Generalised Symmetries and vortices

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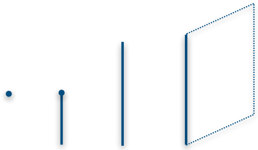
Introduction

Introduction

Recent generalisation of the notion of *symmetry* in quantum field theory.

- Ordinary symmetries are groups and act on *local* operators.
- Generalised symmetries may form n-groups acting on *extended* operators of dimension $0, 1, \cdots, n-1$.
- Symmetries may even by non-invertible!

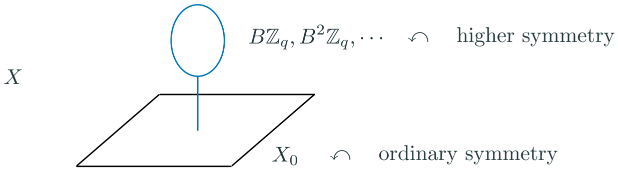


This is the paradigm of *categorical symmetry*.

Introduction II

I will discuss connections with moduli spaces in quantum field theory.

- Groups act by automorphisms of moduli *spaces*.
- Higher groups act by automorphisms of higher *stacks*.
- Captures spontaneous breaking of higher-group symmetries.



Introduction

I would like to present some connections to topics at this workshop.

I will focus on 3d $\mathcal{N} = 2$ supersymmetric gauge theory.

1. Symmetries in 3d gauge theory
2. Moduli stacks in 3d gauge theory[1]
3. Applications to quantum K-theory[2]

[1]WIP: Lakshya Bhardwaj, MB, Andrea Ferrari, Sakura Schäfer-Nameki
[2]WIP: MB, Luca Cassia

Gauge Theory

A supersymmetric gauge theory

Consider the following 3d $\mathcal{N} = 2$ supersymmetric gauge theory.

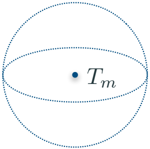
- Gauge group $G = U(1)$.
- Chiral multiplets $\Phi_1, \ldots, \Phi_N$ of charge $q > 0$.
- Critical CS level $k = -\frac{N}{2} q^2$.

The critical CS level means gauge charges of all monopole operators are multiples of N.

0-form symmetries I

Ordinary or 0-form symmetries act on local operators.

- A fundamental role played throughout by the *topological* symmetry.
- This $U(1)$ symmetry measures magnetic flux.
- It is the analogue of θ-angle in two dimensions.
- Monopole operator T_m with $m \geq 0$ has topological charge m.



$$\int_{S^2} F = 2\pi m$$

0-form symmetries II

Additionally, there is a $PSU(N)$ flavour symmetry.

- Chiral multiplets $\Phi_1, \dots, \Phi_N$ in fundamental of $\mathfrak{su}(N)$.
- Gauge invariant local operators:

$$\overline{\Phi_i}\Phi_j, \quad T_{-1}(\Phi_{i_1}\Phi_{j_1})\cdots(\Phi_{i_N}\Phi_{j_N})$$

- Always in powers of adjoint representation of $\mathfrak{su}(N)$.
- The symmetry acting faithfully is $PSU(N)$.

2-group symmetry I

Three-dimensional theories may have generalised symmetries:

- 1-form symmetries.
- 0,1-form symmetries may combine to form a 2-group.[3]

This captures the following phenomena:

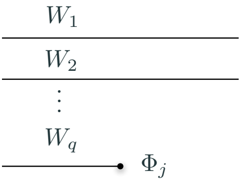
1. Some line operators cannot end.
2. Line operators may end on local operators transforming in central extensions of 0-form symmetry.

[3] Cordova-Dumitrescu-Intriligator '19, Benini-Cordova-Hsin '19

2-group symmetry II

Consider supersymmetric Wilson lines W_n of charge n:

- It cannot end unless $q \mid n \;\Rightarrow\; \mathbb{Z}_q$ 1-form symmetry.
- If $q \mid n$, it ends on operators transforming in $SU(N)$ representations unless $qN \mid n$.



The 2-group symmetry depends on whether q, N are co-prime.

2-group symmetry III

The symmetries combine into a 2-group determined by data:

1. 0-form symmetry $PSU(N)$
2. 1-form symmetry $\mathbb{Z}_q$.
3. Postnikov class $\Theta \in H^3(BPSU(N), \mathbb{Z}_q)$.

Important point:

- The structure depends on short exact sequence

$$1 \to \mathbb{Z}_q \to \mathbb{Z}_{qN} \to \mathbb{Z}_N \to 1$$

- $\Theta = 0$ if sequence splits $\Leftrightarrow \gcd(q, N) = 1$.
- $\Theta \neq 0$ if sequence is extension $\Leftrightarrow \gcd(q, N) > 1$.

Structure Group I

A convenient way to encode symmetries is via the *structure group*:

$$S = \frac{U(1) \times SU(N)}{\mathbb{Z}_{qN}}\,.$$

- This is the most general combination of gauge and flavour symmetries whose bundles may be coupled to consistently.
- Quotient by central subgroup $\mathbb{Z}_{qN} \subset U(1) \times \mathbb{Z}_N$.
- This is generated by central element

$$\left(e^{2\pi i/qN}, e^{2\pi i/N} 1_N\right).$$

Structure Group I

The symmetries are recovered from the structure group as follows.

$$S = \frac{U(1) \times SU(N)}{\mathbb{Z}_{qN}} \,.$$

- The 0-form symmetry is the quotient $PSU(N) = SU(N)/\mathbb{Z}_N$ by elements $e^{2\pi i/N} i_N$ in the image of $\mathbb{Z}_{qN}$ under

$$p : U(1) \times \mathbb{Z}_N \to \mathbb{Z}_N \,.$$

- The 1-form symmetry is intersection $\mathbb{Z}_q = \mathbb{Z}_{qN} \cap U(1)\,.$
- There is natural short exact sequence

$$1 \to \mathbb{Z}_q \to \mathbb{Z}_{qN} \to \mathbb{Z}_N \to 1 \,.$$

Geometry

Sigma model + TQFT

Now introduce real FI parameter $t > 0$:

- Flows to a supersymmetric sigma model $\sigma : \mathbb{R}^3 \to X_0$ to projective space

$$X_0 = \mathbb{P}^{N-1} .$$

- This is coupled to a $\mathbb{Z}_q$ gauge field A,

$$\exp \frac{2\pi i}{q} \int \sigma^* c \cup A ,$$

  with non-trivial class $c \in H^2(\mathbb{P}^{N-1}, \mathbb{Z}_q)$.

- This reflects unbroken $\mathbb{Z}_q$ gauge symmetry.

Sigma model to stack

This is a supersymmetric sigma model to weighted projective stack

$$X = \mathbb{P}^{N-1}(q, \ldots, q) = [(\mathbb{C}^N - \{0\})/\mathbb{C}^\times]$$

- There is a morphism to underlying course moduli space

$$\pi : X \to X_0 = \mathbb{P}^{N-1}\,.$$

- The fibers $\pi^{-1}(x) = B\mathbb{Z}_q$ capture unbroken $\mathbb{Z}_q$ gauge symmetry.
- The stack is a gerbe with non-trivial class $c \in H^2(\mathbb{P}^{N-1}, \mathbb{Z}_q)$.

Action of symmetries

2-group symmetry acts by automorphisms of stack $X = \mathbb{P}^{N-1}(q, \dots, q)$.

- 0-form symmetry $PSU(N)$ acts by Kähler isometries of coarse moduli space $X_0 = \mathbb{P}^{N-1}$.
- 1-form symmetry $B\mathbb{Z}_q$ on fibers $\pi^{-1}(x) = B\mathbb{Z}_q$.
- Postnikov class Θ is obstruction to $PSU(N)$-equivariant structure of $\mathbb{Z}_q$-gerbe $\pi : X \to X_0$.

The moduli stack captures spontaneous symmetry breaking of full symmetry group.

Back to Wilson lines

Let us try to make contact with previous discussion via line operators.

- Wilson lines W_n correspond to line bundles $\mathcal{O}(n)$ on $X = \mathbb{P}^{N-1}(q, \ldots, q)$.
- The local operators on which they end are sections $\Gamma(X, \mathcal{O}(n))$.
- The line bundles are fractional unless $q \mid n$.
- They only admit $PSU(N)$-equivariant structure if $qN \mid n$.

Quasi-map K-theory

Compactification

Now compactify on $S^1 \times \mathbb{R}^2$.

- Introduce mass parameters associated to 0-form symmetries and integrate out chiral multiplets.
- Obtain effective 2d $\mathcal{N} = (2, 2)$ theory for twisted vector multiplet σ.
- Twisted effective superpotential $W(\sigma)$.
- Supersymmetric vacua given by

$$\exp \frac{\partial W}{\partial \sigma} = 1\,.$$

Example

Supersymmetric vacua in example given by solutions to

$$\prod_{j=1}^{N}(1 - P^q X_j^{-1})^q = Q\,.$$

Parameters are supersymmetric Wilson lines wrapping S^1:

- Dynamical Wilson line $P = e^{-\beta\sigma}$.
- Background Wilson lines for symmetries Q, X_j with $\prod_j X_j = 1$.
- The number of solutions $q^2 N$, in agreement with Witten index.

A trick

Consider first the supersymmetric vacuum equations for $q = 1$,

$$\prod_{j=1}^{N}(1 - PX_j^{-1}) = Q\,.$$

- Now gauge subgroup $\mathbb{Z}_q \in U(1)$ of topological symmetry.
- This is equivalence to multiplying charges by q.
- Implemented here by transformation $(P, Q) \mapsto (P^q, Q^{1/q})$.

$$\prod_{j=1}^{N}(1 - P^q X_j^{-1})^q = Q\,.$$

Structure group

Recall the form of the structure group

$$S = \frac{U(1) \times SU(N)}{\mathbb{Z}_{qN}}\,.$$

- P is supersymmetric Wilson line for $U(1)$.
- X_j are supersymmetric Wilson lines for $SU(N)$.
- Denominator acts by

$$(P, X_j) \to (e^{2\pi i/qN} P, e^{2\pi i/N} X_j)\,.$$

- Invariant combinations $P^q X_j^{-1}$ are supersymmetric Wilson lines for structure group S.

Degeneracy

Is there a symmetry based argument for q^2 degeneracy?

- The 1-form symmetry implies two independent $\mathbb{Z}_q$-actions on solutions of the vacuum equations by

$$P \to e^{2\pi i/q} P$$
$$Q \to e^{2\pi i/q} Q$$

- The first is from $\mathbb{Z}_q \subset \mathbb{Z}_{qN}$ transformation on (P, X_j).
- The second is a consequence of fractional gauge vortices / discrete Wilson lines when gauging $\mathbb{Z}_q \subset U(1)$.

Quais-map K-theory

What is the mathematical interpretation of the supersymmetric vacuum equations?

$$\prod_{j=1}^{N}(1 - P^q X_j^{-1})^q = Q\,.$$

- Introduce symplectic form $d\log P \wedge d\log Q$ on $T^*\mathbb{C}^\times$.
- Family of Lagrangians $L_X \subset T^*\mathbb{C}^\times$.
- Equivariant quasi-map K-theory of weighted projective stack

$$X = \mathbb{P}^{N-1}(q, \ldots, q)\,.$$

Quais-map K-theory II

The partition function on $S^1 \times \Sigma_g$ with $g \geq 1$ may be expressed as a sum over supersymmetric vacua:

$$\mathrm{Tr}_{\mathcal{H}_{\Sigma_g}}(-1)^F = \sum_m Q^m \int \widehat{A}(\mathcal{M}_m) = \sum_{P_*} f(P_*; Q, X_j)$$

- Here $\mathcal{M}_m$ is moduli stack of vortices or stable quasi-maps

$$\Sigma_g \to X = \mathbb{P}^{N-1}(q, \ldots, q)\,.$$

- Vortex number or degree $m \in \pi_1(G) = \mathbb{Z}$.
- Equivariant parameters $X_1, \ldots, X_N$.

Quais-map K-theory III

The supersymmetric partition function takes the form

$$q^{2g} \times \{\text{rational function of } Q, X_j\}$$

- Topological lines generating $\mathbb{Z}_q$ 1-form symmetry.
- They may be supported on $2g$ 1-cycles in Σ_g
- Symmetry based argument for q^{2g} degeneracy.

Categorification

Why categorify?

We have seen experimentally that 1-form / 2-group symmetry plays a role in quasi-map K-theory of stacks.

- Systematic understanding?
- 2-groups act via automomorphisms of categories.

The very rough idea is[4]

1. Quantum K-theory of $X \to$
2. K-theory of loop space of $X \to$
3. Coherent sheaves on loop space of X.

[4]Following Givental's loop space formalism.

Line defects

A natural candidate is category of line defects in 3d $\mathcal{N} = 2$ theories.

- Consider vortices or quasi-maps $\mathbb{D}^\times \to X = \mathbb{P}^{N-1}(q, \ldots, q)$.
- Consider sheaves on moduli stack $\mathbb{M}$.
- Pass to universal cover $\widetilde{\mathbb{M}}$ and work equivariantly with respect to deck transformations $\pi_1(G) \cong \mathbb{Z}$.

Claim: 2-group acts by automorphisms of this category.

Example

There is an algebraic model for $\widetilde{\mathbb{M}}$ as toric arc stack ΛX [5].

- Sequence of sub-stacks $\cdots \Lambda^a X \subset \Lambda^{a+1} X \cdots$
- Q^a corresponds to sheaf $\mathcal{O}_{\Lambda^a X}$.
- Resolution of morphism $\mathcal{O}_{\Lambda^0 X} \to \mathcal{O}_{\Lambda^1 X}$ categorifies product

$$\prod_{j=1}^{N} (1 - P^q X_j^{-1})^q$$

[5] Arkhipov-Kapranov '04

Thanks!