Dissipative Euler flows satisfying the local energy inequality

$$\begin{cases} \partial_t v_q + (v_q \cdot \nabla) v_q + \nabla p_q = \nu_q \Delta v_q \\ \operatorname{div} v_q = 0 \end{cases}$$

$$E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v_q|^2 (x,t)\, dx$$

$$E'(t) = - \nu_q \int_{\mathbb{T}^3} |\nabla v_q|^2 (x,t)\, dx$$

$$> 0$$

$$\nu_q \longrightarrow 0$$

Theorem (Isett 2018) $\forall \alpha < \frac{1}{3}$

If $q = +\infty$, i.e. $\nu_q = 0$. $\exists$ weak solutions v s.t.

$$|v(x,t) - v(y,t)| \leq C|x-y|^{\alpha}$$

and $E'(t) \neq 0$

Remark v is achieved as $\lim_{q \to +\infty} v_q$

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0 \end{cases}$$

Compare

$\mathring{R}_q$ to $\nu_q D v_q$

$\operatorname{tr} \mathring{R}_q = 0$ $\qquad$ $\mathring{R}_q$ is symmetric

$$\operatorname{div}\left(Du_q + Du_q^T\right) = \Delta u_q$$

We should compare

$\mathring{R}_q$ to $Du_q + Du_q^T$

Now

$$u_q = \sum_{i=0}^{q} w_i$$

Fourier scale for w_i: λ^i

(actually it is λ^{b^i} $b > 1$)

$$u_{q+1}(x,t) = u_q(x,t) + W(u_q, \mathring{R}_q, \lambda x, \lambda t)$$

+ correction

$$W = \sum_j a_j(u_q, \mathring{R}_q)\, \underbrace{e^{i\lambda k_j \cdot x}}_{W_j}$$

Not so important in many instances

Decide W so that

$$\underbrace{\partial_t v_{q+1} + \operatorname{div}(v_{q+1} \otimes v_{q+1}) + \nabla p_{q+1}} = \operatorname{div} \mathring{R}_{q+1}$$

$$\mathring{R}_{q+1} = \operatorname{div}^{-1}\left(\underbrace{\qquad\qquad\qquad} \right)$$

1) W solves a PDE in the fast variables and kills $\lambda_q(\ )$ terms in the green expressions

2) slow derivatives are killed because

$$\operatorname{div}^{-1}\left(\sum_j b_j(t,x)\, e^{i \lambda_q k_j \cdot x} \right) \sim \frac{1}{\lambda_q}$$

3) Except for $\operatorname{div}_{slow}\left(\sum_{j,k} \underbrace{a_j a_k W_j \otimes W_k + \mathring{R}_q}_{\text{Resonances "cancel" } \mathring{R}_q} \right)$

A) ω_{q+1} interacts "only" with ω_q and ω_{q+2}

B) L^∞ size of $\omega_{q+1} \sim \lambda_{q+1}^{-1/3}$

$$\int |\omega_{q+1}|^2 \sim \lambda_{q+1}^{-2/3}$$

Energy spectrum $\sim \lambda^{-5/3}$

C) Do $\mathring{R}_q$ and v_q Du_q compare in the

$$\|Du_q\|_\infty \sim \lambda_q^{2/3}$$

$$\|\mathring{R}_q\|_\infty \simeq \lambda_{q+1}^{-2/3}$$

$$v_q \sim \lambda_{q+1}^{-2/3} \lambda_q^{-2/3}$$

$$\sim \lambda_q^{-4/3}$$

Kolmogorov's scale !

$\vdash$ Energy identity

$$\partial_t \frac{|u_q|^2}{2} + \operatorname{div}\left(u_q\left(\frac{|u_q|^2}{2} + p_q\right)\right)$$

$$= \left(\Delta \frac{|u_q|^2}{2} - |Du_q|^2\right)\nu_q$$

<u>Suggests</u>

<u>Conjecture</u> (Isett) $\forall \alpha < \frac{1}{3}$ $\exists v$ sol. of Euler s.t.

$$(EI) \quad \partial_t \frac{|v|^2}{2} + \operatorname{div}\left(v\left(\frac{|v|^2}{2} + p\right)\right) \leq 0$$

and $v \in L^\infty_t(C^\alpha_x)$

<u>Theorem</u> (Buckmaster, D, Székelyhidi, Vicol 2018)

$$\frac{d}{dt} \int \frac{|u|^2}{2}(x,t)\,dx < 0 .$$

<u>Theorem</u> (D - Székelyhidi 2007) ∃ weak sol. sat. (EI) in L^∞

<u>Theorem</u> (Isett 2019)

∃ C^α solutions sat. (EI) if $\alpha < \frac{1}{15}$

<u>Theorem</u> (D - Kwon 2020) $\alpha < \frac{1}{7}$

$$\begin{cases} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0 \\ \partial_t |u_q|^2 + \operatorname{div}\big((|u_q|^2 + p_q)u_q\big) \\ \quad \le \operatorname{div}(\mathring{R}_q u_q) + \operatorname{div} \phi_q \end{cases}$$

vanishing

Resonances on the cubic term

well kill ϕ_q