, we have $H^i(\operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} \bar{Q}) = \delta_{i,0}\mathbb{C}$, and

$$H^0(\overline{C}(\mathfrak{g},\mathfrak{m}), \operatorname{ad} \bar{Q}) \cong H^0(\operatorname{gr}_K C(\mathfrak{g},\mathfrak{m})_-, \operatorname{ad} \bar{Q}). \tag{8.5}$$

Proof (of Proposition 8.6) The filtration $K_\bullet U(\mathfrak{g}_0)$ of $U(\mathfrak{g}_0) \cong C(\mathfrak{g},\mathfrak{m})_{-,0}$ induced by the Kazhdan filtration coincides with the usual PBW filtration. By (8.5) and Theorem 7.4, the induced map

$$H^0(\mathrm{gr}_K\, C(\mathfrak{g}, \mathfrak{m})_-, \mathrm{ad}\, Q) \longrightarrow \mathrm{gr}_K\, U(\mathfrak{g}_0)$$

can be identified with the restriction map

$$\bar{\Upsilon}\colon \mathbb{C}[\mathscr{S}_f] = \mathbb{C}[f + \mathfrak{m}^\perp]^M \longrightarrow \mathbb{C}[f + \mathfrak{g}_0]. \tag{8.6}$$

So it is sufficient to show that $\bar{\Upsilon}$ is injective.

If $\varphi \in \mathbb{C}[f + \mathfrak{m}^\perp]^M$ is in the kernel, $\varphi(g.x) = 0$ for all $g \in M$ and $x \in f + \mathfrak{g}_0$. Hence it is enough to show that the image of the the action map

$$M \times (f + \mathfrak{g}_0) \longrightarrow f + \mathfrak{m}^\perp, \quad (g, x) \mapsto g.x, \tag{8.7}$$

is Zariski dense in $f + \mathfrak{m}^\perp$.

The differential of this morphism at $(1, x) \in M \times (f + \mathfrak{g}_0)$ is given by

$$\mathfrak{m} \times \mathfrak{g}_0 \longrightarrow \mathfrak{m}^\perp, \quad (y, z) \mapsto [y, x] + z.$$

This is an isomorphism if $x \in f + (\mathfrak{g}_0)_{\mathrm{ss,reg}}$, where $(\mathfrak{g}_0)_{\mathrm{ss,reg}} = \{x \in (\mathfrak{g}_0)_{\mathrm{ss}} \colon \dim \mathfrak{g}_0^x = r\}$, with $(\mathfrak{g}_0)_{\mathrm{ss}}$ the set of semisimple elements of $\mathfrak{g}_0$. Indeed, if $x \in (\mathfrak{g}_0)_{\mathrm{ss,reg}}$ then $\mathfrak{g}^x = \mathfrak{g}_0^x$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}^x \cap \mathfrak{m} = \{0\}$. Hence (8.7) is a dominant morphism as required, see e.g. [245, Theorem 16.5.7]. □

Remark 8.4 In the case where f is regular, the fact that $\bar{\Upsilon}$ is injective is well-known. In this case $\mathfrak{g}_0$ is the Cartan subalgebra $\mathfrak{h}$ Identifiying $\mathbb{C}[\mathscr{S}_f] \cong \mathbb{C}[\mathfrak{g}]^G$ and $\mathbb{C}[f+\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]$, the map $\bar{\Upsilon}$ is just the Chevalley restriction map $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$, where $\mathcal{W}$ is the Weyl group associated with $(\mathfrak{g}, \mathfrak{h})$.

It is also possible to extend to the case f is not even, for instance, using the construction of Kac, Roan and Wakimoto [163].

The advantage of the above proof is that it applies to a general finite $\mathcal{W}$-algebra ([206]), and also, it generalizes to the affine setting, see §**??**.

Definition 8.3 The map Υ is called the *classical Miura map*.

Chapter 9
Chiral quantized BRST cohomology and $\mathcal{W}$-algebras

In this chapter, we will construct a differential graded vertex algebra, so that its cohomology algebra is a vertex algebra and that will be our main object to study in the next chapters.

We follow the same approach as in Chapter 7.

9.1 Chiral BRST reduction

9.1.1 The BRST complex

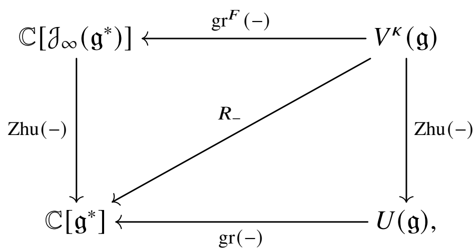
Let G be any connected affine algebraic group with Lie algebra $\mathfrak{g}$, and $\chi \in \mathfrak{g}^*$ a character. We wish to define the vertex algebra analogue of the BRST reduction.

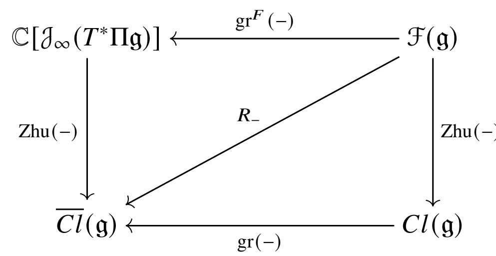
Exercise 9.1 Let V be a vertex superalgebra, and fix an odd element Q of V such that $Q_{(n)}Q = 0$ for all $n \geqslant 0$.

(i) Show that $Q^2_{(0)} = 0$.

(ii) Show that the quotient $\dfrac{\ker Q_{(0)}}{\operatorname{im} Q_{(0)}}$ is naturally a vertex algebra, provided it is nonzero.

Fix a symmetric invariant bilinear form κ on $\mathfrak{g}$, and let $V^{\kappa}(\mathfrak{g})$ be the universal affine vertex algebra associated with $(\mathfrak{g}, \kappa)$ (see Section 3.1). Let also $\mathcal{F}(\mathfrak{g})$ be the fermion Fock space as in §5.6.3. We have the following commutative diagrams.





As before, choose a basis $\{x_i\}_{1\leqslant i\leqslant d}$ of $\mathfrak{g}$ and let $\{x_i^*\}_{1\leqslant i\leqslant d}$ be the corresponding dual basis of $\mathfrak{g}^*$. Denote by $c_{i,j}^k$ the structure constants of $\mathfrak{g}$, that is, $[x_i, x_j] = \sum_{k=1}^d c_{i,j}^k x_k$ for $i, j = 1, \ldots, d$.

Lemma 9.1 *Let $\kappa_\mathfrak{g}$ be the Killing form of $\mathfrak{g}$. The following map defines a vertex algebra homomorphism.*

$$\begin{aligned} \hat{\rho}\colon V^{\kappa_\mathfrak{g}}(\mathfrak{g}) &\longrightarrow \mathcal{F}(\mathfrak{g}) \\ x_i(z) &\longmapsto \sum_{j,k} c_{i,j}^k \,{}^\circ_\circ \psi_k(z)\psi_j^*(z) {}^\circ_\circ \,. \end{aligned}$$

The map $\hat{\rho}$ induces an algebra homomorphism

$$\mathrm{Zhu}\, V^{\kappa_\mathfrak{g}}(\mathfrak{g}) = U(\mathfrak{g}) \longrightarrow \mathrm{Zhu}\, \mathcal{F}(\mathfrak{g}) = Cl(\mathfrak{g})$$

and a Poisson algebra homomorphism

$$R_{V^{\kappa_\mathfrak{g}}(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*] \longrightarrow R_{\mathcal{F}(\mathfrak{g})} = \overline{Cl}(\mathfrak{g})$$

that are identical to ρ and $\bar{\rho}$, see Lemma 7.1 and (7.2), respectively.

Define

$$\hat{C}(\mathfrak{g}) := V^{-\kappa_\mathfrak{g}}(\mathfrak{g}) \otimes \mathcal{F}(\mathfrak{g}).$$

Since it is a tensor product of two vertex algebras, $\hat{C}(\mathfrak{g})$ is a vertex algebra. We have

$$R_{\hat{C}(\mathfrak{g})} = R_{V^{-\kappa_\mathfrak{g}}(\mathfrak{g})} \otimes R_{\mathcal{F}(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl} = \overline{C}(\mathfrak{g}),$$

and

$$\mathrm{Zhu}\, \hat{C}(\mathfrak{g}) = \mathrm{Zhu}\, V^{-\kappa_\mathfrak{g}}(\mathfrak{g}) \otimes \mathrm{Zhu}\, \mathcal{F}(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cl(\mathfrak{g}) = C(\mathfrak{g}).$$

Thus, $\hat{C}(\mathfrak{g})$ is a chiralization of $C(\mathfrak{g})$, considered in §7.1.1. Further we have

$$\mathrm{gr}^F \hat{C}(\mathfrak{g}) = \mathrm{gr}^F V^{-\kappa_\mathfrak{g}}(\mathfrak{g}) \otimes \mathrm{gr}\, \mathcal{F}(\mathfrak{g}) = \mathbb{C}[J_\infty(\mathfrak{g}^*)] \otimes \mathbb{C}[J_\infty(T^*\Pi\mathfrak{n})].$$

Define a gradation

$$\mathcal{F}(\mathfrak{g}) = \bigoplus_{p\in\mathbb{Z}} \mathcal{F}^p(\mathfrak{g}) \tag{9.1}$$

by setting $\deg \psi_{i,m} = -1$, $\deg \psi^*_{j,k} = 1$, for all $i, j \in \{1, \dots, d\}$, $m, k \in \mathbb{Z}$, and $\deg |0\rangle = 0$. This induces a $\mathbb{Z}$-grading (that is different from the conformal grading) on $C^\kappa(\mathfrak{g})$:

$$\hat{C}(\mathfrak{g}) = V^{-\kappa_\mathfrak{g}}(\mathfrak{g}) \otimes \mathcal{F}(\mathfrak{g}) = \bigoplus_{p \in \mathbb{Z}} \hat{C}^p(\mathfrak{g}), \quad \text{where} \quad \hat{C}^p(\mathfrak{g}) := V^\kappa(\mathfrak{g}) \otimes \mathcal{F}^p(\mathfrak{g}). \tag{9.2}$$

Lemma 9.2 *The following defines a vertex algebra homomorphism.*

$$\begin{aligned} \hat{\theta}_\chi \colon V^0(\mathfrak{g}) &\longrightarrow \hat{C}(\mathfrak{g}) \\ x_i(z) &\longmapsto (x_i(z) + \chi(x_i)) \otimes \mathrm{id} + \mathrm{id} \otimes \hat{\rho}(x_i(z)). \end{aligned}$$

The map $\hat{\theta}_\chi$ induces an algebra homomorphism

$$\mathrm{Zhu}\, V^0(\mathfrak{g}) = U(\mathfrak{g}) \longrightarrow \mathrm{Zhu}\, \hat{C}(\mathfrak{g}) = C(\mathfrak{g})$$

and a Poisson algebra homomorphism

$$R_{V^0(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*] \longrightarrow R_{\mathcal{F}(\mathfrak{g})} = \overline{C(\mathfrak{g})}$$

that are identical to θ_χ and $\bar{\theta}$, respectively (see Lemmas 7.3 and 8.1).

The proof of the following assertion is similar to that of Lemma 7.4.

Proposition 9.1 *There exists a unique element $\hat{Q} \in \hat{C}^1(\mathfrak{g})$ such that*

$$[\hat{Q}_\lambda (1 \otimes \psi_i)] = \hat{\theta}_\chi(x_i) \quad \text{for all} \quad i.$$

We have $[\hat{Q}_\lambda \hat{Q}] = 0$.

The field $\hat{Q}(z)$ is given explicitly as

$$\hat{Q}(z) = \sum_i (x_\alpha(z) + \chi(x_i)) \otimes \psi_i^*(z) - \mathrm{id} \otimes \frac{1}{2} \sum_{i,j,k} c_{i,j}^k \,{:}\psi_i^*(z)\psi_j^*(z)\psi_k(z){:}\,.$$

This formula looked strange: not rather $x_\alpha(z)$ or $x_i(z)$ and then ${:}\psi_i^*(z)\psi_j^*(z)\psi_k(z){:}$? I changed: ok?

Since $\hat{Q}$ is odd and $[\hat{Q}_\lambda \hat{Q}] = 0$, we have

$$\hat{Q}_{(0)}^2 = 0.$$

(Recall that we write $\hat{Q}(z) = \sum_{n \in \mathbb{Z}} \hat{Q}_{(n)} z^{-n-1}$.) So $(\hat{C}(\mathfrak{g}), \hat{Q}_{(0)})$ is a cochain complex.

Lemma 9.3 *If it is nonzero, the cohomology $H^\bullet(\hat{C}(\mathfrak{g}), \hat{Q}_{(0)})$ inherits the vertex algebra structure from $\hat{C}(\mathfrak{g})$.*

Proof This follows from Exercise 9.1. □

Remark 9.1 By Proposition 9.1, $\hat{\theta}_\chi$ induces a *trivial* action of $\hat{\mathfrak{g}}$ on the BRST cohomology $H^\bullet(\hat{C}(\mathfrak{g}), \hat{Q}_{(0)})$. This forces to set the level κ of $\hat{\mathfrak{g}}$ to be zero.

More generally, let V be a $V^{-\kappa_{\mathfrak{g}}}(\mathfrak{g})$-vertex algebra, that is, a vertex algebra equipped with a vertex algebra homomorphism $V^{-\kappa_{\mathfrak{g}}}(\mathfrak{g}) \to V$. Then $V \otimes \mathcal{F}(\mathfrak{g})$ is a $\hat{C}(\mathfrak{g})$-vertex algebra, so that $(V \otimes \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is a differential graded vertex algebra in the above sense, and its cohomology $H^{\bullet}(V \otimes \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is naturally a vertex algebra. We set

(9.3) $$H^{\bullet}_{BRST,\chi}(\hat{\mathfrak{g}}, V) := H^{\bullet}(V \otimes \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)}).$$

Let M be a V-module. Then $(M \otimes \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is a differential graded vertex module over $(V \otimes \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$. Thus,

(9.4) $$H^{\bullet}_{BRST,\chi}(\hat{\mathfrak{g}}, M) := H^{\bullet}(M \otimes \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$$

is a module over $H^{\bullet}_{BRST,\chi}(\hat{\mathfrak{g}}, V)$. Restricting the action to the vertex subalgebra $H^{0}_{BRST,\chi}(\hat{\mathfrak{g}}, V)$, we obtain a functor

$$V\text{-Mod} \longrightarrow H^{0}_{BRST,\chi}(\hat{\mathfrak{g}}, V)\text{-Mod}, \quad M \mapsto H^{0}_{BRST,\chi}(\hat{\mathfrak{g}}, M).$$

In the case that $\mathfrak{g}$ is simple (so that $\chi = 0$), it is useful to consider the relative BRST cohomology. For a $V^{-\kappa_{\mathfrak{g}}}(\mathfrak{g})$-module M, set

$$C(\hat{\mathfrak{g}}, \mathfrak{g}, M) := \{c \in M \otimes \mathcal{F}(\mathfrak{g}) : \hat{\theta}(x_i)_{(0)}x = (\psi_i)_{(0)}c = 0 \text{ for all } i\}.$$

By Proposition 9.1, $C(\hat{\mathfrak{g}}, \mathfrak{g}, M)$ is a subcomplex of $M \otimes \mathcal{F}(\mathfrak{g})$. We define

$$H^{\bullet}_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, M) := H^{\bullet}(C(\hat{\mathfrak{g}}, \mathfrak{g}, M), \hat{Q}_{(0)}).$$

Then $H^{\bullet}_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, V)$ is naturally a vertex algebra for a $V^{-\kappa_{\mathfrak{g}}}(\mathfrak{g})$-vertex algebra V, and we have a functor

$$V\text{-Mod} \longrightarrow H^{0}_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, V)\text{-Mod}, \quad M \mapsto H^{0}_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, M).$$

Remark 9.2 The complex $(M \otimes \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is identical to Feigin's standard complex for the semi-infinite $\hat{\mathfrak{g}}$-cohomology $H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, M \otimes \mathbb{C}_{\chi})$ with coefficient in the $\hat{\mathfrak{g}}$-module $M \otimes \mathbb{C}_{\chi}$ ([106]):

(9.5) $$H^{\bullet}_{BRST,\chi}(\hat{\mathfrak{g}}, M) = H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, M \otimes \mathbb{C}_{\chi}),$$

where $\mathbb{C}_{\chi}$ is the one-dimensional representation of $\hat{\mathfrak{g}}$ define by the character

$$\hat{\mathfrak{g}} \longrightarrow \mathbb{C}, \quad xt^n \longmapsto \delta_{n,-1}\chi(x), \ K \longmapsto 0.$$

Similarly, we have

(9.6) $$H^{\bullet}_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, M) = H^{\frac{\infty}{2}+\bullet}(\hat{\mathfrak{g}}, \mathfrak{g}, M).$$

9.1.2 The grading and the conformal structure of BRST reductions

Suppose that a $V^{-\kappa_{\mathfrak{g}}}(\mathfrak{g})$-vertex algebra V is graded by a Hamiltonian H_V (resp. a conformal vector L_V with central charge c_V).

The Fock space $\mathcal{F}(\mathfrak{g})$ has a conformal vector $L_{\mathcal{F}(\mathfrak{g})}$ of central charge $-2\dim\mathfrak{g}$ defined by

$$L_{\mathcal{F}(\mathfrak{g})}(z)=\sum_{n\in\mathbb{Z}}(L_{\mathcal{F}(\mathfrak{g})})_n z^{-n-2}=\sum_i {}^{\circ}_{\circ}(\partial_z\psi_i^*(z))\psi_i(z){}^{\circ}_{\circ}.$$

The corresponding Hamiltonian $(L_{\mathcal{F}(\mathfrak{g})})_0$ satisfies

$$[(L_{\mathcal{F}(\mathfrak{g})})_0,(\psi_i)_{(m)}]=m(\psi_i)_{(m)},\quad [(L_{\mathcal{F}(\mathfrak{g})})_0,(\psi_i)^*_{(m)}]=(m+1)(\psi_i^*)_{(m)},$$

and $(L_{\mathcal{F}(\mathfrak{g})})_0|0\rangle=0$. Thus, $\mathcal{F}(\mathfrak{g})=\bigoplus_{\Delta\geqslant 0}\mathcal{F}(\mathfrak{g})_\Delta$. It follows that

$$H=H_V+(L_{\mathcal{F}(\mathfrak{g})})_0\quad(\text{resp. } L(z)=L_V(z)+L_{\mathcal{F}(\mathfrak{g})}(z)) \tag{9.7}$$

defines a Hamiltonian (resp. a conformal vector with central charge $c=c_V-2\dim\mathfrak{g}$) on $V\otimes\mathcal{F}(\mathfrak{g})$.

If $\chi=0$, $[\hat{Q}_{(0)},H]=0$ (resp. $[\hat{Q}_{(0)},L(z)]=0$), and H (resp. L) defines a Hamiltonian (resp. a conformal vector) on $H^\bullet_{BRST}(\hat{\mathfrak{g}},V)$. In particular, if V is positively graded, then so are $H^\bullet_{BRST}(\hat{\mathfrak{g}},V)$ and $H^\bullet_{BRST}(\hat{\mathfrak{g}},\mathfrak{g},V)$.

This construction has to be modified when $\chi\neq 0$. We continue to assume that there is a grading $\mathfrak{g}=\bigoplus_{i\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_j$ such that $\chi(\mathfrak{g}_j)=0$ unless $j=1$. In addition, we assume that this grading is inner, that is, there exists $h\in V$ such that

$$\begin{aligned}&h_{(0)}x_i=d_ix_i,\\&(L_V)_nh=\delta_{n,0}h,\quad h_{(n)}h=k'\delta_{n,1}|0\rangle\quad n\geqslant 0\end{aligned}$$

for some $k'\in\mathbb{C}$, and $h_{(0)}$ acts semisimply on V with eigenvalues in $\mathbb{Z}$, where $\{x_i\}$ is a homogeneous basis of $\mathfrak{g}$ with $x_i\in\mathfrak{g}_{d_i}$. Set

$$\hat{h}(z)=h(z)+h_{\mathcal{F}(\mathfrak{g})}(z),\quad h_{\mathcal{F}(\mathfrak{g})}(z):=\sum_i 2d_i\,{}^{\circ}_{\circ}\psi_i(z)\psi_i^*(z){}^{\circ}_{\circ}.$$

Lemma 9.4 *We have*

$$\begin{aligned}&[(h_{\mathcal{F}(\mathfrak{g})})_\lambda h_{\mathcal{F}(\mathfrak{g})}]=\sum_i 4d_i^2\lambda,\\&[(L_{\mathcal{F}(\mathfrak{g})})_\lambda h_{\mathcal{F}(\mathfrak{g})}]=h'_{\mathcal{F}(\mathfrak{g})}+h_{\mathcal{F}(\mathfrak{g})}\lambda+\sum_i 2d_i\lambda^2,\end{aligned}$$

and $L_{\mathcal{F}(\mathfrak{g}),new}=L_{\mathcal{F}(\mathfrak{g})}+\frac{1}{2}Th_{\mathcal{F}(\mathfrak{g})}$ *is a conformal vector of* $\mathcal{F}(\mathfrak{g})$ *of central charge*

$$-2\dim\mathfrak{g} - 12\sum_i d_i^2 + 12\sum_i d_i = -\sum_i (12d_i^2 - 12d_i + 2).$$

Define

$$H_{new} = H - \frac{1}{2}\hat{h}_{(0)} \quad (\text{resp. } L_{new}(z) = L(z) + \frac{1}{2}\partial_z \hat{h}(z)).$$

Then H_{new} (resp. $L_{new}(z)$) commutes with $\hat{Q}_{(0)}$ and defines a Hamiltonian (resp. a conformal vector) of $H^\bullet_{BRST,\chi}(\hat{\mathfrak{g}}, V)$. The central charge of L_{new} is given by

$$\begin{aligned}(9.8)\qquad & c_V - 3k' - \sum_i (12d_i^2 - 12d_i + 2)\\ &= c_V - 3k' - \dim\mathfrak{g} + \dim\mathfrak{g}_0 - \frac{1}{2}\dim\mathfrak{g}_{1/2} - 3h^\vee|h|^2 + 12(\rho|h)\\ &= c_V - 3k' + \dim\mathfrak{g}_0 - \frac{1}{2}\dim\mathfrak{g}_{1/2} - 12\left|\frac{\rho}{\sqrt{h^\vee}} - \frac{\sqrt{h^\vee}}{2}h\right|^2.\end{aligned}$$

Note that, as the Hamiltonian is modified, the grading on $H^\bullet_{BRST,\chi}(\hat{\mathfrak{g}}, V)$ is not a priori bounded under below even if V is so with respect to H.

9.1.3 BRST reduction in the Poisson vertex setting

Recall that $\mathrm{gr}^F V^\kappa(\mathfrak{g}) = \mathbb{C}[\mathcal{J}_\infty\mathfrak{g}^*]$. Thus, $\mathrm{gr}\,\hat{C}(\mathfrak{g})$ can be identified with $\mathbb{C}[\mathcal{J}_\infty\mathfrak{g}^*] \otimes \mathrm{gr}^F \mathcal{F}(\mathfrak{g})$. One find that $(\mathbb{C}[\mathcal{J}_\infty\mathfrak{g}^*] \otimes \mathrm{gr}^F \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is a differential graded Poisson vertex algebra, where we denote by the same symbol $\hat{Q}_{(0)}$ the derivation induced by $\hat{Q}_{(0)}$. Thus, $H^\bullet(\mathbb{C}[\mathcal{J}_\infty\mathfrak{g}^*] \otimes \mathrm{gr}^F \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is naturally a Poisson vertex algebra.

More generally, let V be a $\mathbb{C}[\mathcal{J}_\infty\mathfrak{g}^*]$-Poisson vertex algebra, that is, a Poisson vertex algebra equipped with a Poisson vertex algebra homomorphism $\mathbb{C}[\mathcal{J}_\infty\mathfrak{g}^*] \to V$. Then in the same way as above $(V \otimes \mathrm{gr}^F \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is a differential graded Poisson vertex algebra, and $H^\bullet(V \otimes \mathrm{gr}^F \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)})$ is naturally a Poisson vertex algebra. We set

$$H^\bullet_{BRST,\chi}(\hat{\mathfrak{g}}, V) := H^\bullet(V \otimes \mathrm{gr}^F \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)}).$$

Similarly, for a Poisson vertex V-module M, we define a Poisson vertex $H^\bullet_{BRST,\chi}(\hat{\mathfrak{g}}, V)$-module by

$$H^\bullet_{BRST,\chi}(\hat{\mathfrak{g}}, M) := H^\bullet(M \otimes \mathrm{gr}^F \mathcal{F}(\mathfrak{g}), \hat{Q}_{(0)}).$$

If $\mathfrak{g}$ is simple and $\chi = 0$, we define the relative BRST cohomology

$$H^\bullet_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, M)$$

in the same way.

9.1.4 Chiral BRST reduction

Let X be an affine Poisson scheme equipped with a Hamiltonian G-action, $\mu\colon X \to \mathfrak{g}^*$ the moment map, and $\chi \in \mathfrak{g}^*$ a character. The morphism $J_\infty\mu\colon J_\infty X \to J_\infty\mathfrak{g}^*$ gives a Poisson vertex algebra homomorphism

$$(J_\infty\mu)^*\colon \mathbb{C}[J_\infty\mathfrak{g}^*] \longrightarrow \mathbb{C}[J_\infty X].$$

Thus, the Poisson vertex algebra $H^\bullet_{BRST,\chi}(\hat{\mathfrak{g}}, \mathbb{C}[J_\infty X])$ is well defined. We set

$$J_\infty X /\!\!/_{BRST,\chi} J_\infty G := \operatorname{Spec}(H^0_{BRST,\chi}(\hat{\mathfrak{g}}, \mathbb{C}[J_\infty X]).$$

Theorem 9.1 *Let X be an affine Poisson scheme is equipped with a Hamiltonian G-action, $\mu\colon X \to \mathfrak{g}^*$ the moment map, $\chi \in \mathfrak{g}^*$ a character. Suppose that the assumptions of Theorem 7.1 are satisfied, that is,*

(i) *the moment map $\mu\colon X \to \mathfrak{g}^*$ is flat,*
(ii) *there exists a subscheme $\mathscr{S}$ of $\mu^{-1}(\chi)$ such that the action map gives the isomorphism $G \times \mathscr{S} \xrightarrow{\sim} \mu^{-1}(\chi)$.*

Then

$$H^\bullet_{BRST,\chi}(\hat{\mathfrak{g}}, \mathbb{C}[J_\infty X]) \cong \mathbb{C}[J_\infty \mathscr{S}] \otimes H^\bullet_{DR}(G),$$

where $H^\bullet_{DR}(G)$ is equipped with the trivial Poisson vertex algebra structure. In particular,

$$J_\infty X /\!\!/_{BRST,\chi} J_\infty G \cong J_\infty(X /\!\!/_{BRST,\chi} G).$$

Proof The flatness of μ implies the flatness of $J_\infty\mu\colon J_\infty X \to J_\infty\mathfrak{g}^*$. Therefore the assertion can be shown as in same way as Theorem 7.1. □

Actually, the relative BRST cohomology behaves better if G is simple (and so $\chi = 0$).

Theorem 9.2 *In the setting of Theorem 9.1, suppose that G is simple and $\chi = 0$. Then*

$$H^i_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, \mathbb{C}[J_\infty X]) \cong \delta_{i,0}\mathbb{C}[J_\infty \mathscr{S}].$$

9.1.5 Chiral quantization

Let X be an affine Poisson scheme. A *strict chiral quantization* of X is a vertex algebra V such that $\operatorname{gr} V \cong \mathbb{C}[J_\infty X]$ as Poisson vertex algebras. So this is equivalent to that $\tilde{X}_V \cong X$. For instance, $V^\kappa(\mathfrak{g})$ is a chiral quantization of $\mathfrak{g}^*$.

Suppose that X is equipped with a Hamiltonian G-action, and let $\mu\colon X \to \mathfrak{g}^*$ be the moment map. A *chiral quantized moment map* for the G-action on X is a pair $(V, \hat{\mu}^*)$ of a chiral quantization V of X and a vertex algebra homomorphism $\hat{\mu}^*\colon V^k(\mathfrak{g}) \to V$ for some k such that

- the $\mathfrak{g}[t]$-action on V integrates to the action of $\mathcal{J}_\infty G = G[[t]]$,
- the $G[[t]]$-equivariant morphism $\operatorname{gr}^F V^k(\mathfrak{g}) = \mathbb{C}[\mathcal{J}_\infty \mathfrak{g}^*] \to \mathbb{C}[\mathcal{J}_\infty X] = \operatorname{gr}^F V$ induced by $\hat{\mu}^*$ coincides with $(\mathcal{J}_\infty \mu)^*$,

The first condition is satisfied if if V belongs to KL_k as a $\hat{\mathfrak{g}}$-module.

In this setting, the associated graded Poisson vertex algebra $(\operatorname{gr}(V \otimes \mathcal{F}(\mathfrak{g})), \hat{Q}_{(0)})$ is isomorphic to $(\mathbb{C}[X] \otimes \operatorname{gr}^F \mathcal{F}(\mathfrak{g})), \hat{Q}_{(0)})$. Thus, we have a spectral sequence for $H^\bullet_{BRSR,\chi}(\hat{\mathfrak{g}}, V)$ such that

$$E_1^{\bullet,q} = H^q_{BRSR,\chi}(\hat{\mathfrak{g}}, \mathbb{C}[\mathcal{J}_\infty X]).$$

Similarly, there is a spectral sequence for $H^\bullet_{BRSR}(\hat{\mathfrak{g}}, \mathfrak{g}, V)$ such that

$$E_1^{\bullet,q} = H^q_{BRSR}(\hat{\mathfrak{g}}, \mathfrak{g}, \mathbb{C}[\mathcal{J}_\infty X]).$$

The following assertion follows immediately from Theorem 9.1.

Proposition 9.2 *Let X be an affine Poisson scheme is equipped with a Hamiltonian G-action, $\mu\colon X \to \mathfrak{g}^*$ the moment map, $\chi \in \mathfrak{g}^*$ a character. Let V be a chiral quantization of X, $\hat{\mu}^*\colon V^{-\kappa_\mathfrak{g}}(\mathfrak{g}) \to V$ a chiral quantized moment map. Suppose that the assumptions of Theorem 7.1 are satisfied. Assume further the above spectral sequence converges to $H^\bullet_{BRSR,\chi}(\hat{\mathfrak{g}}, V)$. Then*

$$\operatorname{gr} H^\bullet_{BRSR,\chi}(\hat{\mathfrak{g}}, V) \cong H^\bullet_{BRSR,\chi}(\hat{\mathfrak{g}}, \mathbb{C}[\mathcal{J}_\infty X]) = \mathbb{C}[\mathcal{J}_\infty(X /\!\!/_{BRST,\chi} G)] \otimes H^\bullet_{DR}(G).$$

In particular, $H^0_{BRSR,\chi}(\hat{\mathfrak{g}}, V)$ is a chiral quantization of $X /\!\!/_{BRST,\chi} G$. The same is true for $H^\bullet_{BRSR}(\hat{\mathfrak{g}}, \mathfrak{g}, V)$ if G is simple and $\chi = 0$.

The convergency of the spectral sequence can be an issue in the vertex algebra setting, especially when $\chi \neq 0$.

Theorem 9.3 *In the setting of Proposition 9.2, suppose that G is simple, $\chi = 0$ and that V is a direct sum of objects in $\mathbf{KL}_{-\kappa_\mathfrak{g}}$ as a $\hat{\mathfrak{g}}$-module. Here, $\mathbf{KL}_{-\kappa_\mathfrak{g}}$ stands for $\mathbf{KL}_{-2h^\vee}$ in the notation of §6.3. Then $H^i_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, V) = 0$ for $i \neq 0$ and*

$$\operatorname{gr} H^0_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, V) = H^0_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, \operatorname{gr}^F V) = \mathbb{C}[\mathcal{J}_\infty(X /\!\!/_{BRST} G)].$$

Therefore, $H^0_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, V)$ is a chiral quantization of $X /\!\!/_{BRST} G$.

Proof One needs to show that the spectral sequence converges. For $M \in \mathbf{KL}_{-\kappa_\mathfrak{g}}$, the complex for $H^i_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, M)$ is a direct sum of the eigenspaces of the grading operator $-D+(L_{\mathcal{F}(\mathfrak{g})})_0$, which commutes with the action of the differential. Moreover these eigenspaces are finite-dimensional. So the complex for $H^0_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, V)$ is a direct sum of finite-dimensional subcomplexes, and the filtration is regular in the sense of [76] on each of these subcomplexes. □

9.1.6 Chiral differential operators

Assume in this paragraph that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple. Recall from (3.10) and Example 3.7 that the vertex algebra of chiral differential operators on G is defined, for $k \in \mathbb{C}$, by:

$$\mathcal{D}^{\mathrm{ch}}_{G,k} = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathcal{O}(\mathcal{J}_\infty G),$$

where K acts on $\mathcal{O}(\mathcal{J}_\infty G)$ as the multiplication by k and $\mathfrak{g}[t] \subset \mathfrak{g}[[t]] = \mathrm{Lie}(G[[t]])$ acts as left invariant vector fields, with $\mathcal{J}_\infty G = G[[t]]$. Recall from Example 4.7 that the following isomorphisms hold:

$$\mathrm{gr}^F \mathcal{D}^{\mathrm{ch}}_{G,k} \cong \mathcal{O}(\mathcal{J}_\infty T^* G) \quad \text{and} \quad X_{\mathcal{D}^{\mathrm{ch}}_{G,k}} = \tilde{X}_{\mathcal{D}^{\mathrm{ch}}_{G,k}} = T^* G.$$

Furthermore, according to [22] (see also [21]), we have

$$\mathrm{Zhu}(\mathcal{D}^{\mathrm{ch}}_{G,k}) \cong \mathcal{D}_G.$$

Thus, $\mathcal{D}^{\mathrm{ch}}_{G,k}$ is a chiral quantization of $T^* G$. Moreover, the vertex algebra embeddings $\hat{\mu}^*_L := \pi_L \colon V^k(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{G,k}$ and $\hat{\mu}^*_R := \pi_R \colon V^{k^\vee}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{G,k}$, with

$$k + k^\vee = -2h^\vee.$$

defined by Theorem 3.1 and Theorem 3.2, respectively, are chiral quantized moment maps of the moment maps $\mu_L, \mu_R \colon T^* G \cong G \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by $(g, x) \mapsto x$ and $(g, x) \mapsto -g.x = -(\mathrm{Ad}^* g).x$, respectively, since $(\mathcal{J}_\infty \mu_L)^* = \mathrm{gr}\, \hat{\mu}^*_L$, $(\mathcal{J}_\infty \mu_R)^* = \mathrm{gr}\, \hat{\mu}^*_R$.

Exercise 9.2 Show that $\mathcal{D}^{\mathrm{ch}}_{G,k}$ is simple for all $k \in \mathbb{C}$ using the associated scheme of $\mathcal{D}^{\mathrm{ch}}_{G,k}$ (compare with Exercise 3.3).

9.2 $\mathcal{W}$-algebras

In this section, it is required that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple. Let f be a nilpotent element of $\mathfrak{g}$, and $\mathfrak{m}$ as in (7.5). We now apply the above construction to the moment map

$$\mu \colon \mathfrak{g}^* \longrightarrow \mathfrak{m}^* \ni \chi = (f|-)$$

As $\kappa_{\mathfrak{m}} = 0$, we write $L\mathfrak{m}$ for $\widehat{\mathfrak{m}} = \mathfrak{m}[t, t^{-1}]$, and $V(\mathfrak{m})$ for $V^{-\kappa_{\mathfrak{m}}}(\mathfrak{m})$. A natural chiral quantization of $\mathfrak{g}^*$ is $V^k(\mathfrak{g})$, $k \in \mathbb{C}$, and the chiral quantized moment map of μ is the natural emebedding $V(\mathfrak{m}) \hookrightarrow V^k(\mathfrak{g})$. So we have the corresponding BRST reduction $H^{\bullet}_{BRST,\chi}(L\mathfrak{m}, V^k(\mathfrak{g}))$.

9.2.1 Universal $\mathcal{W}$-algebras

Definition 9.1 The universal *(affine) $\mathcal{W}$-algebra* $\mathcal{W}^k(\mathfrak{g}, f)$ associated with $(\mathfrak{g}, f, k)$ is defined to be the vertex algebra

$$\mathcal{W}^k(\mathfrak{g}, f) := H^0_{BRST,\chi}(L\mathfrak{m}, V^k(\mathfrak{g})).$$

In the case that f is regular, we briefly denote by $\mathcal{W}^k(\mathfrak{g})$ the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f)$.

This definition of $\mathcal{W}^k(\mathfrak{g}) = \mathcal{W}^k(\mathfrak{g}, f_{\mathrm{reg}})$ is due to Feigin and Frenkel [108] in the case that f_{reg} is regular, and to Kac, Roan and Wakimoto [163] in the case that $\mathfrak{g}_{1/2} = 0$ (that is, f is an even nilpotent element). One can show that the above definition is equivalent to that of [163] in general, using [74] and the following vanishing result (cf. [30]), see also Chapter 10. This argument also shows that the definition is independent of the choice of a Lagrangian subspace $\ell \subset \mathfrak{g}_{1/2}$.

Theorem 9.4 *Assume that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple and let $\chi = (f|-)$ be as before.*

(i) *We have $H^i_{BRST,\chi}(L\mathfrak{m}, V^k(\mathfrak{g})) = 0$ for $i \neq 0$, and $\mathcal{W}^k(\mathfrak{g}, f)$ is a strict chiral quantization of the Slodowy slice $\mathscr{S}_f$, that is,*

$$\mathrm{gr}^F \mathcal{W}^k(\mathfrak{g}, f) \cong \mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f].$$

In particular, $\tilde{X}_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathscr{S}_f$ and $SS(\mathcal{W}^k(\mathfrak{g}, f)) \cong \mathcal{J}_\infty \mathscr{S}_f$.

(ii) *We have*

$$\mathrm{Zhu}(\mathcal{W}^k(\mathfrak{g}, f)) \cong U(\mathfrak{g}, f).$$

The rest of this section is devoted to the proof of Theorem 9.4, (i). For the proof of (ii) we refer to [15].

Let $C = C(L\mathfrak{m}, V^k(\mathfrak{g})) = V^k(\mathfrak{g}) \otimes \mathcal{F}(\mathfrak{m})$, and let $C_- = C(L\mathfrak{m}, t^{-1}\mathfrak{m}[t^{-1}], V^k(\mathfrak{g}))$ be the quotient of C by the subspace spanned by $\hat{\theta}_\chi((x_i)_{(-n-1)})c$ and $(\psi_i)_{(-n-1)}c$ with $1 \leqslant i \leqslant \dim \mathfrak{m}$, $n \geqslant 0$, $c \in C$. As $[\hat{Q}_{(0)}, (\psi_i)_{(-n-1)}] = \hat{\theta}_\chi((x_i)_{(-n-1)})$ by Proposition 9.1, C_- is a quotient complex of C. Set

$$H^{\bullet}_{BRST,\chi}(L\mathfrak{m}, t^{-1}\mathfrak{m}[t^{-1}], V^k(\mathfrak{g})) = H^{\bullet}(C_-, \hat{Q}_{(0)}).$$

(This is a relative semi-infinite cohomology, cf. [118].)

Note that C_- is a direct sum of finite-dimensional subcomplexes while C is not. Indeed, we have

$$C_- = \bigoplus_{\Delta \geqslant 0} (C_-)_\Delta, \quad \dim(C_-)_\Delta < \infty \text{ for all } \Delta, \tag{9.9}$$

where $(C_-)_\Delta = \{c \in C_- : H_{new}c = \Delta c\}$, and each $\dim(C_-)_\Delta$ is a subcomplex since the differential commutes with H_{new}.

Theorem 9.5 *The natural surjection $C \to C_-$ induces the linear isomorphism*

$$H^\bullet_{BRST,\chi}(L\mathfrak{m}, V^k(\mathfrak{g})) \xrightarrow{\sim} H^\bullet_{BRST,\chi}(L\mathfrak{m}, t^{-1}\mathfrak{m}[t^{-1}], V^k(\mathfrak{g})).$$

Proof Consider the the grading $C = \bigoplus_{\Delta \geqslant 0} C_{\Delta,old}$ with respect to the (old) grading H. Set $G_{-p}C = \bigoplus_{\Delta \leqslant p} C_\Delta$. Then $\dots G_pC \supset G_{p+1}C \supset G_{-1}C \supset G_0C \supset G_1C = 0$, $C = \bigcup_p G_pC$, and $\hat{Q}_{(0)}G_pC \subset G_pC$. Thus there is a corresponding converging spectral sequence $E_1 \Rightarrow H^\bullet(C) = H^\bullet_{BRST,\chi}(L\mathfrak{m}, V^k(\mathfrak{g}))$. We have

$$E_r = H^i_{BRST,0}(L\mathfrak{m}, V^k(\mathfrak{g})),$$

that is, the BRST reduction for $\chi = 0$. Similarly, the filtration $G_\bullet C$ induces a filtration $G_\bullet C_-$ on the quotient complex C_-. Let $E'_r \Rightarrow H^\bullet(C_-)$ be the corresponding spectral sequence. We have

$$E'_1 = H^\bullet_{BRST,0}(L\mathfrak{m}, t^{-1}\mathfrak{m}[t^{-1}], V^k(\mathfrak{g})),$$

the relative BRST cohomology for $\chi = 0$.

Claim The natural surjection $C \to C_-$ induces the isomorphism $E_1 \xrightarrow{\sim} E'_1$. □

Proof (of the claim) Note that the complex for E_1 is a direct sum of finite-dimensional subcomplexes, consisting of weight spaces with respect to the diagonal action of the extended Cartan subalgebra of $\hat{\mathfrak{g}}$. In particular, we have the converging Hochschild-Serre spectral sequence $E''_r \Rightarrow E_1 = H^i_{BRST,0}(L\mathfrak{m}, V^k(\mathfrak{g}))$ for the subalgebra $t^{-1}\mathfrak{m}[t^{-1}] \subset L\mathfrak{m}$ ([250, Theorem 2.3]). By definition, $(E''_1)^{\bullet,q} = H^{Lie}_{-q}(t^{-1}\mathfrak{m}[t^{-1}], V^k(\mathfrak{g})) \otimes \bigwedge^\bullet(\mathfrak{m}^*[t^{-1}]) \cong \delta_{q,0}C_-$ by the freeness of $V^k(\mathfrak{g})$ over $t^{-1}\mathfrak{m}[t^{-1}]$, and we find that the complex (E''_1, d_1) is isomorphic to that for E'_1. It follows that the spectral sequence collapses at the second page $E''_2 \cong E'_1$. Hence we have $E_1 \cong E'_1$ as required. □

It follows that $E_r \xrightarrow{\sim} E'_r$ for any r, and therefore $E_\infty \xrightarrow{\sim} E'_\infty$ and this completes the proof. □

Although $H^\bullet_{BRST,\chi}(L\mathfrak{m}, t^{-1}\mathfrak{m}[t^{-1}], V^k(\mathfrak{g})) = H^\bullet(C_-, \hat{Q}_{(0)})$ does not have a structure of vertex algebras, $H^\bullet(\mathrm{gr}^F C_-, \hat{Q}_{(0)})$ inherits Poisson vertex algebra structure from $H^\bullet(\mathrm{gr}^F C, \hat{Q}_{(0)})$.

Proposition 9.3 *The following statements holds.*

(i) $\mathrm{gr}^F C_- \cong \mathbb{C}[\mathcal{J}_\infty \mu^{-1}(\chi)]$.
(ii) $H^i(\mathrm{gr}^F C_-) \cong 0$ *for* $i \neq 0$ *and* $H^0(\mathrm{gr}^F C) \cong H^0(\mathrm{gr}^F C_-) \cong \mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f]$ *as Poisson vertex algebras.*

(iii) *The natural surjection* $\mathrm{gr}^F C \to \mathrm{gr}^F C_-$ *induces the isomorphism* $H^\bullet(\mathrm{gr}^F C) \cong H^\bullet(\mathrm{gr}^F C_-)$ *of Poisson vertex algebras.*

Part (ii) of the proposition is due to Suh ([242]).

Theorem 9.6 $\mathrm{gr}^F H^i_{BRST,\chi}(L\mathfrak{m}, t^{-1}\mathfrak{m}[t^{-1}], V^k(\mathfrak{g})) \cong \delta_{i,0}\mathbb{C}[\mathcal{J}_\infty\mu^{-1}(\chi)]$.

Proof Since C_- is a direct sum of finite-dimensional subcomplexes, we have the converging spectral sequence

$$E_r''' \Rightarrow H^\bullet_{BRST,\chi}(L\mathfrak{m}, \mathfrak{m}[t^{-1}]t^{-1}, V^k(\mathfrak{g})) = H^\bullet_{BRST,\chi}(L\mathfrak{m}, V^k(\mathfrak{g}))$$

such that the E_1-term is $H^\bullet(\mathrm{gr}^F C_-)$. By Proposition 9.3, it collapses at $E_1''' = E_\infty'''$. □

Proof (of Theorem 9.4) We have the commutative diagram

$$\begin{array}{ccc} \mathrm{gr}^F H^\bullet(C) & \longrightarrow & H^\bullet(\mathrm{gr}^F C) \\ \big\downarrow & & \big\downarrow \\ \mathrm{gr}^F H^\bullet(C_-) & \longrightarrow & H^\bullet(\mathrm{gr}^F C_-), \end{array}$$

where the horizontal arrows are natural maps, the left vertical arrow is the map induced by the isomorphism in Theorem 9.5, and the right vertical arrow is the isomorphism is Proposition 9.3. Since we have shown all the other maps are isomorphisms, the upper horizontal arrow is an isomorphism as well. We have shown that the natural map $\mathrm{gr}^F H^i_{BRST,\chi}(L\mathfrak{m}, V^k(\mathfrak{g})) \to H^i(\mathrm{gr}^F C) \cong \delta_{i,0}\mathbb{C}[J_\infty \mathscr{S}_f]$ is an isomorphism as required. □

By Theorem 9.4, it follows that $\mathcal{W}^k(\mathfrak{g}, f)$ is positively graded:

$$\mathcal{W}^k(\mathfrak{g}, f) = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}} \mathcal{W}^k(\mathfrak{g}, f)_\Delta, \quad \dim \mathcal{W}^k(\mathfrak{g}, f)_\Delta < \infty.$$

If $k \neq -h^\vee$, from (9.8) one deduces that $\mathcal{W}^k(\mathfrak{g}, f)$ is conformal with central charge

$$\dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{1/2} - 12\left|\frac{\rho}{\sqrt{k+h^\vee}} - \frac{\sqrt{k+h^\vee}}{2}h\right|^2. \tag{9.10}$$

More generally, for any $V(\mathfrak{m})$-vertex algebra V we set

$$DS_f(V) := H^\bullet_{BRST,\chi}(L\mathfrak{m}, V).$$

(So $\mathcal{W}^k(\mathfrak{g}, f) = DS_f(V^k(\mathfrak{g}))$). Also, for a V-module M, we set

$$DS_f(M) := H^\bullet_{BRST,\chi}(L\mathfrak{m}, M),$$

which is a $DS_f(V)$-module. So we have a functor

$$V\text{-Mod} \longrightarrow DS_f(V)\text{-Mod}, \quad M \longmapsto DS_f(M).$$

Exercise 9.3 Assume that $\mathfrak{g} = \mathfrak{sl}_2$, let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and set $\mathfrak{n} := \mathbb{C}e$.

The aim of this exercise is to define the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_2, f)$ associated with $\mathfrak{sl}_2$ and f at level $k \in \mathbb{C}$ in a very explicit, and more elementary, way.

(i) Let $\hat{Cl}$ be the Clifford algebra associated with $\mathfrak{n}[t, t^{-1}] \oplus \mathfrak{n}^*[t, t^{-1}]$ and the symmetric bilinear form $(-|-)$ given by:

$$(et^m|et^n) = (e^*t^m|e^*t^n) = 0, \quad (et^m|e^*t^n) = \delta_{m+n,0}.$$

We write ψ_m for $et^m \in \hat{Cl}$ and ψ_m^* for $e^*t^m \in \hat{Cl}$, $m \in \mathbb{Z}$, so that $\hat{Cl}$ is the associative superalgebra with odd generators ψ_m, ψ_m^*, $m \in \mathbb{Z}$, and relations:

$$[\psi_m, \psi_n] = [\psi_m^*, \psi_n^*] = 0, \quad [\psi_m, \psi_n^*] = \delta_{m+n,0}.$$

Define the *charged fermion Fock space* as

$$\mathcal{F} := \frac{\hat{Cl}}{\sum_{m \geqslant 0} \hat{Cl}\psi_m + \sum_{n \geqslant 1} \hat{Cl}\psi_n^*} \cong \bigwedge(\psi_n)_{n<0} \otimes \bigwedge(\psi_n^*)_{n \leqslant 0},$$

where $\bigwedge(a_i)_{i \in I}$ denotes the exterior algebra with generators a_i, $i \in I$.
Show that there is a unique vertex (super)algebra structure on $\mathcal{F}$ such that the image of 1 is the vacuum $|0\rangle$, and

$$\psi(z) := Y(\psi_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) := Y(\psi_0^*|0\rangle, z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}.$$

Let $V^k(\mathfrak{sl}_2)$ be the universal affine vertex algebra associated with $\mathfrak{sl}_2$ at level k, and set

$$\mathcal{C}^k(\mathfrak{sl}_2) := V^k(\mathfrak{sl}_2) \otimes \mathcal{F}.$$

Define a gradation $\mathcal{F} = \bigoplus_{p \in Z} \mathcal{F}^p$ by setting $\deg \psi_m = -1$, $\deg \psi_n^* = 1$ for all $m, n \in \mathbb{Z}$ and $\deg |0\rangle = 0$. Then set $\mathcal{C}^{k,p}(\mathfrak{sl}_2) := V^k(\mathfrak{sl}_2) \otimes \mathcal{F}^p$. Define a vector $\hat{Q}$ of degree 1 in $\mathcal{C}^{k,1}(\mathfrak{sl}_2)$ by:

$$\hat{Q}(z) := (e(z) + 1) \otimes \psi^*(z).$$

(ii) Verify that $\hat{Q}_{(n)}\hat{Q} = 0$ for all $n \geqslant 0$, and deduce from Exercice 9.1 that the cohomology $H^\bullet(\mathcal{C}^k(\mathfrak{sl}_2), \hat{Q}_{(0)})$ inherits a vertex algebra structure from that of $\mathcal{C}^k(\mathfrak{sl}_2)$, provided that it is nonzero.

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_2, f)$ associated with $(\mathfrak{sl}_2, f)$ at level $k \in \mathbb{C}$ is then defined by:

$$\mathcal{W}^k(\mathfrak{sl}_2, f) := H^0(\mathcal{C}^k(\mathfrak{sl}_2), \hat{Q}_{(0)}).$$

(iii) Set

$$L(z) = L_{sug}(z) + \frac{1}{2}h(z) + L_{\mathcal{F}}(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

where

$$L_{sug}(z) = \frac{1}{2(k+2)}\left(:e(z)f(z): + :f(z)e(z): + \frac{1}{2}h(z)^2 \right),$$
$$L_{\mathcal{F}}(z) = :\partial_z\psi(z)\psi^*(z):.$$

(a) Verify that $\hat{Q}_{(0)}L = 0$, with $L := L_{-2}|0\rangle$, so that L defines an element of $\mathcal{W}^k(\mathfrak{sl}_2, f)$.
(b) Check that $L_{-1} = T$ is the translation operator, that L_0 acts semisimply on $\mathcal{W}^k(\mathfrak{sl}_2, f)$ by

$$\begin{aligned} &L_0|0\rangle = 0, && [L_0, h_{(n)}] = -nh_{(n)}, \\ &[L_0, e_{(n)}] = (1-n)e_{(n)}, && [L_0, f_{(n)}] = (-1-n)f_{(n)}, \\ &[L_0, \psi^*_{(n)}] = (-1-n)\psi^*_{(n)}, && [L_0, \psi_{(n)}] = (1-n)\psi_{(n)}, \end{aligned}$$

and that the L_n's verify the Virasoro relations.

(iv) Assume that $k \neq -2$. Show that there exists a unique vertex algebra homomorphism

$$\mathrm{Vir}^{c(k)} \to \mathcal{W}^k(\mathfrak{sl}_2, f), \quad \text{where} \quad c(k) := 1 - \frac{6(k+1)^2}{k+2}.$$

Remark 9.3 It can be shown that the above homomorphism is actually an isomorphism.

9.2.2 Equivariant $\mathcal{W}$-algebras

We now apply the above constructions to the moment map

$$\mu\colon T^*G \longrightarrow \mathfrak{m} \ni \chi = (f|-).$$

Keeping the notation of §9.1.6, the composition of the embeddings $V(\mathfrak{m}) \hookrightarrow V^k(\mathfrak{g}) \xhookrightarrow{\hat{\mu}_L} \mathcal{D}^{\mathrm{ch}}_{G,k}$ is a chiral quantization of μ. Define the *equivariant* $\mathcal{W}$*-algebra* $\widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ by

$$\widetilde{\mathcal{W}}^k(\mathfrak{g}, f) := DS_f(\mathcal{D}^{\mathrm{ch}}_{G,k}) = H^0_{BRST,\chi}(L\mathfrak{m}, \mathcal{D}^{\mathrm{ch}}_{G,k}).$$

Theorem 9.7 *We have* $H^i_{BRST,\chi}(L\mathfrak{m}, \mathcal{D}^{\mathrm{ch}}_{G,k}) = 0$ *for* $i \neq 0$ *and*

$$\mathrm{gr}^F \widetilde{\mathcal{W}}^k(\mathfrak{g}, f) \cong \mathbb{C}[\mathcal{J}_\infty \widetilde{\mathscr{S}_f}],$$

that is, $\widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ *is a chiral quantization of the equivariant Slodowy slice* $\widetilde{\mathscr{S}_f}$ *defined by* (7.10)*. Moreover we have*

$$\mathrm{Zhu}(\widetilde{\mathcal{W}}^k(\mathfrak{g}, f)) \cong \widetilde{U}(\mathfrak{g}, f).$$

By (9.8), $\widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ is conformal with central charge

$$\dim \mathfrak{g} + \dim \mathfrak{g}_0 - \frac{1}{2}\dim \mathfrak{g}_{1/2} - 3(k+h^\vee)|h|^2 + 12(\rho|h).$$

The vertex algebra homomorphism $\hat{\mu}_L^*\colon V^k(\mathfrak{g}) \to \mathcal{D}^{\mathrm{ch}}_{G,k}$ induces the vertex algebra embedding $\mathcal{W}^k(\mathfrak{g}, f) \hookrightarrow \widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ which we denote also by $\hat{\mu}_L^*$. On the other hand, the vertex algebra homomorphism $\hat{\mu}_R^*\colon V^{k^\vee}(\mathfrak{g}) \to \mathcal{D}^{\mathrm{ch}}_{G,k}$ induces the vertex algebra homomorphism $V^{k^\vee}(\mathfrak{g}) \to \widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$, which is also denoted by $\hat{\mu}_R^*$. The induced actions of $\mathcal{W}^k(\mathfrak{g}, f)$ and $V^{k^\vee}$ obviously commute, and we have the following result.

Theorem 9.8 *The map* $\hat{\mu}_L^*$ *gives the isomorphism*

$$\mathcal{W}^k(\mathfrak{g}, f) \cong \widetilde{\mathcal{W}}^k(\mathfrak{g}, f)^{\mathfrak{g}[[t]]}.$$

9.2.3 Moore–Tachikawa operation

Let V, $\mathcal{W}$ be chiral quantizations of Hamiltonian G-schemes X, Y, equipped with chiral quantized moment maps $\hat{\mu}_X^*\colon V^k(\mathfrak{g}) \to V$, $\hat{\mu}_Y^*\colon V^{k^\vee}(\mathfrak{g}) \to W$, for some $k \in \mathbb{C}$. We define the vertex algebra $V \circ W$ by

$$V \circ W := H^0_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, V \otimes W).$$

(Note that $k = k^\vee$ if $k = -h^\vee$.) More generally, if M is a V-module and N is a $\mathcal{W}$-module, we set $M \circ N$

$$M \circ N := H^0_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, M \otimes N),$$

which is a module over $V \circ W$.

Proposition 9.4 *Let* V *be a* $V^k(\mathfrak{g})$*-vertex algebra on which* $\mathfrak{g}[[t]]$ *acts locally finitely. Let* $\hat{\mathfrak{g}}$ *act on* $\mathcal{D}^{\mathrm{ch}}_{G,k}$ *by* $\hat{\mu}_R^*$ *(so it is a level* $k^\vee$*-action).*

(i) $\mathcal{D}^{\mathrm{ch}}_{G,k} \circ V \cong V$ *as vertex algebras.*
(ii) *More generally, for any* V*-module* M *on which* $\mathfrak{g}[[t]]$ *acts locally finitely,* $\mathcal{D}^{\mathrm{ch}}_{G,k} \circ M \cong M$ *as modules over* $\mathcal{D}^{\mathrm{ch}}_{G,k} \circ V = V$.

9.2.4 BRST reduction of associated varieties

Theorem 9.9 *Let V be a $V^k(\mathfrak{g})$-vertex algebra on which $\mathfrak{g}[[t]]$ acts locally finitely.*

(i) $H^i_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, \tilde{W}^k(\mathfrak{g}, f) \otimes V) = 0$ *for $i \neq 0$ and*

$$DS_f(V) \cong \widetilde{\mathcal{W}}^k(\mathfrak{g}, f) \circ V.$$

Moreover, $\mathrm{gr}^F DS_f(V) \cong \mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f] \otimes_{\mathbb{C}[\mathcal{J}_\infty \mathfrak{g}^*]} \mathrm{gr}\, V$, *so that*

$$SS(DS_f(V))) \cong \mathcal{J}_\infty \mathscr{S}_f \times_{\mathcal{J}_\infty \mathfrak{g}^*} SS(V), \quad \tilde{X}_{DS_f(V)} \cong \mathscr{S}_f \times_{\mathfrak{g}^*} \tilde{X}_V.$$

(ii) *More generally, for any V-module M on which $\mathfrak{g}[[t]]$ acts locally finitely,* $H^i_{BRST}(\hat{\mathfrak{g}}, \mathfrak{g}, \tilde{W}^k(\mathfrak{g}, f) \otimes M) = 0$ *for $i \neq 0$ and*

$$DS_f(M) \cong \widetilde{\mathcal{W}}^k(\mathfrak{g}, f) \circ M$$

Moreover, $\mathrm{gr}^F DS_f(M) \cong \mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f] \otimes_{\mathbb{C}[\mathcal{J}_\infty \mathfrak{g}^*]} \mathrm{gr}\, M$.

From the above theorem, one can compute the associated variety of $DS_f(V)$ where V is any quotient of $V^k(\mathfrak{g})$.

Corollary 9.1 *V be a $V^k(\mathfrak{g})$-vertex algebra on which $\mathfrak{g}[[t]]$ acts locally finitely. The functor*

$$V\text{-Mod} \longrightarrow DS_f(V)\text{-Mod}, \quad M \longmapsto DS_f(M)$$

is exact.

In particular, the surjection $V^k(\mathfrak{g}) \twoheadrightarrow L_k(\mathfrak{g})$ induces the surjection

$$\mathcal{W}^k(\mathfrak{g}, f) = DS_f(V^k(\mathfrak{g})) \twoheadrightarrow DS_f(L_k(\mathfrak{g})). \tag{9.11}$$

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. By (9.11), $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of $DS_f(L_k(\mathfrak{g}))$ provided that $DS_f(L_k(\mathfrak{g})) \neq 0$. The following conjecture is formulated in [163, 167].

Conjecture 9.1 The vertex algebra $DS_f(L_k(\mathfrak{g}))$ is either zero or isomorphic to $\mathcal{W}_k(\mathfrak{g}, f)$.

Conjecture 9.1 has been verified in in many cases [9, 10, 11], but not in general.

Chapter 10
An alternative description of the BRST reduction

We give in this section an alternative description of the Poisson structure on Slodowy slices. This slightly different view point gives a better understanding of the definition of $\mathcal{W}$-algebras presented in [163]. Since the arguments and the constructions are similar to the previous chapter, some details and most of the proofs are omitted.

10.1 Poisson structure on Slodowy slices

We assume that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple. Let G_+ be the unipotent subgroup $\exp(\mathfrak{g}_+)$ of G, where $\mathfrak{g}_+ := \bigoplus_{j>0} \mathfrak{g}_j$.

Lemma 10.1 *Let χ be the linear form of $\mathfrak{g}_+$ defined by $\chi = -(f|-)$.*

(i) *The stabilizer of χ in the group G_+ is the subgroup $G_{\geqslant 1} = \exp(\mathfrak{g}_{\geqslant 1})$.*
(ii) *The G_+-coadjoint orbit $\mathbb{O}_\chi$ of χ is equal to $\chi + \mathfrak{g}^*_{\frac{1}{2}} = \kappa(-f + \mathfrak{g}_{-\frac{1}{2}})$.*

Proof It is enough to show that the stabilizer of χ in $\mathfrak{g}_+$ is $\mathfrak{g}_{\geqslant 1}$, and this is easy to check. □

The orbit $\mathbb{O}_\chi$ is symplectic and the Poisson bracket on $\mathbb{C}[\mathbb{O}_\chi]$ is given by $\{x, y\} = \chi([x, y])$ for $x, y \in \mathfrak{g}_{\frac{1}{2}} \subset \mathbb{C}[\mathbb{O}_\chi] \cong S(\mathfrak{g}_{\frac{1}{2}})$.

The action of G_+ on $\mathfrak{g}^*$ is Hamiltionian whose moment map $\mu_1 : \mathfrak{g} \cong \mathfrak{g}^* \to \mathfrak{g}^*_+$ is given by the restriction. The action of G_+ on $\mathbb{O}_\chi$ is Hamiltonian whose moment map $\mu_2 : \mathbb{O}_\chi \hookrightarrow \mathfrak{g}^*_+$ is given by the natural embedding. Consider the diagonal action of G_+ on $\mathfrak{g}^* \times \mathbb{O}_\chi$ with moment map

$$\mu : \mathfrak{g}^* \times \mathbb{O}_\chi \to \mathfrak{g}^*_+, \qquad (\lambda_1, \lambda_2) \mapsto \mu_1(\lambda_1) + \mu_2(\lambda_2).$$

Then $\mu^{-1}(0) = \{(\lambda_1, \lambda_2) \in \mathfrak{g}^* \times \mathbb{O}_\chi : \mu_1(\lambda_1) = -\lambda_2\} = \mu_1^{-1}(\mathbb{O}_{-\chi}) = \kappa(f + \mathfrak{g}_{\geqslant -\frac{1}{2}})$ as G_+-varieties and it follows from [128] (see Remark 7.1) that $\mathscr{S}_f \cong \mu^{-1}(0)/G_+$ inherits a Poisson structure from $\mathfrak{g}^* \times \mathbb{O}_\chi$.

The moment map μ induces a map

$$\mathcal{J}_\infty\mu\colon \mathcal{J}_\infty(\mathfrak{g}^* \times \mathbb{O}_\chi) \cong \mathcal{J}_\infty\mathfrak{g}^* \times \mathcal{J}_\infty\mathbb{O}_\chi \to \mathcal{J}_\infty\mathfrak{g}_+^*.$$

Moreover $\mathbb{C}[(\mathcal{J}_\infty\mu)^{-1}(0)]^{\mathcal{J}_\infty G_+}$ inherits a Poisson vertex algebra from that of $\mathbb{C}[\mathcal{J}_\infty(\mathfrak{g}^* \times \mathbb{O}_\chi)]$ (see Theorem 4.2) which coincides with that of $\mathbb{C}[\mathscr{S}_f]$ under the isomorphism

$$\mathcal{J}_\infty\mathscr{S}_f \cong (\mathcal{J}_\infty\mu)^{-1}(0)/\mathcal{J}_\infty G_+.$$

Similarly to the previous sections, one can consider the classical Clifford algebra $\bar{C}l(\mathfrak{g}_+) = \bigwedge^\bullet(\mathfrak{g}_+^*) \otimes \bigwedge^\bullet(\mathfrak{g}_+)$. There exists a unique Poisson superalgebra structure on $\bar{C}l(\mathfrak{g}_+)$ such that $\bigwedge^\bullet(\mathfrak{g}_+^*) \to \bar{C}l(\mathfrak{g}_+)$, $x_i \mapsto \psi^* = x_i^* \otimes 1$, $\bigwedge^\bullet(\mathfrak{g}_+) \to \bar{C}l(\mathfrak{g}_+)$, $x_i \mapsto \psi_i = 1 \otimes x_i$ are Poisson superalgebra embeddings, and $\{\psi_i, \psi_j^*\} = \delta_{i,j}$, where $\bigwedge^\bullet(\mathfrak{g}_+^*)$, $\bigwedge^\bullet(\mathfrak{g}_+)$ are equipped with the trivial Poisson structure. Here, $\{x_i\}_i$ and $\{x_i^*\}_i$ denote dual basis of $\mathfrak{g}_+$ and $\mathfrak{g}_+^*$, respectively. Then

$$\bar{C} := \mathbb{C}[\mathfrak{g}^*] \otimes \mathbb{C}[\mathbb{O}_\chi] \otimes \bar{C}l(\mathfrak{g}_+) = \bigoplus_{i \in \mathbb{Z}} \bar{C}^n$$

is a Poisson superalgebra, where

$$\bar{C}^n := \mathbb{C}[\mathfrak{g}^*] \otimes \mathbb{C}[\mathbb{O}_\chi] \otimes \Big(\bigoplus_{j-i=n} \bigwedge^j(\mathfrak{g}_+^*) \otimes \bigwedge^i(\mathfrak{g}_+)\Big),$$

with $\deg \psi_i^* = 1 = -\deg \psi_i$. Similarly to Lemma 7.4, setting

$$\bar{Q} := \sum_{i=1}^{\deg \mathfrak{g}_+} (x_i \otimes 1 + 1 \otimes \mu_2(x_i)) \otimes \psi_i^* - 1 \otimes 1 \otimes \frac{1}{2} \sum_{i,j,k=1}^{\deg \mathfrak{g}_+} c_{i,j}^k \psi_i^* \psi_j^* \psi_k \in \bar{C}^1,$$

we get that $\{\bar{Q}, \bar{Q}\} = 0$ because $\bar{Q}$ is odd. Therefore $H^\bullet(\bar{C}, \operatorname{ad} \bar{Q})$ is a Poisson superalgebra.

The following theorem is proven similarly to Theorem 7.1 and Theorem 7.4, so we omit the proof.

Theorem 10.1 *We have* $H^i(\bar{C}, \operatorname{ad} \bar{Q}) = 0$ *for* $i \neq 0$ *and*

$$H^0(\bar{C}, \operatorname{ad} \bar{Q}) = \mathbb{C}[\mathscr{S}_f]$$

as Poisson algebras.

10.2 Chiralization of the Hamiltonian reduction

Recall that $V^k(\mathfrak{g})$ is strict chiralization of $\mathfrak{g}^*$. Consider the vertex subalgebra $V^0(\mathfrak{g}_+)$ of $V^k(\mathfrak{g})$ generated by the fields $x(z)$ for $x \in \mathfrak{g}_+$. It is a strict chiralization of $\mathfrak{g}_+^*$. The natural embedding $\hat{\mu}_1^*\colon V^0(\mathfrak{g}_+) \hookrightarrow V^k(\mathfrak{g})$ is a chiral quantized moment map of the moment map $\mu_1 \colon \mathfrak{g}^* \to \mathfrak{g}_+^*$. Next, consider the *$\beta\gamma$-system vertex algebra* (or *Weyl vertex algebra*) of rank $\frac{1}{2}\dim \mathbb{O}_\chi$ associated with the symplectic variety $\mathbb{O}_\chi$. This is the vertex algebra $\mathcal{F}_\chi$ generated by the fields $\Phi_i(z)$, $i = 1, \ldots, \dim \mathfrak{g}_{\frac{1}{2}}$, with OPE's

$$[(\Phi_i)_\lambda \Phi_j] = \chi([x_i, x_j]),$$

where $\{x_i\}_i$ is a basis of $\mathfrak{g}_{\frac{1}{2}}$. This is a strict chiralization of $\mathbb{O}_\chi$. Moreover, the following vertex algebra morphism

$$\begin{aligned} \hat{\mu}_2^* \colon V^0(\mathfrak{g}_+) &\longrightarrow \mathcal{F}_\chi \\ x_i &\longmapsto \begin{cases} 0, & \text{if } x_i \in \mathfrak{g}_{>1}, \\ \chi(x_i), & \text{if } x_i \in \mathfrak{g}_1, \\ \Phi_i, & \text{if } x_i \in \mathfrak{g}_{\frac{1}{2}}. \end{cases} \end{aligned}$$

is a chiral quantized moment map of the moment $\mu_2 \colon \mathbb{O}_\chi \to \mathfrak{g}_+^*$. Finally, consider the *semi-infinite fermions* (or *free fermions*) $\bigwedge^{\frac{\infty}{2}}(\mathfrak{g}_+^*)$, which is the vertex superalgebra generated by the (odd) fields $\Psi_i(z), \Psi_i^*(z)$ ($i = 1, \ldots, \dim \mathfrak{g}_+$) with λ-bracket:

$$[(\Psi_i)_\lambda \Psi_j^*] = \delta_{i,j}, \qquad [(\Psi_i)_\lambda \Psi_j] = 0 = \quad [(\Psi_i^*)_\lambda \Psi_j^*].$$

It is a chiralization of $\bar{C}l(\mathfrak{g}_+)$. Set

$$C(V^k(\mathfrak{g})) := V^k(\mathfrak{g}) \otimes \mathcal{F}_\chi \otimes \textstyle\bigwedge^{\frac{\infty}{2}}(\mathfrak{g}_+^*),$$

and

$$Q := \sum_{i=1}^{\deg \mathfrak{g}_+} \big(x_i \otimes 1 + 1 \otimes \hat{\mu}_2^*(x_i)\big) \otimes \Psi_i^* - 1 \otimes 1 \otimes \frac{1}{2} \sum_{i,j,k=1}^{\deg \mathfrak{g}_+} c_{i,j}^k :\Psi_i^* \Psi_j^* \Psi_k: .$$

Then $[Q_\lambda Q] = 0$ and so $Q_{(0)}^2 = 0$, where $Q(z) = \sum_{n \in \mathbb{Z}} Q_{(n)} z^{-n-1}$, and, hence, $H^\bullet(C(V^k(\mathfrak{g})), Q_{(0)})$ is a vertex superalgebra.

We refer to [108, 58, 163, 113] to the following theorem.

Theorem 10.2 *We have $H^i(C(V^k(\mathfrak{g})), Q)$ if $i \neq 0$, and*

$$\mathcal{W}^k(\mathfrak{g}, f) = H^0(C(V^k(\mathfrak{g})), Q_{(0)})$$

is the $\mathcal{W}$-algebra associated with $\mathfrak{g}$ and f.

We have $R_{C(V^k(\mathfrak{g}))} = \mathbb{C}[\mathfrak{g}^*] \otimes \mathbb{C}[\mathbb{O}_\chi] \otimes \bar{C}l(\mathfrak{g}_+)$ since $(R_{C(V^k(\mathfrak{g}))}, \operatorname{ad} \bar{Q})$ is identical to the BRST complex that appeared in Theorem 10.1.

The following result was proved in [87]. This is a particular cases of Theorem 9.9 (i).

Theorem 10.3 (De Sole–Kac) *We have $R_{H^0(C(V^k(\mathfrak{g})))} = H^0(R_{C(V^k(\mathfrak{g}))}, \operatorname{ad} \bar{Q}) = \mathbb{C}[\mathscr{S}_f]$. In particular, we recover that $\tilde{X}_{\mathcal{W}^k(\mathfrak{g},f)} = \mathscr{S}_f$.*

Next, $\operatorname{gr} C(V^k(\mathfrak{g}))$ is a Poisson vertex algebra and $(C(V^k(\mathfrak{g})), Q_{(0)}$ is a differential graded Poisson vertex algebra, and we have the following statement ([15]).

Theorem 10.4 *We have* $\operatorname{gr} H^i(C(V^k(\mathfrak{g}))) = H^i(\operatorname{gr} C(V^k(\mathfrak{g})), Q_{(0)}) = 0$ *if* $i \neq 0$. *Moreover,*

$$\operatorname{gr}^F \mathcal{W}^k(\mathfrak{g}, f) = \operatorname{gr} H^0(C(V^k(\mathfrak{g}))) = H^0(\operatorname{gr} C(V^k(\mathfrak{g})), Q_{(0)}) = \mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f].$$

In particular, we recover Theorem 9.4.

Consider now the differential algebra $(\operatorname{Zhu} C(V^k(\mathfrak{g})), \operatorname{ad} Q)$. We now recover Theorem 9.9 (ii) (see [85]).

Theorem 10.5 *We have* $\operatorname{Zhu}(H^i(C(V^k(\mathfrak{g}))) \cong H^i(\operatorname{Zhu} C(V^k(\mathfrak{g}))) = 0$ $i \neq 0$. *Moreover,*

$$\operatorname{Zhu} \mathcal{W}^k(\mathfrak{g}, f) = \operatorname{Zhu}(H^0(C(V^k(\mathfrak{g}))) = H^0(\operatorname{Zhu} C(V^k(\mathfrak{g}))) = U(\mathfrak{g}, f).$$

Part IV
Quasi-lisse vertex algebras and $\mathcal{W}$-algebras

As a Poisson variety, the associated variety of a vertex algebra is a finite disjoint union of smooth analytic Poisson manifolds, and it is stratified by its symplectic leaves. The vertex algebras whose associated variety has finitely many symplectic leaves are particularly matter of interest and were first considered by the authors in [36]. They were then referred to as quasi-lisse vertex algebras in [27].

Definition 10.1 A finitely strongly generated $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra V is called *quasi-lisse* if X_V has only finitely many symplectic leaves.

This part is devoted to the study of quasi-lisse vertex algebras. Lisse vertex algebras appear as special cases quasi-lisse vertex algebras: they are the vertex algebras whose associated variety is just a point (remember that all vertex algebras are assumed to be strongly generated and $\mathbb{Z}_{\geqslant 0}$-graded). By a result of Etingof–Schedler [101], the quasi-lisse condition implies that the zeroth Poisson homology $R_V/\{R_V, R_V\}$ is finite-dimensional. This weaker condition is crucial in many applications.

We give various examples of quasi-lisse vertex algebras, and present remarkable properties of lisse and quasi-lisse vertex algebras. Our first examples are particular cases of simple affine vertex algebras. Using the results of Part III one can construct plenty of other examples in the context of $\mathcal{W}$-algebras by taking the quantized Drinfeld-Sokolov reduction of quasi-lisse affine vertex algebras. There are also other expected examples coming from four dimensional $\mathcal{N} = 2$ superconformal field theories.

As a first motivation, let us comment the quasi-lisse condition in the setting of affine vertex algebras.

We have seen that $V^k(\mathfrak{g})$ plays a role similar to that of the enveloping algebra of $\mathfrak{g}$ for the representation theory of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$. It would be nice to have analogues of the associated varieties of primitive ideals in this context (see Section D.5). Unfortunately, one cannot expect exactly the same theory. One of the main reasons is that the center of $U(\hat{\mathfrak{g}})$ is trivial (unless for the *critical level* $k = -h^\vee$), and so we do not have analogue of the nilpotent cone (for the critical level, the analogue is played by the arc space of the nilpotent cone, see Example 1.3 and Exercice 3.2). So we need some replacements. In this context, the associated variety of the highest weight irreducible representation $L(k\Lambda_0) = L_k(\mathfrak{g})$ of $\hat{\mathfrak{g}}$, $k \in \mathbb{C}$, viewed as a vertex algebra, is a right object. More generally, one can consider the associated variety of any irreducible highest representation $L(\lambda)$ of $\hat{\mathfrak{g}}$, by exploiting the notion of associated variety for any module over a vertex algebra; see Section 6.3. We will see next chapters some analogies between the associated variety of $L_k(\mathfrak{g})$ and the associated variety of primitive ideals. However, there are substantial differences. For example, since $L_k(\mathfrak{g}) \cong V^k(\mathfrak{g})$ for $k \notin \mathbb{Q}$ (cf. [162]), we see that $X_{L_k(\mathfrak{g})}$ is not always contained in the nilpotent cone $\mathscr{N}$ of $\mathfrak{g}$. We observe that $X_{L_k(\mathfrak{g})}$ is contained in the nilpotent cone $\mathscr{N}$ if and only if $L_k(\mathfrak{g})$ is quasi-lisse (see Proposition 12.1). Thus, in this context, the quasi-lisse condition looks very natural.

The part is structured as follows. It starts with Chapter 11 on lisse and rational vertex algebras. Chapter 12 contains various examples of quasi-lisse (simple) vertex algebras, mostly n the context of affine vertex algebras and $\mathcal{W}$-algebras Remarkable properties of quasi-lisses vertex algebras are described in Chapter 13. In fact, the vertex algebras constructed from 4D $\mathcal{N} = 2$ SCFTs are expected to be quasi-lisse, since their associated varieties conjecturally coincide with the Higgs branches of the corresponding four dimensional theories ([231]).

Chapter 11
Lisse and rational vertex algebras

Before discussing the case of quasi-lisse vertex algebras, we focus in this chapter on lisse and rational vertex algebras. Recall that a vertex algebra V is called *lisse* if $\dim X_V = 0$, or equivalently, if R_V is finite-dimensional or, still equivalently, if $\dim SS(V) = 0$ (see Section 4.8). Close to the lisse condition, we have the rationality condition:

Definition 11.1 A conformal vertex algebra V is called *rational* if every $\mathbb{Z}_{\geqslant 0}$-graded V-modules is completely reducible (that is, isomorphic to a direct sum of simple V-modules).

Examples of lisse vertex algebras are given in Section 11.1 and Section 11.2. These examples of lisse vertex algebras are actually also examples of rational vertex algebras. In Section 11.3, we gather together remarkable properties that enjoy lisse and rational vertex algebras.

11.1 Integrable representations of affine Kac-Moody algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra. The irreducible $\mathfrak{g}$-representation $L_{\mathfrak{g}}(\lambda)$, with highest weight $\lambda \in \mathfrak{h}^*$, is finite-dimensional if and only if its associated variety $\mathscr{V}(\mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{g}}(\lambda)))$ is zero (Example D.6). Contrary to irreducible highest weight representations of $\mathfrak{g}$, the irreducible $\hat{\mathfrak{g}}$-representation $L(\lambda)$, where $\lambda \in \hat{\mathfrak{h}}^*$, is finite-dimensional if and only if $\lambda = 0$, that is, $L(\lambda)$ is the trivial representation.

The notion of finite-dimensional representations has to be replaced by the notion of *integrable representations* in the category $\mathscr{O}$. (See Sections A.4 and A.5 for the category $\mathscr{O}$ and the definition of integrable representations for affine Kac-Moody algebras.)

Let $k \in \mathbb{C}$. Recall that the simple affine vertex algebra $L_k(\mathfrak{g})$ is isomorphic to the irreducible highest weight representation $L(k\Lambda_0)$ as a $\hat{\mathfrak{g}}$-module.

Theorem 11.1 *The following are equivalent:*

(i) $L_k(\mathfrak{g})$ *is rational,*
(ii) $L_k(\mathfrak{g})$ *is lisse, that is,* $X_{L_k(\mathfrak{g})} = \{0\}$,
(iii) $L_k(\mathfrak{g})$ *is integrable as a* $\hat{\mathfrak{g}}$*-module (which happens if and only if* $k \in \mathbb{Z}_{\geqslant 0}$*).*

It should be noted a clear analogy between the equivalence (ii) $\iff$ (iii) and the equivalence mentioned in Example D.6.

The equivalence (i) $\iff$ (iii) is known [92], as well as the last equivalence in parenthesis of Part (iii). We explain below only the implication (iii) $\Rightarrow$ (ii).

Lemma 11.1 *Let* (R, ∂) *be a differential algebra over* $\mathbb{Q}$*, and let* I *be a differential ideal of* R*, i.e.,* I *is an ideal of* R *such that* $\partial I \subset I$*. Then* $\partial\sqrt{I} \subset \sqrt{I}$*.*

Proof Let $a \in \sqrt{I}$, so that $a^m \in I$ for some $m \in \mathbb{Z}_{\geqslant 0}$. Since I is ∂-invariant, we have $\partial^m a^m \in I$. But

$$\partial^m a^m \equiv m!(\partial a)^m \pmod{\sqrt{I}}.$$

Hence $(\partial a)^m \in \sqrt{I}$, and therefore, $\partial a \in \sqrt{I}$. □

Recall that a *singular vector* of a $\hat{\mathfrak{g}}$-representation M is a vector $v \in M$ such that $\hat{\mathfrak{n}}.v = 0$, if $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}$ is a trianglar decomposition of $\hat{\mathfrak{g}}$ (see §A.4.3). In particular, regarding $V^k(\mathfrak{g})$ as a $\hat{\mathfrak{g}}$-representation, a vector $v \in V^k(\mathfrak{g})$ is singular if and only if $\hat{\mathfrak{n}}.v = 0$.

In the case where k is a nonnegative integer, the maximal submodule N_k of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$ ([159]), where θ is the highest positive root and $e_\theta \in \mathfrak{g}_\theta \setminus \{0\}$.

Proof (of the implication (iii) ⇒ (ii) in Theorem 11.1) Suppose that $L_k(\mathfrak{g})$ is integrable. This condition is equivalent to that $k \in \mathbb{Z}_{\geqslant 0}$ and, if so, the maximal submodule $N_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$. The exact sequence $0 \to N_k(\mathfrak{g}) \to V^k(\mathfrak{g}) \to L_k(\mathfrak{g}) \to 0$ induces the exact sequence

$$0 \to I_k \to R_{V^k(\mathfrak{g})} \to R_{L_k(\mathfrak{g})} \to 0,$$

where I_k is the image of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$, and so, $R_{L_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_k$. The image of the singular vector in I_k is given by e_θ^{k+1}. Therefore, $e_\theta \in \sqrt{I_k}$. On the other hand, by Lemma 11.1, $\sqrt{I_k}$ is preserved by the adjoint action of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{g} \subset \sqrt{I_k}$. This proves that $X_{L_k(\mathfrak{g})} = \{0\}$ as required. □

The proof of the "only if" part follows from [92]. It can also be proven using W-algebras.

In view of Theorem 11.1, one may regard the lisse condition as a generalization of the integrability condition to an arbitrary vertex algebra.

11.2 Minimal series representations of the Virasoro algebra

Let $c \in \mathbb{C}$. Denote by N_c the unique maximal submodule of the Virasoro vertex algebra Vir^c, and let $\mathrm{Vir}_c := \mathrm{Vir}^c/N_c$ be the simple quotient.

Theorem 11.2 *The following are equivalent:*

(i) Vir_c *is rational,*
(ii) Vir_c *is lisse,*
(iii) $c = 1-\dfrac{6(p-q)^2}{pq}$ *for some* $p, q \in \mathbb{Z}_{\geqslant 2}$ *such that* $(p, q) = 1$. *(These are precisely the central charge of the* minimal series representations *of the Virasoro algebra* Vir*.)*

The equivalence (i) $\Longleftrightarrow$ (iii) is well-known [251].

We explain below the equivalence (ii) $\Longleftrightarrow$ (iii) (see [12, Proposition 3.4.1]).

Proof (of the equivalence (ii) $\Longleftrightarrow$ (iii) ***in Theorem 11.2)*** It is known that the image of N_c in R_{Vir^c} is nonzero if $N_c \neq 0$ (see e.g., [251, Lemmas 4.2 and 4.3] or [140, Proposition 4.3.2]). Therefore $X_{\mathrm{Vir}_c} = \{0\}$ if and only if Vir^c is not irreducible. This happens if and only if the central charge is of the form in (iii) ([158, 110, 140]). □

11.3 On the lisse and the rational conditions

It is known ([89]) that the rationality condition implies that V has finitely many simple $\mathbb{Z}_{\geqslant 0}$-graded modules and that the graded components of each of these $\mathbb{Z}_{\geqslant 0}$-graded modules are finite dimensional. In fact lisse vertex algebras also verify this property (cf. Theorem 5.4). Moreover, the normalized characters of the finitely simple $\mathbb{Z}_{\geqslant 0}$-graded modules over a lisse vertex algebra enjoy remarkable modular invariance properties ([257, 216]). All this is summarized in the below theorem.

Theorem 11.3 (Abe–Buhl–Dong, Miyamoto, Zhu) *Let* V *be a* $\mathbb{Z}_{\geqslant 0}$*-graded conformal lisse vertex algebra.*

(i) *Any simple* V*-module is a positive energy representation, that is, a positively graded* V*-module. Therefore the number of isomorphic classes of simple* V*-modules is finite.*
(ii) *Let* $M_1, \ldots, M_s$ *be representatives of the isomorphic classes of simple* V*-modules, and let for* $i = 1, \ldots, s$,

$$\chi_{M_i}(\tau) = \mathrm{Tr}_{M_i}(q^{L_0-\frac{c}{24}}) = \sum_{n\geqslant 0} \dim(M_i)_n q^{n-\frac{c}{24}}, \quad q = e^{2i\pi\tau},$$

be the normalized character of M_i. *Then* $\chi_{M_i}(\tau)$ *converges in the upper half-plane* $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, *and the vector space generated by* $SL_2(\mathbb{Z}).\chi_{M_i}(\tau)$ *is finite-dimensional.*
(iii) *Assume furthermore that* V *is rational. Then the linear space spanned by the* χ_{M_i}*'s are invariant under the action of* $SL_2(\mathbb{Z})$ *(where we consider* χ_{M_i} *as a holomorphic function on the upper half plane by the transformation* $q = e^{2i\pi\tau}$*).*

If V is as in Theorem 11.3, (iii), it is known by a result of Huang [150] that under some mild assumptions (V must be rational, lisse, simple, self-dual conformal[1]), the category of finitely generated V-modules forms a modular tensor category. which for instance yields an invariant of 3-manifolds, see [43].

It is actually conjectured by Zhu in [257] that rational vertex algebras must be lisse (this conjecture is still open):

Conjecture 11.1 If V is rational (which imply that $\mathrm{Zhu}(V)$ is semisimple), then V must be lisse.

The converse is not true. It is knwon that a lisse vertex algebra is not necessarily rational ([1]).

There are significant vertex algebras that do not satisfy the lisse condition. For instance, an *admissible* affine vertex algebra $L_k(\mathfrak{g})$ (see Section 12.2) has a complete reducibility property ([16]), and the modular invariance property ([165]) in the category $\mathscr{O}$ still holds, although it is not lisse unless it is integrable.

So it is natural to try to relax the lisse condition. This is the purpose of the next chapters.

[1] We refer to §13.1.2 for the notion of *self-dual* vertex algebra.

Chapter 12
Examples of quasi-lisse vertex algebras

This chapter contains various examples of quasi-lisse vertex algebras. Section 12.1 is about general facts on the associated variety of affine vertex algebras. Then we focus on two interesting families of quasi-lisse simple affine vertex algebras: those coming from admissible levels (Section 12.2) and those coming from the Deligne exceptional series (Section 12.3). So far, they are roughly the only known quasi-lisse simple affine vertex algebras (see Remark 12.2 for a couple of other known cases) while they are certainly much more examples. By taking the Drinfeld–Sokolov reduction of all previous examples, we produce in Section 12.4 many other examples of quasi-lisse vertex algebras in the context of $\mathcal{W}$-algebras. A few of other examples are discussed in Section 12.5.

In what follows, $\mathfrak{g}$ is a complex simple Lie algebra with adjoint group G, and $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$, that is, the set of nilpotent elements of $\mathfrak{g}$. We identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using the non-degenerate bilinear form $(-|-) = \dfrac{1}{h^\vee} \times$ Killing form of $\mathfrak{g}$. We shall use the notations of Appendix D, particularly for the nilpotent orbits in $\mathfrak{sl}_n$ in correspondence with the partitions of n.

12.1 General facts on associated varieties of affine vertex algebras

Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with $\mathfrak{g}$ at the level $k \in \mathbb{C}$. Recall first that the associated variety of $V^k(\mathfrak{g})$ is $\mathfrak{g}^* \cong \mathfrak{g}$ (cf. Example 4.5). In particular, $V^k(\mathfrak{g})$ is never quasi-lisse (see Proposition 12.1).

Let us now focus on the associated variety of the simple quotient $L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k$, where N_k is the maximal proper submodule of $V^k(\mathfrak{g})$. Contrary to the associated varieties of primitive ideals of $U(\mathfrak{g})$, the associated variety of $L_k(\mathfrak{g})$ is not always contained in the nilpotent cone $\mathcal{N}$. Indeed, if $V^k(\mathfrak{g})$ is simple, for example if $k \notin \mathbb{Q}$, then $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$ and so $X_{L_k(\mathfrak{g})} = \mathfrak{g} \not\subset \mathcal{N}$. The main result of [28] ensures that the converse is true as well.

Theorem 12.1 *Let $k \in \mathbb{C}$. Then $X_{L_k(\mathfrak{g})} = \mathfrak{g}$ if and only if $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, that is, $V^k(\mathfrak{g})$ is simple.*

On the other hand, we have a simple criterion to check whether $L_k(\mathfrak{g})$ is quasi-lisse:

Proposition 12.1 *The simple affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$.*

Proof Recall that the symplectic leaves of $\mathfrak{g}$ are the adjoint G-orbits of $\mathfrak{g}$, Since the works of Kostant, we know that the nilpotent cone $\mathcal{N}$ of the simple Lie algebra is a finite union of adjoint orbits (see Section D.1). Hence, if $X_{L_k(\mathfrak{g})}$ is contained in $\mathcal{N}$ then $L_k(\mathfrak{g})$ is quasi-lisse.

Conversely, assume that $X_{L_k(\mathfrak{g})}$ contains a non-nilpotent element x, with Jordan decomposition $x = x_s + x_n$. If $x_n = 0$, then $X_{L_k(\mathfrak{g})}$ contains $\overline{G\mathbb{C}^* x} = \overline{G\mathbb{C}^* x_s}$, using the fact that $X_{L_k(\mathfrak{g})}$ is a closed G-invariant cone of $\mathfrak{g}$. But $\overline{G\mathbb{C}^* x_s}$ contains infinitely many symplectic leaves because x_s is semisimple. So $X_{L_k(\mathfrak{g})}$ is not quasi-lisse. If $x_n \neq 0$, choose an $\mathfrak{sl}_2$-triplet (x_n, h, y_n) in $\mathfrak{g}^{x_s}$; one can assume that h and x_x lie in a same Cartan subalgebra so that they commute. Consider the one-parameter subgroup $\rho\colon \mathbb{C}^* \to G$ generated by ad h. We have for all $t \in \mathbb{C}^*$,

$$\rho(t)x = x_s + t^2 x_n.$$

Taking the limit when t goes to 0, we deduce that $x_s \in X_{L_k(\mathfrak{g})}$ and, hence, by the first case, $X_{L_k(\mathfrak{g})}$ is not quasi-lisse. □

The proof of the following proposition uses the infinitesimal character of a highest representation.

Proposition 12.2 *If $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$, then $L_k(\mathfrak{g})$ has only finitely many simple objects in the category $\mathcal{O}$.*

Proof Write $\mathrm{Zhu}(L_k(V))$ as a quotient $U(\mathfrak{g})/I$ (see §5.6.2). By Zhu's correspondence (Theorem 5.2), it suffices to show that there are finitely many simple highest weight $\mathfrak{g}$-modules $L_{\mathfrak{g}}(\lambda)$ annihilated by I since $L(\hat{\lambda})_{\mathrm{top}} = L_{\mathfrak{g}}(\lambda)$, where λ is the restriction to $\mathfrak{h}$ of $\hat{\lambda} \in \hat{\mathfrak{h}}^*$. Indeed, simple objects in the category $\mathcal{O}$ are precisely the simple highest weight modules $L(\lambda)$, [158, Proposition 9.3].

Let us denote by $\chi_\lambda\colon Z(\mathfrak{g}) \to \mathbb{C}$ the *infinitesimal character* associated to $L_{\mathfrak{g}}(\lambda)$ (see §A.1.4). Recall that for $\lambda, \mu \in \mathfrak{h}^*$, $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same W-orbit with respect to the twisted action of W, where $\mathcal{W}$ is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Hence, identifying the maximal spectrum of $Z(\mathfrak{g})$ with the set of all homomorphisms $Z(\mathfrak{g}) \to \mathbb{C}$, it is enough to show that $\mathrm{Specm}\,(Z(\mathfrak{g})/Z(\mathfrak{g}) \cap I)$ is finite or, equivalently, that $Z(\mathfrak{g})/Z(\mathfrak{g}) \cap I$ is finite-dimensional. This will show that the set of possible infinite characters of simple $U(\mathfrak{g})/I$-modules is finite and so is the set of possible simple $U(\mathfrak{g})/I$-modules.

Using the surjective Poisson algebra homomorphism (cf. Lemma 5.11),

$$R_{L_k(\mathfrak{g})} \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}(L_k(\mathfrak{g})) = \mathrm{gr}\,U(\mathfrak{g})/\mathrm{gr}\,I,$$

we get that $(\operatorname{Spec}\operatorname{gr} U(\mathfrak{g})/\operatorname{gr} I)_{\mathrm{red}} \subset X_{L_k(\mathfrak{g}))} \subset \mathcal{N}$, whence the augmentation ideal $(\operatorname{gr} Z(\mathfrak{g}))_+$ of $\operatorname{gr} Z(\mathfrak{g})$ is contained in $\sqrt{\operatorname{gr} I}$ because $(\operatorname{gr} Z(\mathfrak{g}))_+ \cong \mathbb{C}[\mathfrak{g}]_+$ is the defining ideal of $\mathcal{N}$. As a result, $\operatorname{gr} Z(\mathfrak{g})/\operatorname{gr}(Z(\mathfrak{g}) \cap I)$ is finite-dimensional and so is $Z(\mathfrak{g})/Z(\mathfrak{g}) \cap I$. □

In view of the above results, it is natural to ask whether there exist pairs $(\mathfrak{g}, k)$ such that $X_{L_k(\mathfrak{g})}$ is neither $\mathfrak{g}$ nor contained in the nilpotent cone $\mathcal{N}$.

Definition 12.1 The *sheets* of $\mathfrak{g}^*$ are by definition the irreducible components of the locally closed subsets $\mathfrak{g}^{(m)} = \{\xi \in \mathfrak{g}^* :\ \dim G.\xi = 2m\}$, $m \in \mathbb{Z}_{\geqslant 0}$. A sheet of $\mathfrak{g}$ is called *Dixmier* if it contains semisimple elements of $\mathfrak{g}$. We refer to [245, Section 39] for more about this topic.

Example 12.1 In the following examples, the associated variety of $L_k(\mathfrak{g})$ is the closure of some Dixmier sheet. In particular, $X_{L_k(\mathfrak{g})}$ is neither $\mathfrak{g}$ nor contained in the nilpotent cone $\mathcal{N}$ ([35]).

(i) For $n \geqslant 4$,

$$X_{L_{-1}(\mathfrak{sl}_n)} = \overline{G\mathbb{C}^*\check{\varpi}_1} \not\subset \mathcal{N},$$

where $\check{\varpi}_1$ is the fundamental co-weight associated with α_1 if $\alpha_1, \ldots, \alpha_{n-1}$ are the simple roots of $\mathfrak{sl}_n$. Note that $\overline{G\mathbb{C}^*\check{\varpi}_1} = \overline{\mathbb{S}_1}$, where $\mathbb{S}_1$ is the unique sheet of $\mathfrak{sl}_n$ containing the minimal nilpotent orbit $\mathbb{O}_{\min}$ in its closure.

(ii) For $m \geqslant 2$,

$$X_{L_{-m}(\mathfrak{sl}_{2m})} = \overline{G\mathbb{C}^*\check{\varpi}_m} \not\subset \mathcal{N},$$

where $\check{\varpi}_m$ is the fundamental co-weight associated with α_m. Note that $\overline{G\mathbb{C}^*\check{\varpi}_m} = \overline{\mathbb{S}_0}$, where $\mathbb{S}_0$ is the unique sheet of $\mathfrak{sl}_{2m}$ containing the nilpotent orbit $\mathbb{O}_{(2^m)}$ associated with the partition (2^m) in its closure.

Similar examples have been found since, see e.g., [103]. Next, in the light of Theorem 12.1, one can ask whether there is a pair $(\mathfrak{g}, k)$ such that $X_{L_k(\mathfrak{g})}$ is a maximal proper G-invariant closed subcone of $\mathfrak{g}$. Add Cuipo?

For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{N}$ is the unique maximal proper G-invariant closed subcone of $\mathfrak{g}$. In fact the only G-invariant closed subcones of $\mathfrak{sl}_2$ are: $\{0\}$, $\mathcal{N}$, $\mathfrak{g}$. All these subsets can be realized as the associated variety of some $L_k(\mathfrak{sl}_2)$ (see Section 12.2 and more specifically Exercice 12.1).

For $\mathfrak{g} = \mathfrak{sl}_3$, one can construct a maximal proper G-invariant closed subcone of $\mathfrak{g}$ as follows. Let (e, h, f) be a *principal* $\mathfrak{sl}_2$-triple, that is, f is regular nilpotent element of $\mathfrak{g}$. Let $\mathscr{X} = \overline{G\mathbb{C}^* h}$ be the G-invariant closed cone generated by h. This set is referred to as the *principal cone* in [78]. It contains the nilpotent cone $\mathcal{N}$ and has dimension $\dim \mathcal{N} + 1$. So, for $\mathfrak{g} = \mathfrak{sl}_3$, it is maximal for dimension reasons (it has dimension 7 while $\mathfrak{g}$ has dimension 8). More generally, for any regular semisimple element $x \in \mathfrak{g}$, the set $\overline{G\mathbb{C}^* x}$ is a G-invariant closed subcone of $\mathfrak{g}$ containing $\mathcal{N}$ (see [79, Th. 2.9]). The principal cone $\mathscr{X}$ is somehow more canonical: it is precisely the closure of the set of *principal* semisimple elements (that is, the semisimple elements which are central elements of principal $\mathfrak{sl}_2$-triples).

? Open problem

Assume that $\mathfrak{g} = \mathfrak{sl}_3$. Is there a level k such that $X_{L_k(\mathfrak{g})} = \mathscr{X}$, where $\mathscr{X} = \overline{G\mathbb{C}^* h}$ is the principal cone of $\mathfrak{sl}_3$?

In the next two sections, we provide pairs $(\mathfrak{g}, k)$ for which the associated variety $X_k(\mathfrak{g})$ is contained in the nilpotent cone $\mathcal{N}$.

12.2 Admissible representations

Recall that the irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible in the sense of Definition A.6. For example, an irreducible integrable representation of $\hat{\mathfrak{g}}$ is admissible. More generally, the simple affine vertex algebra $L_k(\mathfrak{g})$ is called *admissible* if it is admissible as a $\hat{\mathfrak{g}}$-module, and the level k is called *admissible* if $L_k(\mathfrak{g})$ is addmisible (see Definition A.7). This happens if and only if (Proposition A.3):

$$k = -h^\vee + \frac{p}{q} \text{ with } p, q \in \mathbb{Z}_{\geqslant 1},\ (p, q) = 1,\ p \geqslant \begin{cases} h^\vee & \text{if } (q, \check{r}) = 1, \\ h & \text{if } (q, \check{r}) = \check{r}, \end{cases} \tag{12.1}$$

where $\check{r}$ is the lacing number of $\mathfrak{g}$, that is, the maximal number of edges in the Dynkin diagram of $\mathfrak{g}$.

The fist statement of the following assertion was conjectured by Feigin and Frenkel and proved for the case that $\mathfrak{g} = \mathfrak{sl}_2$ by Feigin and Malikov [111]. The general proof is achieved in [15].

Theorem 12.2 *Assume that k is an admissible level for $\mathfrak{g}$.*

(i) *The singular support $SS(L_k(\mathfrak{g}))$ is contained in $\mathcal{J}_\infty(\mathcal{N})$ or, equivalently, the associated variety $X_{L_k(\mathfrak{g})}$ is contained in $\mathcal{N}$.*
(ii) *A stronger result holds: we have*

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k},$$

where $\mathbb{O}_k$ is a nilpotent orbit which only depends on the denominator q, with k written as in (12.1).

Remark 12.1 More explicitly, we have

$$X_{L_k(\mathfrak{g})} = \begin{cases} \{x \in \mathfrak{g} \colon (\operatorname{ad} x)^{2q} = 0\} & \text{if } (q, \check{r}) = 1, \\ \{x \in \mathfrak{g} \colon \pi_{\theta_s}(x)^{2q/\check{r}} = 0\} & \text{if } (q, \check{r}) \neq 1, \end{cases} \tag{12.2}$$

where π_{θ_s} is the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight θ_s, θ_s is the highest short root of $\mathfrak{g}$.

The irreducibility of the varieties of the right-hand-side of (12.2) was checked in [15] so that the nilpotent orbit $\mathbb{O}_k$ is indeed well-defined. Prior to [15], the irreducibility of variety $\overline{\mathbb{O}_k}$ had been proved in [132] in most cases and the orbits $\mathbb{O}_k$ were studied in [100] as exceptional nilpotent orbits for those of principal Levi type and $(q, \check{r}) = 1$.

Example 12.2 Let us describe explicitly the nilpotent orbit $\mathbb{O}_k$ of Theorem 12.2 in the case where $\mathfrak{g} = \mathfrak{sl}_n$. Recall that the nilpotent orbits of $\mathfrak{sl}_n$ are parameterized by the partitions of n. Let k be an admissible level for $\mathfrak{sl}_n$, that is, $k = -n + \dfrac{p}{q}$, with $p \in \mathbb{Z}$, $p \geqslant n$, and $(p,q) = 1$. Then $\mathbb{O}_k$ is the nilpotent orbit corresponding to the partition (n) is $q \geqslant n$, and to the partition $(q, q, \dots, q, s) = (q^m, s)$, where m and s are the quotient and the rest of the Euclidean division of n by q, respectively, if $q < n$.

Next exercise gives a proof of Theorem 12.2 for $\mathfrak{g} = \mathfrak{sl}_2$. It is based on Feigin and Malikov approach (see also [15, Theorem 5.6]).

Exercise 12.1 Let N_k be the proper maximal ideal of $V^k(\mathfrak{sl}_2)$ so that $L_k(\mathfrak{sl}_2) = V^k(\mathfrak{sl}_2)/N_k$. Let I_k be the image of N_k in $R_{V^k(\mathfrak{sl}_2)} = \mathbb{C}[\mathfrak{sl}_2]$ whence $R_{L_k(\mathfrak{sl}_2)} = \mathbb{C}[\mathfrak{sl}_2]/I_k$. It is known that either N_k is trivial, that is, $V^k(\mathfrak{sl}_2)$ is simple, or N_k is generated by a singular vector v whose image $\overline{v}$ in I_k is nonzero ([164, 211]).

We assume in this exercise that N_k is non trivial. Thus, $N_k = U(\widehat{\mathfrak{sl}}_2)v$.

(i) Using Kostant's Separation Theorem show that, up to a nonzero scalar,

$$\overline{v} = \Omega^m e^n,$$

for some $m, n \in \mathbb{Z}_{>0}$, where $\Omega = 2ef + \frac{1}{2}h^2$ is the Casimir element of the symmetric algebra of $\mathfrak{sl}_2$.

(ii) Deduce from this that $X_{L_k(\mathfrak{sl}_2)}$ is contained in the nilpotent cone of $\mathfrak{sl}_2$.

It is known that N_k is nontrivial if and only if k is an admissible level for $\mathfrak{sl}_2$, or $k = -2$ is critical. Thus we have shown that $X_{L_k(\mathfrak{sl}_2)}$ is contained in the nilpotent cone of $\mathfrak{sl}_2$ if and only if $k = -2$ or k is admissible, i.e., $k = -2 + \frac{p}{q}$, with $(p,q) = 1$ and $p \geqslant 2$.

On the other hand, since $X_{L_k(\mathfrak{sl}_2)} = \{0\}$ if and only if $k \in \mathbb{Z}_{\geqslant 0}$ by Theorem 11.1, we get that $X_{L_k(\mathfrak{sl}_2)} = \mathcal{N}$ if and only if $k = -2$ or k is admissible and $k \notin \mathbb{Z}_{\geqslant 0}$.

The following exercise explains how to compute the associated variety in a concrete example exploiting a singular vector (see §A.4.3). This example is covered by both Theorem 12.2 and Theorem 12.3.

Exercise 12.2 The aim of this exercise is to compute $X_{L_{-3/2}(\mathfrak{sl}_3)}$. It was shown by Perše [226] that the proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the singular vector v given by:

$$v := \frac{1}{3}\Big((h_1t^{-1})(e_{1,3}t^{-1})|0\rangle - (h_2t^{-1})(e_{1,3}t^{-1})|0\rangle\Big) + (e_{1,2}t^{-1})(e_{2,3}t^{-1})|0\rangle - \frac{1}{2}e_{1,3}t^{-2}|0\rangle,$$

where $h_1 := e_{1,1} - e_{2,2}$, $h_2 := e_{2,2} - e_{3,3}$ and $e_{i,j}$ is the elementary matrix of the coefficient (i, j) in $\mathfrak{sl}_3$ identified with the set of traceless 3-size square matrices.

(i) Verify that v is indeed a singular vector for $\widehat{\mathfrak{sl}_3}$, that is, $e_{i,i+1}v = 0$ for $i = 1, 2$ and $(e_{3,1}t)v = 0$.
(ii) Let $\mathfrak{h} := \mathbb{C}h_1 + \mathbb{C}h_2$ be the usual Cartan subalgebra of $\mathfrak{sl}_3$. Show that $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h} = \{0\}$, and deduce from this that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ is contained in the nilpotent cone of $\mathfrak{sl}_3$.
(iii) Show that the nilpotent cone is not contained in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.
(iv) Denoting by $\mathbb{O}_{\min}$ the minimal nilpotent orbit of $\mathfrak{sl}_3$, conclude that

$$X_{L_{-3/2}(\mathfrak{sl}_3)} = \overline{\mathbb{O}_{\min}}.$$

12.3 Exceptional Deligne series

There was actually a "strong Feigin-Frenkel conjecture" stating that k is admissible if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$ Such a statement would be interesting because it would give a geometrical description of the admissible representations $L_k(\mathfrak{g})$. As seen in Exercise 12.1, the equivalence holds for $\mathfrak{g} = \mathfrak{sl}_2$.

12.3.1 Non-admissible quasi-lisse affine vertex algebras

The stronger conjecture is wrong in general, as shown the following result ([34]).

Theorem 12.3 *Assume that* $\mathfrak{g}$ *belongs to the* Deligne exceptional series *,*

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and that $k = -\dfrac{h^\vee}{6} - 1 + n$*, where* $n \in \mathbb{Z}_{\geqslant 0}$ *is such that* $k \notin \mathbb{Z}_{\geqslant 0}$*. Then*

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\min}},$$

where $\mathbb{O}_{\min}$ *is the minimal nilpotent orbit of* $\mathfrak{g}$.

Note that the level $k = -\dfrac{h^\vee}{6} - 1$ is not admissible for the types D_4, E_6, E_7, E_8 (it equals $-2, -3, -4, -6$, respectively). Theorem 12.3 provides the first known examples of associated varieties contained in the nilpotent cone corresponding to non-admissible levels. Theorem 12.3 also allows to produce "new" examples of lisse simple $\mathcal{W}$-algebras (see Example 12.4).

By Proposition 12.2, if $(\mathfrak{g}, k)$ is as in Theorem 12.3, then $L_k(\mathfrak{g})$ has finitely many simple objects in the category $\mathscr{O}$. One can describe them thanks to Joseph's classification [156] of irreducible highest weights representation $L_{\mathfrak{g}}(\lambda)$ whose associated variety is $\overline{\mathbb{O}_{\min}}$ (see Theorem 12.4).

We give in the next section a broad sketch of a proof of Theorem 12.3.

Remark 12.2 There are a couple of other examples of simple quasi-lisse affine vertex algebras $L_k(\mathfrak{g})$, at non-admissible level k ([34, 35, 36]). Namely, for $(\mathfrak{g}, k)$ as below, the simple affine vertex algebras $L_k(\mathfrak{g})$ is quasi-lisse:

- if $\mathfrak{g}$ of type G_2 and $k = -2$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\min}}$,
- if $\mathfrak{g}$ of type D_r, $r \geqslant 5$ and $k = -2, -1$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\min}}$,
- if $\mathfrak{g}$ of type D_r, with r an even integer $\geqslant 4$, and $k = 2-r$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{(2^{r-2},1^4)}}$,
- if $\mathfrak{g}$ of type B_r, $r \geqslant 3$ and $k = -2$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\text{short}}}$, where $\mathbb{O}_{\text{short}}$ is the nilpotent orbit associated with the $\mathfrak{sl}_2$-triple $(e_{\theta_s}, h_{\theta_s}, f_{\theta_s})$, with θ_s the highest short root ε_1 (note that $h_{\theta_s} = 2\varpi_1^\vee$). Notice that $\mathbb{O}_{\text{short}} = \mathbb{O}_{(3,1^{2r-2})}$.
- Finally, for any $\mathfrak{g}$, if $k = -h^\vee$ is critical then $X_{L_k(\mathfrak{g})} = \mathcal{N}$ ([109, 115]).

Except for $\mathfrak{g} = \mathfrak{sl}_2$, the classification problem of quasi-lisse affine vertex algebras is wide open.

12.3.2 Joseph ideal and proof of Theorem 12.3

We refer the reader to Section D.5 for standard facts on primitive ideals and their associated varieties.

If $\mathfrak{g}$ is not of type A, it is known [154, 129] that there exists a unique *completely prime ideal*, that is, the corresponding graded ideal is prime, in $U(\mathfrak{g})$ whose associated variety is the minimal nilpotent orbit $\overline{\mathbb{O}_{\min}}$, which is the unique nilpotent orbit of $\mathfrak{g}$ of minimal dimension $2h^\vee - 2$, with $h^\vee$ the dual Coxeter number of $\mathfrak{g}$ (see Section D.1). See [229] for a more recent review on this topic.

Definition 12.2 If $\mathfrak{g}$ is not of type A, the unique completely prime ideal whose associated variety is $\overline{\mathbb{O}_{\min}}$ is denoted by $\mathcal{J}_0$, and is referred to as the *Joseph ideal* of $U(\mathfrak{g})$.

For $\mathfrak{g}$ of type A, the completely prime primitive ideals I of $U(\mathfrak{g})$ with $\mathscr{V}(I) = \overline{\mathbb{O}_{\min}}$ form a single family parametrized by the elements of $\mathbb{C}$ ([154, 229]). In [154], Joseph has also computed the *infinitesimal character of* $\mathcal{J}_0$, that is, the algebra homomorphism $Z(\mathfrak{g}) \to \mathbb{C}$ through which the centre $Z(\mathfrak{g})$ acts on the primitive quotient $U(\mathfrak{g})/\mathcal{J}_0$. In fact, Joseph has described the set of $\lambda \in \mathfrak{h}^*$ such that such that $\mathcal{J}_0 = \mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{g}}(\lambda))$ (see Table 12.1).

Outside the type A the nilpotent orbit $\mathbb{O}_{\min}$ is *rigid*[1], hence forms a single sheet (cf. Definition 12.1) in $\mathfrak{g}^* \cong \mathfrak{g}$. Indeed, a nilpotent orbit forms a single sheet if and only if it is rigid. So, $\mathcal{J}_0$ cannot be obtained by parabolic induction from a primitive

[1] Rigid nilpotent orbits are those nilpotent orbits which cannot be properly induced from another nilpotent orbit in the sense of Lusztig-Spaltenstein [62, 60].

ideal of a proper Levi subalgebra of $\mathfrak{g}$. Different realizations of $\mathcal{J}_0$ can be found in the literature for various types of $\mathfrak{g}$. Joseph's original proof of the uniqueness of $\mathcal{J}_0$ was incomplete. This led Gan and Savin [129] to give another description of the Joseph ideal $\mathcal{J}_0$. Their argument relies on some invariant theory and earlier results of Garfinkle.

Let us outline their description. Suppose that $\mathfrak{g}$ is not of type A. According to Kostant, $\mathcal{J}_0$ is generated by the $\mathfrak{g}$-submodule $L_{\mathfrak{g}}(0) \oplus W$ in $S^2(\mathfrak{g})$, where $\mathcal{W}$ is such that, as $\mathfrak{g}$-modules,

$$S^2(\mathfrak{g}) = L_{\mathfrak{g}}(2\theta) \oplus L_{\mathfrak{g}}(0) \oplus W.$$

Note that the above decomposition of $S^2(\mathfrak{g})$ still holds in type A ([130, Chap. IV, Prop. 2]). Also, observe that $L_{\mathfrak{g}}(0) = \mathbb{C}\Omega$ where $\Omega = \sum_i x_i x^i$ is the Casimir element in $S(\mathfrak{g})$, with $\{x_i\}_i$ is a basis of $\mathfrak{g}$, and $\{x^i\}_i$ its dual basis with respect to $(\ |\ )$.

Let $\mathcal{J}_W$ be the two-sided ideal of $U(\mathfrak{g})$ generated by $\mathcal{W}$.

Proposition 12.3 *We have the algebra isomorphism*

$$U(\mathfrak{g})/\mathcal{J}_W \cong \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

Proof By the proof of [129, Th. 3.1], $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$. Hence it contains $(\Omega - c_0)\Omega$. Since $c_0 \neq 0$, we have the isomorphism of algebras

$$U(\mathfrak{g})/\mathcal{J}_W \xrightarrow{\sim} U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega\rangle \times U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega - c_0\rangle.$$

As we have explained above, $\langle \mathcal{J}_W, \Omega - c_0\rangle = \mathcal{J}_0$. Also, since $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$, $\langle \mathcal{J}_W, \Omega\rangle$ contains $\mathfrak{g}$. Therefore $U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega\rangle = \mathbb{C}$ as required. □

Lemma 12.1 *Suppose that $\mathfrak{g}$ is not of type A. The ideal J_W in $S(\mathfrak{g})$ generated by $\mathcal{W}$ contains Ω^2, and hence, $\sqrt{J_W} = J_0$, where J_0 is the prime ideal of $S(\mathfrak{g})$ corresponding to the minimal nilpotent orbit closure $\overline{\mathbb{O}_{\min}}$.*

Proof By the proof of [129, Th. 3.1], the ideal $\mathcal{J}_W$ of $U(\mathfrak{g})$ generated by $\mathcal{W}$ contains $\mathfrak{g} \cdot \Omega$, and so the assertion follows. □

The structure of $\mathcal{W}$ was determined by Garfinkle [130]. Consider the $\mathfrak{sl}_2$-triple $(e_\theta, h_\theta, f_\theta)$ of $\mathfrak{g}$ where $f_\theta = e_{-\theta}$ is a θ-root vector (θ denotes the highest positive root) so that it lies in $\mathbb{O}_{\min}$. Set

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \colon [h_\theta, x] = 2jx\}.$$

Then (cf. Remark D.1)

$$\begin{aligned} &\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \\ &\mathfrak{g}_{-1} = \mathbb{C}f_\theta, \quad \mathfrak{g}_1 = \mathbb{C}e_\theta, \quad \mathfrak{g}_0 = \mathbb{C}h_\theta \oplus \mathfrak{g}^\natural, \quad \mathfrak{g}^\natural = \{x \in \mathfrak{g}_0 \colon (h_\theta|x) = 0\}. \end{aligned}$$

The subalgebra $\mathfrak{g}^\natural$ is a reductive subalgebra of $\mathfrak{g}$ whose simple roots are the simple roots of $\mathfrak{g}$ perpendicular to θ. Write

$$[\mathfrak{g}^\natural, \mathfrak{g}^\natural] = \bigoplus_{i \geqslant 1} \mathfrak{g}_i$$

as a direct sum of simple summands, and let θ_i be the highest root of $\mathfrak{g}_i$.

- If $\mathfrak{g}$ is neither of type A_r nor C_r,

$$W = \bigoplus_{i \geqslant 1} L_{\mathfrak{g}}(\theta + \theta_i). \tag{12.3}$$

- If $\mathfrak{g}$ is of type C_r, then $\mathfrak{g}^\natural$ is simple of type C_{r-1}, so that there is a unique θ_1, and we have

$$W = L_{\mathfrak{g}}(\theta + \theta_1) \oplus L_{\mathfrak{g}}\left((\theta + \theta_1)/2\right).$$

By [130, 129], $\mathcal{J}_0$ is generated by $\mathcal{W}$ and $\Omega - c_0$, where $\mathcal{W}$ is identified with a $\mathfrak{g}$-submodule of $U(\mathfrak{g})$ by the $\mathfrak{g}$-module isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$ given by the symmetrization map, and c_0 is the eigenvalue of Ω for the infinitesimal character that Joseph obtained. We have

$$\operatorname{gr} \mathcal{J}_0 = J_0 = \sqrt{J_W}$$

and this shows that $\mathcal{J}_0$ is indeed completely prime.

We now outline the proof of Theorem 12.3 following [34]. The proof is closely related to the Joseph ideal and its description by Gan and Savin. The key point is that this description was successful in constructing singular vectors of $V^k(\mathfrak{g})$ with $\mathfrak{g}, k$ as in Theorem 12.3.

Let $S(\mathfrak{g}) = \bigoplus_d S^d(\mathfrak{g})$ be the usual grading of $S^d(\mathfrak{g})$ and $V^k(\mathfrak{g}) = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_d$ that one of $V^k(\mathfrak{g})$, see (3.2).

Lemma 12.2 *Let $d \in \mathbb{Z}_{>0}$. We have a $\mathfrak{g}$-module embedding*

$$\sigma_d \colon S^d(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g})_d, \quad x_1 \ldots x_d \longmapsto \frac{1}{d!} \sum_{\tau \in \mathfrak{S}_d} (x_{\tau(1)} t^{-1}) \ldots (x_{\tau(d)} t^{-1}) |0\rangle.$$

Notice that if v is a singular vector for $\mathfrak{g}$ in $S^d(\mathfrak{g})$, then $\sigma_d(v)$ is a singular vector of $V^k(\mathfrak{g})$ if and only if $(f_\theta t)\sigma_d(v) = 0$.

For $\mathfrak{g}$ of type A_1, A_2, G_2, F_4, the number $n - h^\vee/6 - 1$ is admissible for $n \in \mathbb{Z}_{\geqslant 0}$, and Theorem 12.3 is a special case of Theorem 12.2. So there is no loss of generality in assuming that $\mathfrak{g}$ is of type D_4, E_6, E_7, or E_8.

Recall that the Joseph ideal $\mathcal{J}_0$ is generated by the $\mathfrak{g}$-submodule $L_{\mathfrak{g}}(0) \oplus W$ in $S^2(\mathfrak{g})$, where $\mathcal{W}$ is such that, as $\mathfrak{g}$-modules,

$$S^2(\mathfrak{g}) = L_{\mathfrak{g}}(2\theta) \oplus L_{\mathfrak{g}}(0) \oplus W.$$

Let $W = \bigoplus_i W_i$ be the decomposition of $\mathcal{W}$ into irreducible submodules, and let w_i be a highest weight vector of W_i. Recall also that by (12.3), we have $W_i = L_{\mathfrak{g}}(\theta + \theta_i)$.

Note that for $\mathfrak{g}$ of type E_6, E_7, E_8, $W = W_1$ is simple. Moreover, according to [130, Chapter IV, Proposition 11] if $\mathfrak{g}$ is not of type E_8, we have[2]

$$w_i = e_\theta e_{\theta_i} - \sum_{j=1}^{\frac{h^\vee}{6}+1} e_{\beta_j+\theta_i} e_{\delta_j+\theta_i},$$

where (β_j, δ_j) runs through the pairs of positive roots such that

$$\beta_j + \delta_j = \theta - \theta_i.$$

The number of such pairs turns out to be equal to $h^\vee/6+1$. Choose a Chevalley basis $\{h_i\}_i \cup \{e_\alpha,\ f_\alpha\}_\alpha$ of $\mathfrak{g}$ so that the following conditions are fulfilled: for all j,

$$[e_{\delta_j}, [e_{\beta_j}, e_{\theta_1}]] = e_\theta, \quad [e_{\beta_j}, e_{\theta_1}] = e_{\beta_j+\theta_1}, \quad [e_{\delta_j}, e_{\theta_1}] = e_{\delta_j+\theta_1}.$$

Exercise 12.3 Assume that $\mathfrak{g}$ is of type D_4, E_6 or E_7, and let $n \in \mathbb{Z}_{\geqslant 0}$. Show that for each i, $\sigma_2(w_i)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if

$$k = n - \frac{h^\vee}{6} - 1.$$

(The statement holds for $\mathfrak{g}$ of type E_8 but one needs to consider a slightly different description of w_1.)

We are now in a position to prove Theorem 12.3.

Proof (of Theorem 12.3) Assume that $\mathfrak{g}$ is of type D_4, E_6, E_7, or E_8 and that

$$k = n - \frac{h^\vee}{6} - 1 \quad \text{with} \quad n \in \mathbb{Z}_{\geqslant 0}.$$

Let N_k be the submodule of $V^k(\mathfrak{g})$ generated by $\sigma_2(w_i)^{n+1}$ for all i, and set

$$\tilde{L}_k(\mathfrak{g}) := V^k(\mathfrak{g})/N_k.$$

The exact sequence $0 \to N_k \to V^k(\mathfrak{g}) \to \tilde{L}_k(\mathfrak{g}) \to 0$ induces an exact sequence

$$N_k/\mathfrak{g}[t^{-1}]t^{-2}N_k \to V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \to \tilde{L}_k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}\tilde{L}_k(\mathfrak{g}) \to 0.$$

Under the isomorphism $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \cong S(\mathfrak{g})$, the image of $N_k/\mathfrak{g}[t^{-1}]t^{-2}N_k$ in $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$ is identified with the ideal J_k of $S(\mathfrak{g})$ generated by w_i for all i. Hence $J_k \subset J_W \subset \sqrt{J_k}$. Therefore by Lemma 12.1,

$$\sqrt{J_k} = \sqrt{J_W} = J_0,$$

[2] The construction is slightly different if $\mathfrak{g}$ is of type E_8 due to the fact that E_8 is not of *depth one* (cf. [130, Chapter IV, Definition 1]), and that $(\theta - \theta_1)/2$ is not a root.

where J_0 is the defining ideal of $\overline{\mathbb{O}_{\min}}$. Hence $X_{\tilde{L}_k(\mathfrak{g})} = \overline{\mathbb{O}_{\min}}$ by Lemma 12.1.

Next, since $L_k(\mathfrak{g})$ is a quotient of $\tilde{L}_k(\mathfrak{g})$, we get that

$$X_{L_k(\mathfrak{g})} \subset X_{\tilde{L}_k(\mathfrak{g})} = \overline{\mathbb{O}_{\min}} = \mathbb{O}_{\min} \cup \{0\}.$$

Therefore $X_{L_k(\mathfrak{g})}$ is either $\{0\}$ and $\mathbb{O}_{\min}$. The theorem follows since $X_{L_k(\mathfrak{g})} = \{0\}$ if and only if $k \in \mathbb{Z}_{\geqslant 0}$ (cf. Theorem 11.1). □

Conjecture 12.1 Assume that $\mathfrak{g}$ is of type D_4, E_6, E_7, or E_8 and that $k = n - h^\vee/6 - 1$. Then $\tilde{L}_k(\mathfrak{g}) = L_k(\mathfrak{g})$, that is, $\tilde{L}_k(\mathfrak{g})$ is simple, if $k < 0$.

Conjecture 12.1 was proven in [34, Proof of Theorem 3.1] for $n = 0$. Note that if $k \geqslant 0$, $\tilde{L}_k(\mathfrak{g})$ is obviously not simple as, if so, the maximal submodule of $V^k(\mathfrak{g})$ is generated by $(e_\theta t^{-1})^{k+1}|0\rangle$.

As a consequence of Lemma 12.1, Lemma 12.2 and the proof of Conjecture 12.1 for $n = 0$, we obtain the following result.

Theorem 12.4 *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series outside the type A and that $k = -\dfrac{h^\vee}{6} - 1$. Let $\mathcal{J}_W$ be the two-sided ideal of $U(\mathfrak{g})$ generated by $\mathcal{W}$ as in Proposition 12.3. Then $L_k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})/\mathcal{J}_W$, that is,*

$$\mathrm{Zhu}(L_k(\mathfrak{g})) \cong U(\mathfrak{g})/\mathcal{J}_W = \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

In particular since $\mathcal{J}_0$ is maximal, the irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ is a $L_k(\mathfrak{g})$-module if and only if

$$\bar{\lambda} = 0 \quad or \quad \mathrm{Ann}_{U(\mathfrak{g})}\, L_{\mathfrak{g}}(\bar{\lambda}) = \mathcal{J}_0,$$

where $\bar{\lambda}$ is the restriction of λ to the Cartan subalgebra of $\mathfrak{g}$.

According to [156, 4.3], The weights μ such that $\mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\mu) = \mathcal{J}_0$ are of the form:

$$\mu = w \circ (\mu_0 - \rho) := w(\mu_0) - \rho, \quad w \in W_0,$$

where λ_0 and W_0, a subset of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$, are described in Table 12.1. Here we adopt the standard Bourbaki numbering for the simple roots $\{\alpha_1, \dots, \alpha_\ell\}$ of $\mathfrak{g}$, and we denote by $\varpi_1, \dots, \varpi_\ell$ and $s_1, \dots, s_\ell$ the corresponding fundamental weights and simple reflections of $\mathcal{W}$.

12.4 Lisse and quasi-lisse $\mathcal{W}$-algebras

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be a simple Lie algebra, f a nilpotent element of $\mathfrak{g}$ and k a complex number. Recall from Theorem 9.4 that

$\mathfrak{g}$	μ_0	W_0
G_2	$\varpi_1 + \frac{1}{3}\varpi_2$	$\{1, s_2\}$
D_4	$\varpi_1 + \varpi_3 + \varpi_4$	$\{1, s_1, s_3, s_4\}$
F_4	$\frac{1}{2}(\varpi_1 + \varpi_2) + \varpi_3 + \varpi_4$	$\{1, s_1, s_2\}$
E_6	$\varpi_1 + \varpi_2 + \varpi_3 + \varpi_5 + \varpi_6$	$\{1, s_2, s_3, s_1s_3, s_5, s_6s_5\}$
E_7	$\varpi_1 + \varpi_2 + \varpi_3 + \varpi_5 + \varpi_6 + \varpi_7$	$\{1, s_2, s_3, s_1s_3, s_5, s_6s_5, s_7s_6s_5\}$
E_8	$\varpi_1 + \varpi_2 + \varpi_3 + \varpi_5 + \varpi_6 + \varpi_7 + \varpi_8$	$\{1, s_2, s_3, s_1s_3, s_5, s_6s_5, s_7s_6s_5, s_8s_7s_6s_5\}$

Table 12.1 The weight μ_0 and the subset $W_0 \subset W$ such that for $\mu = w(\mu_0) - \rho$, we have $\mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\mu) = \mathcal{J}_0$.

$$\tilde{X}_{\mathcal{W}^k(\mathfrak{g},f)} = \mathscr{S}_f,$$

where $\mathscr{S}_f$ is the Slodowy slice $\mathscr{S}_f = f + \mathfrak{g}^e$ associated with an $\mathfrak{sl}_2$-triple (e, f, h). We write $\mathcal{W}_k(\mathfrak{g}, f)$, as in §9.2.4, the simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$. The associated variety $X_{\mathcal{W}_k(\mathfrak{g},f)}$ of $\mathcal{W}_k(\mathfrak{g}, f)$ is a $\mathbb{C}^*$-invariant Poisson subvariety of $\mathscr{S}_f$. Since it is $\mathbb{C}^*$-invariant, $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse if and only if $X_{\mathcal{W}_k(\mathfrak{g},f)} = \{f\}$.

Recall from Theorem 9.9 (i) that for any quotient V of the universal affine vertex algebra $V^k(\mathfrak{g})$, the associated scheme $\tilde{X}_{DS_f(V)}$ is isomorphic to the scheme theoretic intersection $\tilde{X}_V \times_{\mathfrak{g}^*} \mathscr{S}_f$. In particular,

$$X_{DS_f(V)} = X_V \cap \mathscr{S}_f.$$

Therefore, using Proposition 12.1, we get the following result.

Theorem 12.5 *Let V be a quotient of the universal affine vertex algebra $V^k(\mathfrak{g})$.*

(i) *$DS_f(V)$ is nonzero if and only if $\overline{G.f} \subset X_V$,*
(ii) *$DS_f(V)$ is lisse, and so is $\mathcal{W}_k(\mathfrak{g}, f)$, if $\overline{G.f} = X_V$,*
(iii) *$DS_f(V)$ is quasi-lisse, and so is $\mathcal{W}_k(\mathfrak{g}, f)$, if X_V is contained in $\mathcal{N}$ and $f \in X_V$.*

The fact that $X_V \cap \mathscr{S}_f$ is nonempty if and only if $f \in X_V$, that is, $\overline{G.f} \subset X_V$ is seen using the $\mathbb{C}^*$-contracting action on $\mathscr{S}_f$ (see Lemma D.4) because X_V is closed and G-invariant. Indeed, if $f \in X_V$, clearly $X_V \cap \mathscr{S}_f$ is nonempty. Conversely, if $x \in X_V \cap \mathscr{S}_f$, then $f = \lim_{t \to 0} \tilde{\rho}(t)x$ belongs to $X_V \cap \mathscr{S}_f$, where $\tilde{\rho}$ is the one-parameter subgroup defined as in (D.4).

As a consequence of Theorems 12.1 and 12.5, we get the following analogous result for the Drinfeld–Sokolov reduction of $V^k(\mathfrak{g})$.

Theorem 12.6 *The equality $DS_f(L_k(\mathfrak{g})) = DS_f(V^k(\mathfrak{g}))$ holds if and only if $X_{DS_f(L_k(\mathfrak{g}))} = \mathscr{S}_f$.*

Proof For the critical level $k = -h^\vee$, it is known that $X_{DS_f(L_k(\mathfrak{g}))} = \mathcal{N} \cap \mathscr{S}_f \neq \mathscr{S}_f$. Hence, there is no loss of generality in assuming that $k + h^\vee \neq 0$. Suppose

$X_{DS_f(L_k(\mathfrak{g}))} = \mathscr{S}_f$. We wish to show $DS_f(V^k(\mathfrak{g})) = DS_f(L_k(\mathfrak{g}))$. By Theorem 12.1, it is enough to show that $X_{L_k(\mathfrak{g})} = \mathfrak{g}$. Assume the contrary. Then $X_{L_k(\mathfrak{g})}$ is contained in a proper G-invariant closed subset of $\mathfrak{g}$. On the other hand, by Theorem 12.5 (i) and our hypothesis, we have

$$\mathscr{S}_f = X_{DS_f(L_k(\mathfrak{g}))} = X_{L_k(\mathfrak{g})} \cap \mathscr{S}_f .$$

Hence, $\mathscr{S}_f$ must be contained in a proper G-invariant closed subset of $\mathfrak{g}$. But this contradicts Theorem 7.5. The proof of the theorem is completed. □

The simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient vertex algebra of $DS_f(L_k(\mathfrak{g}))$, provided it is nonzero.

Theorem 12.5 (iii) implies that if $L_k(\mathfrak{g})$ is quasi-lisse and if $f \in X_{L_k(\mathfrak{g})}$, then the $\mathcal{W}$-algebra $DS_f(L_k(\mathfrak{g}))$ is quasi-lisse as well (and nonzero), and so is its simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$. In this way we obtain a huge number of quasi-lisse $\mathcal{W}$-algebras. Furthermore, if $X_{L_k(\mathfrak{g})} = \overline{G.f}$, then $X_{DS_f(L_k(\mathfrak{g}))} = \{f\}$, so that $\mathcal{W}_k(\mathfrak{g}, f)$ is in fact lisse.

Example 12.3 If k is an admissible level, then one knows that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$ for some nilpotent orbit $\mathbb{O}_k$ (cf. Theorem 12.2). Picking $f \in \mathbb{O}$, we obtain that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. Moreover, for any $f \in \overline{\mathbb{O}_k}$, we obtain that $\mathcal{W}_k(\mathfrak{g}, f)$ is quasi-lisse.

Example 12.4 By Theorem 12.3, there exist other lisse simple $\mathcal{W}$-algebras, not coming from admissible levels. Namely, fix $\mathfrak{g}, k$ as in Theorem 12.3, and choose $f_{\min} \in \mathbb{O}_{min}$. Then $\mathcal{W}_k(\mathfrak{g}, f_{\min})$ is lisse. In [34], we actually obtained a stronger result.

Theorem 12.7 *Assume that* $\mathfrak{g}$ *is of type* D_4, E_6, E_7 *or* E_8 *and that* $k = -h^\vee/6 - 1 + n$, *where* $n \in \mathbb{Z}_{\geqslant 0}$, *then* $\mathcal{W}_k(\mathfrak{g}, f_{\min})$ *is lisse.*

In fact, for $n = 0$, we have that $\mathcal{W}_k(\mathfrak{g}, f_{\min}) \cong \mathbb{C}$ and so $\mathcal{W}_k(\mathfrak{g}, f_{\min})$ is also rational.

The first example of lisse simple affine $\mathcal{W}$-algebra not coming from an admissible level was discovered by Kawasetsu [171]. Specifically, Kawasetsu showed that $\mathcal{W}_k(\mathfrak{g}, f_{\min})$ is lisse for $\mathfrak{g}$ of type D_4, E_6, E_7 or E_8 and $k = -h^\vee/6$. Furthermore, for such $\mathfrak{g}, k$, $\mathcal{W}_k(\mathfrak{g}, f_{\min})$ is rational.

Conjecture 12.2 Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series and that $k = -h^\vee/6 - 1 + n$, where $n \in \mathbb{Z}_{\geqslant 0}$. Then $\mathcal{W}_k(\mathfrak{g}, f_{\min})$ is rational if and only if $k \notin \mathbb{Z}_{\geqslant 0}$.

Conjecture 12.2 for admissible k, that is, for $\mathfrak{g}$ of type A_1, A_2, G_2 or F_4 is known by Kac–Wakimoto [167]. We refer to Conjecture 14.1 and Conjecture 14.4 for other conjectures on the same theme.

Talk about the conjecture for G_2 at level -2 somewhere!!!

12.5 Other examples

Let us mention in this paragraph other examples of quasi-lisse vertex algebras.

Given a smooth affine variety X, the global section of the *chiral differential operators*[3] $\mathscr{D}_X^{\mathrm{ch}}$ ([207, 138, 53]) is quasi-lisse because its associated scheme is canonically isomorphic to the cotangent bundle T^*X which is symplectic. In particular, the vertex algebra of chiral differential operators $\mathscr{D}_{G,k}^{\mathrm{ch}}$ associated with a reductive group G at level k (see Section 3.4 and §9.1.6), is quasi-lisse.

Assume now that $\mathfrak{g} = \mathrm{Lie}(G)$ is simple. The equivariant $\mathcal{W}$-algebra $\widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ associated with the simple Lie algebra $\mathfrak{g}$ and the nilpotent element $f \in \mathfrak{g}$ at level $k \in \mathbb{C}$ is quasi-lisse. Indeed, by Theorem 9.7, its associated scheme is the equivariant Slodowy slice $\widetilde{\mathscr{S}_f}$ defined by (7.10).

Obviously, if X is any affine Poisson variety with finitely many symplectic leaves, then $\mathcal{J}_\infty X$ is a quasi-lisse vertex algebra, with associated scheme X.

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[3] If X is a smooth scheme, there is a non-trivial obstruction of cohomological nature to the construction of $\mathscr{D}_X^{\mathrm{ch}}$ which can be expressed in terms of a certain homotopy Lie algebra.

Chapter 13
Properties of quasi-lisse vertex algebras

In this chapter, it is assumed that V is a strongly generated $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra such that $V_0 \cong \mathbb{C}|0\rangle$. Such vertex algebras are sometimes said of CFT-type. Recall that X_V is called quasi-lisse is X_V has only finitely many symplectic leaves. We have already noticed that lisse vertex algebras are very nice (see e.g., Lemma 4.9 and Theorem 11.3). This chapter explores interesting properties of the larger class of quasi-lisse vertex algebras.

It is known that Poisson varieties with only finitely many symplectic leaves have special properties. For example, Brown and Gordon have proved [69] that a finite number of symplectic leaves in a Poisson variety X implies that the symplectic leaf $\mathscr{L}_x$ at $x \in X$ coincides with the regular locus of the zero variety of the maximal Poisson ideal contained in the maximal ideal $\mathfrak{m}_x$ corresponding to x (see Section **??** for more details about this). Thus, each symplectic leaf $\mathscr{L}_x$ is a smooth connected locally-closed algebraic subvariety in X. In particular, every irreducible component of X is the closure of a symplectic leaf [135, Corollary 3.3]. On the other hand, it has been established by Etingof and Schedler [101] that if R is a finitely generated Poisson algebra such that $X = \mathrm{Specm}(R)$ has finitely many symplectic leaves, then the 0-th Poisson cohomology $R/\{R, R\}$ is finite-dimensional. As one can expect, these important facts play an important role in the study of quasi-lisse vertex algebras. This is what we shall see in this chapter.

Section 13.1 is about the modular invariance properties of quasi-lisse vertex algebras. In Section 13.3, we introduce the notion of *chiral symplectic core* and exploit them to show that any quasi-lisse vertex algebra V is a quantization of the reduced arc space of its associated variety, in the sense that its reduced singular support $\mathrm{Spec}(\mathrm{gr}^F V)$ coincides with $\mathcal{J}_\infty(X_V)$ as topological spaces (Theorem 13.4). Finally, Section 13.2 concerns the irreducibility conjecture for the associated variety of quasi-lisse vertex algebras (Conjecture 13.1) and its connection with the *Higgs branch conjecture* in four-dimensional $\mathcal{N} = 2$ super-conformal theories.

13.1 Modular invariance property of quasi-lisse vertex algebras

Let V be a conformal vertex algebra and M a V-module M. As usual, we set $M_d := \{m \in M : L_0 m = dm\}$. The L_0-eigenvalue of a nonzero L_0-eigenvector $m \in M$ is called its *conformal weight*. For an element $a \in V$ of conformal weight Δ we write $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-\Delta}$.

Recall that a finitely generated V-module M is called *ordinary* (Definition 3.2) if L_0 acts semisimply, $\dim M_d < \infty$ for all d, and the conformal weights of M are bounded from below. The minimum conformal weight of a simple ordinary V-module M is called the *conformal dimension* of M. The *normalized character* of an ordinary representation M is defined by

$$\chi_M(\tau) = \mathrm{tr}_M q^{L_0 - c_V/24} = q^{-c_V/24} \sum_{d\in\mathbb{C}} (\dim M_d) q^d, \quad q = e^{2\pi i \tau} \text{ with } \tau \in \mathbb{C}.$$

13.1.1 Analogues to Theorems 5.4 and 11.3

If V is quasi-lisse, the finiteness of the symplectic leaves of X_V implies that the 0-th Poisson cohomology $R_V/\{R_V, R_V\}$ of the Zhu C_2-algebra R_V is finite-dimensional ([101]). Zhu's proof of modular invariance property for lisse and rational vertex algebras [257] can be adapted for the vertex algebras V such that the following condition holds:

$$\dim R_V/\{R_V, R_V\} < +\infty. \tag{13.1}$$

Starting out from this observation, the following result was established in [27].

Theorem 13.1 *Let V be a quasi-lisse conformal vertex algebra. Then V admits only finitely many simple ordinary representations. Moreover, the normalized character of any ordinary module has satisfies a modular linear differential equation.*

The proof of [27] only uses the condition (13.1). So the condition "quasi-lisse" can be replaced by the weaker condition (13.1) in the above theorem.

Remark 13.1 In general, the condition $\dim R/\{R, R\} < +\infty$ does not imply that $X = \operatorname{Spec} R$ has finitely many symplectic leaves.

Using a modular linear differential equation, the explicit character formulas of the simple quasi-lisse affine vertex algebras associated with the Deligne exceptional series at level $k = -h^\vee/6 - 1$ (cf. Theorem 12.3) are obtained in [27].

It is known that the admissible representations $L_k(\mathfrak{g})$ have only finitely many simple objects in the category $\mathcal{O}$ and that their normalized characters satisfy a modular invariance property. Note that the above result has a different meaning (simple objects in the category $\mathcal{O}$ does not coincide with simple ordinary modules), and was new even for an admissible affine vertex algebras.

A key point in the proof of the above result is the following fact (compare with Corollary 4.3).

Theorem 13.2 *Let ω be a conformal vector a simple affine vertex algebra $L_k(\mathfrak{g})$, and $\bar{\omega}$ its image in the quotient $R_{L_k(\mathfrak{g})}$. The following two conditions are equivalent.*

(i) $L_k(\mathfrak{g})$ *is quasi-lisse.*
(ii) $\bar{\omega}$ *is nilpotent in $R_{L_k(\mathfrak{g})}$.*

Proof The direction (i) ⇒ (ii) is true in general [27]. Let us show (ii) ⇒ (i). Recall that the Poisson center of $\mathbb{C}[\mathfrak{g}^*]$ is the ring $\mathbb{C}[\mathfrak{g}^*]^G$ of G-invariant polynomials. By Theorem 4.4, the image of $\mathbb{C}[\mathfrak{g}^*]^G_+$ in $R_{L_k(\mathfrak{g})}$ is contained in the nilradical, and this implies that $X_{L_k(\mathfrak{g})}$ is contained in the nilpotent cone of $\mathfrak{g}^*$. Thus the assertion follows from Proposition 12.1. □

Remark 13.2 The same statement as Corollary 13.2 holds for simple affine W-algebras.

13.1.2 Asymptotic behavior of the normalized characters

The phenomenon of modular invariance of characters for quasi-lisse vertex algebras yields to the notion of asymptotic datum.

Let V be a vertex algebra of CFT-type, that is, a $\mathbb{Z}$-graded conical vertex algebra. Then V is called *self-dual* if $V \cong V'$ as V-modules, where M' denotes the contragredient dual [119] of the V-module M. Equivalently V is self-dual if and only if it admits a non-degenerate symmetric invariant bilinear form.

The following definition goes back to [164, Conjecture 1][1].

Definition 13.1 A conformal vertex algebra V is said to admit an *asymptotic datum* if there exist $\mathbf{A}_V \in \mathbb{R}$, $\mathbf{w}_V \in \mathbb{R}$, $\mathbf{g}_V \in \mathbb{R}$ such that

$$\chi_V(\tau) \sim \mathbf{A}_V(-\mathrm{i}\tau)^{\frac{\mathbf{w}_V}{2}} e^{\frac{\pi \mathrm{i}}{12\tau}\mathbf{g}_V} \quad \text{as } \tau \downarrow 0.$$

The numbers $\mathbf{A}_V$, $\mathbf{w}_V$ and $\mathbf{g}_V$ are called the *asymptotic dimension* of V, the *asymptotic weight*, and the *asymptotic growth*, respectively. Similarly, an ordinary V-module M is said to admit an asymptotic datum if there exist $\mathbf{A}_M \in \mathbb{C}$, $\mathbf{w}_M, \mathbf{g}_M \in \mathbb{R}$ such that

$$\chi_M(\tau) \sim \mathbf{A}_M(-\mathrm{i}\tau)^{\frac{\mathbf{w}_M}{2}} e^{\frac{\pi \mathrm{i}}{12\tau}\mathbf{g}_M} \quad \text{as } \tau \downarrow 0.$$

Proposition 13.1 *Let V be a finitely strongly generated, rational, lisse self-dual simple vertex operator algebra of CFT-type. Then any simple V-module M admits an asymptotic datum with $\mathbf{w}_M = 0$.*

---

[1] In [164] the triple $(\mathbf{A}_V, \mathbf{w}_V, \mathbf{g}_V)$ was called the asymptotic dimension.

Proof By Theorem 11.3 (i) any simple V-module is ordinary, and there exist finitely many simple V-modules, say, $\{L_i : i = 0, \dots, r\}$. Let h_i be the conformal dimension of L_i. Then

$$\chi_{L_i}(\tau) = q^{h_i - c/24} \sum_{d \geqslant 0} (\dim(L_i)_{h_i+d}) q^d.$$

By Theorem 11.3 (iii) the vector space spanned by $\chi_{L_i}(\tau)$, $i = 0, \dots, r$, is invariant under the natural action of the modular group $\mathrm{SL}_2(\mathbb{Z})$. Hence,

$$\chi_{L_i}(\tau) = \sum_{j=1}^{r} S_{i,j} \chi_{L_j}(-1/\tau)$$

for some $S_{i,j} \in \mathbb{C}$, $j = 0, \dots, r$. The assertion follows. □

The following result is proved in [38].

Proposition 13.2 *Let V be a finitely strongly generated, quasi-lisse vertex operator algebra of CFT-type. Then any simple ordinary V-module L admits an asymptotic datum.*

Proof By Theorem 13.1, the set $\{L_i\}$ of simple ordinary V-modules is finite, and the characters $\chi_{L_i}(\tau)$ are solutions of a modular linear differential equation. Since the space spanned by the solutions of a modular linear differential equation is invariant under the natural action of $\mathrm{SL}_2(\mathbb{Z})$, the assertion follows in a similar manner as Proposition 13.1, except that a solution of a modular linear differential equation may have logarithmic terms, that is, it has the form

$$q^{\beta_i} \sum_{i=1}^{e} f_i(q)(\log q)^{e-i}, \quad f_i(q) \in \mathbb{C}[[q]].$$

This concludes the proof. □

Similarly to Theorem 13.1, the condition "quasi-lisse" can be replaced by the weaker condition (13.1).

The asymptotic datum can be explicitly computed for some classes of quasi-lisse vertex algebras.

Example 13.1 A quotient of a universal Virasoro vertex algebra admits an asymptotic datum. Indeed, it is well-known that Vir^c has length two if $c = 1 - 6(p-q)^2/pq$ for some $p, q \in \mathbb{Z}_{\geqslant 2}$, $(p, q) = 1$, and otherwise $\mathrm{Vir}^c = \mathrm{Vir}_c$. Hence, a quotient V of Vir^c is either Vir^c or Vir_c.

- If $V = \mathrm{Vir}^c$, then

$$\chi_V(\tau) = \frac{(1-q)q^{(1-c_V)/24}}{\eta(q)},$$

where $\eta(q) = q^{1/24} \prod_{j \geqslant 1}(1 - q^j)$. Hence (indicating by $+\cdots$ terms of lower growth)

$$\chi_V(e^{2\pi i\tau}) = (1 - e^{2\pi i\tau})e^{2\pi i\tau(1-c_V)/24}(-i\tau)^{\frac{1}{2}}\left(e^{2\pi i(-\frac{1}{\tau})(-\frac{1}{24})} + \cdots\right),$$
$$\sim (-2\pi i\tau)(-i\tau)^{\frac{1}{2}}e^{\frac{\pi i}{12\tau}},$$

where we have used l'Hopital's rule. So V admits an asymptotic datum with $\mathbf{A}_V = 2\pi$, $\mathbf{w}_V = 3$, $\mathbf{g}_V = 1$.

- If $V = \mathrm{Vir}_c$ with $c = 1 - 6(p-q)^2/pq$, $p, q \in \mathbb{Z}_{\geqslant 2}$, $(p,q) = 1$, then as it is well-known [110, 164], V admits an saymptotic datum with $\mathbf{w}_V = 0$,

$$\mathbf{A}_V = \left(\frac{8}{pq}\right)^{1/2} \sin\left(\frac{\pi a(p-q)}{q}\right)\sin\left(\frac{\pi b(p-q)}{p}\right),$$

where (a,b) is the unique solution of $pa - qb = 1$ in integers $1 \leqslant a \leqslant q$ and $1 \leqslant b \leqslant p$, and

$$\mathbf{g}_V = 1 - \frac{6}{pq}.$$

Example 13.2 Let $k = -h^\vee + p/q$ be an admissible number for the simple Lie algebra $\mathfrak{g}$ of the form (12.1), defined by a root system Δ. Then the simple affine vertex algebra $L_k(\mathfrak{g})$ admits an asymptotic datum ([165, 38]).

- If $(\check{r}, q) = 1$, then

$$\mathbf{g}_{L_k(\mathfrak{g})} = \left(1 - \frac{h^\vee}{pq}\right)\dim\mathfrak{g}, \quad \mathbf{w}_{L_k(\mathfrak{g})} = 0,$$
$$\mathbf{A}_{L_k(\mathfrak{g})} = \frac{1}{q^{|\Delta_+|}|P/(pq)Q^\vee|^{\frac{1}{2}}}\prod_{\alpha\in\Delta_+} 2\sin\frac{\pi(\rho|\alpha)}{p},$$

where ρ is a half-sum of the positive root, P is the weight lattice and $Q^\vee$ is the dual root lattice.

- If $(\check{r}, q) = \check{r}$, then

$$\mathbf{g}_{L_k(\mathfrak{g})} = \left(1 - \frac{\check{r}h^\vee_{^L\mathfrak{g}}}{pq}\right)\dim\mathfrak{g}, \quad \mathbf{w}_{L_k(\mathfrak{g})} = 0,$$
$$\mathbf{A}_{L_k(\mathfrak{g})} = \frac{\check{r}^{|\Delta_+^{\mathrm{short}}|}}{q^{|\Delta_+|}|P^\vee/(pq)Q|^{\frac{1}{2}}}\prod_{\alpha\in\Delta_+} 2\sin\frac{\pi(\rho|\check{\alpha})}{p},$$

where $\Delta_+^{\mathrm{short}}$ is the set of positive short roots, $h^\vee_{^L\mathfrak{g}}$ is the dual Coxeter number of simple Lie algebra ${}^L\mathfrak{g}$ defined by the dual root system $\check{\Delta} = \{\check{\alpha} : \alpha \in \Delta\}$, $P^\vee$ is the dual weight lattice and Q is the root lattice.

Moreover, the Drinfeld–Sokolov reduction $DS_f(L_k(\mathfrak{g}))$ also admits an asymptotic data, where f is a nonzero nilpotent element of $\mathfrak{g}$.

- If $(\check{r}, q) = 1$, then

$$\mathbf{g}_{DS_f(L_k(\mathfrak{g}))} = \mathbf{g}_{L_k(\mathfrak{g})} - \dim \overline{G.f} = \dim \mathfrak{g}^f - \frac{h^\vee \dim \mathfrak{g}}{pq},$$
$$\mathbf{A}_{DS_f(L_k(\mathfrak{g}))} = \frac{1}{2^{\frac{|\Delta^{1/2}|}{2}} q^{|\Delta_+^0|} |P/(pq)Q^\vee|^{\frac{1}{2}}} \prod_{\alpha\in\Delta_+} 2\sin\frac{\pi(\rho|\alpha)}{p} \prod_{\alpha\in\Delta_+\setminus\Delta_+^0} 2\sin\frac{\pi(h|\alpha)}{q},$$

where h is the neutral element of an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$, that we assume to be in the Cartan associated with the choice of Δ, and $\Delta^i := \{\alpha \in \Delta\colon [h, e_\alpha] = ie_\alpha\}$, with e_α a nonzero element in the α-root space.

- If $(\check{r}, q) = \check{r}$, then

$$\mathbf{g}_{DS_f(L_k(\mathfrak{g}))} = \mathbf{g}_{L_k(\mathfrak{g})} - \dim \overline{G.f} = \dim \mathfrak{g}^f - \frac{\check{r} h^\vee_{L_\mathfrak{g}} \dim \mathfrak{g}}{pq},$$
$$\mathbf{A}_{DS_f(L_k(\mathfrak{g}))} = \frac{\check{r}^{|\Delta_+^{\text{short}}\cap\Delta^0|}}{2^{\frac{|\Delta_\Gamma^{1/2}|}{2}} q^{|\Delta^0_{\Gamma,+}|} |P^\vee/(pq)Q|^{\frac{1}{2}}} \prod_{\alpha\in\Delta_+} 2\sin\frac{\pi(\rho|\check{\alpha})}{p} \prod_{\alpha\in\Delta_+\setminus\Delta_+^0} 2\sin\frac{\pi(h|\check{\alpha})}{q}.$$

13.2 Irreducibility conjecture

Taking all examples of quasi-lisse vertex algebras as in Chapter 12 into consideration we formulate in [35] a conjecture.

Conjecture 13.1 Let $V = \oplus_{d\geqslant 0} V_d$ be a simple, finitely strongly generated, positively graded conformal vertex operator algebra such that $V_0 \cong \mathbb{C}|0\rangle$. Assume that X_V has finitely many symplectic leaves, that is, V is quasi-lisse. Then X_V is irreducible.

In particular, if the associated variety $X_{L_k(\mathfrak{g})}$ of the simple affine vertex algebra $L_k(\mathfrak{g})$ at level k is contained in the nilpotent cone $\mathcal{N}$, the conjecture stipulates that $X_{L_k(\mathfrak{g})}$ must be the closure of some nilpotent orbit in $\mathfrak{g}$.

Thus, Conjecture 13.1 in particular says that any quasi-lisse simple affine vertex algebra produces exactly one lisse simple $\mathcal{W}$-algebra. Indeed, assume that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}$ for some nilpotent orbit $\mathbb{O} = G.f$ in $\mathfrak{g}$. Then the $\mathcal{W}$-algebra $DS_f(L_k(\mathfrak{g}))$ is lisse by Theorem 12.5 (ii), and so is the simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$.

13.2.1 On the irreducibility of nilpotent Slodowy slices

Conjecture 13.1 has been verified for the known cases of quasi-lisse vertex algebras in [36] as described in Chapter 12. This is a subtle problem for the associated varieties of the Drinfeld–Sokolov reduction of examples of known simple quasi-lisse affine vertex algebras (see Section 12.4) for which deep results on the geometry of nilpotent orbits are needed for the verifications. We refer the reader to Section D.6 in appendix for more details about the geometry of nilpotent Slodowy slices, in particular the unibranchness.

To be more specific, by Theorem 12.5 (iii), the $\mathcal{W}$-algebra $DS_f(L_k(\mathfrak{g}))$ is quasi-lisse if $X_{L_k(\mathfrak{g})}$ is contained in the nilpotent cone $\mathcal{N}$. Assuming Conjecture 13.1 true, the associated variety of $DS_f(L_k(\mathfrak{g}))$ is in fact the nilpotent Slodowy slice $\overline{\mathbb{O}} \cap \mathscr{S}_f$ for some nilpotent orbit $\mathbb{O}$. This variety is known to be irreducible for all $f \in \overline{\mathbb{O}}$ if and only if $\overline{\mathbb{O}}$ is unibranch at all $f \in \overline{\mathbb{O}}$.

It is verified in [36] that all nilpotent orbit closures appearing as associated varieties of known quasi-lisse simple affine vertex algebras, that is, the simple affine vertex algebras at admissible levels (for which the associated varieties are described in Tables 2–10 of [15, Table 4]) plus those listed in §12.3.1, are unibranch. They are all normal (and so unibranch) in the simple Lie algebras of classical types, and all unibranch in the simple Lie algebras of exceptional types. Note that the nilpotent orbit $\tilde{A}_1$ of G_2 of dimension 8 is not normal [178], but unibranch, and appears as associated variety of $L_k(G2)$ for some k with denominator 2 ([15, Table 4]).

Many other simple affine vertex algebras (at non-admissible levels) are expected to be quasi-lisse. So the above discussion is a sort of verification that Conjecture 13.1 is highly plausible.

13.2.2 Ginzburg's irreducibility theorem

In another direction, Conjecture 13.1 is a natural affine analog of the irreducibility theorem (cf. Theorem D.5) for the associated variety of primitive ideals of $U(\mathfrak{g})$, which has been generalized to a larger class of Noetherian algebras in [134]:

Theorem 13.3 (Ginzburg) *Let A be a filtered unital $\mathbb{C}$-algebra. Assume furthermore that* $\operatorname{gr} A \cong \mathbb{C}[X]$ *is the coordinate ring of a reduced irreducible affine algebraic variety X, and assume that the Poisson variety* $\operatorname{Spec}(\operatorname{gr} A)$ *has only finitely many symplectic leaves. Then for any primitive ideal $I \subset A$, the zero locus $\mathcal{V}(I)$ of* $\operatorname{gr} I$ *in X is the closure of a single symplectic leaf. In particular, it is irreducible.*

Ginzburg's proof of Theorem 13.3 is an adaptation of a more direct proof of Theorem D.5 discovered subsequently by Vogan [249], combined with the results by Brown and Gordon [69] on symplectic cores. Motivated by Ginzburg's theorem and Conjecture 13.1, we introduced in [37] the notion of *chiral symplectic cores* (see Definition 13.3). This is the topic of the next section.

13.3 Chiral symplectic cores and applications

Let $X = \operatorname{Spec} R$ be a reduced Poisson scheme. For I an ideal of R, we denote by $\mathscr{P}_R(I)$ the biggest Poisson ideal of R contained in I. The *symplectic core* $\mathscr{C}_R(x)$ of a point $x \in X$ is the equivalence class of x for $\sim$, with

$$x \sim y \iff \mathscr{P}_R(\mathfrak{m}_x) = \mathscr{P}_R(\mathfrak{m}_y).$$

Here, $\mathfrak{m}_x$ stands for the maximal ideal of R corresponding to x. The notion of symplectic cores, introduced in [69], are expected to be the finest possible algebraic stratification in which the Hamiltonian vector fields are tangent. Brown and Gordon showed that the symplectic cores coincide with the symplectic leaves, if there is only finitely many numbers of symplectic leaves.

13.3.1 Chiral symplectic cores

It is natural to try to adapt this notion to the context of Poisson vertex algebras. Assume for awhile that V is a Poisson vertex algebra. Let I be an ideal of V in the associative sense.

Definition 13.2 We say that I is a *chiral Poisson ideal* of V if $a_{(n)}I \subset I$ for all $a \in V$, $n \in \mathbb{Z}_{\geqslant 0}$.

Thus a Poisson vertex ideal of V is a chiral Poisson ideal that is stable under the action of the translation operator T. The quotient space V/I inherits a Poisson vertex algebra structure from V if I is a Poisson vertex ideal. Note that if I is a vertex (resp. chiral) Poisson ideal of V, then so is its radical $\sqrt{I}$ ([88, §3.3.2]).

Denote by $\mathscr{P}_V(I)$ the biggest chiral Poisson ideal of V contained in I. It exists since the sum of two chiral Poisson ideals is chiral Poisson. Set

$$\mathcal{L} := \mathrm{Specm}(V),$$

and define a relation $\sim$ on $\mathcal{L}$ by

$$x \sim y \iff \mathscr{P}_V(\mathfrak{m}_x) = \mathscr{P}_V(\mathfrak{m}_y),$$

where $\mathfrak{m}_x$ denotes the maximal ideal corresponding to $x \in \mathcal{L}$. Clearly $\sim$ is an equivalence relation. We will write $\mathscr{C}_{\mathcal{L}}(x)$ for the equivalence class in $\mathcal{L}$ of x, so that

$$\mathcal{L} = \bigsqcup_x \mathscr{C}_{\mathcal{L}}(x).$$

Definition 13.3 Call the set $\mathscr{C}_{\mathcal{L}}(x)$ the *chiral symplectic core* of x in $\mathcal{L}$.

Chiral symplectic cores are expected to be the finest possible algebraic stratification in which the *chiral* Hamiltonian vector fields are tangent.

Let us return to the case where V is arbitrary (not necessarily a vertex Poison algebra). Recall that $SS(V)$ stands for the singular support of V, that is, the spectrum of $\mathrm{gr}^F V$ (cf. Definition 4.8). By Lemma 4.9, the vertex algebra V is lisse if and only if $\dim SS(V) = 0$ ($\iff$ $\dim X_V = 0$). For the quasi-lisse vertex algebras, we have the following result.

Theorem 13.4 *Assume that V is a quasi-lisse vertex algebra. Then $SS(V) \cong \mathcal{J}_\infty X_V$ as topological spaces, that is,*

$$SS(V)_{\mathrm{red}} \cong (\mathcal{J}_\infty X_V)_{\mathrm{red}}.$$

Moreover, the reduced singular support $SS(V)_{\mathrm{red}}$ have finitely many irreducible components, and each of them is the closure of some chiral symplectic leaf.

Proof Set

$$\mathcal{L} := \mathrm{Specm}(\mathrm{gr}^F V) = SS(V)_{\mathrm{red}},$$

and let $X_1, \ldots, X_r$ be the irreducible components of X_V. By the quasi-lisse hypothesis, the symplectic leaves in X_V are algebraic and coincide with the symplectic cores ([69]). Moreover, each component irreducible component X_i is the Zariski closure of some symplectic core $X_i = \overline{\mathscr{C}_{X_V}(x_i)}$ with $x_i \in X_i$. By [37, Theorem 9.1], we have

$$(\mathcal{J}_\infty X_V)_{\mathrm{red}} = \overline{\mathscr{C}_{\mathcal{J}_\infty(X_V)}(x_{1,\infty})} \cup \ldots \cup \overline{\mathscr{C}_{\mathcal{J}_\infty(X_V)}(x_{r,\infty})}, \tag{13.2}$$

where x_∞ stands for the image of $x \in X_V$ by the canonical embedding $\iota_\infty : X_V \hookrightarrow (\mathcal{J}_\infty X_V)_{\mathrm{red}}$. By Proposition 4.5, $\mathrm{gr}^F V$ is a Poisson vertex algebra quotient of $\mathcal{J}_\infty R_V$, that is, $\mathrm{gr}^F V = \mathcal{J}_\infty R_V / I$ with I a Poisson vertex ideal of $\mathcal{J}_\infty(R_V)$. Furthermore, the surjective morphisms,

$$\mathcal{J}_\infty R_V \twoheadrightarrow \mathrm{gr}^F V \twoheadrightarrow R_V,$$

induce the embeddings of varieties,

$$X_V \hookrightarrow \mathcal{L} \hookrightarrow (\mathcal{J}_\infty X_V)_{\mathrm{red}},$$

and the composition map is nothing but ι_∞. Hence for any $x \in X_V$, the maximal ideal of $\mathcal{J}_\infty R_V$ corresponding to x_∞ contains $\sqrt{I}$. So x_∞ is a point of $\mathcal{L}$.

Since $\mathcal{L}$ is a reduced closed vertex Poisson subscheme of $\mathcal{J}_\infty(X_V)$, we know that $\mathscr{C}_{\mathcal{J}_\infty(X_V)}(x_{i,\infty}) = \mathscr{C}_{\mathcal{L}}(x_{i,\infty})$ for any $i = 1, \ldots, r$ ([37, Lemma 6.5]). Then use (13.2) and Proposition 4.5 to obtain that

$$\mathcal{L} \subset (\mathcal{J}_\infty X_V)_{\mathrm{red}} = \overline{\mathscr{C}_{\mathcal{L}}(x_{1,\infty})} \cup \ldots \cup \overline{\mathscr{C}_{\mathcal{L}}(x_{r,\infty})} \subset \mathcal{L},$$

since $\mathcal{L}$ is closed. This proves all statements. □

Denote, as usual, by $\tilde{X}_V = \mathrm{Spec}\, R_V$ the associated scheme of V (cf. Definition 4.7).

Corollary 13.1 *Suppose that $\tilde{X}_V$ is smooth, reduced and symplectic. Then $\mathrm{gr}^F V \cong \mathcal{J}_\infty R_V$, that is, $SS(V) \cong \mathcal{J}_\infty \tilde{X}_V$ as schemes and $\mathrm{gr}^F V$ is simple as a Poisson vertex algebra. In particular, V is simple.*

Proof If X_V is a smooth symplectic variety then $\mathcal{J}_\infty X_V$ consists of a single chiral symplectic leaf. So $\mathcal{J}_\infty X_V = \mathscr{C}_{\mathcal{J}_\infty X_V}(x)$ for any $x \in \mathcal{J}_\infty X_V$. It follows that there is no nonzero proper chiral Poisson subscheme in $\mathcal{J}_\infty X_V$. From Theorem 13.4, we conclude that there is no nonzero proper chiral Poisson subscheme in $\mathrm{Spec}\, \mathrm{gr}^F V$, too. Therefore $\mathrm{gr}^F V$ is simple as a Poisson vertex algebra. This shows that V is simple, because any vertex ideal $I \subset V$ yields a Poisson vertex (and so chiral Poisson) ideal $\mathrm{gr}^F I$ in $\mathrm{gr}^F V$. □

Example 13.3 If X is a smooth affine variety X, the global section of the chiral differential operators $\mathscr{D}_X^{\mathrm{ch}}$ is a simple vertex algebra, and $\mathrm{gr}^F \mathscr{D}_X^{\mathrm{ch}}$ is a simple Poisson vertex algebra because the associated scheme of $\mathscr{D}_X^{\mathrm{ch}}$ is canonically isomorphic to the cotangent bundle T^*X which is symplectic (see Section 12.5). In the case where G is a reductive group, the simplicity of $\mathscr{D}_{G,k}^{ch}$, for $k \in \mathbb{C}$n can be shown by other methods (see Exercise 3.3 or, for instance, [237]).

Example 13.4 The equivariant $\mathcal{W}$-algebra $\widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ associated with a simple Lie algebra $\mathfrak{g}$, a nilpotent element f and a level $k \in \mathbb{C}$ is simple, and $\mathrm{gr}^F \widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ is a simple Poisson vertex algebra, because the associated scheme of $\widetilde{\mathcal{W}}^k(\mathfrak{g}, f)$ is the equivariant Slodowy slice $\widetilde{\mathscr{S}_f}$ which is symplectic (see Section 12.5).

13.3.2 Poisson vertex center of the arc spaces of Slodowy slices

Assume that $\mathfrak{g}$ is a simple Lie algebra with adjoint group G, and identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using the nondegenerate bilinear form $(-|-)$. Recall that the *Slodowy slice* $\mathscr{S}_f := f + \mathfrak{g}^e$ associated with an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$ has a Poisson structure obtained from that of $\mathfrak{g}^*$ by Hamiltonian reduction (see Section 7.2). Consider the adjoint quotient morphism

$$\psi_f : \mathscr{S}_f \to \mathfrak{g}^* /\!\!/ G. \tag{13.3}$$

It is known [227] that any fiber $\psi_f^{-1}(\xi)$ of this morphism is the closure of a symplectic leaf, which is irreducible and reduced. An analogue statement is true for the arc space of the Slodowy slices ([37]).

Theorem 13.5 *Let $\mathfrak{g}$ be a simple Lie algebra, f a nilpotent element of $\mathfrak{g}$ and ψ_f the quotient morphism* (13.3)*.*

(i) *Any fiber of the induced Poisson vertex algebra morphism*

$$\mathcal{J}_\infty \psi_f : \mathcal{J}_\infty \mathscr{S}_f \to \mathcal{J}_\infty(\mathfrak{g}^* /\!\!/ G)$$

is an irreducible and reduced chiral Poisson subscheme of $\mathcal{J}_\infty \mathscr{S}_f$.

(ii) *The comorphism $(\mathcal{J}_\infty \psi_f)^*$ induces an isomorphism of Poisson vertex algebras between $\mathbb{C}[\mathcal{J}_\infty \mathfrak{g}^*]^{\mathcal{J}_\infty G}$ and the Poisson vertex center of $\mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f]$.*

(iii) *The Poisson vertex algebra $\mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f]$ is free over its Poisson vertex center.*

From the above theorem we recover the Poisson algebra isomorphism

$$Z(\mathbb{C}[\mathscr{S}_f]) \cong \mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}]^G \tag{13.4}$$

(see Theorem 7.6) using $\mathbb{C}[\mathcal{J}_\infty \mathfrak{g}]^{\mathcal{J}_\infty G} \cap \mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}]^G$ and $Z(\mathbb{C}[\mathcal{J}_\infty \mathscr{S}_f]) \cap \mathbb{C}[\mathscr{S}_f) = Z(\mathbb{C}[\mathscr{S}_f])$.

Parts (i) and (ii) of the theorem are proved similarly to Theorem 7.6, thanks to the properties of chiral symplectic cores (see [37, Section 11]). Let us explain

Part (iii). The intersection $\mathscr{S}_f \cap \mathcal{N}$ enjoys the same geometrical properties as $\mathcal{N}$. In particular $\mathscr{S}_f \cap \mathcal{N}$ is reduced, irreducible and is a complete intersection with rational singularities (see Theorem D.3), the arguments of [97, Theorem A.4] can be applied in order to get that $\mathbb{C}[J_\infty \mathscr{S}_f]$ is free over its Poisson vertex center (see also [80, Proposition 2.5 (ii)]).

13.3.3 Center of $\mathcal{W}$-algebras

Let $\mathfrak{g}$ be a simple Lie algebra with adjoint group G, f a nilpotent element of $\mathfrak{g}$ and k a complex number. It is known from [13] that the embedding $Z(V^k(\mathfrak{g})) \hookrightarrow V^k(\mathfrak{g})$ induces an obvious vertex algebra homomorphism

$$Z(V^k(\mathfrak{g})) \longrightarrow Z(\mathcal{W}^k(\mathfrak{g}, f)),$$

where $Z(V)$ denotes the vertex center of a vertex algebra V (see Section 2.11). As mentioned in Remark 3.1, $Z(V^k(\mathfrak{g}))$ and $Z(\mathcal{W}^k(\mathfrak{g}, f))$ are trivial unless $k = -h^\vee$, and $\mathfrak{z}(\hat{\mathfrak{g}}) := Z(V^{-h^\vee}(\mathfrak{g}))$ is the Feigin–Frenkel center [109].

Theorem 13.6 *Let $\mathfrak{g}$ be a simple Lie algebra and f a nilpotent element of $\mathfrak{g}$. The embedding $\mathfrak{z}(\hat{\mathfrak{g}}) \hookrightarrow V^{-h^\vee}(\mathfrak{g})$ induces an isomorphism*

$$\mathfrak{z}(\hat{\mathfrak{g}}) \xrightarrow{\sim} Z(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f)),$$

and we have $\mathrm{gr}^F Z(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f)) \cong Z(\mathbb{C}[J_\infty \mathscr{S}_f])$.

Proof Since there is a vertex algebra homomorphism $\mathfrak{z}(\hat{\mathfrak{g}}) \to Z(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f))$, it is sufficient to show that the induced homomorphism $\mathrm{gr}^F \mathfrak{z}(\hat{\mathfrak{g}}) \to \mathrm{gr}^F Z(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f))$ is an isomorphism.

First, by Remark 3.1,

$$\mathrm{gr}^F \mathfrak{z}(\hat{\mathfrak{g}}) \cong \mathbb{C}[J_\infty \mathfrak{g}]^{J G}.$$

On the other hand, by Theorem 9.4 (i), we have

$$\mathrm{gr}^F \mathcal{W}^{-h^\vee}(\mathfrak{g}, f) \cong \mathbb{C}[J_\infty \mathscr{S}_f],$$

and so

$$Z(\mathrm{gr}^F \mathcal{W}^{-h^\vee}(\mathfrak{g}, f)) \cong Z(\mathbb{C}[J \mathscr{S}_f]).$$

By Theorem 13.5 (i), $Z(\mathbb{C}[J_\infty \mathscr{S}_f]) \cong \mathbb{C}[J_\infty \mathfrak{g}]^{J_\infty G}$, which forces the compound map

$$\begin{aligned} Z(\mathbb{C}[J_\infty \mathscr{S}_f]) \cong \mathrm{gr}^F \mathfrak{z}(\hat{\mathfrak{g}}) &\longrightarrow \mathrm{gr}^F Z(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f)) \\ &\hookrightarrow Z(\mathrm{gr}^F \mathcal{W}^{-h^\vee}(\mathfrak{g}, f)) \cong Z(\mathbb{C}[J_\infty \mathscr{S}_f]) \end{aligned}$$

to be an isomorphism. This completes the proof. □

Theorem 13.6 was stated in [13], but the proof of the surjectivity was incomplete.

Similar arguments as above using the isomorphism (13.4) recovers Premet's result [228] stating that the center of the finite $\mathcal{W}$-algebra $U(\mathfrak{g}, f)$ (see Definition 8.1) associated with $(\mathfrak{g}, f)$ is isomorphic to the center of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

Chapter 14
Representation theory of $\mathcal{W}$-algebras

Since the Drinfeld–Sokolov reduction applies to any V-module for V a $V^k(\mathfrak{g})$-vertex algebra, with $k \in \mathbb{C}$, it allows us to consider the associated variety of any V-module (see Theorem 9.9). In particular, from the knowledge of the associated variety of the simple affine vertex algebra $L_k(\mathfrak{g})$, one can compute the associated variety of $DS_f(L_k(\mathfrak{g}))$. This chapter develops several applications of the associated variety of the Drinfeld–Sokolov reduction to the representation theory of $\mathcal{W}$-algebras.

Section 14.1 is about the rationality question for $\mathcal{W}$-algebras at admissible levels. Section 14.2 gathers together a few remarks about the non-admissible levels. Section 14.3 covers technics to study the collapsing levels, that is, the level for which a simple $\mathcal{W}$-algebra is isomorphic to its affine vertex algebra. We mainly focus on the admissible case while there are at the present time some progresses outside the admissible case, see for instance [6, 104, 23].

Throughout this chapter, $\mathfrak{g}$ will be a simple Lie algebra, with nilpotent cone $\mathcal{N}$. The universal $\mathcal{W}$-algebra associated with $\mathfrak{g}$ and $f \in \mathcal{N}$ at the level $k \in \mathbb{C}$ is denoted by $\mathcal{W}^k(\mathfrak{g}, f)$. Unless otherwise specified, we keep the related notation of Chapter 9.

14.1 Rationality of $\mathcal{W}$-algebras

Recall from Theorem 12.2 (ii) that if k is an admissible level for $\mathfrak{g}$, then there is a nilpotent orbit $\mathbb{O}_k$ of $\mathfrak{g}$ such that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$. Then by Theorem 12.5 (ii), the simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse if $f \in \mathbb{O}_k$.

Let us detail the $\mathfrak{sl}_2$ case. Assume that $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Combining Theorem 11.1 and Exercise 12.1, we get

$$X_{L_k(\mathfrak{sl}_2)} = \begin{cases} \{0\}, & \text{if } k \in \mathbb{Z}_{\geqslant 0}, \\ \overline{G.f}, & \text{if } k \text{ is admissible and } k \notin \mathbb{Z}_{\geqslant 0}, \\ \mathfrak{sl}_2^*, & \text{if } k \text{ if not admissible.} \end{cases}$$

When k is admissible and not an integer, we know that $DS_f(L_k(\mathfrak{sl}_2)) = \mathcal{W}_k(\mathfrak{sl}_2, f) = \mathrm{Vir}_{c(k)}$, where $c(k) := 1 - \dfrac{6(k+1)^2}{k+2}$ (cf. Exercise 9.3). These simple Virasoro vertex algebra are precisely the minimal series Virasoro vertex algebra appearing in Theorem 11.2, which are known to be rational by this theorem. In other words, in this situation, the converse of Zhu's Conjecture 11.1 holds.

There was a long-standing conjecture generalizing this observation ([117, 167, 17]). This conjecture is now proved in full generality by McRae ([210]), and, hence, one can state the following remarkable result.

Theorem 14.1 *Assume that k is an admissible level for the simple Lie algebra $\mathfrak{g}$, and choose $f \in \mathbb{O}_k$ so that the simple affine $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. Then $\mathcal{W}_k(\mathfrak{g}, f)$ is also rational.*

In [117, 167], the geometrical condition $f \in \mathbb{O}_k$ for admissible k was stated in a combinatorial way. Before the general proof of McRae, many important steps was done toward to Theorem 14.1 by several authors.

- for $\mathfrak{g}$ arbitrary and $f = f_{\mathrm{reg}}$ a regular nilpotent element ([16]),
- for $\mathfrak{g} = \mathfrak{sl}_3$ and $f = f_{\mathrm{min}}$ a minimal nilpotent element, in which case the corresponding $\mathcal{W}$-algebra is the *Bershadsky–Polyakov vertex algebra* ([14]),
- for $\mathfrak{g} = \mathfrak{sl}_n$ and arbitrary f, or $\mathfrak{g}$ of type A, D, E and $f = f_{\mathrm{subreg}}$ a subregular nilpotent element ([24]),
- for $\mathfrak{g} = \mathfrak{sl}_4$ and $f = f_{\mathrm{subreg}}$ a subregular nilpotent element ([84]),
- for $\mathfrak{g} = \mathfrak{sp}_4$ and $f = f_{\mathrm{subreg}}$ a subregular nilpotent element ([102]), in which case the component group nontrivially acts.

14.2 Lisse, rational and quasi-lisse $\mathcal{W}$-algebras outside the admissible levels

There are lisse and quasi-lisse $\mathcal{W}$-algebras outside the admissible levels (see Section 12.4).

Conjecture 14.1 Let $k \in \mathbb{C}$ and assume that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}$ for some nilpotent orbit $\mathbb{O}$ of $\mathfrak{g}$. Choose $f \in \mathbb{O}$ so that the simple affine $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. Then $\mathcal{W}_k(\mathfrak{g}, f)$ is also rational.

Conjecture 14.1 is known for $\mathfrak{g}$ belonging to the Deligne exceptional series, and $k = -h^\vee/6 - 1$ according to the remark following Theorem 12.7. Conjecture 14.1 is in fact a generalization of Conjecture 12.2.

? Open problems

How to classify all lisse/quasi-lisse $\mathcal{W}$-algebras? How to find, and classify, lisse but not rational $\mathcal{W}$-algebras?

Example 14.1 Fasquel's computations of the OPE's in rank two ([104]) show that the simple $\mathcal{W}$-algebra associated with $\mathfrak{g} = G_2$ at level -2 and a subregular nilpotent element $f = f_{\text{subreg}}$ is trivial: $\mathcal{W}_{-2}(G_2, f_{\text{subreg}}) = \mathbb{C}$.

In view of Conjecture 9.1, Conjecture 13.1 and Theorem 12.5 (ii), this suggests the following conjecture ([103]).

Conjecture 14.2 Let $L_{-2}(G_2)$ be the simple affine vertex algebra associated with G_2 at level -2. We have $X_{L_{-2}(G_2)} = \mathbb{O}_{\text{subreg}}$, where $\mathbb{O}_{\text{subreg}}$ is the 10-dimensional subregular nilpotent orbit of G_2.

The above conjecture is deeply related to the computation of the associated variety of the simple affine vertex algebra $L_{-2}(D_4)$ (cf. Theorem 12.3). Note that the Dynkin diagram automorphism of D_4 identified with the symmetric group $\mathfrak{S}_3$. Its action on D_4 induces an action on $V^k(D_4)$ for any k. According to [7], this action passes through the simple quotient $L_{-2}(D_4)$ at level -2. Furthermore, we have the following statement can be deduced from ([7, Theorem 5]).

Proposition 14.1 *The simple affine vertex algebra $L_{-2}(G_2)$ is isomorphic to the fixed point vertex subalgebra $(L_{-2}(D_4))^{\mathfrak{S}_3}$.*

By a result of Miyamoto ([217]), if G is a finite solvable automorphism group of acting on a lisse vertex operator algebra V, then the fixed point vertex operator subalgebra V^G is lisse as well. The following generalization is expected to be true[1].

Conjecture 14.3 Let V be a quasi-lisse vertex operator algebra, and G a finite automorphism group of acting on V. Then the fixed point vertex operator subalgebra V^G is quasi-lisse.

By Proposition 14.1, Conjecture 14.2 is a particular case of Conjecture 14.3.

Remark 14.1 The representation theory of quasi-lisse $\mathcal{W}$-algebras is wide open. It appears in the 4D/2D duality (see Chapter 15) and is expected to have interesting connections with affine Springer fibers ([234]).

14.3 Collapsing levels for $\mathcal{W}$-algebras

Let G be a complex connected, simple algebraic group of adjoint type with Lie algebra $\mathfrak{g}$, and let $k \in \mathbb{C}$ be a complex number. Let also (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, and $\mathfrak{g}^\natural$ its centralizer in $\mathfrak{g}$. It is known that $\mathcal{W}^k(\mathfrak{g}, f)$ contains an embedded copy of the affine vertex algebra $V^{k^\natural}(\mathfrak{g}^\natural)$, where $k^\natural$ is defined as follows. In more details, the reductive Lie algebra $\mathfrak{g}^\natural$ decomposes as $\mathfrak{g}^\natural = \bigoplus_{i=0}^{s} \mathfrak{g}_i^\natural$, where $\mathfrak{g}_0^\natural$ is the center of $\mathfrak{g}^\natural$ and $\mathfrak{g}_1^\natural, \dots, \mathfrak{g}_s^\natural$ are the simple factors of $[\mathfrak{g}^\natural, \mathfrak{g}^\natural]$. Recall that the neutral element h induces a grading on $\mathfrak{g}$, see (D.1):

[1] This was suggested to the second author by Adamović in a priviate communication.

$$\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_i, \quad \mathfrak{g}_j = \{x \in \mathfrak{g} \colon [h,x] = 2jx\}.$$

Note that we have $\mathfrak{g}^\natural \subset \mathfrak{g}_0$. Define an invariant bilinear form on $\mathfrak{g}_i^\natural$, for $i = 0, \ldots, s$, by (cf. [166]):

$$(x|y)_i^\natural := k(x|x)_{\mathfrak{g}} + (\kappa_{\mathfrak{g}}(x,x) - \kappa_{\mathfrak{g}_0}(x,x) - \kappa_{\frac{1}{2}}(x,x))/2, \quad x, y \in \mathfrak{g}_i^\natural,$$

where $\kappa_{\mathfrak{g}_0}$ denotes the Killing form of $\mathfrak{g}_0$, and $\kappa_{\frac{1}{2}}(x,y) := \operatorname{tr}\left(\operatorname{ad}_{\mathfrak{g}_{1/2}}(x)\operatorname{ad}_{\mathfrak{g}_{1/2}}(y)\right)$, for $x, y \in \mathfrak{g}_0$, with $\operatorname{ad}_{\mathfrak{g}_{1/2}}(x)$ the endomorphism of $\mathfrak{g}_{1/2}$ sending z to $(\operatorname{ad} x)z$. For $i \neq 0$ there exists a polynomial $k_i^\natural$ in k of degree one such that

$$(-|-)_i^\natural = k_i^\natural(-|-)_i, \qquad i = 1, \ldots, s,$$

where $(-|-)_i$ is the normalized inner product of $\mathfrak{g}_i^\natural$.

Set

$$V^{k^\natural}(\mathfrak{g}^\natural) := V^{(-|-)_0^\natural}(\mathfrak{g}_0^\natural) \otimes \bigotimes_{i=1}^{s} V^{k_i^\natural}(\mathfrak{g}_i^\natural). \tag{14.1}$$

By [166, Theorem 2.1], there exists an embedding

$$\iota \colon V^{k^\natural}(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f) \tag{14.2}$$

of vertex algebras.

We denote by $\mathcal{V}^k(\mathfrak{g}^\natural)$ the image in $\mathcal{W}^k(\mathfrak{g}, f)$ of the embedding ι, and by $\mathcal{V}_k(\mathfrak{g}^\natural)$ the image of $\mathcal{V}^k(\mathfrak{g}^\natural)$ by the canonical projection $\pi \colon \mathcal{W}^k(\mathfrak{g}, f) \twoheadrightarrow \mathcal{W}_k(\mathfrak{g}, f)$. Next definition appears in [4] for f a minimal nilpotent element.

Definition 14.1 If $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathcal{V}_k(\mathfrak{g}^\natural)$, we say that the level k is *collapsing*.

Lemma 14.1 *The level k is collapsing if and only if*

$$\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural), \tag{14.3}$$

where $L_{k^\natural}(\mathfrak{g}^\natural)$ stands for $L_{(-|-)_0^\natural}(\mathfrak{g}_0^\natural) \otimes \bigotimes_{i=0}^{s} L_{k_i^\natural}(\mathfrak{g}_i^\natural)$. Equivalently, k is collapsing for $\mathcal{W}^k(\mathfrak{g}, f)$ if and only if there exists a surjective vertex algebra homomorphism

$$\mathcal{W}^k(\mathfrak{g}, f) \twoheadrightarrow L_{k^\natural}(\mathfrak{g}^\natural).$$

For example, if $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}$, then k is collapsing.

Proof If $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathcal{V}_k(\mathfrak{g}^\natural)$, then $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to the quotient of $V^{k^\natural}(\mathfrak{g})$ by the kernel of the composition $\pi \circ \iota$. Since $\mathcal{W}_k(\mathfrak{g}, f)$ is simple we deduce that this quotient is isomorphic to $L_{k^\natural}(\mathfrak{g})$. Conversely, if $\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g})$, then $\pi \circ \iota$

factorises through $L_{k^\natural}(\mathfrak{g})$, and so $\mathcal{W}_k(\mathfrak{g}, f)$ is isomorphic to the image of this induced map, so $\mathcal{W}_k(\mathfrak{g}, f) \cong \mathcal{V}_k(\mathfrak{g}^\natural)$. □

In [5] it was shown that collapsing levels have remarkable applications to the representation theory of affine vertex algebras. They are also useful in elucidating the structure of modular tensor categories of representations of simple W-algebras at admissible, not necessarily collapsing, levels [24]. Furthermore it has recently been observed [254] that many collapsing levels for quasi-lisse W-algebras should come from non-trivial isomorphisms of 4D $N = 2$ SUSY field theories, via the 4D/2D duality [51] (see Section 15.3).

There is a full classification of collapsing levels for the case that f is a minimal nilpotent element $f_{\min}$, including the case in which $\mathfrak{g}$ is a simple Lie superalgebra ([3, 4]). It can be summarized as follows: for $f = f_{\min}$, k is collapsing if and only if $k \neq -h^\vee$ and $p(k) = 0$, where p is a polynomial of degree two with coefficients in $\mathbb{Q}$.

Example 14.2 If $\mathfrak{g} = \mathfrak{sl}(m|n)$, $n \neq m$, then k is collapsing if and only if $(k+1)(k+(m-n)/2) = 0$. If $\mathfrak{g}$ is of type E_6, then k is collapsing if and only if $(k+3)(k+4) = 0$, etc.

Furthermore, there is a full classification of pairs $(\mathfrak{g}, k)$ such that $\mathcal{W}_k(\mathfrak{g}, f_{\min}) \cong \mathbb{C}$. It was obtained by the authors in [34], and then extended to the super case in [5]. For the non super case, the statement is the following.

Theorem 14.2 $\mathcal{W}_k(\mathfrak{g}, f_{\min}) \cong \mathbb{C}$ *if and only if either* $\mathfrak{g}$ *belongs to the Deligne exceptional series and* $k = -h^\vee/6 - 1$*, or* $\mathfrak{g} = \mathfrak{sp}_{2r}$*,* $r \geqslant 2$*,* $k = -1/2$*, or* $\mathfrak{g} = \mathfrak{sl}_2$ *and* $k + 2 = 2/3$ *or* $3/2$.

In particular, for $\mathfrak{g} = D_4, E_6, E_7, E_8$ and $k = -h^\vee/6 - 1$, $\mathcal{W}_k(\mathfrak{g}, f_{\min})^{\mathfrak{g}^\natural[t]} \cong \mathbb{C}$ is lisse. Kawasetsu's description of the vertex algebra $\mathcal{W}_{-h^\vee/6}(\mathfrak{g}, f_{\min})$ implies that $\mathcal{W}_{k+1}(\mathfrak{g}, f_{\min})^{\mathfrak{g}^\natural[t]}$ is lisse (and rational). See Example 12.4 and Conjecture 12.2 for related topics.

Conjecture 14.4 If $\mathcal{W}_k(\mathfrak{g}, f_{\min})^{\mathfrak{g}^\natural[[t]]}$ is lisse for some k, then $\mathcal{W}_{k+n}(\mathfrak{g}, f_{\min})^{\mathfrak{g}^\natural[[t]]}$ is lisse for all $n \in \mathbb{Z}_{\geqslant 0}$.

For more general nilpotent elements f, since the commutation relations in $\mathcal{W}^k(\mathfrak{g}, f)$ are unknown, almost nothing is known about collapsing levels. To discover them, more indirect methods were used in [26], and two important invariants of vertex algebras have been exploited: associated varieties and asymptotic data (see Definition 13.1).

14.3.1 Asymptotic data

Plainly the isomorphism (14.3) entailed by a collapsing level induces an equality of asymptotic data. Recall that the asymptotic data for $L_k(\mathfrak{g})$ and its Drinfeld–Sokolov reduction associated with any nilpotent element f are given in Example 13.2. Next theorem ([26, Theorem 3.10 and Proposition 6.6]) is a sort of converse for the admissible levels.

Theorem 14.3 *Assume that k and $k^\natural$ are admissible levels for $\mathfrak{g}$ and $\mathfrak{g}^\natural$, respectively, that $f \in \overline{\mathbb{O}_k}$ and that $\chi_{DS_f(L_k(\mathfrak{g}))}(\tau) \sim \chi_{L_{k^\natural}(\mathfrak{g}^\natural)}(\tau)$, as $\tau \downarrow 0$, i.e.,*

$$\mathbf{g}_{DS_f(L_k(\mathfrak{g}))} = \mathbf{g}_{L_{k^\natural}(\mathfrak{g}^\natural)} = \sum_{i=1}^{s} \mathbf{g}_{L_{k_i}(\mathfrak{g}_i)},$$

$$\mathbf{A}_{DS_f(L_k(\mathfrak{g}))} = \mathbf{A}_{L_{k^\natural}(\mathfrak{g}^\natural)} = \prod_{i=1}^{s} \mathbf{A}_{L_{k_i}(\mathfrak{g}_i)}.$$

Then k is a collapsing level, that is, $\mathcal{W}_k(\mathfrak{g}, f) \cong L_{k^\natural}(\mathfrak{g}^\natural)$.

Remark 14.2 In Theorem 14.3 suppose further that $f \in \mathbb{O}_k$ so that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. Then we get that

$$\mathcal{W}_k(\mathfrak{g}, f) \cong DS_f(L_k(\mathfrak{g})).$$

Indeed, $f \in \mathbb{O}_k$ implies that $DS_f(L_k(\mathfrak{g}))$ is lisse. Hence $L_{k^\natural}(\mathfrak{g}^\natural)$ must be integrable and the homomorphism $V^{k^\natural} \to DS_f(L_k(\mathfrak{g}))$ must factors through the embedding $L_{k^\natural}(\mathfrak{g}^\natural) \hookrightarrow DS_f(L_k(\mathfrak{g}))$ ([92]). In particular, $DS_f(L_k(\mathfrak{g}))$ is a direct sum of integrable representations of the affine Kac-Moody algebra associated with $\mathfrak{g}^\natural$. It follows that the proof of [26, Theorem 3.10] goes through to obtain that $DS_f(L_k(\mathfrak{g})) \cong L_{k^\natural}(\mathfrak{g}^\natural)$.

The following proposition is useful to obtain explicit decompositions of finite extensions of admissible simple affine vertex algebras (see [26]).

Proposition 14.2 *Let k and $k^\natural$ be admissible, $f \in \overline{\mathbb{O}_k}$. Suppose that the associated varieties of $DS_f(L_k(\mathfrak{g}))$ and $L_{k^\natural}(\mathfrak{g}^\natural)$ have the same dimension and are not isomorphic. Then k is not collapsing.*

Proof The assumption ensures that $DS_f(L_k(\mathfrak{g}))$ is nonzero. Hence, $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of $DS_f(L_k(\mathfrak{g}))$ and its associated variety is a Zariski closed subvariety of that of $DS_f(L_k(\mathfrak{g}))$. On the other hand, since k is admissible, the associated variety of $DS_f(L_k(\mathfrak{g}))$ is irreducible, $\overline{\mathbb{O}_k}$ being unibranch (see §D.6.4). If k were collapsing, then the associated variety of $\mathcal{W}_k(\mathfrak{g}, f)$ would be isomorphic to the (irreducible) variety $X_{L_{k^\natural}(\mathfrak{g}^\natural)}$, and so, would have the same dimension as $X_{DS_f(L_k(\mathfrak{g}))}$ by the hypothesis. Then it would be isomorphic to the variety $X_{DS_f(L_k(\mathfrak{g}))}$, since $X_{DS_f(L_k(\mathfrak{g}))}$ is irreducible. This contradicts the non-isomorphism hypothesis. □

14.3.2 Associated varieties

The nature of the formulas for asymptotic data are such that it is not feasible to find collapsing levels by a naive search for coincidences between respective asymptotic data. For this reason we turn to the more refined invariant given by the associated variety.

Lemma 14.2 *Suppose that $\mathcal{W}_k(\mathfrak{g}, f)$ is quasi-lisse. If k is collapsing then $(-|-)_0^\natural$ is identically zero on $\mathfrak{g}_0^\natural$. In particular, if k is admissible and $f \in \overline{\mathbb{O}}_k$, then k can only be collapsing if $(-|-)_0^\natural = 0$.*

Our convention is that $(-|-)_0^\natural = 0$ when $\mathfrak{g}_0^\natural = \{0\}$.

Proof The associated variety $X_{L_{k^\natural}(\mathfrak{g}^\natural)}$ is a subvariety of $(\mathfrak{g}_0^\natural)^* \times (\mathfrak{g}_1^\natural)^* \times \cdots \times (\mathfrak{g}_s^\natural)^*$ and the symplectic leaves of $(\mathfrak{g}_0^\natural)^* \times (\mathfrak{g}_1^\natural)^* \times \cdots \times (\mathfrak{g}_s^\natural)^*$ are the coadjoint orbits of $G_0^\natural \times \cdots \times G_s^\natural$, where $G_i^\natural$ is the adjoint group of $\mathfrak{g}_i^\natural$. Recall that $V^{(-|-)_0^\natural}(\mathfrak{g}_0^\natural)$ is a Heisenberg vertex algebra of rank $\dim \mathfrak{g}_0^\natural$. The associated variety of its simple quotient is $\mathbb{C}^{\operatorname{rank}(-|-)_0^\natural}$, provided that $(-|-)_0^\natural \neq 0$. Hence, $X_{L_{k^\natural}(\mathfrak{g}^\natural)}$ has finitely many symplectic leaves if and only if it is contained in $\mathcal{N}_{\mathfrak{g}_1} \times \cdots \times \mathcal{N}_{\mathfrak{g}_s}$, where $\mathcal{N}_{\mathfrak{g}_i}$ is the nilpotent cone of $\mathfrak{g}_i^\natural$. In particular, we must have $X_{L_{(-|-)_0^\natural}(\mathfrak{g}_0^\natural)} = \{0\}$. This happens if and only if $(-|-)_0^\natural = 0$. □

We are thus led to consider levels k for which $(-|-)_0^\natural = 0$. Next, the levels $k_i^\natural$ can then be expressed as functions of the level k. Tables can be found in [26, Tables 2-3-4, 11–17].

Recall that whenever k is an admissible level for $\mathfrak{g}$, the associated variety $X_{DS_f(L_k(\mathfrak{g}))}$ is the nilpotent Slodowy slice $\overline{\mathbb{O}_k} \cap \mathscr{S}_f$. It is conjectured in general (and confirmed in many cases) that $DS_f(L_k(\mathfrak{g}))$ is simple, so that $\mathcal{W}_k(\mathfrak{g}, f) = DS_f(L_k(\mathfrak{g}))$ in fact (see Conjecture 9.1). A collapsing level thus induces, in these cases, an isomorphism

$$\overline{\mathbb{O}_k} \cap \mathscr{S}_f \cong \overline{\mathbb{O}_{k^\natural}}. \tag{14.4}$$

In particular, the singularity of the nilpotent Slodowy slice in $\mathfrak{g}$ on the left-hand-side should be of the same type as that of the nilpotent orbit closure in $\mathfrak{g}^\natural$ on the right-hand-side. We may therefore apply known results on the geometry of nilpotent Slodowy slices (see Section D.6) to find candidates for collapsing levels. Using the row/column removal rule (see §D.6.3), it is possible to identify classes of nilpotent orbits for which an isomorphism of the type (14.4) holds.

Definition 14.2 We say that a nilpotent Slodowy slice $\mathscr{S}_{\mathbb{O},f}$ is *collapsing* if $\mathscr{S}_{\mathbb{O},f}$ is isomorphic to a product of nilpotent orbits closures in $\mathfrak{g}^\natural$.

Conjecture 14.5 If k is admissible and if $f \in \overline{\mathbb{O}}_k$ is such that $\mathscr{S}_{\mathbb{O}_k,f}$ is collapsing, then $k^\natural$ is admissible, provided that $k_0^\natural = 0$.

Conjecture 14.5 has been case-by-case proved if $\mathfrak{g}$ is simple of exceptional type, and has been verified in the classical cases in the cases where $\mathscr{S}_{\mathbb{O}_k,f}$ is collapsing and the isomorphism between $\mathscr{S}_{\mathbb{O}_k,f}$ and a product of nilpotent orbit closures in $\mathfrak{g}^\natural$ is obtained from the row/column removal rule of Kraft–Procesi (Lemma D.5 and Lemma D.6), see [26, Section 6].

? Open problem

Do the cases from the row/column removal rule of Kraft–Procesi exhaust all possible cases of collapsing nilpotent Slodowy slices?

14.3.3 Main results

Using Theorem 14.3, many new infinite families of (admissible) collapsing levels for $\mathfrak{g}$ of classical type, and a huge number of (admissible) collapsing levels for $\mathfrak{g}$ of exceptional type have been detected. Roughly speaking, a collapsing nilpotent Slodowy slice will yield a collapsing level if the asymptotic data of the vertex algebras corresponding to the two sides of (14.4) can be shown to coincide. In fact asymptotic data is rather difficult to compute, so the strategy in many examples is to first compare the central charges.

Assume that the simple Lie algebra $\mathfrak{g}$ is of classical types $\mathfrak{sl}_n$, $\mathfrak{sp}_n$ or $\mathfrak{so}_n$. We detect collapsing nilpotent Slodowy slices using the row/column removal rule of Kraft–Procesi as follows. Let k be an admissible level for $\mathfrak{g}$, and let $\boldsymbol{\lambda_0}$ the partition of n such that

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k} = \overline{\mathbb{O}_{\boldsymbol{\lambda_0}}}.$$

We consider all partitions $\boldsymbol{\mu}$ such that $\mathbb{O}_{\boldsymbol{\mu}} \subset \overline{\mathbb{O}_{\boldsymbol{\lambda_0}}}$ and, if $\boldsymbol{\lambda_0'}$ and $\boldsymbol{\mu}'$ denote the partitions obtained from $\boldsymbol{\lambda_0}$ and $\boldsymbol{\mu}$, respectively, by erasing the common rows and columns, $\boldsymbol{\mu}'$ corresponds to a zero nilpotent orbit in the corresponding classical Lie algebra. In particular the nilpotent Slodowy slice $\mathscr{S}_{\boldsymbol{\lambda_0},\boldsymbol{\mu}}$ is collapsing in the notation (D.5).

The work in the exceptional types is more exhaustive in a sense, since one can directly exploit the description of the $k_i^\natural$ and the central charge to detect all possible collapsing levels. In this way, many nontrivial isomorphisms between a Slodowy slice $\mathscr{S}_{\mathbb{O},f}$ and a product of nilpotent orbit closures in $\mathfrak{g}^\natural$ are obtained. Many of these were observed in [123] already, though others seem to be new. On the other hand, of course, many of isomorphisms between Slodowy slices obtained in [123] do not correspond to collapsing levels.

Following this strategy, the results for the classical and exceptional types can be summarized as follows. We refer to [26] for more statements, including the finite exetensions.

Theorem 14.4 *Assume that $k = -n + p/q$ is admissible for $\mathfrak{g} = \mathfrak{sl}_n$, and write $\boldsymbol{\lambda_0} = (q^{m_0}, s_0)$.*

(i) *Pick a nilpotent element $f \in \mathbb{O}_k$, so that $\mathcal{W}_k(\mathfrak{g}, f)$ is rational.*

(a) *If* $n \equiv \pm 1 \pmod q$*, then*

$$\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong \mathbb{C}.$$

(b) *If* $n \equiv 0 \pmod q$*,*

$$\mathcal{W}_{-n+(n+1)/q}(\mathfrak{sl}_n, f) \cong L_1(\mathfrak{sl}_{m_0}).$$

(ii) *Pick a nilpotent element* $f \in \overline{\mathbb{O}}_k$ *corresponding to the partition* $(q^m, 1^s)$*, with* $m \geqslant 0$ *and* $s > 0$*. Then*

$$\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-s+s/q}(\mathfrak{sl}_s).$$

(iii) *Assume that* $s_0 = q - 2$ *and pick a nilpotent element* $f \in \overline{\mathbb{O}}_k$ *corresponding to the partition* $(q^m, (q-1)^2)$*, with* $m = m_0 - 1$*. Then*

$$\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong L_{-2+2/q}(\mathfrak{sl}_2).$$

Theorem 14.5 *Assume that* $k = -\left(\frac{n}{2} + 1\right) + p/q$ *is an admissible level for* $\mathfrak{g} = \mathfrak{sp}_n$*.*

(i) *Pick a nilpotent element* $f \in \mathbb{O}_k$ *so that* $\mathcal{W}_k(\mathfrak{g}, f)$ *is rational.*

(a) *Assume that* q *is odd. If* $n \equiv 0, -1 \mod q$*, then*

$$\mathcal{W}_{-\left(\frac{n}{2}+1\right)+\left(\frac{n}{2}+1\right)/q}(\mathfrak{sp}_n, f) \cong \mathbb{C}.$$

(b) *Assume that* q *is even. If* $n \equiv 0, 1 \mod q/2$ *with odd* $q/2$*, then*

$$\mathcal{W}_{-\left(\frac{n}{2}+1\right)+(n+1)/q}(\mathfrak{sp}_n, f) \cong \mathbb{C},$$

and if $n \equiv 0, 1 \mod a/2$ *with even* $q/2$*, then*

$$\mathcal{W}_{-\left(\frac{n}{2}+1\right)+(n+1)/q}(\mathfrak{sp}_n, f) \cong L_1(\mathfrak{so}_m).$$

(ii) *Assume that* q *is odd.*

(a) *Pick a nilpotent element* $f \in \overline{\mathbb{O}}_k$ *corresponding to the partition* $(q^m, 1^s)$*, with* m, s *even. If* $p = \frac{n}{2} + 1$*, then*

$$\mathcal{W}_{-\left(\frac{n}{2}+1\right)+\left(\frac{n}{2}+1\right)/q}(\mathfrak{sp}_n, f) \cong L_{-\left(\frac{s}{2}+1\right)+\left(\frac{s}{2}+1\right)/q}(\mathfrak{sp}_s).$$

(iii) *Assume that* q *is even.*

(i) *Pick a nilpotent element* $f \in \overline{\mathbb{O}}_k$ *corresponding to the partition* $(\frac{q}{2} + 1, (\frac{q}{2})^m, 1^s)$*, with odd* $q/2$*, even* m*, even* s*. If* $s = 2$*, then*

$$\mathcal{W}_{-\left(\frac{n}{2}+1\right)+(n+1)/q}(\mathfrak{sp}_n, f) \cong L_{-2+2/(q/2)}(\mathfrak{sl}_2).$$

Theorem 14.6 *Assume that* $k = -(n-2) + p/q$ *is admissible for* $\mathfrak{g} = \mathfrak{so}_n$*.*

(i) *Pick a nilpotent element $f \in \mathbb{O}_k$ so that $\mathcal{W}_k(\mathfrak{g}, f)$ is rational.*

(a) *Assume that q is odd. If $n \equiv 0, 1 \mod q$, then*

$$\mathcal{W}_{-(n-2)+(n-2)/q}(\mathfrak{so}_n, f) \cong \mathbb{C}.$$

(b) *Assume that n and q are even. If $n \equiv 0, 2 \mod q$, then*

$$\mathcal{W}_{-(n-2)+(n-1)/q}(\mathfrak{so}_n, f) \cong \mathbb{C}.$$

(c) *Assume that n is odd and that q is even. If $n \equiv -1, 1 \mod q$, then*

$$\mathcal{W}_{-(n-2)+n/q}(\mathfrak{so}_n, f) \cong \mathbb{C}.$$

(ii) *Assume that q is odd so that k is principal.*

(a) *Pick a nilpotent element $f \in \overline{\mathbb{O}}_k$ corresponding to the partition $(q^m, 1^s)$ with $s \geqslant 3$. Then*

$$\mathcal{W}_{-(n-2)+(n-2)/q}(\mathfrak{so}_n, f) \cong L_{-(s-2)+(s-2)/q}(\mathfrak{so}_s).$$

(b) *Pick a nilpotent element $f \in \overline{\mathbb{O}}_k$ corresponding to the partition $(q^m, (q-1)^2)$. Then,*

$$\mathcal{W}_{-(n-2)+(n-2)/q}(\mathfrak{so}_n, f) \cong L_{-2+2/q}(\mathfrak{sl}_2),$$

(iii) *Assume that q and n are even so that k is principal.*

(a) *Pick a nilpotent element $f \in \overline{\mathbb{O}}_k$ corresponding to the partition $(q+1, q^m, 1^s)$, with even m, odd s. Then,*

$$\mathcal{W}_{-(n-2)+(n-1)/q}(\mathfrak{so}_n, f) \cong L_{-(s-2)+s/q}(\mathfrak{so}_s).$$

(iv) *Assume that n is odd and q is even.*

(a) *Pick a nilpotent element $f \in \overline{\mathbb{O}}_k$ corresponding to the partition $(q^m, 1^s)$, with even m and odd s. Then,*

$$\mathcal{W}_{-(n-2)+n/q}(\mathfrak{so}_n, f) \cong L_{-(s-2)+s/q}(\mathfrak{so}_s).$$

For the exceptional types, the following isomorphisms hold. Here, in place of f we write the label of $G.f$ in the Bala–Carter classification, and $\mathfrak{g}$ and $\mathfrak{g}^\natural$ are denoted by their types.

(i) The following isomorphisms hold, providing collapsing levels for $\mathfrak{g} = E_6$.

$\mathcal{W}_{-12+12/13}(E_6, E_6) \cong \mathbb{C}$, $\mathcal{W}_{-12+13/12}(E_6, E_6) \cong \mathbb{C}$,
$\mathcal{W}_{-12+13/9}(E_6, E_6(a_1)) \cong \mathbb{C}$, $\mathcal{W}_{-12+13/6}(E_6, E_6(a_3)) \cong \mathbb{C}$,
$\mathcal{W}_{-12+13/6}(E_6, A_5) \cong L_{-2+2/3}(A_1)$, $\mathcal{W}_{-12+12/7}(E_6, D_4) \cong L_{-3+3/7}(A_2)$,
$\mathcal{W}_{-12+13/6}(E_6, D_4) \cong L_{-2+4/3}(A_2)$, $\mathcal{W}_{-12+12/5}(E_6, A_4) \cong L_{-2+2/5}(A_1)$,
$\mathcal{W}_{-12+13/3}(E_6, 2A_2 + A_1) \cong \mathbb{C}$, $\mathcal{W}_{-12+13/3}(E_6, 2A_2) \cong L_{-4+7/3}(G_2)$,
$\mathcal{W}_{-12+13/2}(E_6, 3A_1) \cong L_1(A_2)$.

(ii) The following isomorphisms hold, providing collapsing levels for $\mathfrak{g} = E_7$.

$\mathcal{W}_{-18+18/19}(E_7, E_7) \cong \mathbb{C}$, $\mathcal{W}_{-18+19/18}(E_7, E_7) \cong \mathbb{C}$,
$\mathcal{W}_{-18+19/14}(E_7, E_7(a_1)) \cong \mathbb{C}$, $\mathcal{W}_{-18+18/13}(E_7, E_6) \cong L_{-2+2/13}(A_1)$,
$\mathcal{W}_{-18+19/12}(E_7, E_6) \cong L_{-2+3/4}(A_1)$, $\mathcal{W}_{-18+19/10}(E_7, D_6) \cong L_{-2+2/5}(A_1)$,
$\mathcal{W}_{-18+18/7}(E_7, A_6) \cong \mathbb{C}$, $\mathcal{W}_{-18+19/7}(E_7, A_6) \cong L_1(A_1)$,
$\mathcal{W}_{-18+18/7}(E_7, (A_5)'') \cong L_{-4+4/7}(G_2)$, $\mathcal{W}_{-18+18/7}(E_7, D_4) \cong L_{-4+4/7}(C_3)$,
$\mathcal{W}_{-18+19/6}(E_7, E_7(a_5)) \cong \mathbb{C}$, $\mathcal{W}_{-18+19/6}(E_7, E_6(a_3)) \cong L_{-2+3/2}(A_1)$,
$\mathcal{W}_{-18+19/6}(E_7, D_6(a_2)) \cong L_{-2+2/3}(A_1)$, $\mathcal{W}_{-18+19/6}(E_7, (A_5)') \cong L_{-2+2/3}(A_1) \otimes L_{-2+3/2}(A_1)$,
$\mathcal{W}_{-18+19/6}(E_7, (A_5)'') \cong L_{-4+7/6}(G_2)$, $\mathcal{W}_{-18+19/6}(E_7, D_4) \cong L_{-4+7/6}(C_3)$,
$\mathcal{W}_{-18+19/5}(E_7, A_4 + A_2) \cong L_3(A_1)$, $\mathcal{W}_{-18+18/5}(E_7, A_4) \cong L_{-3+3/5}(A_2)$,
$\mathcal{W}_{-18+19/4}(E_7, A_3 + A_2 + A_1) \cong L_2(A_1)$, $\mathcal{W}_{-18+19/4}(E_7, D_4(a_1)) \cong L_{-2+3/4}(A_1)^{\otimes 3}$,
$\mathcal{W}_{-18+19/3}(E_7, 2A_2 + A_1) \cong L_1(A_1)$, $\mathcal{W}_{-18+19/3}(E_7, A_2 + 3A_1) \cong L_{-4+8/3}(G_2)$,
$\mathcal{W}_{-18+19/3}(E_7, 2A_2) \cong L_1(A_1) \otimes L_{-4+7/3}(G_2)$, $\mathcal{W}_{-18+19/2}(E_7, 4A_1) \cong \mathbb{C}$,
$\mathcal{W}_{-18+19/2}(E_7, (3A_1)') \cong L_{-4+7/2}(C_3)$, $\mathcal{W}_{-18+19/2}(E_7, (3A_1)'') \cong L_{-9+13/2}(F_4)$.

(iii) The following isomorphisms hold, providing collapsing levels for $\mathfrak{g} = E_8$.

$\mathcal{W}_{-30+30/31}(E_8, E_8) \cong \mathbb{C}$, $\mathcal{W}_{-30+31/30}(E_8, E_8) \cong \mathbb{C}$,
$\mathcal{W}_{-30+30/24}(E_8, E_8(a_1)) \cong \mathbb{C}$, $\mathcal{W}_{-30+31/20}(E_8, E_8(a_2)) \cong \mathbb{C}$,
$\mathcal{W}_{-30+31/18}(E_8, E_7) \cong L_{-2+2/9}(A_1)$, $\mathcal{W}_{-30+31/15}(E_8, E_8(a_4)) \cong \mathbb{C}$,
$\mathcal{W}_{-30+31/12}(E_8, D_7) \cong L_{-2+2/3}(A_1)$, $\mathcal{W}_{-30+31/12}(E_8, E_8(a_5)) \cong \mathbb{C}$,
$\mathcal{W}_{-30+30/13}(E_8, E_6) \cong L_{-4+4/13}(G_2)$, $\mathcal{W}_{-30+31/12}(E_8, E_6) \cong L_{-4+7/12}(G_2)$,
$\mathcal{W}_{-30+31/10}(E_8, E_8(a_6)) \cong \mathbb{C}$, $\mathcal{W}_{-30+31/10}(E_8, D_6) \cong L_{-3+3/5}(B_2)$,
$\mathcal{W}_{-30+31/9}(E_8, E_6(a_1)) \cong L_{-3+4/9}(A_2)$, $\mathcal{W}_{-30+31/8}(A_7, A_7) \cong \mathbb{C}$,
$\mathcal{W}_{-30+30/7}(E_8, A_6) \cong L_{-2+2/7}(A_1)$, $\mathcal{W}_{-30+31/6}(E_8, E_8(a_7)) \cong \mathbb{C}$,
$\mathcal{W}_{-30+31/6}(E_8, E_8(a_7)) \cong \mathbb{C}$, $\mathcal{W}_{-30+31/6}(E_8, D_6(a_2)) \cong L_{-2+2/3}(A_1) \otimes L_{-2+2/3}(A_1)$,
$\mathcal{W}_{-30+31/6}(E_8, E_6(a_3)) \cong L_{-4+7/6}(G_2)$, $\mathcal{W}_{-30+31/6}(E_8, D_4) \cong L_{-9+13/6}(F_4)$,
$\mathcal{W}_{-30+30/7}(E_8, D_4) \cong L_{-9+9/7}(F_4)$, $\mathcal{W}_{-30+31/5}(E_8, A_4 + A_3) \cong \mathbb{C}$,
$\mathcal{W}_{-30+32/5}(E_8, A_4 + A_3) \cong L_2(A_1)$, $\mathcal{W}_{-30+31/4}(E_8, 2A_3) \cong \mathbb{C}$,
$\mathcal{W}_{-30+31/4}(E_8, D_4(a_1) + A_2) \cong L_{-3+3/2}(A_2)$, $\mathcal{W}_{-30+31/3}(E_8, 2A_2 + 2A_1) \cong \mathbb{C}$,
$\mathcal{W}_{-30+32/3}(E_8, 2A_2 + 2A_1) \cong L_1(B_2)$, $\mathcal{W}_{-30+31/3}(E_8, 2A_2) \cong L_{-4+7/3}(G_2) \otimes L_{-4+7/3}(G_2)$,
$\mathcal{W}_{-30+31/2}(E_8, 4A_1) \cong \mathbb{C}$, $\mathcal{W}_{-30+31/2}(E_8, 3A_1) \cong L_{-9+13/2}(F_4)$.

(iv) The following isomorphisms hold, providing collapsing levels for $\mathfrak{g} = F_4$.

$$\begin{aligned}
&\mathcal{W}_{-9+13/18}(F_4, F_4) \cong \mathbb{C}, && \mathcal{W}_{-9+9/13}(F_4, F_4) \cong \mathbb{C}\\
&\mathcal{W}_{-9+13/12}(F_4, F_4(a_1)) \cong \mathbb{C}, && \mathcal{W}_{-9+9/7}(F_4, C_3) \cong L_{-2+2/7}(A_1)\\
&\mathcal{W}_{-9+9/7}(F_4, B_3) \cong L_{-2+2/7}(A_1), && \mathcal{W}_{-9+13/8}(F_4, B_3) \cong L_1(A_1)\\
&\mathcal{W}_{-9+13/6}(F_4, F_4(a_3)) \cong \mathbb{C}, && \mathcal{W}_{-9+13/6}(F_4, C_3(a_1)) \cong L_{-2+2/3}(A_1)\\
&\mathcal{W}_{-9+13/6}(F_4, B_3) \cong L_{-2+2/3}(A_1) \otimes L_{-2+2/3}(A_1), && \mathcal{W}_{-9+13/6}(F_4, \tilde{A}_2)) \cong L_{-4+7/6}(G_2)\\
&\mathcal{W}_{-9+13/4}(F_4, A_2 + \tilde{A}_1) \cong \mathbb{C}, && \mathcal{W}_{-9+13/4}(F_4, A_2)) \cong L_{-3+3/2}(A_2),\\
&\mathcal{W}_{-9+13/2}(F_4, A_1)) \cong \mathbb{C}.
\end{aligned}$$

(v) The following isomorphisms hold, providing collapsing levels for G_2.

$$\begin{aligned}
&\mathcal{W}_{-4+7/12}(G_2, G_2) \cong \mathbb{C}, && \mathcal{W}_{-4+4/7}(G_2, G_2) \cong \mathbb{C}\\
&\mathcal{W}_{-4+7/6}(G_2, G_2(a_1)) \cong \mathbb{C}, && \mathcal{W}_{-4+7/6}(G_2, \tilde{A}_1) \cong L_{-2+2/3}(A_1),\\
&\mathcal{W}_{-4+7/3}(G_2, A_1) \cong \mathbb{C}, && \mathcal{W}_{-4+8/3}(G_2, A_1) \cong L_1(A_1).
\end{aligned}$$

It is conjectured in [26] that all the above results give the exhaustive list of admissible collapsing levels.

14.3.4 Related problems and further works

Aside from determination of collapsing levels, the methods exploiting asymptotic data can be used to prove other results of a similar flavour. For instance, if $\mathfrak{g} = \mathfrak{sp}_n$ and $k = -h^\vee + p/q$ where q is twice an odd integer and $p = h+1$, with h the Coxeter number, then for $f \in \mathbb{O}_k = \mathbb{O}_{\boldsymbol{\lambda_0}}$ where $\boldsymbol{\lambda_0} = (\frac{q}{2}+1, (\frac{q}{2})^m, 2)$,

$$\mathcal{W}_k(\mathfrak{g}, f) \cong DS_f(L_k(\mathfrak{g})) \cong \mathrm{Vir}_{2,q/2}\,.$$

We propose the following conjectural extension of Theorem 14.4.

Conjecture 14.6 Let $f \in \mathfrak{sl}_n$ be a nilpotent element associated with a partition $(q^m, \boldsymbol{\nu})$, where $1 < m \leqslant m_0$ in the notation of Theorem 14.4, and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_t)$ is a partition of $s := n - qm$ such that $\nu_1 < q$. Then

$$\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f) \cong \mathcal{W}_{-s+s/q}(\mathfrak{sl}_s, f'),$$

where f' is a nilpotent element in $\mathfrak{sl}_s$ associated with the partition $\boldsymbol{\nu}$.

Note that Conjecture 14.6 has been proven in the special case where $n = 7$, $q = 3$ and $s = 4$ by Francesco Allegra [8]. This case is in fact a particular case of Theorem 14.4 used with f corresponding to the partitions $(3^2, 1)$, $(3, 2^2)$ and $(3, 1^4)$. It seems that this conjecture has been stated in [254].

The associated variety of $\mathcal{W}_{-n+n/q}(\mathfrak{sl}_n, f)$ is $\mathscr{S}_{(q^{m_0}, s_0), (q^m, \nu)}$ while the associated variety of $\mathcal{W}_{-s+s/q}(\mathfrak{sl}_s, f')$ is $\mathscr{S}_{(q^{m_0-m}, s_0), \boldsymbol{\nu}}$. These two nilpotent Slodowy slices are isomorphic by Lemma D.5.

In another direction, the above methods allow to obtain many cases where $\mathcal{W}_k(\mathfrak{g}, f)$ is merely a finite extension of its simple affine vertex algebra $L_{k^\natural}(\mathfrak{g}^\natural)$. Here, by *finite extension* we mean here a vertex algebra W which decomposes as a finite direct sum of irreducible modules over its conformal vertex subalgebra V. For instance, if $\mathfrak{g} = \mathfrak{sp}_n$, and if $f \in \overline{\mathbb{O}}_k$ is a nilpotent element corresponding to the partition $(q^m, q-1, 1^s)$, with odd q and even s, then the following inclusion is a finite extension:

$$L_{-(\frac{s}{2}+1)+(s+1)/(2q)}(\mathfrak{sp}_s) \hookrightarrow \mathcal{W}_{-(\frac{n}{2}+1)+(\frac{n}{2}+1)/q}(\mathfrak{sp}_n, f). \tag{14.5}$$

Another example is the following for $\mathfrak{g} = E_6$:

$$L_{-3+4/3}(A_2) \otimes L_{-3+4/3}(A_2) \hookrightarrow \mathcal{W}_{-12+13/3}(E_6, A_2). \tag{14.6}$$

We refer to [26] for more examples in both classical types and exceptional types. In the isomorphism (14.5), note that the associated varieties of both sides are isomorphic, while it is not the case for the isomorphism (14.6). Indeed, the associated variety of $\mathcal{W}_{-12+13/2}(E_6, A_1)$ is the nilpotent Slodowy slice[2] $\mathscr{S}_{2A_2+A_1,A_1}$ which is not isomorphic to the the product $\mathcal{N}_{A_2} \times \mathcal{N}_{A_2}$, the associated variety of $L_{-3+4/3}(A_2) \otimes L_{-3+4/3}(A_2)$. However, these varieties share the same dimension 12.

The following conjecture is formulated in [26]:

Conjecture 14.7 If W is a finite extension of the vertex algebra V then the corresponding morphism of Poisson algebraic varieties $\pi\colon X_W \to X_V$, is a dominant morphism.

The validity of Conjecture 14.7 would imply the widely-believed fact that if a finite extension of a vertex algebra is lisse then so is the vertex algebra.

Note that Conjecture 14.2 is a particular case of the above conjecture. Indeed, as proved in [7], the simple affine vertex algebra $L_{-2}(D_4)$ is a finite extension of $L_{-2}(G_2)$, and we know that $X_{L_{-2}(D_4)} = \overline{\mathbb{O}_{\min}}$ by Theorem 12.3. On the other hand, the inclusion $G_2 \cong D_4^{\mathfrak{S}_3} \hookrightarrow D_4$ induces a projection morphism $\pi\colon D_4 \twoheadrightarrow G_2$, identifying $(D_4)^*$ and $(G_2)^*$ with D_4 and G_2 using the Killing form of D_4 and G_2, respectively. By [178], the image of $\overline{\mathbb{O}_{\min}}$ (of dimension 10) is the the closure of the subregular nilpotent orbit in G_2 so that the restriction to $\overline{\mathbb{O}_{\min}}$ of π is dominant onto $\overline{\mathbb{O}_{\text{subreg}}}$.

[2] A_2 being an even nilpotent orbit, Conjecture 9.1 holds.

Chapter 15
4D/2D duality

The vertex algebras (or vertex operator algebras) appear in string theory in physics. They give the rigorous mathematical definition of the chiral part of two-dimensional quantum field theories, that were intensively studied since the pioner works of Beliakov, Polyakov and Zamolodchikov [56]. By a recent discovery, which goes back to Nakajima [222] the vertex operator algebras also appear in higher dimensional quantum field theories as well in several ways. This gives a new insights for the representation theory of vertex operator algebras. We explore in this chapter interesting problems that are motivated by results and conjectures in the higher dimensional quantum field theory.

15.1 The Higgs branch conjecture

In the study of four-dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs) in physics, Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees [51] have constructed a remarkable map

$$\mathbb{V}\colon \{4\text{D } \mathcal{N} = 2 \text{ SCFTs}\} \longrightarrow \{\text{vertex operator algebras}\} \tag{15.1}$$

that canonically associates to any four-dimensional $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}$ a (two-dimensional) vertex operator algebra $\mathbb{V}(\mathcal{T})$ that encodes the spectrum and the OPE coefficients of the Schur operators. This correspondence between four-dimensional superconformal field theories and two-dimensional vertex operator algebras, that we refer to as the *4D/2D duality*, illuminates both sides.

This map must verify, among other properties, that the character of the vertex algebra $\mathbb{V}(\mathcal{T})$ coincides with the Schur index $\mathcal{I}_{\mathrm{Schur}(\mathcal{T})}$, which is a formal q-series, of the corresponding four-dimensional $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}$:

$$\mathcal{I}_{\mathrm{Schur}(\mathcal{T})}(q) = \chi_{\mathbb{V}(\mathcal{T})}(q). \tag{15.2}$$

According to [51], we have

$$c_{2D}(\mathbb{V}(\mathcal{T})) = -12c_{4D}(\mathcal{T}),$$

where $c_{4D}(\mathcal{T})$ and $c_{2D}(\mathbb{V}(\mathcal{T}))$ are the central charges of the four-dimensional $\mathcal{N}=2$ superconformal field theory $\mathcal{T}$ and the corresponding vertex algebra $\mathbb{V}(\mathcal{T})$, respectively. Since the central charge is positive for a unitary theory, this implies that the vertex algebras obtained by $\mathbb{V}$ are never unitary. In particular, integrable affine vertex algebras never appear by this correspondence. The main examples of vertex algebras considered in [51] are the simple affine vertex algebras $L_k(\mathfrak{g})$ of types D_4, E_6, E_7, E_8 at level $k = -\frac{h^\vee}{6} - 1$ that appear in Theorem 12.3, which are non-rational, non-admissible affine vertex algebras studied for different motivations (see §12.3.2). One can find more examples in this book (cf. e.g., §§9.2.2 and 9.1.6), see also [52, 71, 83, 72, 255, 241, 70].

There is another important invariant (or, observable) of a four-dimensional $\mathcal{N}=2$ superconformal field theory $\mathcal{T}$, called the *Higgs branch*, which we denote by $\mathrm{Higgs}(\mathcal{T})$. The Higgs branch $\mathrm{Higgs}(\mathcal{T})$ is an affine algebraic variety that has the hyperKähler structure in its smooth part. In particular, $\mathrm{Higgs}(\mathcal{T})$ is a (possibly singular) symplectic variety. Let $\mathcal{T}$ be one of the four-dimensional $\mathcal{N}=2$ superconformal field theorys studied in [51] such that $\mathbb{V}(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = -\frac{h^\vee}{6} - 1$ for types D_4, E_6, E_7, E_8 as above. It is known that $\mathrm{Higgs}(\mathcal{T})$ equals $\mathbb{O}_{\mathrm{min}}$, the minimal nilpotent orbit of the corresponding simple Lie algebra, and this is equal (cf. Theorem 12.3) to the associated variety $X_{\mathbb{V}(\mathcal{T})}$. It is expected that this is not just a coincidence [49].

Conjecture 15.1 For all four-dimensional $\mathcal{N}=2$ superconformal field theory $\mathcal{T}$, we have

$$\mathrm{Higgs}(\mathcal{T}) = X_{\mathbb{V}(\mathcal{T})}.$$

15.2 Class $\mathcal{S}$

There is a distinguished class of four-dimensional $\mathcal{N}=2$ superconformal field theories, called the *theory of class* $\mathcal{S}$ [126, 127]. Class $\mathcal{S}$ theories are those that arise from compactification of any of the $\mathcal{N}=(2,0)$ six-dimensional superconformal theories on a Riemann surface. They are labeled by a complex semisimple group G and a punctured Riemann surface.

The vertex algebras associated to the theory of class $\mathcal{S}$ by the above 4D/2D duality are called the *chiral algebras of class* $\mathcal{S}$ [52]. In [21], the first author constructed the chiral algebra $\mathbb{V}^{\mathcal{S}}_{G,b}$ of class $\mathcal{S}$ corresponding to the group G and a b-punctured Riemann surface of genus zero. Chiral algebras of class $\mathcal{S}$ associated with higher genus Riemann surfaces are obtained by glueing such vertex algebras; see [52], or [243, 244] for mathematical expositions.

According to Moore and Tachikawa [219], the Higgs branches of the theory of class $\mathcal{S}$ associated with such G and b are exactly the Moore–Tachikawa symplectic varieties $W_{G,b}$ that have been mathematically constructed by Braverman, Finkelberg and Nakajima [65] in terms of two-dimensional topological quantum field theories. Moreover, by [21] the associated variety of the chiral algebra $\mathbb{V}^{\mathcal{S}}_{G,b}$ of class $\mathcal{S}$ is precisely the Moore–Tachikawa symplectic variety $W_{G,b}$. This proves Conjecture 15.1 for the genus zero class $\mathcal{S}$ theories.

Although it is very difficult to describe the vertex algebra $\mathbb{V}^{\mathcal{S}}_{G,b}$ explicitly in general, conjectural descriptions of $\mathbb{V}^{\mathcal{S}}_{G,b}$ have been given in several cases in [51]. For instance, $\mathbb{V}^{\mathcal{S}}_{G=\mathrm{SL}_2,b=4}$ is isomorphic to the simple affine vertex algebra $L_{-2}(D_4)$, and $\mathbb{V}^{\mathcal{S}}_{G=\mathrm{SL}_3,b=3}$ is isomorphic to the simple affine vertex algebra $L_{-3}(E_6)$. In fact, the constructions of [21] allow to recover that $\mathrm{Higgs}(\mathcal{T}) = \mathbb{O}_{\min}$ for the theories $\mathcal{T}$ from the class $\mathcal{S}$ that correspond to $\mathbb{V}(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = -\frac{h^\vee}{6} - 1$ for $\mathfrak{g}$ of types D_4, E_6, E_7, E_8.

The simply-laced simple Lie algebras of the Deligne series appear in Kodaira's classification of isotrivial elliptic fibrations. Moreover, the integers $\frac{h^\vee}{6}+1 = 2, 3, 4, 6$ for the labels D_4, E_6, E_7, E_8 are precisely the multiplicities of the fibers corresponding to the node of the affine Dynkin diagrams (that is, the fibers with the highest multiplicity). The corresponding four-dimensional $\mathcal{N} = 2$ superconformal field theories are obtained by applying the F-theory to these isotrivial elliptic fibrations [51]. In other words, they come from four-dimensional superconformal field theories

Remark 15.1 The vertex algebra $\mathbb{V}^{\mathcal{S}}_{G=\mathrm{SL}_n,b=3}$ is called a *chiral algebra for the trinion theory*, and is denoted by $\chi(T_n)$ in the literature. A conjectural description of the list of strong generators has been given in [52, 191].

More examples of $\mathbb{V}^{\mathcal{S}}_{G,b}$ are computed in the appendix of [21].

Remark 15.2 There is a close relationship between the Higgs branches of four-dimensional $\mathcal{N} = 2$ superconformal field theories and the Coulomb branches [65] of three-dimensional $\mathcal{N} = 4$ SUSY gauge theories. Indeed, it is known [65, Theorem 5.1] that in type A the Moore-Tachikawa variety $W_{G,b}$ is isomorphic to the Coulomb branch of a star shaped quiver gauge theory. In fact, we have $\mathrm{Higgs}(\mathcal{T}) = \mathrm{Higgs}(\mathcal{T}_{3\mathrm{D}}) = \mathrm{Coulomb}(\check{\mathcal{T}}_{3\mathrm{D}})$ by the three-dimensional mirror symmetry, where $\mathcal{T}_{3\mathrm{D}}$ is the three-dimensional theory obtained from $\mathcal{T}$ by the $\mathbb{S}^1$-compactification.

15.3 Argyres–Douglas theory

We observe that many of our examples of collapsing levels (see §14.3.3) are of the form

$$-h^\vee + \frac{h^\vee}{q},$$

with $(h^\vee, q) = (\check{r}, q) = 1$, or of the form

$$-h^\vee + \frac{h+1}{q},$$

with $(h+1, q) = 1$, $(\check{r}, q) = \check{r}$, where h is the Coxeter number, $h^\vee$ is the dual Coxeter number, $\check{r}$ is the lacing number of $\mathfrak{g}$, respectively. A level of the first form is called a *boundary principal admissible level* [163] (see also [168]). The vertex algebras $L_k(\mathfrak{g})$ and $\mathcal{W}_k(\mathfrak{g}, f)$ at boundary principal admissible level k appear as vertex algebras associated with Argyres–Douglas theories via the 4D/2D duality ([241, 255, 256]). Collapsing levels which are boundary principal admissible have been studied by Xie and Yan [254] in this connection, and some results from §14.3.3 confirm their conjectures [254, Section 3.4]. For example, the isomorphism of Theorem 14.4 (ii) was conjectured.

In fact, as observed in [241], a given Argyres–Douglas theory can sometimes be realized in several ways. Since the map $\mathbb{V}$ is well-defined, whenever this happens, it means that we have an isomorphisms between $\mathcal{W}$-algebras. Such a phenomenon essentially reflects that the level is collapsing, provided that one of the $\mathcal{W}$-algebras in an affine vertex algebra.

15.4 Other conjectures and open problems

It immediately from [65] and [21] that the Poisson structure of the associated variety of $\mathbb{V}^{\mathcal{S}}_{G,b}$ is symplectic on its smooth locus. The Higgs branch of a four-dimensional $\mathcal{N} = 2$ superconformal field theory is expected have a finitely many symplectic leaves ([49]). Hence, it is natural to expect the following.

Conjecture 15.2 The vertex algebra $\mathbb{V}(\mathcal{T})$ is quasi-lisse for any four-dimensional $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}$.

Conjecture 15.2 is consistent with Conjecture 13.1 because the Higgs branch of a four-dimensional $\mathcal{N} = 2$ superconformal field theory is also expected to be irreducible. Also, Conjecture 15.2 is consistent with Theorem 13.1. Indeed, in view of Conjecture 15.1 and (15.2), Theorem 13.1 implies that the Schur index $\mathcal{I}_{\mathrm{Schur}(\mathcal{T})}(q)$ of a four-dimensional $\mathcal{N} = 2$ superconformal field theory satisfies a modular linear differential equation, which is something that has been conjectured in physics ([49]).

Many four-dimensional $\mathcal{N} = 2$ superconformal field theories have trivial Higgs branch. So we cannot expect that the associated variety $X_{\mathbb{V}(\mathcal{T})}$ is a complete invariant of a four-dimensional $\mathcal{N} = 2$ superconformal field theory even assuming Conjecture 15.1 true.

More recently, Beem and Rastelli conjectured the following:

Conjecture 15.3 ([50]) The Zhu's C_2-algebra $R_{\mathbb{V}(\mathcal{T})}$, or equivalently $\tilde{X}_{\mathbb{V}(\mathcal{T})}$, is a complete invariant of any four-dimensional $\mathcal{N} = 2$ superconformal field theory.

Conjecture 15.3 suggests that the vertex operator algebras coming from the 4D/2D duality are very special. It means that for such vertex algebras V, it is expected that *many* invariants (e.g., the normalized character) must be recovered from R_V. In particular, we expect that for such vertex operator algebras, the natural surjective Poisson vertex algebra morphism $\mathcal{J}_\infty R_V \to \mathrm{gr}^F V$ is an isomorphism, that is,

$$\mathcal{J}_\infty R_V \cong \mathrm{gr}^F V. \tag{15.3}$$

By Corollary 13.1, the isomorphism (15.3) holds for vertex algebras whose associated variety is smooth, reduced and symplectic. In general, for an arbitrary quasi-lisse vertex algebra, the isomorphism (15.3) does not hold. Using the chiral homology of elliptic curves with coefficients in a conformal vertex algebra V, Van Ekeren and Heluani have proved the following surprising result [95].

Theorem 15.1 *Let* $V = \mathrm{Vir}_{p,q}$ *be the Virasoro minimal model with central charge*

$$c(p,q) := 1 - 6\frac{(p-q)^2}{pq}.$$

If $(p,q) = (2, 2k+1)$ *where* $k \geqslant 1$*, then the isomorphism* (15.3) *holds. If* $p, q \geqslant 3$ *then* (15.3) *does not hold.*

In all cases, in the setting of the above theorem, the reduced schemes of $SS(V)$ and $\mathcal{J}_\infty X_V$ are isomorphic, both consisting of a single closed point. This is consistent with Theorem 13.4.

Note that the Virasoro minimal models do not appear as vertex algebras from the 4D/2D duality.

Further, Beem and Rastelli states that vertex operator algebras coming from the 4D/2D duality should carry a filtration, called the *R-filtration*, induced from the R-charge of the 4D theory. In general that filtration has not been understood from a mathematical point of view (see [240] for some partial progress in this direction).

? Open problem

Let $V = \mathbb{V}(\mathcal{T})$ be a vertex operator algebra coming from a four-dimensional $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}$. How to define the R-filtration on V in a purely mathematical way?

Appendices

Appendix A
Simple Lie algebras and affine Kac-Moody algebras

The ground field is the field $\mathbb{C}$ of complex numbers. Recall that a Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear form $[\ ,\ ]\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ satisfying the following conditions:

- (skew-symmetry) $[x, y] = [y, x]$, for all $x, y \in \mathfrak{g}$,
- (Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in \mathfrak{g}$.

It is assumed that the reader is familiar with the basics on Lie algebras. We review in this appendix some of the standard facts on the structure of semisimple Lie algebras, and corresponding affine Kac-Moody algebras. This appendix is also used to fix the main notations relative to these structures.

Recall that the *enveloping algebra* of a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is the quotient

$$U(\mathfrak{g}) := T(\mathfrak{g})/\mathscr{J}(\mathfrak{g}),$$

where

$$T(\mathfrak{g}) := \bigoplus_{i=0}^{\infty} T^i(\mathfrak{g}), \qquad T^i(\mathfrak{g}) = \underbrace{\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}}_{i \text{ times}},$$

is the tensor algebra of $\mathfrak{g}$ and $\mathscr{J}(\mathfrak{g})$ is the two-sided ideal of $T\mathfrak{g}$ generated by elements $x \otimes y - y \otimes -[x, y]$, for $x, y \in \mathfrak{g}$. It is a unital associative $\mathbb{C}$-algebra. The enveloping algebra $U(\mathfrak{g})$ is naturally filtered by the *PBW filtration* $U_\bullet(\mathfrak{g})$, where $U_i(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ spanned by the products of at most i elements of $\mathfrak{g}$, for $i \geqslant 0$, and $U_0(\mathfrak{g}) = \mathbb{C}1$. By the PBW theorem, we have

$$\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}), \tag{A.1}$$

as graded commutative algebras, where $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$ is the *symmetric algebra* of $\mathfrak{g}$, that is, the quotient $T(\mathfrak{g})/J(\mathfrak{g})$, where $J(\mathfrak{g})$ is the two-sided ideal of $T(\mathfrak{g})$ generated by elements $x \otimes y - y \otimes x$, for $x, y \in \mathfrak{g}$.

Our main references for this chapter are [77, 187, 188, 245, 218, 158, 162]. See also [145] for a survey.

A.1 Quick review on semisimple Lie algebras

Let $\mathfrak{g}$ be a complex finite dimensional *semisimple* Lie algebra, i.e., $\{0\}$ is the only abelian ideal of $\mathfrak{g}$. Its *adjoint group* G is the smallest algebraic subgroup of $GL(\mathfrak{g})$ whose Lie algebra contains ad $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $G = \mathrm{Aut}_e(\mathfrak{g})$, where $\mathrm{Aut}_e(\mathfrak{g})$ is the subgroup of $GL(\mathfrak{g})$ generated by the elements $\exp(\mathrm{ad}\, x)$ with x a *nilpotent element* of $\mathfrak{g}$ (i.e., $(\mathrm{ad}\, x)^n = 0$ for n large enough). Hence

$$\mathrm{Lie}(G) = \mathrm{ad}\, \mathfrak{g} \cong \mathfrak{g}$$

since the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $x \mapsto (\mathrm{ad}\, x)(y) = [x, y]$ is faithful, $\mathfrak{g}$ being semisimple.

A.1.1 Main notations

For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$ and $x \in \mathfrak{g}$, we shall denote by $\mathfrak{a}^x$ the *centralizer* of x in $\mathfrak{a}$, that is,

$$\mathfrak{a}^x = \{y \in \mathfrak{a} \colon [x, y] = 0\},$$

which is also the intersection of $\mathfrak{a}$ with the kernel of the map

$$\mathrm{ad}\, x \colon \mathfrak{g} \longrightarrow \mathfrak{g}, \quad y \longmapsto [x, y].$$

Let $\kappa_{\mathfrak{g}}$ be the *Killing form* of $\mathfrak{g}$,

$$\kappa_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y).$$

It is a nondegenerate symmetric bilinear form of $\mathfrak{g}$ which is G-invariant, that is,

$$\kappa_{\mathfrak{g}}(g.x, g.y) = \kappa_{\mathfrak{g}}(x, y) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } g \in G,$$

or else,

$$\kappa_{\mathfrak{g}}([x, y], z) = \kappa_{\mathfrak{g}}(x, [y, z]) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Since $\mathfrak{g}$ is semisimple, any other such bilinear form is a nonzero multiple of the Killing form.

Example A.1 Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n$, for $n \geqslant 2$, which is the set of traceless complex n-size square matrices, with bracket $[A, B] = AB - BA$. The Lie algebra $\mathfrak{sl}_n$ is known to be *simple* , that is, $\{0\}$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$ and $\dim \mathfrak{g} \geqslant 3$. Its Killing form is given by

$$(A, B) \mapsto 2n\, \mathrm{tr}(AB).$$

The bilinear form $(A, B) \mapsto \mathrm{tr}(AB)$ is more naturally used.

In fact, mathematicians usually consider a certain normalization $(\ |\ )$ of the Killing form which will coincide with this bilinear form for $\mathfrak{sl}_n$ (see Section A.2).

A.1.2 Cartan matrix and Chevalley generators

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha := \{y \in \mathfrak{g} \colon [x, y] = \alpha(x) y \text{ for all } x \in \mathfrak{h}\},$$

be the corresponding root decomposition of $(\mathfrak{g}, \mathfrak{h})$, where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a basis of Δ, with r the *rank* of $\mathfrak{g}$, and let $\alpha_1^\vee, \ldots, \alpha_r^\vee$ be the coroots of $\alpha_1, \ldots, \alpha_r$, respectively. The element $\alpha_i^\vee$, for $i = 1, \ldots, r$, viewed as an element of $(\mathfrak{h}^*)^* \cong \mathfrak{h}$, will be often denoted by h_i.

Recall that the *Cartan matrix* of Δ is the matrix $C = (C_{i,j})_{1 \leqslant i,j \leqslant r}$, where $C_{i,j} := \alpha_j(h_i)$. The Cartan matrix C does not depend on the choice of the basis Π. It verifies the following properties:

(A.2) $\quad C_{i,j} \in \mathbb{Z}$ for all i, j,

(A.3) $\quad C_{i,i} = 2$ for all i,

(A.4) $\quad C_{i,j} \leqslant 0$ if $i \neq j$,

(A.5) $\quad C_{i,j} = 0$ if and only if $C_{j,i} = 0$.

Moreover, all principal minors of C are strictly positive,

$$\det\big((C_{i,j})_{1 \leqslant i,j \leqslant s}\big) > 0 \quad \text{for} \quad 1 \leqslant s \leqslant r.$$

The semisimple Lie algebra $\mathfrak{g}$ has a presentation in term of Chevalley generators. Namely, consider the generators $(e_i)_{1 \leqslant i \leqslant r}$, $(f_i)_{1 \leqslant i \leqslant r}$, $(h_i)_{1 \leqslant i \leqslant r}$ with relations

(A.6) $\quad [h_i, h_j] = 0,$

(A.7) $\quad [e_i, f_j] = \delta_{i,j} h_i,$

(A.8) $\quad [h_i, e_j] = C_{i,j} e_j,$

(A.9) $\quad [h_i, f_j] = -C_{i,j} f_j,$

(A.10) $\quad (\operatorname{ad} e_i)^{1 - C_{i,j}} e_j = 0$ for $i \neq j$,

(A.11) $\quad (\operatorname{ad} f_i)^{1 - C_{i,j}} f_j = 0$ for $i \neq j$,

where $\delta_{i,j}$ is the Kronecker symbol. The last two relations are called the *Serre relations*. By (A.8) and (A.9), $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ for all i.

It is well-known that $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in \Delta$. One can choose nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$, for all α, such that $\{h_i \colon i = 1, \ldots, r\} \cup \{e_\alpha \colon \alpha \in \Delta\}$ forms a *Chevalley basis* of $\mathfrak{g}$. This means, apart from the above relations, that:

(A.12) $$[e_\beta, e_\gamma] = \pm(p+1)e_{\beta+\gamma},$$

for all $\beta, \gamma \in \Delta$, where p is the greatest positive integer such that $\gamma - p\beta$ is a root. Here we consider that $e_{\beta+\gamma} = 0$ if $\beta+\gamma$ is not a root, and that $e_{\alpha_i} = e_i$, $e_{-\alpha_i} = f_i$ for $i = 1, \dots, r$.

Let Δ_+ be the positive root system corresponding to Π, and let

(A.13) $$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

be the corresponding triangular decomposition. Thus $\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha$ are both nilpotent Lie subalgebras of $\mathfrak{g}$.

A.1.3 Verma modules

We refer, for instance, to [77, Section 10] for the main results of this paragraph. Let $\lambda \in \mathfrak{h}^*$ and set

$$K_\mathfrak{g}(\lambda) := U(\mathfrak{g})\mathfrak{n}_+ + \sum_{x\in\mathfrak{h}} U(\mathfrak{g})(x - \lambda(x)).$$

Since $K_\mathfrak{g}(\lambda)$ is a left $U(\mathfrak{g})$-module,

$$M_\mathfrak{g}(\lambda) := U(\mathfrak{g})/K_\mathfrak{g}(\lambda)$$

is naturally a left $U(\mathfrak{g})$-module, called a *Verma module* .

Theorem A.1 (i) *Each element of $M_\mathfrak{g}(\lambda)$ is uniquely written in the form um_λ for some $u \in U(\mathfrak{g})$ where $m_\lambda := 1 + K_\mathfrak{g}(\lambda)$.*
(ii) *The elements $f_{\beta_1}^{n_1} \dots f_{\beta_s}^{n_s} m_\lambda$, for all $n_i \geqslant 0$, form a basis of $M_\mathfrak{g}(\lambda)$.*

Note that $M_\mathfrak{g}(\lambda)$ can also be described as follows (up to isomorphism of $U(\mathfrak{g})$-modules):

$$M_\mathfrak{g}(\lambda) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda =: \mathrm{Ind}_\mathfrak{b}^\mathfrak{g}(\mathbb{C}_\lambda),$$

where $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathbb{C}_\lambda$ is a one-dimensional $\mathfrak{b}$-module whose $\mathfrak{b}$-action is given by: $(x+n).z = \lambda(x)z$ for $x \in \mathfrak{h}$, $n \in \mathfrak{n}_+$ and $z \in \mathbb{C}_\lambda$. Then, up to scalars, $m_\lambda = 1 \otimes 1$.

For each $\mu \in \mathfrak{h}^*$, set

$$M_\mathfrak{g}(\lambda)_\mu := \{m \in M_\mathfrak{g}(\lambda) : xm = \mu(x)m \text{ for all } x \in \mathfrak{h}\}.$$

For $\lambda, \mu \in \mathfrak{h}^*$ we write $\mu \preccurlyeq \lambda$ if $\lambda - \mu$ belongs to the *root lattice*

$$Q := \sum_{i=1}^r \mathbb{Z}\alpha_i.$$

This defines a partial order on $\mathfrak{h}^*$.

Theorem A.2 (i) $M_\mathfrak{g}(\lambda) = \bigoplus_{\mu\in\mathfrak{h}^*} M_\mathfrak{g}(\lambda)_\mu$.

(ii) $M_{\mathfrak{g}}(\lambda)_\mu \neq 0$ *if and only if* $\mu \preccurlyeq \lambda$, *and* $\dim M_{\mathfrak{g}}(\lambda)_\mu$ *is the number of ways of expressing* $\lambda - \mu$ *as a sum of positive roots. In particular,* $\dim M_{\mathfrak{g}}(\lambda)_\lambda = 1$.

If $M_{\mathfrak{g}}(\lambda)_\mu \neq 0$, then μ called a *weight* of $M_{\mathfrak{g}}(\lambda)$, and $M_{\mathfrak{g}}(\lambda)_\mu$ is called the *weight space* of $M_{\mathfrak{g}}(\lambda)$ with weight μ.

Theorem A.2 says that the weights of $M_{\mathfrak{g}}(\lambda)$ are precisely the elements $\mu \in \mathfrak{h}^*$ such that $\mu \preccurlyeq \lambda$. Thus λ is the *highest weight* of $M_{\mathfrak{g}}(\lambda)$ with respect to the partial order $\preccurlyeq$. We say that $M_{\mathfrak{g}}(\lambda)$ is the *Verma module with highest weight* λ.

One of the important fact about $M_{\mathfrak{g}}(\lambda)$ is that it has a unique maximal submodule $N_{\mathfrak{g}}(\lambda)$. It is constructed as follows: since $M_{\mathfrak{g}}(\lambda)_\lambda = \mathbb{C}m_\lambda$ and that $M_{\mathfrak{g}}(\lambda)$ is generated by m_λ, any proper submodule N of $M_{\mathfrak{g}}(\lambda)$ satisfy $N_\lambda = 0$. In particular the sum $N_{\max}$ of all proper submodules of M satisfies $(N_{\max})_\lambda = 0$. This proves the existence and the unicity of the maximal proper submodule of $M_{\mathfrak{g}}(\lambda)$: just set $N_{\mathfrak{g}}(\lambda) := N_{\max}$.

Since $N_{\mathfrak{g}}(\lambda)$ is a maximal submodule of $M_{\mathfrak{g}}(\lambda)$,

$$L_{\mathfrak{g}}(\lambda) := M_{\mathfrak{g}}(\lambda)/N_{\mathfrak{g}}(\lambda)$$

is a simple $U(\mathfrak{g})$-module, that is, an irreducible representation of $\mathfrak{g}$. There is $v_\lambda \in L_{\mathfrak{g}}(\lambda) \setminus \{0\}$ such that

- $h_i v = \lambda(h_i)v$ for all $i = 1, \dots, r$,
- $e_i v = 0$ for all $i = 1, \dots, r$, that is, $\mathfrak{n}_+ v = 0$,
- $L_{\mathfrak{g}}(\lambda) = U(\mathfrak{n}_-)v_\lambda$,
- λ is the highest weight of $L_{\mathfrak{g}}(\lambda)$.

Let

$$P := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z} \text{ for all } i = 1, \dots, r\},$$
$$P^+ := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z}_{\geqslant 0} \text{ for all } i = 1, \dots, r\},$$

be the *weight lattice* of $\mathfrak{h}^*$ and the set of *dominant integral weights* , respectively. The elements $\varpi_i \in \mathfrak{h}^*$, $i = 1, \dots, r$, satisfying $\varpi_i(h_j) = \delta_{i,j}$ for all j are called the *fundamental weights*. We denote by $\varpi_1^\vee, \dots, \varpi_r^\vee$ the fundamental coweights. They are the elements of $\mathfrak{h}$ such that $\{\varpi_1^\vee, \dots, \varpi_r^\vee\}$ is the dual basis of $\{\alpha_1, \dots, \alpha_r\}$.

We conclude this section by the following crucial result.

Theorem A.3 *The simple* $U(\mathfrak{g})$*-module* $L_{\mathfrak{g}}(\lambda)$ *is finite dimensional if and only if* $\lambda \in P^+$. *Moreover, all simple finite dimensional* $U(\mathfrak{g})$*-modules are of the form* $L_{\mathfrak{g}}(\lambda)$ *for some* $\lambda \in P^+$. *These modules are pairwise non-isomorphic.*

The highest weight modules $M_{\mathfrak{g}}(\lambda)$ and $L_{\mathfrak{g}}(\lambda)$ are both elements of the *category* $\mathscr{O}$ of $\mathfrak{g}$. To avoid repetitions, we will define the category $\mathscr{O}$ only for affine Kac-Moody algebras (see Section A.4); the definition and properties are very similar.

For more about semisimple Lie algebras and their representations, possible references are [77, 187, 245]; see [188] about the category $\mathscr{O}$.

For the category $\mathscr{O}$ in the affine Kac-Moody algebras setting, we refer to Moody-Pianzola's book [218].

A.1.4 Infinitesimal characters

In this paragraph, we recall how to deduce the *infinitesimal character* of $L_{\mathfrak{g}}(\lambda)$ from the knowledge of $\lambda \in \mathfrak{h}^*$.

Identify Specm $Z(\mathfrak{g})$ with the set of all homomorphisms $Z(\mathfrak{g}) \to \mathbb{C}$. Such morphisms are called *infinitesimal characters*. Consider the projection map from $U(\mathfrak{g})$ to $U(\mathfrak{h}) = S(\mathfrak{h})$ with respect to the decomposition

$$U(\mathfrak{g}) = S(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+).$$

It is not a morphism of algebras in general, but its restriction to $U(\mathfrak{g})^{\mathfrak{h}} = \{u \in U(\mathfrak{g}) \colon (\operatorname{ad} x)u = 0 \text{ for all } x \in \mathfrak{h}\}$ is. In particular, we get a morphism

$$p \colon Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]$$

using $S(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$, usually called the *Harish-Chandra morphism*. Its comorphism gives a map

$$\chi \colon \mathfrak{h}^* \to \operatorname{Specm}(Z(\mathfrak{g})), \quad \lambda \mapsto (\chi_\lambda Z(\mathfrak{g}) \to \mathbb{C}) \tag{A.14}$$

where $\chi_\lambda(z) = p(z)(\lambda + \rho)$ for $z \in Z(\mathfrak{g})$ with ρ the half-sum of positive roots.

Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ which acts on $\mathfrak{h}^*$ with respect to the twisted action of W:

$$w \circ \lambda = w.(\lambda + \rho) - \rho, \qquad w \in W,\ \lambda \in \mathfrak{h}^*,$$

where $\cdot$ stands for the usual action of W on $\mathfrak{h}^*$.

An important consequence of the Harish-Chandra Theorem is that the map χ induces a bijection

$$\mathfrak{h}^*/W \xrightarrow{\sim} \operatorname{Spec}(Z(\mathfrak{g})).$$

In particular, $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same W-orbit with respect to the twisted action of W, and the infinitesimal character associated with the irreducible representation $L_{\mathfrak{g}}(\lambda)$ is just χ_λ.

A.2 Affine Kac-Moody algebras

Our basic reference about affine Kac-Moody algebras is [158]. We assume from now on that $\mathfrak{g}$ is simple, that is, the only ideals of $\mathfrak{g}$ are $\{0\}$ or $\mathfrak{g}$ and $\dim \mathfrak{g} \geqslant 3$.

A.2.1 The loop algebra

Consider the *loop algebra of* $\mathfrak{g}$ which is the Lie algebra

$$L\mathfrak{g} := \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n}, \qquad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z},$$

where xt^m stands for $x \otimes t^m$.

Remark A.1 The Lie algebra $L\mathfrak{g}$ is the Lie algebra of polynomial functions from the unit circle to $\mathfrak{g}$. This is the reason why it is called the loop algebra.

A.2.2 Definition of affine Kac-Moody algebras

Define the bilinear form $(-|-)$ on $\mathfrak{g}$ by:

$$(-|-) = \frac{1}{2h^\vee}\kappa_{\mathfrak{g}},$$

where $h^\vee$ is the *dual Coxeter number* (see §A.3.3 for the definition). For example, if $\mathfrak{g} = \mathfrak{sl}_n$ then $h^\vee = n$. Thus, with respect to the induced bilinear form on $\mathfrak{h}^*$, $(\theta|\theta) = 2$, where θ is the *highest positive root* of $\mathfrak{g}$, that is, the unique (positive) root $\theta \in \Delta$ such that $\theta + \alpha_i \notin \Delta \cup \{0\}$ for $i = 1, \ldots, r$.

Definition A.1 We define a bilinear map ν on $L\mathfrak{g}$ by setting:

$$\nu(x \otimes f, y \otimes g) := (x|y)\mathrm{Res}_{t=0}(\frac{df}{dt}g),$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t, t^{-1}]$, where the linear map $\mathrm{Res}_{t=0}\colon \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ is defined by $\mathrm{Res}_{t=0}(t^m) = \delta_{m,-1}$ for $m \in \mathbb{Z}$.

The bilinear ν is a *2-cocycle* on $L\mathfrak{g}$, that is, for any $a, b, c \in L\mathfrak{g}$,

$$\nu(a, b) = -\nu(b, a), \tag{A.15}$$

$$\nu([a, b], c) + \nu([b, c], a) + \nu([c, a], b) = 0. \tag{A.16}$$

Definition A.2 We define the *affine Kac-Moody algebra* $\hat{\mathfrak{g}}$ as the vector space $\hat{\mathfrak{g}} := L\mathfrak{g} \oplus \mathbb{C}K$, with the commutation relations $[K, \hat{\mathfrak{g}}] = 0$ (so K is a central element), and

$$[x \otimes f, y \otimes g] = [x, y]_{L\mathfrak{g}} + \nu(x \otimes f, y \otimes g)K, \quad x, y \in \mathfrak{g},\ f, g \in \mathbb{C}[t, t^{-1}],$$

where $[\ ,\ ]_{L\mathfrak{g}}$ is the Lie bracket on $L\mathfrak{g}$. In other words the commutation relations are given by:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K,$$
$$[K, \hat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

A.2.3 Chevalley generators

The following result shows that affine Kac-Moody algebras are natural generalizations of finite dimensional semisimple Lie algebras.

Theorem A.4 *The Lie algebra $\hat{\mathfrak{g}}$ can be presented by generators $(E_i)_{0\leqslant i\leqslant r}$, $(F_i)_{0\leqslant i\leqslant r}$, $(H_i)_{0\leqslant i\leqslant r}$, and relations*

$$[H_i, H_j] = 0, \tag{A.17}$$

$$[E_i, F_j] = \delta_{i,j} H_i, \tag{A.18}$$

$$[H_i, E_j] = C_{i,j} E_j, \tag{A.19}$$

$$[H_i, F_j] = -C_{i,j} F_j, \tag{A.20}$$

$$(\operatorname{ad} E_i)^{1-C_{i,j}} E_j = 0 \textit{ for } i \neq j, \tag{A.21}$$

$$(\operatorname{ad} F_i)^{1-C_{i,j}} F_j = 0 \textit{ for } i \neq j, \tag{A.22}$$

where $\hat{C} = (C_{i,j})_{0\leqslant i\leqslant r}$ is an affine Cartan matrix, *that is, $\hat{C}$ satisfies the relations (A.2)–(A.5) of a Cartan matrix, all proper principal minors are strictly positive,*

$$\det\left((C_{i,j})_{0\leqslant i,j\leqslant s}\right) > 0 \quad \textit{for} \quad 0 \leqslant s \leqslant r-1,$$

and $\det(\hat{C}) = 0$.

Moreover, we can choose the labelling $\{0, \ldots, r\}$ so that the subalgebra generated by $(E_i)_{1\leqslant i\leqslant r}$, $(F_i)_{1\leqslant i\leqslant r}$, $(H_i)_{1\leqslant i\leqslant r}$ is isomorphic to $\mathfrak{g}$, that is, $(C_{i,j})_{1\leqslant i\leqslant r}$ is the Cartan matrix C of $\mathfrak{g}$.

Let us give the general idea of the construction of the Chevalley generators of $\hat{\mathfrak{g}}$ (see [145]). Set for $i = 1, \ldots, r$,

$$E_i := e_i = e_i \otimes 1, \quad F_i := f_i = f_i \otimes 1, \quad H_i := h_i = h_i \otimes 1.$$

The point is to define E_0, F_0, H_0. Recall that θ is the highest root of Δ. Consider the *Chevalley involution* ω which is the linear involution map of $\mathfrak{g}$ defined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h_i) = -h_i$ for $i = 1, \ldots, r$. Then pick $f_0 \in \mathfrak{g}_\theta$ such that

$$(f_0|\omega(f_0)) = -\frac{h^\vee}{(\theta|\theta)} = -\frac{h^\vee}{2}.$$

Then we set $e_0 := -\omega(f_0) \in \mathfrak{g}_{-\theta}$ and,

$$E_0 := e_0 t = e_0 \otimes t, \quad F_0 := f_0 t^{-1} = f_0 \otimes t^{-1}, \quad H_0 := [E_0, F_0].$$

Example A.2 Assume that $\mathfrak{g} = \mathfrak{sl}_2$. Then the Cartan matrix C is $C = (2)$. Let us check that the affine Cartan matrix of $\hat{\mathfrak{sl}}_2$ is $\hat{C} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We have

$$\hat{\mathfrak{sl}}_2 = e \otimes \mathbb{C}[t, t^{-1}] \oplus f \otimes \mathbb{C}[t, t^{-1}] \oplus h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K,$$

where

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We follow the above construction. We set $E_1 := e$, $F_1 := f$ and $H_1 := h$. We have $h^\vee = 2$ and $\Delta = \{\alpha, -\alpha\}$ with $\alpha(h) = 2$. The highest root is $\theta = \alpha$ and $(\mathfrak{sl}_2)_\theta = \mathbb{C}e$. So f_0 is of the form $f_0 = \lambda e$, $\lambda \in \mathbb{C}^*$ and verifies:

$$-1 = (f_0, \omega(f_0)) = -\lambda^2,$$

whence $\lambda^2 = \pm 1$. Let us fix $\lambda = 1$. So we have

$$E_0 = ft \quad \text{and} \quad F_0 = et^{-1}.$$

Then

$$H_0 = [E_0, F_0] = [f, e] + (f|e)K = K - H_1.$$

We can verify the relations of Chevalley generators. In particular, $[H_1, E_0] = -2E_0$ and $[H_0, E_1] = -2E_1$, whence the expected affine Cartan matrix $\hat{C}$.

A.3 Root systems and triangular decomposition

In order to construct analogs of highest weight representations, we need a triangular decomposition for $\hat{\mathfrak{g}}$ and the corresponding combinatoric, that is, a system of roots.

A.3.1 Triangular decomposition

Recall the triangular decomposition (A.13) of $\mathfrak{g}$, and consider the following subspaces of $\hat{\mathfrak{g}}$:

$$\begin{aligned}
\hat{\mathfrak{n}}_+ &:= (\mathfrak{n}_- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t] \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t] = \mathfrak{n}_+ + t\mathfrak{g}[t], \\
\hat{\mathfrak{n}}_- &:= (\mathfrak{n}_+ \oplus \mathfrak{h}) \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}] = \mathfrak{n}_- + t^{-1}\mathfrak{g}[t^{-1}], \\
\hat{\mathfrak{h}} &:= (\mathfrak{h} \otimes 1) \oplus \mathbb{C}K = \mathfrak{h} + \mathbb{C}K.
\end{aligned}$$

They are Lie subalgebras of $\hat{\mathfrak{g}}$ and we have

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+. \tag{A.23}$$

In fact, $\hat{\mathfrak{n}}_+$ (resp., $\hat{\mathfrak{n}}_-$, $\hat{\mathfrak{h}}$) is generated by the E_i (resp., F_i, H_i), for $i = 0, \dots, r$. The verifications are left to the reader.

A.3.2 Extended affine Kac-Moody algebras

We now intend to define a corresponding root system, and simple roots. The simple roots $\alpha_i \in \hat{\mathfrak{h}}^*$ are defined by $\alpha_j(H_i) = C_{i,j}$ for $0 \leqslant i, j \leqslant r$. As $\det(\hat{C}) = 0$, the simple roots $\alpha_0, \dots, \alpha_r$ are not linearly independent. For example, for $\hat{\mathfrak{sl}}_2$, we have $\alpha_0 + \alpha_1 = 0$.

For the following constructions, we need linearly independent simple roots. This is the reason why we consider the *extended affine Lie algebra* :

$$\tilde{\mathfrak{g}} := \hat{\mathfrak{g}} \oplus \mathbb{C}D,$$

with commutation relations (apart from those of $\hat{\mathfrak{g}}$),

$$[D, x \otimes f] = x \otimes t\frac{df}{dt}, \qquad [D, K] = 0, \quad x \in \mathfrak{g},\ f \in \mathbb{C}[t, t^{-1}],$$

that is,

$$[D, xt^m] = mxt^m, \qquad [D, K] = 0, \qquad x \in \mathfrak{g},\ m \in \mathbb{Z}.$$

We have the new Cartan subalgebra

$$\tilde{\mathfrak{h}} := \hat{\mathfrak{h}} \oplus \mathbb{C}D.$$

It is a commutative Lie subalgebra of $\tilde{\mathfrak{g}}$ of dimension $r + 2$, and we have the corresponding triangular decomposition :

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+.$$

Let us define the new simple roots $\alpha_i \in \tilde{\mathfrak{h}}^*$, for $i = 0, \dots, r$. The action of α_i on $\hat{\mathfrak{h}}$ has already been defined, and so we only have to specify $\alpha_i(D)$, for $i = 0, \dots, r$. From the relations

$$\alpha_i(D)E_i = [D, E_i] = [D, e_i] = 0, \qquad i = 1, \dots, r,$$

we deduce that $\alpha_i(D) = 0$ for $i = 1, \dots, r$. From the relation

$$\alpha_0(D)E_0 = [D, E_0] = [D, e_0 t] = E_0,$$

we deduce that $\alpha_0(D) = 1$.

A.3.3 Root system

The bilinear form $(\ |\ )$ extends from $\mathfrak{g}$ to a symmetric bilinear form on $\tilde{\mathfrak{g}}$ by setting for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$:

$$(xt^m|yt^n) = \delta_{m+n,0}(x|y), \qquad (L\mathfrak{g}|\mathbb{C}K \oplus \mathbb{C}D) = 0,$$
$$(K|K) = (D|D) = 0, \qquad (K|D) = 1.$$

Since the restriction of the bilinear form $(\ |\ )$ to $\tilde{\mathfrak{h}}$ is nondegenerate, we can identify $\tilde{\mathfrak{h}}^*$ with $\tilde{\mathfrak{h}}$ using this form. Through this identification, $\alpha_0 = K - \theta$. For $\alpha \in \tilde{\mathfrak{h}}^*$ such that $(\alpha|\alpha) \neq 0$, we set $\alpha^\vee = \dfrac{2\alpha}{(\alpha|\alpha)}$. Note that $\alpha^\vee$ obviously corresponds to $\alpha_i^\vee = h_i$ for $\alpha = \alpha_i$, $i = 1, \ldots, r$.

The set of roots $\hat{\Delta}$ of $\tilde{\mathfrak{g}}$ with basis $\hat{\Pi} := \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ is

$$\hat{\Delta} = \hat{\Delta}^{\mathrm{re}} \cup \hat{\Delta}^{\mathrm{im}},$$

where the set of *real roots* is

$$\hat{\Delta}^{\mathrm{re}} := \{\alpha + nK \colon \alpha \in \Delta,\ n \in \mathbb{Z}\},$$

and the set of *imaginary roots* is

$$\hat{\Delta}^{\mathrm{im}} := \{nK \colon n \in \mathbb{Z},\ n \neq 0\}.$$

Then we set $\hat{\Delta}^\vee := \hat{\Delta}^{\vee,\mathrm{re}} \cup \hat{\Delta}^{\vee,\mathrm{im}}$, with

$$\hat{\Delta}^{\vee,\mathrm{re}} := \{\alpha^\vee \colon \alpha \in \hat{\Delta}^{\mathrm{re}}\}, \qquad \hat{\Delta}^{\vee,\mathrm{im}} := \{\alpha^\vee \colon \alpha \in \hat{\Delta}^{\mathrm{im}}\}.$$

The positive integers

$$h := (\rho^\vee|\theta) + 1 \quad \text{and} \quad h^\vee = (\rho|\theta^\vee) + 1$$

are called the *Coxeter number* and the *dual Coxeter number* of $\mathfrak{g}$, respectively, where ρ (resp., $\rho^\vee$) is the half sum of positive roots (resp., coroots), that is defined by $(\rho|\alpha_i^\vee) = 1$ (resp., $(\rho^\vee|\alpha_i) = 1$), for $i = 1, \ldots, r$. Defining $\hat{\rho} := h^\vee D + \rho \in \tilde{\mathfrak{h}}$ and $\hat{\rho}^\vee := hD + \rho^\vee \in \tilde{\mathfrak{h}}$ we have the following formulas: $(\hat{\rho}|\alpha_i^\vee) = 1$ and $(\hat{\rho}^\vee|\alpha_i) = 1$, for $i = 0, \ldots, r$.

A.4 Representations of affine Kac-Moody algebras, category $\mathscr{O}$

We extend some notations and definitions of Section A.1 to $\tilde{\mathfrak{g}}$. For example, for M a $\tilde{\mathfrak{g}}$-module and $\lambda \in \tilde{\mathfrak{h}}^*$, we set

$$M_\lambda := \{m \in M \colon xm = \lambda(x)m \text{ for all } x \in \tilde{\mathfrak{h}}\}.$$

The space M_λ is called the *weight space* of weight λ of M. The set of weights of M is

$$\mathrm{wt}(M) := \{\lambda \in \tilde{\mathfrak{h}}^* : M_\lambda \neq 0\}.$$

The partial order $\preccurlyeq$ is extended to $\tilde{\mathfrak{h}}^*$ as follows: we write $\mu \preccurlyeq \lambda$ if $\lambda - \mu = \sum_{i=0}^{r} m_i \alpha_i$ with $m_i \in \mathbb{Z}$, $m_i \geqslant 0$. For $\lambda \in \tilde{\mathfrak{h}}^*$, we set $D(\lambda) := \{\mu \in \tilde{\mathfrak{h}}^* : \mu \preccurlyeq \lambda\}$.

A.4.1 The category $\mathscr{O}$

Let $U(\tilde{\mathfrak{g}})$ -Mod be the category of left $U(\tilde{\mathfrak{g}})$-modules.

Definition A.3 The *category* $\mathscr{O}$ is defined to be the full subcategory of $U(\tilde{\mathfrak{g}})$ -Mod whose objects are the modules M satisfying the following conditions:

($\mathscr{O}$1) M is $\tilde{\mathfrak{h}}$-diagonalizable, that is, $M = \oplus_{\lambda \in \tilde{\mathfrak{h}}^*} M_\lambda$,
($\mathscr{O}$2) all weight spaces of M are finite dimensional,
($\mathscr{O}$3) there exists a finite number of $\lambda_1, \dots, \lambda_s \in \tilde{\mathfrak{h}}^*$ such that

$$\mathrm{wt}(M) \subset \bigcup_{1 \leqslant i \leqslant s} D(\lambda_i).$$

The category $\mathscr{O}$ is stable by submodules and quotients. For M_1, M_2 two representations of $\tilde{\mathfrak{g}}$ we can define a structure of $\tilde{\mathfrak{g}}$-module on $M_1 \otimes M_2$ by using the coproduct $\tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}}$, $x \longmapsto x \otimes 1 + 1 \otimes x$ for $x \in \tilde{\mathfrak{g}}$. Then if M_1 and M_2 are objects of $\mathscr{O}$, then so are $M_1 \oplus M_2$ and $M_1 \otimes M_2$.

A.4.2 Verma modules

We now give important examples of modules in the category $\mathscr{O}$. For $\lambda \in \tilde{\mathfrak{h}}^*$, set:

$$K(\lambda) := U(\tilde{\mathfrak{g}})\hat{\mathfrak{n}}_+ + \sum_{x \in \tilde{\mathfrak{h}}^*} U(\tilde{\mathfrak{g}})(x - \lambda(x)).$$

As it is a left ideal of $U(\tilde{\mathfrak{g}})$,

$$M(\lambda) := U(\tilde{\mathfrak{g}})/K(\lambda)$$

has a natural structure of a left $U(\tilde{\mathfrak{g}})$-module. It is called a *Verma module*.

Proposition A.1 *The $U(\tilde{\mathfrak{g}})$-module $M(\lambda)$ is in the category $\mathscr{O}$ and has a unique proper submodule $N(\lambda)$.*

We construct $N(\lambda)$ in the same way as $N_{\mathfrak{g}}(\lambda)$ for $\mathfrak{g}$ (see §A.1.3).

As a consequence of the proposition, $M(\lambda)$ has a unique simple quotient

$$L(\lambda) := M(\lambda)/N(\lambda).$$

Proposition A.2 *The simple module $L(\lambda)$ is in the category $\mathcal{O}$ and all simple modules of the category $\mathcal{O}$ are of the form $L(\lambda)$ for some $\lambda \in \tilde{\mathfrak{h}}^*$.*

The *character* of a module M in the category $\mathcal{O}$ is by definition

$$\mathrm{ch}(M) = \sum_{\lambda \in \tilde{\mathfrak{h}}^*} (\dim M_\lambda) e(\lambda),$$

where the $e(\lambda)$ are formal elements.

In general a representation M in $\mathcal{O}$ does not have a finite composition series. However, the multiplicity $[M : L(\lambda)]$ of $L(\lambda)$ in M makes sense ([162]). As a consequence, we have

$$\mathrm{ch}\, M = \sum_{\lambda} [M : L(\lambda)]\, \mathrm{ch}\, L(\lambda), \quad [M : L(\lambda)] \in \mathbb{Z}_{\geqslant 0},$$

A.4.3 Singular vectors

A *singular vector* of a $\mathfrak{g}$-representation M is a vector $v \in M$ such that $\mathfrak{n}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \ldots, r$. A *singular vector* of a $\hat{\mathfrak{g}}$-representation M is a vector $v \in M$ such that $\hat{\mathfrak{n}}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \ldots, r$, and $(f_\theta t).v = 0$, with $f_\theta \in \mathfrak{g}_{-\theta} \setminus \{0\}$.

A.5 Integrable and admissible representations

A.5.1 Integrable representations

The representation $L(\lambda)$, for $\lambda \in \tilde{\mathfrak{h}}^*$, is finite dimensional if and only if $\lambda = 0$, that is, $L(\lambda)$ is the trivial representation. The notion of finite dimensional representations has to be replaced by the notion of integrability in the category $\mathcal{O}$.

Definition A.4 A representation M of $\tilde{\mathfrak{g}}$ is said to be *integrable* if

(1) M is $\tilde{\mathfrak{h}}$-diagonalizable,
(2) for $\lambda \in \tilde{\mathfrak{h}}^*$, M_λ is finite dimensional,
(3) for all $\lambda \in \mathrm{wt}(M)$, for all $i = 0, \ldots, r$, there is $N \geqslant 0$ such that for $m \geqslant N$, $\lambda + m\alpha_i \notin \mathrm{wt}(M)$ and $\lambda - m\alpha_i \notin \mathrm{wt}(M)$.

Remark A.2 As an $\mathfrak{a}_i$-module, $i = 0, \ldots, r$, an integrable representation M decomposes into a direct sum of finite dimensional irreducible $\hat{\mathfrak{h}}$-invariant modules, where

$\mathfrak{a}_i \cong \mathfrak{sl}_2$ is the Lie algebra generated by the Chevalley generators E_i, F_i, H_i. Hence the action of $\mathfrak{a}_i$ on M can be "integrated" to the action of the group $SL_2(\mathbb{C})$.

The character of the simple integrable representations in the category $\mathcal{O}$ satisfy remarkable combinatorial identities (related to Macdonald identities).

A.5.2 Level of a representation

According to the well-known Schur Lemma, any central element of a Lie algebra acts as a scalar on a simple finite dimensional representation L. As the Schur Lemma extends to a representation with countable dimension, the result holds for highest weight $\tilde{\mathfrak{g}}$-modules. In particular, $K \in \tilde{\mathfrak{g}}$ acts as a scalar $k \in \mathbb{C}$ on the simple representations of the category $\mathcal{O}$.

Definition A.5 A representation M is said to be *level* k if K acts as $k\mathrm{Id}$ on M.

All simple representations of the category $\mathcal{O}$ have a level. Namely, $L(\lambda)$ has level $k = \lambda(K) \in \mathbb{C}$, and so $k = \mu(K)$ for all $\mu \in \mathrm{wt}(L(\lambda))$. Note that

$$k = \lambda(K) = \sum_{i=0}^{r} a_i \lambda(\alpha_i^\vee)$$

where the a_i are defined by $K = \sum_{i=0}^{r} a_i \alpha_i^\vee$.

Lemma A.1 *The simple representation $L(\lambda)$ is integrable if and only if λ is dominant and integrable, that is, $\lambda(H_i) \in \mathbb{Z}_{\geqslant 0}$ for all $i = 0, \ldots, r$. It has level 0 if and only if* $\dim L(\lambda) = 1$.

Recall that $\tilde{\mathfrak{h}}^*$ is identified with $\tilde{\mathfrak{h}}$ through $(\ |\ )$, and that through this identification the dual of K is D. Then, as a particular case of Lemma A.1, $L(kD)$ is integrable if and only if $k \in \mathbb{Z}_{\geqslant 0}$.

The category of modules of the category $\mathcal{O}$ of level k will be denoted by $\mathcal{O}_k$ ([158]).

The level $k = -h^\vee$ is particular since the center of $\tilde{U}(\hat{\mathfrak{g}})/\tilde{U}(\hat{\mathfrak{g}})(K-k)$ is large and the representation theory changes drastically at this level. Here, $\tilde{U}(\hat{\mathfrak{g}})$ is the completion of the enveloping algebra $U(\hat{\mathfrak{g}})$. This level is called the *critical level*.

Although the category $\mathcal{O}$ is stable by tensor product, the category $\mathcal{O}_k$ is not stable by tensor product (except for $k = 0$). Indeed from the coproduct, we get that for M_1, M_2 representations in $\mathcal{O}_{k_1}$, $\mathcal{O}_{k_2}$ respectively, the module $M_1 \otimes M_2$ is in $\mathcal{O}_{k_1+k_2}$. This is one motivation to study the *fusion product*; see [43], [145, Section 5] for more details on this topic.

A.5.3 Admissible representations

We now introduce a class of representations, called *admissible representations*, which includes the class of integrable representations. The definition goes back to Kac and Wakimoto [165]. While the notion of integrable representations has a geometrical meaning, the notion of admissible representations is purely combinatorial. However, conjecturally, admissible representations are precisely the representations which satisfy a certain modular invariant property (see below).

Retain the notations of §A.3.3, and recall the definition of the affine and extended affine Weyl groups (see e.g., [167]). Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and extend it to $\hat{\mathfrak{h}}$ by setting $w(K) = K$, $w(D) = D$ for all $w \in W$. Let $Q^\vee = \sum_{i=1}^r \mathbb{Z}\alpha_i^\vee$ be the coroot lattice of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}$, define the translation ([159]),

$$t_\alpha(v) = v + (v|K)\alpha - \left(\frac{1}{2}|\alpha|^2(v|K) + (v|\alpha)\right)K, \qquad v \in \hat{\mathfrak{h}},$$

and for a subset $L \subset \mathfrak{h}$, let

$$t_L := \{t_\alpha : \alpha \in L\}.$$

The *affine Weyl groups* $\hat{W}$ and the *extended affine Weyl group* $\tilde{W}$ are then defined by:

$$\hat{W} := W \ltimes t_{Q^\vee}, \qquad \tilde{W} := W \ltimes t_{P^\vee},$$

so that $\hat{W} \subset \tilde{W}$. Here $P^\vee = \{\lambda \in \mathfrak{h} : \langle \lambda, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in Q\}$, with $Q = \sum_{i=1}^r \mathbb{Z}\alpha_i$ the root lattice.

The group $\tilde{W}_+ := \{w \in \tilde{W} : w(\hat{\Pi}^\vee) = \hat{\Pi}^\vee\}$ acts transitively on orbits of $\mathrm{Aut}\,\hat{\Pi}^\vee$ and simply transitively acts on the orbit of $\alpha_0^\vee$. Moreover $\tilde{W} = \tilde{W}_+ \ltimes \hat{W}$. Here, $\hat{\Pi}^\vee := \{\alpha^\vee : \alpha \in \hat{\Pi}\}$.

Definition A.6 ([165, 167]) A weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if

(1) λ is *regular dominant*, that is,

$$\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \notin -\mathbb{Z}_{\geqslant 0} \quad \text{for all} \quad \alpha \in \hat{\Delta}_+^{\mathrm{re}},$$

(2) the $\mathbb{Q}$-span of $\hat{\Delta}_\lambda$ contains $\hat{\Delta}^{\mathrm{re}}$, where $\hat{\Delta}_\lambda := \{\alpha \in \hat{\Delta}^{\mathrm{re}} : (\lambda|\alpha^\vee) \in \mathbb{Z}\}$.

The irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible. Note that an irreducible integrable representation of $\hat{\mathfrak{g}}$ is admissible.

Proposition A.3 ([167, Prop. 1.2]) *For $k \in \mathbb{C}$, the weight $\lambda = kD$ is admissible if and only if k satisfies one of the following conditions:*

(i) $k = -h^\vee + \dfrac{p}{q}$ *where* $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, *and* $p \geqslant h^\vee$,

(ii) $k = -h^\vee + \dfrac{p}{r^\vee q}$ *where* $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, $(p, r^\vee) = 1$ *and* $p \geqslant h$.

Here $r^\vee$ is the lacety *of $\mathfrak{g}$ (i.e., $r^\vee = 1$ for the types A, D, E, $r^\vee = 2$ for the types B, C, F and $r^\vee = 3$ for the type G_2), h and $h^\vee$ are the Coxeter and dual Coxeter numbers.*

Definition A.7 If k satisfies one of the conditions of Proposition A.3, we say that k is an *admissible level*.

For an admissible representation $L(\lambda)$ we have [164]

$$\mathrm{ch}(L(\lambda)) = \sum_{w \in \hat{W}(\lambda)} (-1)^{\ell_\lambda(w)} \mathrm{ch}(M(w \circ \lambda)) \tag{A.24}$$

since λ is regular dominant, where $\hat{W}(\lambda)$ is the *integral Weyl group* ([180, 218]) of λ, that is, the subgroup of $\hat{W}$ generated by the reflections s_α associated with $\alpha \in \hat{\Delta}_\lambda$, $w \circ \lambda = w(\lambda + \rho) - \rho$, and ℓ_λ is the length function of the Coxeter group $\hat{W}(\lambda)$. Further, Condition (ii) of Proposition A.3 implies that $\mathrm{ch}(L(\lambda))$ is written in terms of certain *theta functions* [159, Chap. 13]. Kac and Wakimoto [165] showed that admissible representations are *modular invariant*, that is, the characters of admissible representations form an $SL_2(\mathbb{Z})$ invariant subspace.

Let λ, μ be distinct admissible weights. Then Condition (1) of Proposition A.3 implies that

$$\mathrm{Ext}^1_{\hat{\mathfrak{g}}}(L(\lambda), L(\mu)) = 0.$$

Further, the following fact is known by Gorelik and Kac [139].

Theorem A.5 *Let λ be admissible. Then*

$$\mathrm{Ext}^1_{\hat{\mathfrak{g}}}(L(\lambda), L(\lambda)) = 0.$$

Therefore admissible representations form a semisimple full subcategory of the category of $\hat{\mathfrak{g}}$-modules.

Appendix B
Poisson algebras, Poisson varieties and Hamiltonian reduction

We have compiled in this appendix some basic facts on Poisson algebras and Poisson varieties.

B.1 Poisson algebras and Poisson varieties

Let A be a commutative associative $\mathbb{C}$-algebra with unit.

Definition B.1 Suppose that A is endowed with an additional $\mathbb{C}$-bilinear bracket $\{-,-\} : A \times A \to A$. Then A is called a *Poisson algebra* if the following conditions holds:

(i) A is a Lie algebra with respect to $\{-,-\}$,
(ii) (*Leibniz rule*) $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$, for all $a, b, c \in A$.

The Lie bracket $\{\ ,\ \}$ is called a *Poisson bracket* on A.

Similarly, one can define the notion of *Poisson superalgebra*: see Appendix E.

Example B.1 Let (X, ω) be a symplectic variety. Then the algebra $(\mathscr{O}(X), \{\ ,\ \})$ of regular functions, with pointwise multiplication, is a Poisson algebra.

As an example, let $\mathfrak{g} = \mathrm{Lie}(G)$ be a complex algebraic finite-dimensional Lie algebra. and pick a coadjoint orbit $\mathbb{O} = G.\xi$ of $\mathfrak{g}^*$. Then $\mathbb{O}$ has a natural structure of symplectic structure, see e.g. [81, Proposition 1.1.5]; for $\xi \in \mathfrak{g}^*$, we have

$$T_\xi(\mathbb{O}) = T_\xi(G/G^\xi) \simeq \mathfrak{g}/\mathfrak{g}^\xi$$

and the bilinear form $\omega_\xi \colon (x, y) \mapsto \xi([x, y])$ descends to $\mathfrak{g}/\mathfrak{g}^\xi$. This gives the symplectic structure. Hence, together with a coadjoint orbit in $\mathfrak{g}^*$, we have a natural Poisson algebra.

B.2 Tensor products of Poisson algebras

B.3 Almost commutative algebras

In another direction, we have examples of Poisson algebras coming from some noncommutative algebras. Let B be an associative filtered (noncommutative) algebra with unit,

$$0 = B_{-1} \subset B_0 \subset B_1 \subset \cdots, \quad \bigcup_{i \geqslant 0} B_i = B,$$

such that $B_i.B_j \subset B_{i+j}$ for any $i, j \geqslant 0$. Let

$$A := \operatorname{gr} B = \bigoplus_i B_i/B_{i-1}$$

be its graded algebra (the multiplication in B gives rise a well-defined product $B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j}/B_{i+j-1}$, making A an associative algebra). Say that B is *almost commutative* if A is commutative: this means that $a_i b_j - b_j a_i \in B_{i+j-1}$ for $a_i \in B_i, b_j \in B_j$.

Assume that B is almost commutative. Then $\operatorname{gr} B$ has a natural structure of Poisson algebra. We define the Poisson bracket

$$\{-,-\} : B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j-1}/B_{i+j-2}$$

as follows: for $a_1 \in B_i/B_{i-1}$ and $a_2 \in B_j/B_{j-1}$, let b_1 (resp. b_2) be a representative of a_1 in B_i (resp. B_j) and set

$$\{a_1, a_2\} := b_1 b_2 - b_2 b_1 \mod B_{i+j-2}\,.$$

Then we can check the required properties.

Example B.2 Let $\mathfrak{g}$ be any complex finite-dimensional Lie algebra. Let $U_\bullet\mathfrak{g}$ be the PBW filtration of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, that is, $U_i(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ spanned by the products of at most i elements of $\mathfrak{g}$, and $U(\mathfrak{g})_0 = \mathbb{C}1$ (see Appendix A). Then

$$0 = U_{-1}(\mathfrak{g}) \subset U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \ldots, \quad U(\mathfrak{g}) = \bigcup_i U_i(\mathfrak{g}),$$

$$U_i(\mathfrak{g}) \cdot U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g}), \quad [U_i(\mathfrak{g}), U_j(\mathfrak{g})] \subset U_{i+j-1}(\mathfrak{g}).$$

The associated graded space $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i \geqslant 0} U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g})$ is naturally a Poisson algebra, and the PBW Theorem states that

$$\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$$

as Poisson algebras, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$.

Let us describe explicitly the Poisson bracket on $\mathbb{C}[\mathfrak{g}^*]$ (see [81, Proposition 1.3.18]). Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$, with structure constants $c_{i,j}^k$, that is, $[x_i, x_j] = \sum_k c_{i,j}^k x_k$. Through the canonical isomorphism $(\mathfrak{g}^*)^* \cong \mathfrak{g}$, any element of $\mathfrak{g}$ is regarded as a linear functions on $\mathfrak{g}^*$, and thus as an element of $\mathbb{C}[\mathfrak{g}^*]$. We get for $f, g \in \mathbb{C}[\mathfrak{g}^*]$,

$$\{f, g\} = \sum_{i,j,k} c_{i,j}^k x_k \frac{\partial f}{\partial x_i^*} \frac{\partial g}{\partial x_j^*}.$$

In a more concise way, we have:

$$\{f, g\} \colon \mathfrak{g}^* \longrightarrow \mathbb{C}, \quad \xi \longmapsto \langle \xi, [d_\xi f, d_\xi g] \rangle$$

where $d_\xi f, d_\xi g \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$ denote the differentials of f and g at ξ. In particular, if $x, y \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*]$, then

$$\{x, y\} = [x, y].$$

Moreover, if $\mathbb{O}$ is a coadjoint orbit of $\mathfrak{g}^*$,

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{\text{symplectic}}.$$

The above Poisson structure on $\mathbb{C}[\mathfrak{g}^*]$ is referred to as the *Kirillov–Kostant–Souriau Poisson structure*.

Example B.3 Let G be an affine algebraic group, $\mathfrak{g} = \operatorname{Lie}(G)$, and $\mathcal{D}(G)$ the algebra of (global) differential operators on G. It is filtered by the order filtration $F_\bullet \mathcal{D}(G)$, see Section C.2. According to Proposition C.1 (i), the filtered algebra $\mathcal{D}(G)$ is almost commutative. It fact, by (C.1), one knows that

$$\operatorname{gr} \mathcal{D}(G) \cong \pi_* \mathcal{O}_{T^*G},$$

where T^*G is the cotangent bundle of G. Thus $\mathcal{O}_{T^*G}$ inherits a Poisson algebra structure from that of $\operatorname{gr} \mathcal{D}(G)$.

One the other hand, T^*G is a symplectic variety and therefore $\mathcal{O}_{T^*G}$ has a Poisson algebra structure arising from its symplectic structure. It turns out that these two Poisson structures coincide (see, for instance, [81, Theorem 1.3.10]).

A *affine Poisson scheme* (resp., *affine Poisson variety)* is an affine scheme $X = \operatorname{Spec} A$ (resp. $X = \operatorname{Specm} A$) such that A is a Poisson algebra. A *Poisson scheme* (resp. *Poisson variety)* is a scheme (resp. reduced scheme) such that the structure sheaf $\mathcal{O}_X$ is a sheaf of Poisson algebras.

For example, let B be as above and continue to assume that B is almost commutative, that is, $A = \operatorname{gr} B$ is commutative. Assume furthermore that A is a finitely generated commutative algebra without zero-divisors. In other words, $A = \mathbb{C}[X]$ is the coordinate ring of a (reduced) irreducible affine algebraic variety X. So the Poisson structure on A makes X a Poisson variety.

B.4 Symplectic leaves

If X is smooth, then one may view X as a complex-analytic manifold equipped with a holomorphic Poisson structure. For each point $x \in X$ one defines the *symplectic leaf* $\mathscr{L}_x$ through x to be the set of points that could be reached from x by going along Hamiltonian flows[1].

If X is not necessarily smooth, let $\mathrm{Sing}(X)$ be the singular locus of X, and for any $k \geqslant 1$ define inductively $\mathrm{Sing}^k(X) := \mathrm{Sing}(\mathrm{Sing}^{k-1}(X))$. We get a finite partition $X = \bigsqcup_k X^k$, where the strata $X^k := \mathrm{Sing}^{k-1}(X) \setminus \mathrm{Sing}^k(X)$ are smooth analytic varieties (by definition we put $X^0 = X \setminus \mathrm{Sing}(X)$). It is known (cf. e.g., [69]) that each X^k inherits a Poisson structure. So for any point $x \in X^k$ there is a well defined symplectic leaf $\mathscr{L}_x \subset X^k$. In this way one defines symplectic leaves on an arbitrary Poisson variety. In general, each symplectic leaf is a connected smooth analytic (but not necessarily algebraic) subset in X. However, if the algebraic variety X consists of finitely many symplectic leaves only, then it was shown in [69] that each leaf is a smooth irreducible locally-closed algebraic subvariety in X, and the partition into symplectic leaves gives an algebraic stratification of X.

Example B.4 If $\mathfrak{g} = \mathrm{Lie}(G)$ is an algebraic Lie algebra, the space $\mathfrak{g}^*$ is a (smooth) Poisson variety and the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits of $\mathfrak{g}^*$, cf. [248, Proposition 3.1].

If $\mathfrak{g}$ is simple, the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$, which is the (reduced) subscheme of $\mathfrak{g}^*$ associated with the augmentation ideal $\mathbb{C}[\mathfrak{g}^*]^G_+$ of the ring of invariants $\mathbb{C}[\mathfrak{g}^*]$, is an example of Poisson variety with finitely many symplectic leaves. These are precisely the nilpotent orbits of $\mathfrak{g}^* \cong \mathfrak{g}$.

B.5 Induced Poisson structures and Hamiltonian reduction

There are roughly two ways to construct a new Poisson variety from a Poisson manifold X: the *induction* and the *Hamiltonian reduction*. First recall a result of Weinstein about the induction; see [248, Proposition 3.10]:

Theorem B.1 (Weinstein) *Let Y be a submanifold of a Poisson manifold X such that:*

(i) *Y is transversal to the symplectic leaves, i.e., for any symplectic leaf S and any $x \in Y \cap S$, $T_xY + T_xS = T_xX$,*
(ii) *for any $x \in Y$, $T_xY \cap T_xS$ is a symplectic subspace of T_xS, where S is the leaf of X containing x.*

---

[1] A *Hamiltonian flow* in X from x to x' is a curve γ defined on an open neighborhood of $[0, 1]$ in $\mathbb{C}$, with $\gamma(0) = x$ and $\gamma(1) = x'$, which is an integral curve of a Hamiltonian vector field ξ_f, for some $f \in \mathscr{O}(X)$, defined on an open neighborhood of $\gamma([0, 1])$. We refer to [189, Chapter 1] for more details.

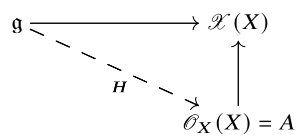
Then, there is a natural induced Poisson structure on Y, and the symplectic leaf of Y through $x \in Y$ is $Y \cap S$ if S is the symplectic leaf through x in X.

We now follow [189, §5.2,5.4] for the algebraic treatment of Hamiltonian reduction. Fix $(A, \{-,-\})$ a Poisson algebra. Set $X = \mathrm{Spec}(A)$ and let G be an affine algebraic group with a left action $G \times X \to X$. Assume that the corresponding left action of G on A is by Poisson automorphisms. We obtain an action by derivations on A of the Lie algebra $\mathfrak{g}$ of G, which is denoted by $y.f$ for $y \in \mathfrak{g}$ and $f \in A$. Note that $y.(A^G) = 0$.

Definition B.2 The action of G in X is said to be *Hamiltonian* if there is a Lie algebra homomorphism

$$H\colon \mathfrak{g} \longrightarrow \mathscr{O}_X(X) = A, \quad x \longmapsto H_x$$

such that the following diagram is commutative:



where $\mathscr{X}(X)$ is the Lie algebra of vector fields on X and the vertical map is the natural map from A to $\mathscr{X}(X)$ given by $f \mapsto \{f,-\}$. As for the horizontal map, it comes from the G-action on X. Namely, it is the map

$$\mathfrak{g} \longrightarrow \mathscr{X}(X), \quad y \longmapsto \Big(x \mapsto \frac{d}{dt}(\exp(t \operatorname{ad} y).x)|_{t=0} \in T_x X\Big).$$

The map $H\colon \mathfrak{g} \to A$ is called the *Hamiltonian*. Define the *moment map*

$$\mu\colon X \longrightarrow \mathfrak{g}^*$$

by assigning to $x \in X$ the linear function $\mu(x)\colon \mathfrak{g} \to \mathbb{C}$, $a \mapsto H_a(x)$. The moment map induces a Poisson algebra homomorphism, called the *comoment map*

$$\mu^*\colon \mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathscr{O}_X(X) = A.$$

Moroever, if the group G is connected, then μ is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$.

The restriction to $\mathfrak{g}$ of the comoment map is the morphism of Lie algebras

$$\tilde{\mu}\colon \mathfrak{g} \to A$$

which is equivariant (for the adjoint action on $\mathfrak{g}$) and satisfies

$$x.F = \{\tilde{\mu}(x), f\}, \qquad x \in \mathfrak{g},\ f \in A.$$

We refer to [248, Theorem 7.31] or [189, Proposition 5.39 and Definition 5.9] for the following result.

Theorem B.2 (Marsden–Weinstein) *Assume that G is connected and that the action of G in X is Hamiltonian. Let $\gamma \in \mathfrak{g}^*$. Assume that γ is a regular value*[2] *of μ, that $\mu^{-1}(\gamma)$ is G-stable and that $\mu^{-1}(\gamma)/G$ is a variety. Let $\iota\colon \mu^{-1}(\gamma) \hookrightarrow X$ and $\pi\colon \mu^{-1}(\gamma) \twoheadrightarrow \mu^{-1}(\gamma)/G$ be the natural maps: ι is the inclusion and π is the quotient map. Then the triple*

$$(X, \mu^{-1}(\gamma), \mu^{-1}(\gamma)/G)$$

is Poisson-reducible*, i.e., there exists a Poisson structure $\{-,-\}'$ on $\mu^{-1}(\gamma)/G$ such that for all open subset $U \subset X$ and for all $f, g \in \mathscr{O}_X(\pi(U \cap \mu^{-1}(\gamma))$, on has*

$$\{f, g\}' \circ \pi(u) = \{\tilde{f}, \tilde{g}\} \circ \iota(u)$$

at any point $u \in U \cap \mu^{-1}(\gamma)$, where $\tilde{f}, \tilde{g} \in \mathscr{O}_X(U)$ are arbitrary extensions of $f \circ \pi|_{U\cap\mu^{-1}(\gamma)}, g \circ \pi|_{U\cap\mu^{-1}(\gamma)}$ to U.

B.6 Poisson modules

Let R be a Poisson algebra. A *Poisson R-module* is a R-module M in the usual associative sense equipped with a bilinear map

$$R \times M \to M, \quad (r, m) \mapsto \operatorname{ad} r(m) = \{r, m\},$$

which makes M a Lie algebra module over R satisfying

$$\{r_1, r_2 m\} = \{r_1, r_2\}m + r_2\{r_1, m\}, \quad \{r_1 r_2, m\} = r_1\{r_2, m\} + r_2\{r_1, m\}$$

for $r_1, r_2 \in R$, $m \in M$.

Lemma B.1 *For any Lie algebra $\mathfrak{g}$, a Poisson module over $\mathbb{C}[\mathfrak{g}^*]$ is the same as a $\mathbb{C}[\mathfrak{g}^*]$-module N in the usual associative sense equipped with a Lie algebra module structure $\mathfrak{g} \to \operatorname{End} M$, $x \mapsto \operatorname{ad}(x)$, such that*

$$\operatorname{ad}(x)(fm) = \{x, f\}.m + f.\operatorname{ad}(x)(m)$$

for $x \in \mathfrak{g}$, $f \in \mathbb{C}[\mathfrak{g}^*]$, $m \in M$.

Example B.5 If $\mathfrak{g} = \operatorname{Lie}(G)$ is a simple Lie algebra, let $\overline{\mathcal{HC}}(\mathfrak{g})$ be the full subcategory of the category of Poisson $\mathbb{C}[\mathfrak{g}^*]$-modules on which the Lie algebra $\mathfrak{g}$-action is *integrable*, that is, locally finite. If X is an affine Poisson scheme equipped with a Hamiltonian G-action, then $\mathbb{C}[X]$ is an object of $\overline{\mathcal{HC}}(\mathfrak{g})$. Note that the action of $\mathbb{C}[\mathfrak{g}^*]$ on $\mathbb{C}[X]$ is given by $\{f, g\} = \{\mu^*(f), g\}$, for $f \in \mathbb{C}[\mathfrak{g}^*]$ and $g \in \mathbb{C}[X]$, where $\mu^*\colon \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[X]$ is the comorphism of the moment map $\mu\colon X \to \mathfrak{g}^*$.

---

[2] If $f : X \to Y$ is a smooth map between varieties, we say that a point y is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $d_x f : T_x(X) \to T_y(Y)$ is surjective. If so, then $f^{-1}(y)$ is a subvariety of X and the codimension of this variety in X is equal to the dimension of Y.

Appendix C
Differential operators

In this chapter, $X = \operatorname{Spec} A$ is an affine algebraic variety over the complex number field of dimension n. We are particularly interested in the case where X is an affine algebraic group G.

Our main references are [149, 208].

C.1 Tangent sheaf and cotangent sheaf

Let $\mathscr{O}_X$ be the sheaf of rings of regular functions, that is, the structure sheaf on X. We denote briefly the algebra $\mathscr{O}_X(X)$ of global sections by $\mathscr{O}(X)$.

We say that a section $\theta \in (\operatorname{End}_{\mathbb{C}} \mathscr{O}_X)(X)$ is a *vector field* on X if for each open subset $U \subset X$, $\theta(U) := \theta|_U \in (\operatorname{End}_{\mathbb{C}} \mathscr{O}_X)(U)$ satisfies the condition

$$\theta(U)(fg) = \theta(U)(f)g + f\theta(U)(g), \qquad f, g \in \mathscr{O}_X(U).$$

For an open subset U of X, denote the set of vector fields θ on U by $\Theta(U)$. Then $\Theta(U)$ is an $\mathscr{O}_X(U)$-module, and the presheaf $U \mapsto \Theta(U)$ turns out to be a sheaf of $\mathscr{O}_X$-modules. We denote this sheaf by Θ_X and call it the *tangent sheaf* of X. Thus

$$\Theta_X = \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_X).$$

It is a coherent sheaf of $\mathscr{O}_X$-modules. Indeed, if $X = \operatorname{Spec} A$, with $A = \mathbb{C}[x_1, \ldots, x_n]/I$, with I an ideal of $\mathbb{C}[x_1, \ldots, x_n]$, then

$$\operatorname{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n]) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n]\partial_i, \quad \text{where} \quad \partial_i := \frac{\partial}{\partial x_i},$$

is a free $\mathbb{C}[x_1, \ldots, x_n]$-module of rank n, and

$$\operatorname{Der}_{\mathbb{C}}(A) \cong \{\theta \in \operatorname{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n]) \colon \theta(I) \subset I\}.$$

Hence $\mathrm{Der}_{\mathbb{C}}(A)$ is finitely generated over A.

We define the *cotangent sheaf* of X by $\Omega^1_X = \delta^{-1}(\mathcal{I}/\mathcal{I}^2)$, where $\delta\colon X \to X \times X$ is the diagonal embeddings, $\mathcal{I}$ is the ideal sheaf of $\delta(X)$ in $X \times X$ defined by

$$\mathcal{I}(V) = \{f \in \mathcal{O}_{X\times X}(V)\colon f(V \cap \delta(X)) = 0\}$$

for any open subset V of $X \times X$, and δ^{-1} stands for the sheaf-theoretical inverse image functor. (We usually keep the notation Ω_X for the sheaf $\bigwedge^n \Omega^1_X$ of differential forms of top degree.)

Sections of the sheaf Ω^1_X are called *differential forms*. By the canonical morphism $\mathcal{O}_X \to \delta^{-1}\mathcal{O}_{X\times X}$ of sheaf of $\mathbb{C}$-algebras, Ω^1_X is naturally an $\mathcal{O}_X$-module.

We have a morphism $d\colon \mathcal{O}_X \to \Omega^1_X$ of $\mathcal{O}_X$-modules defined by

$$df = f \otimes 1 - 1 \otimes f \mod \delta^{-1}\mathcal{I}^2.$$

It satisfies $d(fg) = d(f)g + f(dg)$ for any $f, g \in \mathcal{O}_X$.

We denote briefly the $\mathcal{O}(X)$-modules $\Theta_X(X) = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(X))$ and $\Omega^1_X(X)$ by $\Theta(X)$ and $\Omega^1(X)$, respectively.

Thus $\Omega^1(X) = \mathcal{I}(X)/\mathcal{I}(X)^2$, and $\mathcal{I}(X)$ is the kernel of the morphism

$$\varepsilon\colon \mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(X) \longrightarrow \mathcal{O}(X), \quad f \otimes g \longmapsto fg.$$

The $\mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(X)$-modules $\mathcal{I}(X)$, $\mathcal{I}(X)^2$ and $\Omega^1(X)$ are viewed as $\mathcal{O}(X)$-modules via the homomorphism $\mathcal{O}(X) \to \mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(X)$, $f \mapsto f \otimes 1$.

In $\mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(X)$ we have

$$f \otimes g = fg \otimes 1 + f(1 \otimes g - g \otimes 1) = \varepsilon(f \otimes g) + f(dg \mod \mathcal{I}(X)^2).$$

Therefore, if $\sum_i f_i \otimes g_i \in \mathcal{I}(X) = \ker \varepsilon$, then

$$\sum_i f_i \otimes g_i = \sum_i f_i dg_i \mod \mathcal{I}(X)^2,$$

and so any element of $\Omega^1(X) = \mathcal{I}(X)/\mathcal{I}(X)^2$ has the form $\sum_i f_i dg_i$, for $f_i, g_i \in \mathcal{O}(X)$.

In conclusion, we obtain the following fact.

Lemma C.1 *As $\mathcal{O}(X)$-module, Ω^1_X is generated by df, for $f \in \mathcal{O}(X)$.*

For $\alpha \in \mathrm{Hom}_{\mathcal{O}(X)}(\Omega^1_X, \mathcal{O}_X)$ we have $\alpha \circ d \in \Theta_X$, which gives an isomorphism

$$\mathrm{Hom}_{\mathcal{O}(X)}(\Omega^1_X, \mathcal{O}_X) \cong \Theta_X$$

as $\mathcal{O}_X$-modules.

Theorem C.1 *Assume that X is smooth. For each point $x \in X$, there exist an affine open neighbourhood V of x, regular functions $x_i \in \mathcal{O}_X(V)$, and vector fields $\partial_i \in \Theta_X(V)$, for $i \in \{1, \ldots, n\}$, satisfying the conditions:*

$$[\partial_i, \partial_j] = 0, \quad \partial_i(x_j) = \delta_{i,j}, \quad 1 \leqslant i, j \leqslant n,$$
$$\Theta_V = \bigoplus_{i=1}^{n} \mathscr{O}_V \partial_i.$$

Moreover, one can choose the functions $x_1, x_2, \dots, x_n$ *so that they generate the maximal ideal* $\mathfrak{m}_x$ *of the local ring* $\mathscr{O}_{X,x}$ *at* x.

Proof Since the local ring $\mathscr{O}_{X,x}$ is regular, there exist $n = \dim X$ functions $x_1, \dots, x_n \in \mathfrak{m}_x$ generating the ideal $\mathfrak{m}_x$. Then $dx_1, \dots, dx_n$ is a basis of the free $\mathscr{O}_{X,x}$-module $\Omega^1_{X,x}$. Hence we can take an affine open neighbourhood V of x such that $\Omega^1_X(V)$ is a free module with basis $dx_1, \dots, dx_n$ over $\mathscr{O}_X(V)$. Taking the dual basis $\partial_1, \dots, \partial_n \in \Theta_X(V) \cong \mathrm{Hom}_{\mathscr{O}_X(V)}(\Omega^1_X(V), \mathscr{O}_X(V))$ we get $\partial_i(x_j) = \delta_{i,j}$. Write $[\partial_i, \partial_j]$ as $[\partial_i, \partial_j] = \sum_{l=1}^{n} g^l_{i,j} \partial_l \in \mathscr{O}_X(V)$. Then we have $g^l_{i,j} = [\partial_i, \partial_j] x_l = \partial_i \partial_j x_l - \partial_j \partial_i x_l = 0$. Hence $[\partial_i, \partial_j] = 0$. □

The set $\{x_i, \partial_i : 1 \leqslant i \leqslant n\}$ defined over an affine open neighborhood of x satisfying the conditions of Theorem C.1 is called a *local coordinate system* at x.

C.2 Sheaf of differential operators

We define the sheaf $\mathcal{D}_X$ as the sheaf of $\mathbb{C}$-subalgebras of $\mathrm{End}_{\mathbb{C}}(\mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X. Here we identify $\mathscr{O}_X$ with a subsheaf of $\mathrm{End}_{\mathbb{C}}(\mathscr{O}_X)$ by identifying $f \in \mathscr{O}_X$ with the element $g \mapsto fg$ of $\mathrm{End}_{\mathbb{C}}(\mathscr{O}_X)$.

We call the sheaf $\mathcal{D}_X$ the *sheaf of differential operators* on X. For any point of X we can take an affine open neighborhood U and a local coordinate system $\{x_i, \partial_i : 1 \leqslant i \leqslant n\}$. Hence we have

$$\mathcal{D}_U := \mathcal{D}_X(U) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} \mathscr{O}_U \partial_x^\alpha, \quad \partial_x^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

We define the *order filtration* $F_\bullet \mathcal{D}_U$ of $\mathcal{D}_U$ by

$$F_l \mathcal{D}_U = \sum_{|\alpha| \leqslant l} \mathscr{O}_U \partial_x^\alpha, \qquad l \in \mathbb{Z}_{\geqslant 0}, \ |\alpha| := \sum_i \alpha_i.$$

More generally, for an arbitrary open subset V of X we define the order filtration $F_\bullet \mathcal{D}_X$ over V by

$$(F_l \mathcal{D}_X)(V) = \{P \in \mathcal{D}_X(V) : \mathrm{res}^V_U P \in (F_l \mathcal{D}_X)(U) \text{ for any affine open subset } U \text{ of } V\},$$

where $\mathrm{res}^V_U : \mathcal{D}_X(V) \to \mathcal{D}_X(U)$ is the restriction map.

For convenience we set $F_p \mathcal{D}_X = 0$ for $p < 0$.

Proposition C.1

(i) *$F_\bullet \mathcal{D}_X$ is an increasing filtration of $\mathcal{D}_X$ such that $\mathcal{D}_X = \bigcup_{l \geqslant 0} F_l \mathcal{D}_X$ and each $F_l \mathcal{D}_X$ is a locally free module over $\mathscr{O}_X$.*
(ii) *$F_0 \mathcal{D}_X := \mathscr{O}_X$ and $(F_l \mathcal{D}_X)(F_m \mathcal{D}_X) = F_{l+m} \mathcal{D}_X$.*
(iii) *$[F_l \mathcal{D}_X, F_m \mathcal{D}_X] \subset F_{l+m-1} \mathcal{D}_X$.*

Remark C.1 One can alternatively define $F_\bullet \mathcal{D}_X$ by the recursive formula:

$$F_l \mathcal{D}_X = \{P \in \operatorname{End}_{\mathbb{C}}(\mathscr{O}_X) \colon [P, f] \in F_{l-1} \mathcal{D}_X \text{ for all } f \in \mathscr{O}_X\}, \quad l \in \mathbb{Z}_{\geqslant 0}.$$

Let us consider the sheaf of graded rings

$$\operatorname{gr} \mathcal{D}_X = \operatorname{gr}^F \mathcal{D}_X = \bigoplus_{l \geqslant 0} \operatorname{gr}_l \mathcal{D}_X,$$

where $\operatorname{gr}_l \mathcal{D}_X := F_l \mathcal{D}_X / F_{l-1} \mathcal{D}_X$, $F_{-1} \mathcal{D}_X = 0$. By Proposition C.1 (iii), $\operatorname{gr} \mathcal{D}_X$ is a sheaf of commutative algebras finitely generated over $\mathscr{O}_X$. Take an affine chart U with a coordinate system $\{x_i, \partial_i\}$ and set

$$\xi_i := (\partial_i \mod F_0 \mathcal{D}_U = \mathscr{O}_U) \in \operatorname{gr}_1 \mathcal{D}_U.$$

Then we have

$$\operatorname{gr}_l \mathcal{D}_U = \bigoplus_{|\alpha| = l} \mathscr{O}_U \xi^\alpha,$$
$$\operatorname{gr} \mathcal{D}_U = \mathscr{O}_U [\xi_1, \dots, \xi_n].$$

We can globalize this notion as follows. Let $T^* X$ be the cotangent bundle of X and let $\pi \colon T^* X \to X$ be the projection. We may regard $\xi_1, \dots, \xi_n$ as the coordinate system of the cotangent space $\bigoplus_{i=1}^n \mathbb{C} dx_i$ and hence $\mathscr{O}_U [\xi_1, \dots, \xi_n]$ is canonically identified with the sheaf $\pi_* \mathscr{O}_{T^* X}$ of algebras. Thus we obtain a canonical identification

$$\operatorname{gr} \mathcal{D}_X \cong \pi_* \mathscr{O}_{T^* X}. \tag{C.1}$$

The algebra $\mathcal{D}(X) := \mathcal{D}_X(X)$ is called the *algebra of differential operators* on X.

C.3 Derivations and differential forms on a group

Let G be an affine algebraic group. By definition, the Lie algebra of G is the Lie algebra of *left invariant vector fields* on G, that is,

$$\operatorname{Lie}(G) = \{\theta \in \operatorname{Der}_{\mathbb{C}}(\mathscr{O}(G)) \colon \Delta \circ \theta = (1 \otimes \theta) \circ \Delta\},$$

(see e.g. [215, Proposition 10.29]), where $\Delta\colon \mathscr{O}(G) \to \mathscr{O}(G) \otimes \mathscr{O}(G)$ is the coproduct induced by the multiplication G. Thus, $\theta \in \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ is in $\mathrm{Lie}(G)$ if and only if for all $g \in G$, $\lambda_g \theta = \theta \lambda_g$, where $(\lambda_g f)(y) = f(g^{-1}y)$ for $f \in \mathscr{O}(G)$ and $y \in G$.

The Lie algebra of G is canonically isomorphic, as a Lie algebra, to the Lie algebra $\mathrm{Lie}_r(G)$ of right invariant vector fields θ, that is, the Lie algebra consisted of $\theta \in \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ such that $\rho_g \theta = \theta \rho_g$, where $(\rho_g f)(y) = f(yg)$ for $f \in \mathscr{O}(G)$ and $y \in G$. It is also canonically isomorphic to $T_e(G)$, the tangent space at the identity to G, via the isomorphism,

$$\mathrm{Lie}(G) \longrightarrow T_e(G),$$

sending $\theta \in \mathrm{Lie}(G)$ to $\mathrm{ev}_e \circ \theta$, where ev_e is the evaluation map at the neutral element e, in which we have identified $\Theta(G) = \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ with the tangent bundle TG. We denote by $\mathfrak{g}$ this Lie algebra.

Thus, we have

$$TG \cong G \times \mathfrak{g} \quad \text{and} \quad T^*G \cong G \times \mathfrak{g}^*.$$

For $x \in \mathfrak{g}$, we write x_L (resp., x_R) the corresponding left (resp., right) invariant vector field on G. Note that $(x_L f)(a) = x(\lambda_{a^{-1}} f)$ for $f \in \mathscr{O}(G)$ and $a \in G$.

Remark C.2 Concretely, viewing G as a complex analytic space, we have for $x \in \mathfrak{g}$ and $f \in \mathscr{O}(G)$,

$$(x_L f)(a) = \frac{d}{dt}\bigg|_{t=0} f(a \exp(tx)), \qquad a \in G,$$
$$(x_R f)(a) = \frac{d}{dt}\bigg|_{t=0} f(\exp(tx) a), \qquad a \in G,$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map.

The embedding $\mathfrak{g} \hookrightarrow \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$, $x \mapsto x_L$, induces an isomorphism of left $\mathscr{O}(G)$-modules

$$\mathscr{O}(G) \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)). \tag{C.2}$$

Indeed, both sides are free $\mathscr{O}(G)$-modules of rank the dimension of $\mathfrak{g}$ since G is smooth.

We denote by $\langle -, - \rangle\colon \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)) \times \Omega^1(G) \to \mathscr{O}(G)$ the canonical $\mathscr{O}(G)$-bilinear pairing.

Let us collect some useful identities. Let $\{x^1, \ldots, x^d\}$ be a basis of $\mathfrak{g}$, and $\{\omega^1, \ldots, \omega^d\}$ the dual $\mathscr{O}(G)$-basis of $\Omega^1(G)$. Write

$$[x^i, x^j] = \sum_p c_p^{i,j} x^p, \quad \text{for} \quad i, j = 1, \ldots, d,$$

with $(c_p^{i,j}) \in \mathbb{C}$. The isomorphism (C.2) tells that $\{x^1, \ldots, x^d\}$ forms an $\mathscr{O}(G)$-basis of $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$. In particular,

(C.3)
$$x^i_R = \sum_p f^{i,p} x^p, \quad i = 1, \dots, d,$$

for some invertible matrix $(f^{i,p})_{1\leqslant i,p\leqslant d}$ over $\mathscr{O}(G)$.

Lemma C.2 *The following identities hold:*

(i) *for all* $i, j, s \in \{1, \dots, d\}$,
$$x^i_L f^{j,s} + \sum_p c^{i,p}_s f^{j,p} = 0,$$

(ii) *for all* $i, j, s \in \{1, \dots, d\}$,
$$\sum_p f^{i,p}(x^p_L f^{j,s}) = \sum_q c^{i,j}_q f^{q,s}.$$

Proof The identities of (i) hold because they are equivalent to the commutation relations

(C.4)
$$[x^i_L, x^j_R] = 0$$

for all i, j, s.

To prove (ii), we write down the relations

(C.5)
$$[x^i_R, x^j_R] = [x^i, x^j]_R,$$

for $i, j = 1, \dots, d$, in coordinates. We have
$$[x^i_R, x^j_R] = \sum_s [x^i_R, f^{j,s} x^s] = \sum_s (x^i_R f^{j,s}) x^s = \sum_{s,p} f^{i,p}(x^p_L f^{j,s}) x^s$$

by (C.4) and (C.3). Plugging this into (C.5), we get the identities of (ii). □

The Lie algebra $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ acts on $\Omega^1(G)$ by the Lie derivative as follows:

(C.6)
$$((\mathrm{Lie}\,\theta).\omega)(\theta_1) = \theta(\langle \theta_1, \omega\rangle) - \langle [\theta, \theta_1], \omega\rangle,$$

for $\omega \in \Omega^1_G$ and $\theta, \theta_1 \in \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$. (In fact, the Lie algebra $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ acts on $\Omega(G) = \bigwedge^d \Omega^1(G)$ by the Lie derivative; see [149, §1.2].)

So for $i, j = 1, \dots, d$, we have
$$(\mathrm{Lie}\, x^i).\omega^j = \sum_s \alpha^{i,j}_s \omega^p,$$

for some $\alpha^{i,j}_s \in \mathbb{C}$.

Lemma C.3 *The following identities hold:*

(i) *for all* $i, j \in \{1, \dots, d\}$,

$$(\mathrm{Lie}\, x^i).\omega^j = \sum_s c_j^{s,i} \omega^s,$$

(ii) *for all* $i, j \in \{1, \dots, d\}$,

$$(\mathrm{Lie}\, x_R^i).\omega^j = 0.$$

Proof For $i, j, s \in \{1, \dots, d\}$, we have

$$\alpha_s^{i,j} = \langle x^s, (\mathrm{Lie}\, x^i).\omega^j \rangle = (\mathrm{Lie}\, x^i).\langle x^s, \omega^j \rangle + \langle [x^s, x^i], \omega^j \rangle = c_j^{s,i},$$

whence (i).

Similarly, for $i, j, s \in \{1, \dots, d\}$, we have

$$\begin{aligned} \langle x_R^s, (\mathrm{Lie}\, x_R^i).\omega^j \rangle &= (\mathrm{Lie}\, x_R^i).\langle x_R^s, \omega^j \rangle + \langle [x^s, x^i]^R, \omega^j \rangle \\ &= \sum_p f^{s,p}(x_L^p f^{i,j}) + \sum_p c_p^{i,s} f^{p,s} = 0 \end{aligned}$$

by (C.3) and Lemma C.2, whence (ii). whence (ii). □

By the Frobenius reciprocity, we have

$$\mathrm{Hom}_{\mathscr{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)), \mathscr{O}(G)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$$

since $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)) \cong \mathscr{O}(G) \otimes_{\mathbb{C}} \mathfrak{g}$. Hence, as a $\mathbb{C}$-vector spaces,

$$\Omega^1(G) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G)).$$

The above isomorphism has to be understand as follows. Write $\omega \in \Omega^1(G)$ as $\omega = \sum_i f_i dg_i$ by Lemma C.1. To such an ω we attach the element of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$ which maps an element $x \in \mathfrak{g}$ to $\sum_i f_i(x_L g_i) \in \mathscr{O}(G)$.

As a consequence we obtain the following proposition.

Proposition C.2 *The linear map from* Ω_G^1 *to* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$ *sending* dg *to the element* $x \mapsto x_L g$ *of* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$ *is an isomorphism of* $\mathscr{O}(G)$*-modules.*

C.4 The algebra of differential operators on a group

We keep the notation of the previous section.

Let $\mathcal{D}(G)$ be the algebra of differential operators on G. We have a natural embedding

$$\mathscr{O}(G) \hookrightarrow \mathcal{D}(G).$$

Moreover, from the embedding $\mathfrak{g} \hookrightarrow \Theta(G)$, $x \mapsto x_L$, given by the left invariant vector fields, we get an embedding

$$U(\mathfrak{g}) \hookrightarrow \mathcal{D}(G),$$

where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$ (see Appendix A). This induces a map

(C.7) $$\iota\colon U(\mathfrak{g}) \otimes \mathscr{O}(G) \hookrightarrow \mathcal{D}(G)$$

of $\mathscr{O}(G)$-modules. We have a structure of G-equivariant sheaf on both sides, with respect to the left translation action of G on itself. The G-equivariant structure on the left-hand-side comes from the G-action on $\mathscr{O}(G)$ induces by the left translation action of G on itself, that is, the G-action on $U(\mathfrak{g})$ is trivial; the G-action on the right-hand-side is described as follows: for $g \in G$, $f \in \mathscr{O}(G)$ and $\theta \in \mathfrak{g} \subset \Theta(G)$ then $g.(f\theta) = (g.f)\theta$.

Let $\mathcal{D}_l(G)$ be the algebra of left invariant differential operators on G, that is, the algebra of elements $\alpha \in \mathcal{D}(G)$ such that for all $g \in G$ and all $f \in \mathscr{O}(G)$, $\lambda_g(\alpha f) = \alpha(\lambda_g f)$.

Proposition C.3 *The map ι induces an isomorphism of $\mathscr{O}(G)$-modules,*

$$\mathscr{O}(G) \otimes U(\mathfrak{g}) \cong \mathcal{D}(G).$$

Moreover,

$$U(\mathfrak{g}) \cong \mathcal{D}_l(G) \cong \mathcal{D}(G)^G.$$

Proof Let us first show that ι is an isomorphism. The algebra $\mathcal{D}(G)$ is filtered by the order filtration $F_\bullet\mathcal{D}(G)$. On the other hand, the PBW filtration $F_\bullet U(\mathfrak{g})$ on $U(\mathfrak{g})$ induces a filtration $F_\bullet(\mathscr{O}(G) \otimes U(\mathfrak{g}))$ on $\mathscr{O}(G) \otimes U(\mathfrak{g})$ by setting

$$F_l(\mathscr{O}(G) \otimes U(\mathfrak{g})) := \mathscr{O}(G) \otimes F_l U(\mathfrak{g}), \quad l \in \mathbb{Z}_{\geqslant 0}.$$

The map ι sends $F_l(\mathscr{O}(G) \otimes U(\mathfrak{g}))$ to $F_l\mathcal{D}(G)$, and both filtrations are exhaustive. So it suffices to check that the map on associated graded space is an isomorphism. The associated graded of the right-hand-side is

$$\mathscr{O}_{T^*G} \cong \mathscr{O}_{G\times\mathfrak{g}^*} \cong \mathscr{O}(G) \otimes \mathscr{O}(\mathfrak{g}^*),$$

by (C.1), while by the PBW theorem the associated graded of the left-hand-side is

$$\mathscr{O}(G) \otimes S(\mathfrak{g}),$$

whence the statement since $\mathscr{O}(\mathfrak{g}^*) \cong S(\mathfrak{g})$.

Next, since the map ι is G-equivariant,

$$(\mathcal{D}(G))^G \cong (\mathscr{O}(G) \otimes U(\mathfrak{g}))^G \cong \mathscr{O}(G)^G \otimes U(\mathfrak{g}) \cong U(\mathfrak{g}) \hookrightarrow \mathcal{D}_l(G).$$

To show the other inclusion, observe that $\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \mathscr{O}(\mathfrak{g}^*)$ while

$$\operatorname{gr} \mathcal{D}_l(G) \cong (\mathscr{O}(G) \otimes \mathscr{O}(\mathfrak{g}^*))^G \cong \mathscr{O}(G)^G \otimes \mathscr{O}(\mathfrak{g}^*) \cong \mathscr{O}(\mathfrak{g}^*),$$

where G acts on on $\mathscr{O}(G)$ by λ_g, $g \in G$, and trivially on $\mathscr{O}(\mathfrak{g}^*)$. Hence, $\mathcal{D}_l(G) \cong (\mathcal{D}(G))^G \cong U(\mathfrak{g})$ as desired.

Appendix D
Nilpotent orbits of a simple Lie algebra

Let G be a complex connected, simple algebraic group of adjoint type with Lie algebra $\mathfrak{g}$, We keep all the related notations used in Appendix A.

D.1 Nilpotent cone

Our main references for the results of this section are [153, 82, 245].

Let $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ be the *nilpotent cone of* $\mathfrak{g}$, that is, the set of all nilpotent elements of $\mathfrak{g}$:

$$\mathcal{N} = \{x \in \mathfrak{g} \colon (\operatorname{ad} x)^m = 0 \text{ for some } m\}.$$

If $\mathfrak{g}$ is a simple Lie algebra of matrices, the nilpotent cone $\mathcal{N}$ coincides with the set of nilpotent matrices of $\mathfrak{g}$. For $e \in \mathfrak{g}$, we denote by $G.e = \{(\operatorname{Ad} g)(e) \colon g \in G\}$ its adjoint G-orbit. The nilpotent cone is a finite union of nilpotent G-orbits and it is itself the closure of the *regular nilpotent orbit*, denoted by $\mathbb{O}_{\mathrm{reg}}$. It is the unique nilpotent orbit of codimension the rank r of $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is *regular* if its centralizer $\mathfrak{g}^x$ has the minimal dimension, that is, the rank ℓ of $\mathfrak{g}$. Thus, $\mathbb{O}_{\mathrm{reg}}$ is the set of all regular nilpotent elements of $\mathfrak{g}$. Regular nilpotent elements are sometimes called *principal* and the regular nilpotent orbit the *principal nilpotent orbit*.

Example D.1 If $\mathfrak{g} = \mathfrak{sl}_n$, then the rank of $\mathfrak{g}$ is $n-1$ and $\mathbb{O}_{\mathrm{reg}}$ is the conjugacy class of the n-size Jordan block J_n, i.e., $\mathbb{O}_{\mathrm{reg}} = \{gJ_ng^{-1} \colon g \in \mathrm{SL}_n\}$ with

$$J_n := \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = \sum_{i=1}^{n-1} e_{i,i+1},$$

where $e_{i,j}$ is the elementary matrix whose entries are all zero, except the one in position (i, j) which equals 1.

Next, there is a unique dense open orbit in $\mathcal{N} \setminus \mathbb{O}_{\text{reg}}$ which is called the *subregular nilpotent orbit* of $\mathfrak{g}$, and denoted by $\mathbb{O}_{\text{subreg}}$. Its codimension in $\mathfrak{g}$ is the rank of $\mathfrak{g}$ plus two. At the extreme opposite, there is a unique nilpotent orbit of smallest positive dimension called the *minimal nilpotent orbit* of $\mathfrak{g}$, and denoted by $\mathbb{O}_{\text{min}}$. Its dimension is $2(h^\vee - 1)$ ([252]), where $h^\vee$ is the dual Coxeter number.

D.2 Chevalley order

The set of nilpotent orbits in $\mathfrak{g}$ is naturally a poset $\mathscr{P}$ with partial order $\leqslant$, called the *Chevalley order*, or *closure order*, defined as follows: $\mathbb{O}' \leqslant \mathbb{O}$ if and only if $\mathbb{O}' \subseteq \overline{\mathbb{O}}$. If $\mathbb{O}' \subsetneq \overline{\mathbb{O}}$ we say that $\mathbb{O}'$ is a *degeneration* of $\mathbb{O}$. The degeneration is called *minimal* if $\mathbb{O}'$ is open in $\overline{\mathbb{O}} \setminus \mathbb{O}$.

The regular nilpotent orbit $\mathbb{O}_{\text{reg}}$ is maximal and the zero orbit is minimal with respect to this order. Moreover, $\mathbb{O}_{\text{subreg}}$ is maximal in the poset $\mathscr{P} \setminus \mathbb{O}_{\text{reg}}$ and $\mathbb{O}_{\text{min}}$ is minimal in the poset $\mathscr{P} \setminus \{0\}$.

The Chevalley order on $\mathscr{P}$ corresponds to a partial order on the set $\mathscr{P}(n)$ of partitions of n, for $n > 1$, for $\mathfrak{g} = \mathfrak{sl}_n$, first described by Gerstenhaber. More generally, the Chevalley order corresponds to a partial order on some subset of $\mathscr{P}(n)$ when $\mathfrak{g}$ is of classical type as we explain below.

The notions of *degeneration* and *minimal degeneration* can be expressed in term of partitions.

Definition D.1 Let $\boldsymbol{\lambda} \in \mathscr{P}(n)$. A *degeneration* of $\boldsymbol{\lambda}$ is an element $\boldsymbol{\mu} \in \mathscr{P}(n)$ such that $\mathbb{O}_{\boldsymbol{\mu}} < \overline{\mathbb{O}}_{\boldsymbol{\lambda}}$, that is, $\boldsymbol{\mu} < \boldsymbol{\lambda}$. A degeneration $\boldsymbol{\mu}$ of $\boldsymbol{\lambda}$ is said to be *minimal* if $\mathbb{O}_{\boldsymbol{\mu}}$ is open in $\overline{\mathbb{O}}_{\boldsymbol{\lambda}} \setminus \mathbb{O}_{\boldsymbol{\lambda}}$.

Let $n \in \mathbb{Z}_{>0}$. As a rule, unless otherwise specified, we write an element $\boldsymbol{\lambda}$ of $\mathscr{P}(n)$ as a decreasing sequence $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ omitting the zeroes. Thus,

$$\lambda_1 \geqslant \cdots \geqslant \lambda_s \geqslant 1 \quad \text{and} \quad \lambda_1 + \cdots + \lambda_s = n.$$

We shall denote the dual partition of a partition $\boldsymbol{\lambda} \in \mathscr{P}(n)$ by $\boldsymbol{\lambda}^T$.

Let us denote by $\geqslant$ the partial order on $\mathscr{P}(n)$ relative to the dominance. More precisely, given $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_s), \boldsymbol{\eta} = (\mu_1, \dots, \mu_t) \in \mathscr{P}(n)$, we have $\boldsymbol{\lambda} \geqslant \boldsymbol{\eta}$ if

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i \quad \text{for} \quad 1 \leqslant k \leqslant \min(s, t).$$

D.2.1 Case $\mathfrak{sl}_n$

Every nilpotent matrix in $\mathfrak{sl}_n$ is conjugate to a Jordan block diagonal matrix. Therefore, the nilpotent orbits in $\mathfrak{g}$ are parameterized by $\mathscr{P}(n)$. We shall denote by $\mathbb{O}_{\boldsymbol{\lambda}}$

the corresponding nilpotent orbit of $\mathfrak{sl}_n$. Then $\mathbb{O}_\lambda$ is represented by the standard Jordan form $\operatorname{diag}(J_{\lambda_1}, \ldots, J_{\lambda_s})$, where J_k is the k-size Jordan block. If we write $\boldsymbol{\lambda}^T = (d_1, \ldots, d_t)$ the transpose matrix, then

$$\dim \mathbb{O}_\lambda = n^2 - \sum_{i=1}^{t} d_i^2.$$

If $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}(n)$, then $\mathbb{O}_{\boldsymbol{\eta}} \subset \overline{\mathbb{O}_{\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$.

The regular, subregular, minimal and zero nilpotent orbits of $\mathfrak{sl}_n$ correspond to the partitions (n), $(n-1, 1)$, $(2, 1^{n-2})$ and (1^n) of n, respectively.

We give in Figure D.1 the description of the poset $\mathscr{P}(n)$ for $n = 6$. The column on the right indicates the dimension of the orbits appearing in the same row. Such diagram is called a *Hasse diagram*.

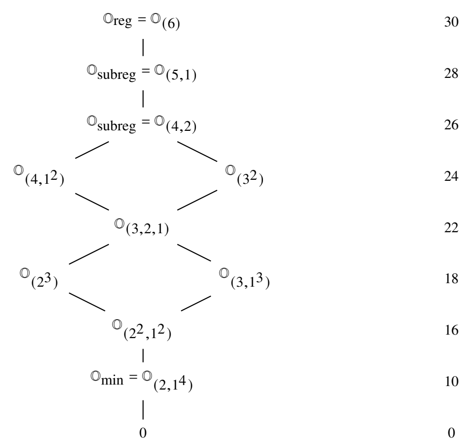


Fig. D.1 Hasse diagram for $\mathfrak{sl}_6$

D.2.2 Cases $\mathfrak{o}_n$ and $\mathfrak{so}_n$

For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_1(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n) : \text{number of parts of each even number is even}\}.$$

The nilpotent orbits of $\mathfrak{so}_n$ are parametrized by $\mathscr{P}_1(n)$, with the exception that each *very even* partition $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$ (i.e., $\boldsymbol{\lambda}$ has only even parts) corresponds to two nilpotent orbits. For $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, not very even, we shall denote by $\mathbb{O}_{\mathbf{1},\lambda}$, or simply by $\mathbb{O}_\lambda$ when there is no possible confusion, the corresponding nilpotent orbit of $\mathfrak{so}_n$. For very even $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, we shall denote by $\mathbb{O}^{I}_{1,\lambda}$ and $\mathbb{O}^{II}_{1,\lambda}$ the two corresponding nilpotent orbits of $\mathfrak{so}_n$. In fact, their union forms a single $O(n)$-orbit. Thus nilpotent orbits of $\mathfrak{o}_n$ are parametrized by $\mathscr{P}_1(n)$.

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s) \in \mathscr{P}_1(n)$ and $\boldsymbol{\lambda}^T = (d_1, \dots, d_t)$, then

$$\dim \mathbb{O}^{\bullet}_{1,\lambda} = \frac{n(n-1)}{2} - \frac{1}{2}\left(\sum_{i=1}^{t} d_i^2 - \#\{i : \lambda_i \text{ odd}\}\right),$$

where $\mathbb{O}^{\bullet}_{1,\lambda}$ is either $\mathbb{O}_{1,\lambda}$, $\mathbb{O}^{I}_{1,\lambda}$ or $\mathbb{O}^{II}_{1,\lambda}$ according to whether $\boldsymbol{\lambda}$ is very even or not. Using the same notations, If $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}_1(n)$, then $\overline{\mathbb{O}^{\bullet}_{1,\eta}} \subsetneq \overline{\mathbb{O}^{\bullet}_{1,\lambda}}$ if and only if $\boldsymbol{\eta} < \boldsymbol{\lambda}$, where $\mathbb{O}^{\bullet}_{1,\lambda}$ is either $\mathbb{O}_{1,\lambda}$, $\mathbb{O}^{I}_{1,\lambda}$ or $\mathbb{O}^{II}_{1,\lambda}$ according to whether $\boldsymbol{\lambda}$ is very even or not.

Given $\boldsymbol{\lambda} \in \mathscr{P}(n)$, there exists a unique $\boldsymbol{\lambda}^+ \in \mathscr{P}_1(n)$ such that $\boldsymbol{\lambda}^+ \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\eta} \in \mathscr{P}_1(n)$ verifies $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$, then $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}^+$. More precisely, let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ (adding zeroes if necessary). If $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, then $\boldsymbol{\lambda}^+ = \boldsymbol{\lambda}$. Otherwise if $\boldsymbol{\lambda} \notin \mathscr{P}_1(n)$, set

$$\boldsymbol{\lambda}' = (\lambda_1, \dots, \lambda_s, \lambda_{s+1} - 1, \lambda_{s+2}, \dots, \lambda_{t-1}, \lambda_t + 1, \lambda_{t+1}, \dots, \lambda_n),$$

where s is maximum such that $(\lambda_1, \dots, \lambda_s) \in \mathscr{P}_1(\lambda_1 + \cdots + \lambda_s)$, and t is the index of the first even part in $(\lambda_{s+2}, \dots, \lambda_n)$. Note that $s = 0$ if such a maximum does not exist, while t is always defined. If $\boldsymbol{\lambda}'$ is not in $\mathscr{P}_1(n)$, then we repeat the process until we obtain an element of $\mathscr{P}_1(n)$ which will be our $\boldsymbol{\lambda}^+$.

D.2.3 Case $\mathfrak{sp}_n$

For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_{-1}(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n) : \text{number of parts of each odd number is even}\}.$$

The nilpotent orbits of $\mathfrak{sp}_n$ are parametrized by $\mathscr{P}_{-1}(n)$. For $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}_{-1}(n)$, we shall denote by $\mathbb{O}_{-1,\lambda}$, or simply by $\mathbb{O}_{\lambda}$ when there is no possible confusion, the corresponding nilpotent orbit of $\mathfrak{sp}_n$, and if we write $\boldsymbol{\lambda}^T = (d_1, \dots, d_t)$, then

$$\dim \mathbb{O}_{-1,\lambda} = \frac{n(n+1)}{2} - \frac{1}{2}\left(\sum_{i=1}^{s} d_i^2 + \#\{i : \lambda_i \text{ odd}\}\right).$$

As in the case of $\mathfrak{sl}_n$, if $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}_{-1}(n)$, then $\mathbb{O}_{-1,\eta} \subset \overline{\mathbb{O}_{-1,\lambda}}$ if and only if $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$.

Given $\boldsymbol{\lambda} \in \mathscr{P}(n)$, there exists a unique $\boldsymbol{\lambda}^- \in \mathscr{P}_{-1}(n)$ such that $\boldsymbol{\lambda}^- \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\eta} \in \mathscr{P}_{-1}(n)$ verifies $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$, then $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}^-$. The construction of $\boldsymbol{\lambda}^-$ is the same as in the orthogonal case except that t is the index of the first odd part in $(\lambda_{s+2}, \dots, \lambda_n)$.

Similarly to Definition D.1, we have the following definition for $\mathfrak{g} = \mathfrak{o}_n$, $\mathfrak{so}_n$ or $\mathfrak{sp}_\mathfrak{n}$.

Definition D.2 Let $\varepsilon \in \{\pm 1\}$, and $\boldsymbol{\lambda} \in \mathscr{P}_\varepsilon(n)$. An *$\varepsilon$-degeneration* of $\boldsymbol{\lambda}$ is an element $\boldsymbol{\mu} \in \mathscr{P}_\varepsilon(n)$ such that $\mathbb{O}_{\varepsilon,\boldsymbol{\mu}} < \overline{\mathbb{O}}_{\varepsilon,\boldsymbol{\lambda}}$. An ε-degeneration $\boldsymbol{\mu}$ of $\boldsymbol{\lambda}$ is said to be *minimal* if $\mathbb{O}_{\varepsilon,\boldsymbol{\mu}}$ is open in $\overline{\mathbb{O}}_{\varepsilon,\boldsymbol{\lambda}} \setminus \mathbb{O}_{\varepsilon,\boldsymbol{\lambda}}$.

D.3 Jacobson–Morosov Theorem and Dynkin grading

A $\frac{1}{2}\mathbb{Z}$*-grading* of the Lie algebra $\mathfrak{g}$ is a decomposition $\Gamma\colon \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ which verifies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j. The following result is proved for instance in [245, Proposition 20.1.5].

Lemma D.1 *If Γ is a $\frac{1}{2}\mathbb{Z}$-grading of $\mathfrak{g}$, then for some semisimple element h_Γ of $\mathfrak{g}$,*

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \colon [h_\Gamma, x] = 2jx\}.$$

Let $(-|-) = \dfrac{1}{2h^\vee}\kappa_{\mathfrak{g}}$, where $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. This is the nondegenerate symmetric bilinear form on $\mathfrak{g}$ as in Appendix A. Since the bilinear form $(-|-)$ of $\mathfrak{g}$ is $(\operatorname{ad} h_\Gamma)$-invariant and nondegenerate, we get

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \iff i + j \neq 0.$$

Hence $\mathfrak{g}_j$ and $\mathfrak{g}_{-j}$ are in pairing. In particular, they have the same dimension.

Theorem D.1 (Jacobson–Morosov) *Fix a nonzero nilpotent element $e \in \mathfrak{g}$. There exists $h, f \in \mathfrak{g}$ such that the triple (e, h, f) verifies the* $\mathfrak{sl}_2$-triple *relations:*

$$[h, e] = 2e, \qquad [e, f] = h, \qquad [h, f] = -2f.$$

In particular, h is semisimple and the eigenvalues of $\operatorname{ad} h$ *are integers. Moreover, e and f belong to the same nilpotent G-orbit.*

Example D.2 Let $\mathfrak{g} = \mathfrak{sl}_n$, and set,

$$e := J_n, \qquad h := \sum_{i=1}^{n} (n + 1 - 2i)e_{i,i}, \qquad f := \sum_{i=1}^{n-1} i(n - i)e_{i+1,i}.$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple. From this observation, we readily construct $\mathfrak{sl}_2$-triples for any standard Jordan form $\operatorname{diag}(J_{\lambda_1}, \ldots, J_{\lambda_n})$ with $(\lambda_1, \ldots, \lambda_n) \in \mathscr{P}(n)$.

The group G acts on the collection of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ by simultaneous conjugation. This defines a natural map:

$$\Omega\colon \{\mathfrak{sl}_2\text{-triples}\}/G \longrightarrow \{\text{nonzero nilpotent orbits}\}, \quad (e, h, f) \mapsto G.e.$$

Theorem D.2 *The map Ω is bijective.*

The map Ω is surjective according to Jacobson–Morosov Theorem (Theorem D.1. The injectivity is a result of Kostant ([82, Theorem 3.4.10]). We refer to [253, §2.6] for a sketch of proof of Theorem D.2.

Since h is semisimple and the eigenvalues of ad h are integers, we get a $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$ defined by h, called the *Dynkin grading* associated with h:

$$\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \qquad \mathfrak{g}_j := \{x \in \mathfrak{g} \colon [h,x] = 2jx\}. \tag{D.1}$$

We have $e \in \mathfrak{g}_1$. Moreover, it follows from the representation theory of $\mathfrak{sl}_2$ that $\mathfrak{g}^e \subset \bigoplus_{j\geqslant 0} \mathfrak{g}_j$ and that

$$\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}}.$$

One can draw a picture to visualize the above properties. Decompose $\mathfrak{g}$ into simple $\mathfrak{sl}_2$-modules $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ and denote by d_k the dimension of V_k for $k = 1,\ldots,s$. We can assume that $d_1 \geqslant \cdots \geqslant d_s \geqslant 1$. We have $\dim V_k \cap \mathfrak{g}_j \leqslant 1$ for any $j \in \frac{1}{2}\mathbb{Z}$. We represent the module V_k on the k-th row with d_k boxes, each box corresponding to a nonzero element of $V_k \cap \mathfrak{g}_j$ for j such that $V_k \cap \mathfrak{g}_j \neq \{0\}$. We organize the rows so that the $2j$-th column corresponds to a generator of $V_k \cap \mathfrak{g}_j$. Then the boxes appearing on the right position of each row lie in $\mathfrak{g}^e$.

Example D.3 Consider the element $e = \mathrm{diag}(J_3, J_1)$ of $\mathfrak{sl}_4$ corresponding to the partition $(3,1)$. Here, we get $\dim \mathfrak{g}_0 = 5$, $\dim \mathfrak{g}_{\frac{1}{2}} = 0$, $\dim \mathfrak{g}_1 = 4$ and $\dim \mathfrak{g}_2 = 1$. The corresponding picture is given in Figure D.2.

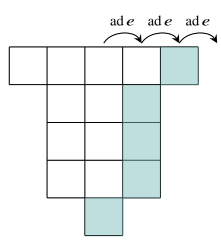


Fig. D.2 Decomposition into $\mathfrak{sl}_2$-modules for $(3,1)$

In Figure D.2, the with boxes correspond to nonzero elements lying in $[f,\mathfrak{g}]$. The coloured boxes correspond to nonzero elements lying in $\mathfrak{g}^e$.

This is an example of *even nilpotent element*, which means that $\mathfrak{g}_i = \{0\}$ for all half-integers i. The nilpotent orbit of an even nilpotent element is called an *even nilpotent orbit*. Note that the regular nilpotent orbit is always even.

Example D.4 Consider the element $e = \mathrm{diag}(J_2, J_1, J_1)$ of $\mathfrak{sl}_4$ corresponding to the partition $(2,1,1)$. It lies in the minimal nilpotent orbit of $\mathfrak{sl}_4$. Here, we get $\dim \mathfrak{g}_0 = 5$, $\dim \mathfrak{g}_{\frac{1}{2}} = 4$, $\dim \mathfrak{g}_1 = 1$. The corresponding picture is given in Fig. D.3:

We observe that $\bigoplus_{i\geqslant 1} \mathfrak{g}_i$ equals $\mathfrak{g}_1$ and has dimension 1.

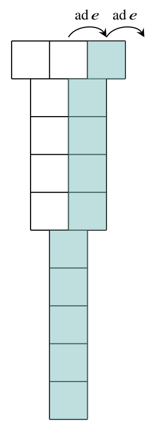


Fig. D.3 Decomposition into $\mathfrak{sl}_2$-modules for $(2, 1^2)$

Remark D.1 This is a general fact: if e is a minimal nilpotent element of $\mathfrak{g}$, then $\oplus_{i\geqslant 1}\mathfrak{g}_i = \mathfrak{g}_1 = \mathbb{C}e$, and

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1.$$

One can assume that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is also a Cartan subalgebra of the reductive Lie algebra $\mathfrak{g}_0$.

Lemma D.2 *Fix an $\mathfrak{sl}_2$-triple (e, h, f) with corresponding Dynking grading as in* (D.1)*, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_0$.*

(i) *For any $\alpha \in \Delta$, the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_j$ for some $j \in \frac{1}{2}\mathbb{Z}$.*
(ii) *Fix a root system Δ_0 of $(\mathfrak{g}_0, \mathfrak{h})$, and set $\Delta_{0,+} = \Delta_+ \cap \Delta_0$. Then*

$$\Delta_+ = \Delta_{0,+} \cup \{\alpha \colon \mathfrak{g}_\alpha \subset \mathfrak{g}_{>0}\}.$$

Denoting by Π the set of simple roots of Δ_+, we get

$$\Pi = \bigcup_{j\in\frac{1}{2}\mathbb{Z}} \Pi_j \quad \text{with} \quad \Pi_j := \{\alpha \in \Pi \colon \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}.$$

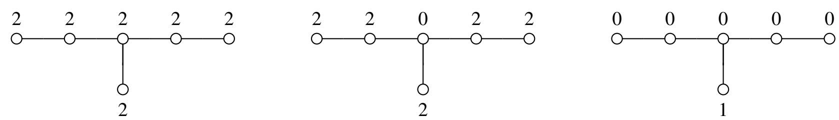
Lemma D.3 *We have* $\Pi = \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$.

Proof Assume that there exists $\beta \in \Pi_s$ for $s > 1$. A contradiction is expected. Since $e \in \mathfrak{g}_1$ and since $\mathfrak{g}_1$ is contained in the subalgebra generated by the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$, we get $[e, \mathfrak{g}_{-\beta}] = \{0\}$. In other words, $\mathfrak{g}_{-\beta} \subset \mathfrak{g}^e$. This contradicts the fact that $\mathfrak{g}^e \subset \mathfrak{g}_{\geqslant 0}$. □

From Lemma D.3 we define the *weighted Dynkin diagram*, or *characteristic*, of the nilpotent orbit $G.e$ when $\mathfrak{g}$ is simple as follows. Consider the Dynkin diagram of the simple Lie algebra $\mathfrak{g}$. Each node of this diagram corresponds to a simple root $\alpha \in \Pi$. Then the weighted Dynkin diagram is obtained by labeling the node corresponding to α with the value $\alpha(h) \in \{0, 1, 2\}$.

By convention, the zero orbit has a weighted Dynkin diagram with every node labeled with 0.

Example D.5 In type E_6, the characteristics of the regular, subregular and minimal nilpotent orbits are respectively:



An important consequence of Lemma D.3 is that there are only finitely many nilpotent orbits, namely at most $3^{\operatorname{rank}\mathfrak{g}}$. Also, the weighted Dynkin diagram is a complete invariant, i.e., two such diagrams are equal if and only if the corresponding nilpotent orbits are equal, [82, Theorem 3.5.4].

The regular nilpotent orbit always corresponds to the weighted Dynkin diagram with only 2's (this result is not obvious, cf. e.g., [82, Theorem 4.1.6] for a proof). More generally, a nilpotent orbit is even if and only if the weighted Dynkin diagram have only 2's or 0's.

D.4 The nilpotent cone

The nilpotent cone enjoys remarkable properties. We collect some of them in this section, omitting the proofs.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and denote by W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. The restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ induces an algebra isomorphism $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$, referred to as the *Chevalley isomorphism*. As a consequence, the algebra $\mathbb{C}[\mathfrak{g}]^G$ is a is a polynomial algebra of dimension ℓ, the rank of $\mathfrak{g}$ (which is the dimension of $\mathfrak{h}$), since W is a finite group.

The cone nilpotent turns out to be the zero locus of the augmentation ideal $\mathbb{C}[\mathfrak{g}]^G_+$ of $\mathbb{C}[\mathfrak{g}]^G$, that is, the zero locus of homogeneous generators $p_1, \ldots, p_r$ of $\mathbb{C}[\mathfrak{g}]^G$:

$$\text{(D.2)} \qquad \mathcal{N} = \{x \in \mathfrak{g} \colon p(x) = 0 \text{ for all } p \in \mathbb{C}[\mathfrak{g}]^G_+\} = \operatorname{Spec} \mathbb{C}[\mathfrak{g}]/(p_1, \ldots, p_\ell).$$

Theorem D.3 *Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$.*

(i) *The scheme $\mathcal{N}$ is reduced, irreductible and it is a complete intersection.*
(ii) *The scheme $\mathcal{N}$ has rational singularities. Namely, there exists a proper birational morphism $\tau \colon Y \to \mathcal{N}$ such that Y is smooth over $\mathbb{C}$, $\tau_*(\mathcal{O}_Y) = \mathcal{O}_{\mathcal{N}}$ and $R^p\tau_*(\mathcal{O}_Y) = 0$ for all $p \geqslant 1$.*

Part (i) of the theorem is due to Kostant [184], Part (ii) is due to Hesselink [148].

D.5 Associated varieties of primitive ideals

Let I be a two-sided ideal of the enveloping algebra $U(\mathfrak{g})$. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on I, so that $\operatorname{gr} I$ becomes a graded Poisson ideal in $\mathbb{C}[\mathfrak{g}^*]$ (see Example B.2). Denote by $\mathcal{V}(I)$ the zero locus of $\operatorname{gr} I$ in $\mathfrak{g}^*$,

$$\mathcal{V}(I) := \{\lambda \in \mathfrak{g}^* : p(\lambda) = 0 \text{ for all } p \in \operatorname{gr} I\} = \operatorname{Specm}(\mathbb{C}[\mathfrak{g}^*]/\operatorname{gr} I).$$

The set $\mathcal{V}(I)$ is usually referred to as the *associated variety* of I. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through a nondegenerate bilinear symmetric form on $\mathfrak{g}$, we shall often view associated varieties of two-sided ideals of $U(\mathfrak{g})$ as subsets of $\mathfrak{g}$.

A proper two-sided ideal I of $U(\mathfrak{g})$ is called *primitive* if it is the annihilator in $U(\mathfrak{g})$ of a simple left $U(\mathfrak{g})$-module. There are two fundamental results ([93, 61, 179, 155]) on primitive ideals of $U(\mathfrak{g})$.

Theorem D.4 (Duflo) *Any primitive ideal in* $U(\mathfrak{g})$ *is the annihilator* $\operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)$ *in* $U(\mathfrak{g})$ *of some irreducible highest weight representation* $L_{\mathfrak{g}}(\lambda)$ *of* $\mathfrak{g}$*, where* $\lambda \in \mathfrak{h}^*$.

Theorem D.5 (Joseph) *Let* I *be a primitive ideal of* $U(\mathfrak{g})$.

(i) *The associated variety* $\mathcal{V}(I)$ *is contained in the nilpotent cone* $\mathcal{N}$ *of* $\mathfrak{g}$.
(ii) *The associated variety* $\mathcal{V}(I)$ *is irreducible and, hence, it is the closure* $\overline{\mathbb{O}}$ *of some nilpotent orbit* $\mathbb{O}$ *in* $\mathfrak{g}$.

Part (i) of the theorem is not hard to prove. By Theorem D.4, there is $\lambda \in \mathfrak{h}^*$ such that $I = \operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)$. Consider the central character $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$, associated to λ defined by (A.14). Its kernel is a maximal ideal of $Z(\mathfrak{g})$ contained in $\operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)$. We deduce that $\operatorname{gr} I$ contains the symbols of all elements in $\ker \chi_\lambda$. Since $\operatorname{gr} Z(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$, we easily see that these elements generate the augmentation ideal $S(\mathfrak{g})^{\mathfrak{g}}_+$. As a consequence, $\operatorname{gr} I \supset S(\mathfrak{g})^{\mathfrak{g}}_+$, whence $\mathcal{V}(I) \subset \mathcal{N}$ by (D.2).

Part (ii) is much harder to establish. Theorem D.5 (ii) was first partially proved (by a case-by-case argument) in [61], and in a more conceptual way in [179] and [155] (independently), using many earlier deep results due to Joseph, Gabber, Lusztig, Vogan and others.

Different primitive ideals may share the same associated variety. At the same time, not all nilpotent orbit closures appear as associated variety of some primitive ideal of $U(\mathfrak{g})$.

Let $\lambda \in \mathfrak{h}^*$, and set

$$J_\lambda := \operatorname{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{g}}(\lambda)).$$

The *associated variety* of the irreducible highest weight representation $L_{\mathfrak{g}}(\lambda)$ of $\mathfrak{g}$ is $\mathcal{V}(J_\lambda)$. By Theorem D.5,

$$\mathcal{V}(J_\lambda) = \overline{\mathbb{O}_\lambda}$$

for some nilpotent orbit $\mathbb{O}_\lambda$. Naturally, the geometry of $\mathbb{O}_\lambda$ is expected to reflect some properties of the representation $L_{\mathfrak{g}}(\lambda)$. The nilpotent orbits $\mathbb{O}_\lambda$ for which λ is *integral*, that is, $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ are called *special* ([205]). For a special nilpotent orbit $\mathbb{O}$, one can describe the weights λ for which $\mathcal{V}(J_\mu) = \mathbb{O}$ ([45, 46]).

Example D.6 The irreducible highest weight representation $L_\mathfrak{g}(\lambda)$ is finite-dimensional if and only if its associated variety $\mathcal{V}(J_\lambda)$ is reduced to $\{0\}$. It is well-known that this happens if and only if the weight λ is integral and dominant.

D.6 Nilpotent Slodowy slices

We collect in this section geometrical properties of nilpotent Slodowy slices.

D.6.1 Smoothly equivalent singularities

Consider two varieties X, Y and two points $x \in X$, $y \in Y$. We refer to [147] for the following definition.

Definition D.3 The singularity of X at x is called *smoothly equivalent* to the singularity of Y at y if there is a variety Z, a point $z \in Z$ and two maps $\varphi\colon Z \to X$, $\psi\colon Z \to Y$, such that $\varphi(z) = x$, $\psi(z) = y$, and φ and ψ are smooth in z. This clearly defines an equivalence relation between pointed varieties (X, x). We denote the equivalence class of (X, x) by $\mathrm{Sing}(X, x)$.

Various geometric properties of X at x only depends on the equivalence class $\mathrm{Sing}(X, x)$, for example: the smoothness, the normality, the unibranchness (cf. §D.6.4), the Cohen–Macaulay or rational singularities, etc.

Assume that an algebraic group G acts regularly on the variety X. Then $\mathrm{Sing}(X, x) = \mathrm{Sing}(X, x')$ if x and x' belongs to the same G-orbit $\mathbb{O}$. In this case, we denote the equivalence class also by $\mathrm{Sing}(X, \mathbb{O})$.

A *transverse slice* at the point $x \in X$ is defined to be a locally closed subvariety $S \subset X$ such that $x \in S$ and the map

$$G \times S \longrightarrow X, \quad (g, s) \longmapsto g.s,$$

is smooth at the point $(1, x)$. We have $\mathrm{Sing}(S, x) = \mathrm{Sing}(X, x)$.

D.6.2 Transverse slices for nilpotent orbit closures

Let G be a complex connected, simple algebraic group of adjoint type with Lie algebra $\mathfrak{g}$ and f a nilpotent element of $\mathfrak{g}$. By the Jacobson–Morosov Theorem, we can embed f into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. The affine space

$$\mathscr{S}_f := f + \mathfrak{g}^e$$

is a transverse slice of $\mathfrak{g}$ at f, called the *Slodowy slice* associated with (e, h, f).

The variety

$$\mathscr{S}_{\mathbb{O},f} = \overline{\mathbb{O}} \cap \mathscr{S}_f, \tag{D.3}$$

where $\overline{\mathbb{O}}$ is a nilpotent orbit of $\mathfrak{g}$, is referred to as a *nilpotent Slodowy slice*.

Lemma D.4 *The variety $\mathscr{S}_{\mathbb{O},f}$ is nonempty if and only if $f \in \overline{\mathbb{O}}$, that is, $G.f \leqslant \overline{\mathbb{O}}$.*

Proof If $f \in \overline{\mathbb{O}}$, clearly $f \in \mathscr{S}_{\mathbb{O},f}$. To show the converse implication, consider the one-parameter subgroup defined as follows. The embedding $\mathrm{span}_{\mathbb{C}}\{e, h, f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ exponentiates to a homomorphism $SL_2 \to G$. By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup $\rho\colon \mathbb{C}^* \to G$. Thus $\rho(t)x = t^{2j}x$ for any homogeneous $x \in \mathfrak{g}_j$ with respect to the Dynkin grading (D.1). For $t \in \mathbb{C}^*$ and $x \in \mathfrak{g}$, set

$$\tilde{\rho}(t)x := t^2\rho(t)(x). \tag{D.4}$$

So, for any $x \in \mathfrak{g}_j$, $\tilde{\rho}(t)x = t^{2+2j}x$. In particular, $\tilde{\rho}(t)f = f$ and the $\mathbb{C}^*$-action of $\tilde{\rho}$ stabilizes $\mathscr{S}_f$. Moreover, it is contracting to f on $\mathscr{S}_f$, that is,

$$\lim_{t\to 0} \tilde{\rho}(t)x = f$$

for any $x \in \mathscr{S}_f$, because $\mathfrak{g}^e \subseteq \mathfrak{g}_{\geqslant 0}$. Therefore, if $x \in \mathscr{S}_{\mathbb{O},f}$, then $f = \lim_{t\to 0} \tilde{\rho}(t)x$ belongs to $\mathscr{S}_{\mathbb{O},f}$ using that $\overline{\mathbb{O}}$ is G-invariant and closed. □

If $f \in \overline{\mathbb{O}}$, the variety $\mathscr{S}_{\mathbb{O},f}$ is a transverse slice of $\overline{\mathbb{O}}$ at f. It is equidimensional, and

$$\dim \mathscr{S}_{\mathbb{O},f} = \mathrm{codim}_{\overline{\mathbb{O}}}(\overline{G.f}).$$

Since any two $\mathfrak{sl}_2$-triples containing f are conjugate by an element of the isotropy group of f in G, the isomorphism type of $\mathscr{S}_{\mathbb{O},f}$ is independent of the choice of such $\mathfrak{sl}_2$-triples. Moreover, the isomorphism type of $\mathscr{S}_{\mathbb{O},f}$ is independent of the choice of f in its nilpotent orbit $\mathbb{O}' = G.f$. By focussing on $\mathscr{S}_{\mathbb{O},f}$, we reduce the study of $\mathrm{Sing}(\overline{\mathbb{O}}, \mathbb{O}')$ to the study of the singularity of $\mathscr{S}_{\mathbb{O},f}$ at f.

Recall that nilpotent orbits in simple Lie algebras of classical type are parametrized by certain classes of partitions (see §D.2). We set

$$\mathscr{S}_{\lambda,\mu} := \mathscr{S}_{\mathbb{O}_\lambda,f} \quad \text{for} \quad f \in \mathbb{O}_\mu, \tag{D.5}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are partitions of nilpotent orbits in the corresponding simple classical Lie algebra.

Normalizations of nilpotent Slodowy slices are symplectic singularities in the sense of Beauville [48] and, like nilpotent orbit closures, these varieties are studied for their role in representation theory and in the theory of symplectic singularities.

D.6.3 Minimal degenerations

The nilpotent Slodowy slices are best understood in the case of minimal degeneration (cf Definition D.1) in which case $G.f$ is an open subvariety of the boundary[1] of $\mathbb{O}$ in $\overline{\mathbb{O}}$. In the context of this class of examples, one has the celebrated result of Brieskorn and Slodowy ([66, 236]) confirming a conjecture of Grothendieck, that the nilpotent Slodowy slice associated with the principal (or, regular) nilpotent orbit $\mathbb{O} = \mathbb{O}_{\text{reg}}$ and a subregular nilpotent element f_{subreg} has a simple singularity of the same type as G, for G of type A, D, E.

Kraft and Procesi studied nilpotent Slodowy slices for minimal degenerations in the classical types [176, 177], motivated by the normality problem for nilpotent orbit closures (see §D.6.5). They described the smooth equivalences of singularities between two Slodowy slices $\mathscr{S}_{\lambda,\mu}$ and $\mathscr{S}_{\lambda',\mu'}$, where $\mathscr{S}_{\lambda',\mu'}$ is obtained from $\mathscr{S}_{\lambda,\mu}$ by the *row/column removal rule* that we briefly describe below. It turns out that these smooth equivalences actually yield isomorphisms of varieties, see [196, Proposition 7.3.2][2].

The following lemma is a refinement of [176, Propositions 4.4 and 5.4]. We refer to [196, Propositions 7.3.1 and 7.3.2] for a proof.

Lemma D.5 *Let $\boldsymbol{\lambda} \in \mathscr{P}(n)$ and $\boldsymbol{\mu}$ a degeneration (cf. Definition D.1) of $\boldsymbol{\lambda}$. Assume that the first l rows and the first m columns of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ coincide. Denote by $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}'$ the partitions obtained by erasing these l common rows and m common columns. Then*

$$\mathscr{S}_{\lambda,\mu} \cong \mathscr{S}_{\lambda',\mu'},$$

as algebraic varieties. In particular, if $\boldsymbol{\mu}' = 0$, then $\mathscr{S}_{\lambda,\mu} \cong \overline{\mathbb{O}}_{\lambda'}$.

We illustrate in Figure D.4 the row removal rule in the case where $\boldsymbol{\lambda} = (3^2, 1)$, $\boldsymbol{\mu} = (3, 2^2)$.

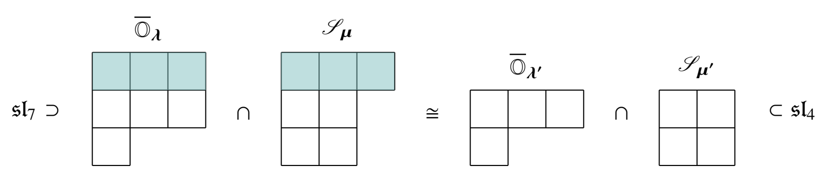


Fig. D.4 Row removal rule for $\boldsymbol{\lambda} = (3^2, 1)$ and $\boldsymbol{\mu} = (3, 2^2)$

Remark D.2 Thibault Juillard recently gave a BSRT interpretation of the row removal rule in type A ([157]) . This proves that the isomorphism in Lemma D.5 is Poisson,

[1] The boundary of $\mathbb{O}$ in $\overline{\mathbb{O}}$ is precisely the singular locus of $\overline{\mathbb{O}}$ as was shown by Namikawa [223] using results of Kaledin and Panyushev [169, 225]; this can also be deduced from [176, 177, 123].

[2] This is also mentioned without detail in [123, §1.8.1].

where the Poisson structure on nilpotent Slodowy slices is the BRST structures (see Section 7.2).

The following lemma is a refinement of [177, Theorem 12.3]. We refer to [196, Propositions 8.5.1 and 8.5.2] for a proof.

Lemma D.6 *Let $\boldsymbol{\lambda} \in \mathscr{P}_\varepsilon(n)$, for $\varepsilon \in \{\pm 1\}$, and $\boldsymbol{\mu}$ an ε-degeneration (cf. Definition D.2) of $\boldsymbol{\lambda}$. Assume that the first l rows and the first m columns of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ coincide and denote by $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}'$ the partitions obtained by erasing these l common rows and common m columns. Then*

$$\mathscr{S}_{\lambda,\mu} \cong \mathscr{S}_{\lambda',\mu'},$$

as algebraic varieties. In particular, if $\boldsymbol{\mu}' = 0$, then $\mathscr{S}_{\lambda,\mu} \cong \overline{\mathbb{O}}_{\lambda'}$.

Fu, Juteau, Levy and Sommers [123] have complemented the work of Kraft and Procesi by determining the generic singularities of nilpotent orbit closures $\overline{\mathbb{O}}$ in exceptional types, which they did through a study of the nilpotent Slodowy slices $\mathscr{S}_{\mathbb{O},f}$ at minimal degenerations $G.f$.

D.6.4 Branching

Let X be an irreducible algebraic variety, and $x \in X$. We say that X is *unibranch at x* if the normalization $\pi\colon (\tilde{X}, x) \to (X, x)$ of (X, x) is locally a homeomorphism at x ([143, Chapter III, §4.3]; see also [123, §2.4]). Otherwise, we say that X *has branches at x* and the number of branches of X at x is the number of connected components of $\pi^{-1}(x)$ [57, §5,(E)].

As it is explained in [123, §2.4], the number of irreducible components of $\mathscr{S}_{\mathbb{O},f}$ is equal to the number of branches of $\overline{\mathbb{O}}$ at f. If an irreducible algebraic variety X is normal, then it is obviously unibranch at any point $x \in X$. The converse is not true. For instance, there is no branching in type G_2 but one knows that the nilpotent orbit $\tilde{A}_1$ of G_2 of dimension 8 is not normal [178].

The number of branches of $\overline{\mathbb{O}}$ at f, and so the number of irreducible components of $\mathscr{S}_{\mathbb{O},f}$, can be determined from the tables of Green functions in [235], as discussed in [57, Section 5,(E)–(F)]. We indicate in Table D.1 the nilpotent orbits $\mathbb{O}$ which have branchings in types F_4, E_6, E_7 and E_8 (there is no branching in type G_2). The nilpotent orbits are labelled using the Bala–Carter classification.

We indicate in Table D.2 the (conjectural) list of non-normal nilpotent orbit closures in the exceptional types. These results are extracted from [178, 174, 67, 68, 238]. The list is known to be exhaustive for the types G_2, F_4 and E_6. It is only conjecturally exhaustive for the types E_7 and E_8.

Type F_4	C_3, $C_3(a_1)$.
Type E_6	A_4, $2A_2$, $A_2 + A_1$.
Type E_7	$D_6(a_1)$, $(A_5)''$, A_4, $A_3 + A_2$, $D_4(a_1) + A_1$.
Type E_8	$E_7(a_1)$, E_6, $E_6(a_1)$, $E_7(a_4)$, A_6, $D_6(a_1)$, $D_5 + A_1$, $E_7(a_5)$, A_4, $A_3 + A_2$, D_4, $D_4(a_1)$, $A_3 + A_1$, $2A_2 + A_1$.

Table D.1 Branchings for nilpotent orbit closures in the simple Lie algebras of exceptional types

Type G_2	$\tilde{A}_1$.
Type F_4	C_3, $C_3(a_1)$, $\tilde{A}_2 + A_1$, $\tilde{A}_2$, B_2.
Type E_6	A_4, $A_3 + A_1$, A_3, $2A_2$, $A_2 + A_1$.
Type E_7	$D_6(a_1)$, $D_6(a_2)$, $(A_5)''$, A_4, $A_3 + A_2$, $D_4(a_1) + A_1$, $A_3 + 2A_1$, $(A_3 + A_1)'$, $(A_3 + A_1)''$, A_3.
Type E_8	$E_7(a_1)$, $E_7(a_2)$, $D_7(a_1)$, $E_7(a_3)$, E_6, D_6, $E_6(a_1)$, $E_7(a_4)$, $D_6(a_1)$, A_6, $D_5 + A_1$, $E_7(a_5)$, $E_6(a_3) + A_1$, $D_6(a_2)$, $D_5(a_1) + A_2$, $A_5 + A_1$, D_5, $E_6(a_3)$, $D_4 + A_2$, $D_5(a_1) + A_1$, A_5, $D_5(a_1)$, $D_4 + A_1$, A_4, $A_3 + A_2$, $A_3 + 2A_1$, D_4, $D_4(a_1)$ $A_3 + A_1$, $2A_1 + A_1$, A_3.

Table D.2 Non-normal nilpotent orbit closures in the simple Lie algebras of exceptional types

D.6.5 Normality of nilpotent orbit closures in the classical types

The normality question of nilpotent orbit closures in the classical types is completely answered ([147, 175, 177, 239]). First of all, if $\mathfrak{g} = \mathfrak{sl}_n$, then all nilpotent orbit closures are normal ([175]). To explain the results in the other types, we now focus on the orthogonal and symplectic Lie algebras. We assume in the rest of the section that $\mathfrak{g}$ is either the Lie algebra $\mathfrak{o}_n$ of the orthogonal group $O(n)$, or the Lie algebra $\mathfrak{so}_n$ of the special orthogonal group $SO(n)$, or the Lie algebra $\mathfrak{sp}_n$ of the symplectic group $SP(n)$.

The following result is due to Kraft and Procesi ([177, Theorem 1]).

Theorem D.6 *Let $\mathbb{O}$ be a nilpotent orbit in $\mathfrak{o}_n$ or $\mathfrak{sp}_n$.*

(i) *$\overline{\mathbb{O}}$ is normal if and only if it is unibranch.*
(ii) *$\overline{\mathbb{O}}$ is normal if and only if it is normal in codimension 2.*

In particular, $\overline{\mathbb{O}}$ is normal if it does not contain a nilpotent orbit $\mathbb{O}' \leqslant \mathbb{O}$ of codimension 2. Theorem D.6 does not hold if $\mathfrak{g} = \mathfrak{so}_{2n}$ and if $\mathbb{O} = \mathbb{O}_{1,\boldsymbol{\lambda}}$, with $\boldsymbol{\lambda}$ very even.

By Theorem D.6, for the normality question, it is enough to consider minimal ε-degeneration of codimension 2 (except for the very even nilpotent orbits in $\mathfrak{so}_{2n}$). In the setting of Lemma D.6 we say that the ε-degeneration $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$ *is obtained from the ε'-degeneration $\boldsymbol{\eta}' \leqslant \boldsymbol{\lambda}'$ by adding rows and columns*. An ε-degeneration $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$ is called *irreducible* if it cannot be obtained by adding rows and columns in a non trivial way.

When we obtain an irreducible pair $(\boldsymbol{\eta}', \boldsymbol{\lambda}')$, such a pair is called the *type of* $(\mathbb{O}_{\varepsilon,\boldsymbol{\eta}}, \mathbb{O}_{\varepsilon,\boldsymbol{\lambda}})$.

So for the classification of the minimal ε-degenerations, one needs to describe the minimal irreducible ε-degenerations. They are given in [177, Table 3.4]. We reproduce it in Table D.3. In the last column, X_ℓ refers to the type of a simple surface singularity of type X_ℓ, which corresponds to the type of the isolated surface singularity of a nilpotent Slodowy slice $\mathscr{S}_{\mathcal{N}, f_{\text{subreg}}}$ in a simple (simply-laced) Lie algebra of type X_ℓ associated with a subregular nilpotent element f_{subreg} in $\mathcal{N} = \overline{\mathbb{O}_{\text{reg}}}$. The singularity $A_\ell \cup A_\ell$ refers to a non-normal union of two surface singularities of type A_ℓ meeting transversally in a singular point. The singularity x_ℓ refers to a minimal singularity, that is, the type of singularity at 0 of $\overline{\mathbb{O}_{\min}}$ in a simple Lie algebra of type X_ℓ.

Note that, except for the type $A_\ell \cup A_\ell$, all types of singularities appearing in the last column are normal.

Lie algebra	ε	$\boldsymbol{\lambda}$	$\boldsymbol{\eta}$	$\operatorname{codim}_{\overline{\mathbb{O}_{\varepsilon',\boldsymbol{\lambda}'}}}(\overline{\mathbb{O}_{\varepsilon',\boldsymbol{\eta}'}})$	$\operatorname{Sing}(\overline{\mathbb{O}_{\varepsilon,\boldsymbol{\lambda}}}, \mathbb{O}_{\varepsilon,\boldsymbol{\eta}})$
$\mathfrak{sp}_2$	-1	(2)	$(1,1)$	2	A_1
$\mathfrak{sp}_{2n}$, $n>1$	-1	$(2n)$	$(2n-1,2)$	2	D_{n+1}
$\mathfrak{so}_{2n+1}$, $n>0$	1	$(2n+1)$	$(2n-1,1,1)$	2	A_{2n-1}
$\mathfrak{sp}_{4n+2}$, $n>0$	-1	$(2n+1,2n+1)$	$(2n,2n,2)$	2	A_{2n-1}
$\mathfrak{so}_{4n}$, $n>0$	1	$(2n,2n)$	$(2n-1,2n-1,1,1)$	2	$A_{2n-1}\cup A_{2n-1}$
$\mathfrak{so}_{2n+1}$, $n>1$	1	$(2,2,1^{2n-1})$	(1^{2n+1})	$4n-4$	b_n
$\mathfrak{sp}_{2n}$, $n>1$	-1	$(2,1^{2n-2})$	(1^{2n})	$2n$	c_n
$\mathfrak{so}_{2n}$, $n>2$	1	$(2,2,1^{2n-4})$	(1^{2n})	$4n-6$	d_n

Table D.3 Irreducible minimal ε-degenerations

The main result is the following ([177, Theorem 12.3]).

Theorem D.7 *Let $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$ be the ε-degeneration obtained from the ε'-degeneration $\boldsymbol{\eta}' \leqslant \boldsymbol{\lambda}'$ by adding rows and columns. Then*

$$\operatorname{Sing}(\overline{\mathbb{O}_{\varepsilon,\boldsymbol{\lambda}}}, \mathbb{O}_{\varepsilon,\boldsymbol{\eta}}) = \operatorname{Sing}(\overline{\mathbb{O}_{\varepsilon',\boldsymbol{\lambda}'}}, \mathbb{O}_{\varepsilon',\boldsymbol{\eta}'}).$$

Appendix E
Superalgebras and Clifford algebras

A *superspace* is a $\mathbb{C}$-vector space E equipped with a $\mathbb{Z}_2$-grading, $E = E^{\bar{0}} \oplus E^{\bar{1}}$. Elements in $E^{\bar{0}}$ are called *even*, elements of $E^{\bar{1}}$ are called *odd*. We denote by $|v| \in \{\bar{0}, \bar{1}\}$ the parity of homogeneous elements $v \in E$. A *morphism of superspaces* is a linear map preserving $\mathbb{Z}_2$-gradings. It is itself a superspace by:

$$\mathrm{Hom}(E,F)^{\bar{0}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{0}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{1}}),$$
$$\mathrm{Hom}(E,F)^{\bar{1}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{1}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{0}}).$$

The category of superspaces is a tensor category. Then one may define superalgebras, Lie superalgebras, Poisson superalgebras, etc. as the algebra objects, Lie algebra objects, Poisson algebra objects etc. in this tensor category.

For example, a *Lie superalgebra* is a superspace A together with a bracket $[-,-]\colon A \times A \to A$ such that for all homogeneous elements $a, b \in A$,

$$[a,b] = -(-1)^{|a||b|}[b,a],$$
$$[[a,b],c] = [a,[b,c]] - (-1)^{|a||b|}[b,[a,c]].$$

Note that any superalgebra A is naturally a Lie superalgebra by setting for all homogeneous elements $a, b \in A$,

$$[a,b] = ab - (-1)^{|a||b|}ba.$$

It is *supercommutative* if $[A, A] = 0$.

A superspace A is a *Poisson superalgebra* if it is equipped with a bracket $\{-,-\}\colon A \times A \to A$ such that $(A, \{-,-\})$ is a Lie superalgebra and for any $a \in A$, the operator $\{a,-\}\colon A \to A$ is a *superderivation*: for all homogeneous elements $a, b \in A$,

$$\{a, bc\} = \{a,b\}c + (-1)^{|a||b|}b\{a,c\}.$$

Let E be a $\mathbb{C}$-vector space. The *exterior algebra* $\bigwedge E$ is the quotient of the tensor algebra $T(E) = \bigoplus_{k\in\mathbb{Z}} T^k(E)$, with $T^k(E) = E \otimes \cdots \otimes E$ the k-fold tensor product, by the two-sided ideal $I(E)$ generated by elements of the form $v \otimes w + w \otimes v$ with $v, w \in E$. The product in $\bigwedge E$ is usually denoted by $v \wedge w$. Since $I(E)$ is graded, the exterior algebra inherits a grading

$$\bigwedge E = \bigoplus_{k\in\mathbb{Z}} \bigwedge{}^k E.$$

Clearly, $\bigwedge^0 E = \mathbb{C}$ and $\bigwedge^1 E = E$. We may thus think of $\bigwedge E$ as the associative algebra linearly generated by E, subject to the relations $v \wedge w + w \wedge v = 0$. We will regard $\bigwedge E$ as a graded superalgebra, where the $\mathbb{Z}_2$-grading is the mod 2 reduction of the $\mathbb{Z}$-grading. Since

$$[u_1, u_2] = u_1 \wedge u_2 - (-1)^{k_1 k_2} u_2 \wedge u_1 = 0$$

for $u_1 \in \bigwedge^{k_1} E$ and $u_2 \in \bigwedge^{k_2} E$, we see that $\bigwedge E$ is supercommutative.

Assume that E is endowed with a symmetric bilinear form $B\colon E \times E \to E$ (possibly degenerate).

Definition E.1 The *Clifford algebra*[1] $Cl(E, B)$ is the quotient of $T(E)$ by the two-sided ideal $\mathscr{I}(E, B)$ generated by all elements of the form

$$v \otimes w + w \otimes v - B(v, w)1, \qquad v, w \in E.$$

Clearly, $Cl(E, 0) = \bigwedge V$.

The inclusions $\mathbb{C} \to T(E)$ and $E \to T(E)$ descend to inclusions $\mathbb{C} \to Cl(E, B)$ and $E \to Cl(E, B)$ respectively. We will always view E as a subspace of $Cl(E, B)$.

Let us view $T(E) = \bigoplus_{k\in\mathbb{Z}} T^k(E)$ as a filtered superalgebra, with the $\mathbb{Z}_2$-grading and filtration inherited from the $\mathbb{Z}$-grading. Since the elements $v\otimes w + w\otimes v - B(v, w)1$ are even, of filtration degree 2, the ideal $\mathscr{I}(E, B)$ is a filtered super subspace of $T(E)$, and hence $Cl(E, B)$ inherits the structure of a filtered superalgebra. The $\mathbb{Z}_2$-grading and filtration on $Cl(E, B)$ are defined by the condition that the generators $v \in E$ are odd, of filtration degree 1. In the decomposition

$$Cl(E, B) = Cl(E, B)^{\bar{0}} \oplus Cl(E, B)^{\bar{1}}$$

the two summands are spanned by products $v_1 \ldots v_k$ with k even, respectively odd. We will always regard $Cl(E, B)$ as a filtered superalgebra. Then the defining relations for the Clifford algebra become

$$[v, w] = vw + wv = B(v, w), \qquad v, w \in E.$$

The *quantization map*, given by the anti-symmetrization:

[1] In [214], there is a factor 2. For some reasons, we prefer here a different normalization.

$$q\colon \bigwedge(E) \to Cl(E,B), \quad v_1 \wedge \ldots \wedge v_k \mapsto \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) v_{\sigma 1} \ldots v_{\sigma k},$$

with $\mathfrak{S}_k$ the permutation group of order k, is an isomorphism of superspaces. Its inverse is called the *symbol map*.

Proposition E.1 *The symbol map $\sigma\colon Cl(E,B) \to \bigwedge E$ induces an isomorphism of graded superalgebras,*

$$\operatorname{gr} Cl(E,B) \xrightarrow{\sim} \bigwedge E.$$

Since $\bigwedge(E)$ is supercommutative, $\operatorname{gr} Cl(E,B)$ inherits a Poisson superalgebra structure[2], and the graded symbol map is an isomorphism of graded Poisson superalgebras. The Poisson bracket on $\bigwedge E$ can be described by:

$$\{v,w\} = B(v,w), \qquad v,w \in E = \bigwedge\nolimits^1 E.$$

For more about Clifford algebras, we refer to the recent book of Eckhard Meinrenken (it also addresses *Weil algebras* and *quantized Weil algebras*) [214].

[2] The arguments are similar to the case of almost commutative algebras; see §B.1.

Hints for the exercises

2.3 Use the proof of Theorem 2.2.

2.4 Notice that the locality axiom is automatically satisfied by the OPE: cf. Proposition 2.1, (i) $\Rightarrow$ (ii).

3.2 Apply the "Frobenius reciprocity", which asserts that

$$\operatorname{Hom}_{\hat{\mathfrak{g}}}(U(\hat{\mathfrak{g}}) \otimes_{\mathfrak{g}[t]\oplus\mathbb{C}K} \mathbb{C}_k, V^k(\mathfrak{g})) \cong \operatorname{Hom}_{\mathfrak{g}[t]\oplus\mathbb{C}K}(\mathbb{C}_k, V^k(\mathfrak{g})).$$

4.1
(i) Describe $F^p \mathrm{Vir}^c_\Delta$, where $\Delta \in \mathbb{Z}_{\geqslant 0}$, using the PBW Theorem.
(ii) Just use (i).
(iii) Remember that by Remark 4.3, one can go one step further, and then compute $\sigma_1(L_{(0)}L)$, $\sigma_0(L_{(1)}L)$ using the commuting relations.

5.1 Note that the maximal submodule of $L_1(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^2|0\rangle$ to show that $R_V \cong \mathrm{Zhu}(\mathrm{gr}^F V)$, and use Example 5.1.

7.1
(i) Introduce the set Y of $y \in f + \mathfrak{g}^e$ such that $[y, [e, \mathfrak{g}]] \cap \mathfrak{g}^e \neq 0$, and use the $\mathbb{C}^*$-contracting action $\tilde{\rho}$ to show that Y must be empty.
(ii) Remember that the symplectic form on $T_\xi(\mathbb{O})$ was described in Example B.1 and observe that the annihilator of $T_\xi(\mathscr{S}_f) \simeq \mathfrak{g}^e$ in $\mathfrak{g}$ is $(\mathfrak{g}^e)^\perp = [e, \mathfrak{g}]$.
(iii) Notice that Part (i) of Theorem B.1 is known by Theorem 7.2. Part (ii) follows from previous questions.

8.1 Use $\mathrm{gr}_K\, \widetilde{U}(\mathfrak{g}, f) \cong \mathbb{C}[\widetilde{\mathscr{S}_f}]$ and note that $\mathbb{C}[\widetilde{\mathscr{S}_f}]$ is simple as a Poisson algebra since $\widetilde{\mathscr{S}_f}$ is symplectic.

9.1
(i) Remember that Q is odd and, hence, observe that $Q^2_{(0)} = \dfrac{1}{2}[Q_{(0)}, Q_{(0)}]$. Then use Borcherds identities.

(ii) Show that $\ker Q_{(0)}$ is a vertex subalgebra of V, and that $\operatorname{im} Q_{(0)}$ is a vertex ideal of it.

9.2 Use Corollary 13.1.

9.3
(i) The main point is the locality axiom.
(ii) Observe that $\hat{Q} = (e_{(-1)}|0\rangle + |0\rangle) \otimes e^*_{(0)}|0\rangle$ and then compute $\hat{Q}(z)\hat{Q} = 0$.
(iv) Use the action of the Virasoro given by (iii).

12.1
(i) Kostant's Separation Theorem [184, Th. 0.2 and 0.11] says that $S = ZH$, where $Z \cong \mathbb{C}[\Omega]$ is the center of the symmetric algebra S of $\mathfrak{sl}_2$, and H is the space of invariant harmonic polynomials which decomposes, as an $\mathfrak{sl}_2$-module, as $H = \bigoplus_{\lambda\in\mathbb{Z}_{\geqslant 0}} V_\lambda^{m_\lambda}$, with $m_\lambda = 1$ for all λ. Therefore, $S^{\operatorname{ad} e} = \bigoplus_{\lambda\in\mathbb{Z}_{\geqslant 0}} ZV_\lambda^{\operatorname{ad} e}$. To conclude, observe that, v being a singular vector, it has a fixed weight and, hence, a fixed degree.
(ii) Note that from (i), $\Omega e \in \sqrt{I_k}$ and, so, $\Omega\, \mathfrak{sl}_2 \in \sqrt{I_k}$, whence $\Omega \in \sqrt{I_k}$. But in $\mathfrak{sl}_2$, the nilpotent cone is precisely the zero locus of Ω.

12.2
(i) Just use the commuting relations in $V^{-3/2}(\mathfrak{sl}_3)$.
(ii) Observe that the image I_k of the maximal proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the vector $\bar{v}$ as an $(\operatorname{ad}\mathfrak{sl}_3)$-module, where

$$\bar{v} = \frac{1}{3}\,(h_1 - h_2)\, e_{1,3} + e_{1,2}e_{2,3}$$

is the image of v in $R_{V^{-3/2}(\mathfrak{sl}_3)} \cong \mathbb{C}[h_i, e_{k,l}\,;\, i = 1, 2, k \neq l]$. Verify that

$$(\operatorname{ad} e_{3,2})(\operatorname{ad} e_{2,1})\bar{v} = -e_{1,2}e_{2,1} + e_{1,3}e_{3,1} + \frac{1}{3}\,(2h_1 + h_2)\, h_2,$$
$$(\operatorname{ad} e_{2,1})(\operatorname{ad} e_{3,2})\bar{v} = -e_{2,3}e_{3,2} + e_{1,3}e_{3,1} + \frac{1}{3}\,(h_1 + 2h_2)\, h_1,$$

and deduce from this that the intersection $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h}$ is zero. For the last part, resume the arguments of the proof of Proposition 12.1.
(iii) Verify that $e_{1,2} + e_{2,3}$ is not in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.
(iv) Observe that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ cannot be reduced to zero.

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