Correlation functions from hydrodynamics beyond the Boltzmann-Gibbs paradigm

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Hydrodynamics and fluctuations

- microscopic approaches in condensed matter systems

(ICTS Bengaluru, India)

One-dimensional many-body interacting systems with conservation laws

We are given some Hamiltonian H for an extensive (infinitely long) system in one dimension, which admits a certain number of conservation laws,

$$\partial_t q_i + \partial_x j_i = 0, \qquad \partial_t Q_i = 0, \qquad Q_i = \sum_{x\in\mathbb{Z}} q_i(x)$$

I will consider throughout a thermal state

$$\langle\cdots\rangle = Z^{-1}\mathrm{Tr}\Big(e^{-\beta H}\cdots\Big)$$

but all results / ideas hold for more general states such as GGEs $e^{-\sum_i \beta^i Q_i}$.

Correlations at large scales of space and time

The problem under study: **dominant correlations at large scales of space and time**:

$$\langle \mathcal{a}(x,t)\mathcal{b}(0,0)\rangle^{\rm c} \sim \quad ??? \qquad (|x|, t\to\infty)$$

where $\langle \mathcal{a}(x,t)\mathcal{b}(0,0)\rangle^{\rm c} = \langle \mathcal{a}(x,t)\mathcal{b}(0,0)\rangle - \langle \mathcal{a}(x,t)\rangle\langle \mathcal{b}(0,0)\rangle$ is the covariance.

I will be concentrating on **hyperbolic scaling**,

$$|x|, t\to\infty, \quad x/t \text{ fixed.}$$

Correlations at large scales of space and time

For illustration purposes, I concentrate on the XX model:

$$H = -\sum_{x\in\mathbb{Z}} \left[\sigma^1_x\sigma^1_{x+1} + \sigma^2_x\sigma^2_{x+1} + h\sigma^3_x\right]$$

By Wick's theorem and saddle point analysis ($|x|, t \to \infty,\ x/t = \xi \in (-4,4)$)

$$\langle \sigma^3_x(t)\sigma^3_0(0)\rangle^{\rm c} \sim \frac{2}{\pi|t|\sqrt{16-\xi^2}} \sum_{a=\pm} \times$$

$$\times\ n_a\Big(1 - n_a + a\mathrm{i}\,(1-n_{-a})(-1)^x e^{-2a\mathrm{i}(x\arcsin(\xi/4) + t\sqrt{16-\xi^2})}\Big)$$

where

$$n_\pm = \frac{1}{1+\exp\Big[-\beta\Big(2h \mp \sqrt{16-\xi^2}\Big)\Big]}$$

Correlations at large scales of space and time

Asymptotics of Fredholm determinants ($|x|, t \to \infty,\ x/t = \xi \in \mathbb{R}$)

$$\langle \sigma_x^+(t)\sigma_0^-(0)\rangle \qquad (E(k) = 4h - 2\cos k,\ v(k) = 4\sin k)$$

$$\asymp \begin{cases} \exp\left[|t|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,|\xi - v(k)|\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi| \le 4) \\ \exp\left[|x|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi| > 4,\ |h| \le 2) \\ \text{[Its, Izergin, Korepin, Slavnov 1992; Jie (PhD thesis) 1998]} & \\ e^{\mathrm{i}\Phi(x,t)}\exp\Big[-|x|\min\Big(\operatorname{arccosh}(|h|/2), & \\ \qquad\qquad \operatorname{arccosh}(|\xi|/4) - \sqrt{1 - \frac{16}{\xi^2}}\Big)\Big]\times & \\ \qquad \times\ \exp\left[|x|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi| > 4,\ |h| > 2) \\ \text{[BD, Del Vecchio Del Vecchio 2021 – from hydrodynamics]} & \end{cases}$$

Correlations at large scales of space and time

Three different types of behaviours are seen:

- Monotonic algebraic decay
- Oscillating algebraic decay
- Exponential decay

Can we explain these results using **general hydrodynamic principles** applied to the XX model? What are these principles?

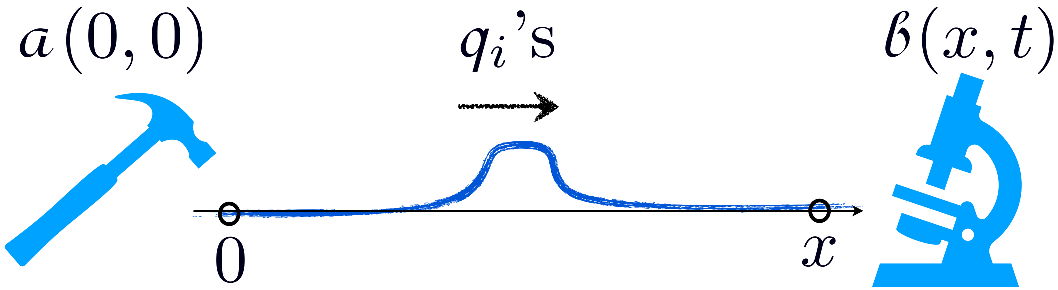
Boltzmann-Gibbs principle (conventional hydrodynamic projections)

There is a **reduction of the number of degrees of freedom**, a projection onto slowly decaying hydrodynamic modes (or ballistic waves),

$$\lim_{x,t\to\infty} \langle a(0,0)b(x,t)\rangle^{\mathrm{c}} \sim \text{projection of } a \text{ onto modes } q_i$$

$$\times \quad \text{propagation } \langle q_i(0,0)q_j(x,t)\rangle^{\mathrm{c}}$$

$$\times \quad \text{projection of modes } q_j \text{ onto } b$$



Boltzmann-Gibbs principle (conventional hydrodynamic projections)

Define inner product ("susceptibilities" with wavenumber k)

$$\langle a\,, b\rangle_k = \sum_x e^{\mathrm{i}kx}\langle a^\dagger(x) b(0)\rangle^{\mathrm{c}}$$

Define matrix of susceptibilities (or static correlation matrix)

$$\mathsf{C}_{ij} = \langle q_i, q_j\rangle_0$$

inverse $\mathsf{C}_{ij}\mathsf{C}^{jm} = \delta_i^m$. In limit $k\to 0, t\to\infty$ with kt fixed, **hydrodynamic projection principle** [Zwanzig 1961; Mori 1965; .... ; Spohn 2014; Mendl, Spohn 2015; BD, Spohn 2017; BD 2018]

$$\lim_{kt}\langle a\,(t), b\rangle_k = \sum_{i,j,m,n}\langle a\,, q_i\rangle_0\,\mathsf{C}^{ij}\,\mathsf{S}_{jm}(kt)\,\mathsf{C}^{mn}\,\langle q_n, b\rangle_0$$

where

$$\mathsf{S}_{jm}(kt) = \lim_{kt}\langle q_j(t), q_m\rangle_k$$

Theorem [BD 2020]: under an appropriate definition of $\lim_{kt}$ this holds in every short-range homogeneous quantum spin chain.

Boltzmann-Gibbs principle (conventional hydrodynamic projections)

Remarks:

- $\langle \cdot, \cdot \rangle_0$ gives rise to an inner product and a Hilbert space, interpreted as the **Hilbert space of extensive homogeneous observables** $\mathcal{H}$.
- Time evolution $\tau_t = e^{\mathrm{i}t[H,\cdot]}$ gives rise to a **unitary operators on** $\mathcal{H}$.
- The space of conserved charges is mathematically defined as the $\bigcap_t$ **of kernel of** τ_t.

Boltzmann-Gibbs principle (conventional hydrodynamic projections)

Apply linearised response to Euler equation for **averages $\mathsf{q}_i(x,t),\ \mathsf{j}_i(x,t)$ in space-time dependent, entropy-maximised fluid cells**:

$$\partial_t \mathsf{q}_i(x,t) + \partial_x \mathsf{j}_i(x,t) = 0$$

This leads to linearised equation for perturbations of state

$$\partial_t \delta q_i(x,t) + \sum_j \mathsf{A}_i^{\ j} \partial_x \delta q_j(x,t) = 0, \qquad \mathsf{A}_i^{\ j} = \frac{\partial \langle j_i \rangle}{\partial \langle q_j \rangle}$$

and translates into equations for correlations

$$\frac{\partial}{\partial(kt)} \mathsf{S}_{jm}(kt) = \mathrm{i} \sum_n \mathsf{A}_j^{\ n} \mathsf{S}_{nm}(kt)$$

[Similar techniques for higher-point functions BD 2019; Fava, Biswas, Gopalakrishnan, Vasseur, Parameswaran 2021]

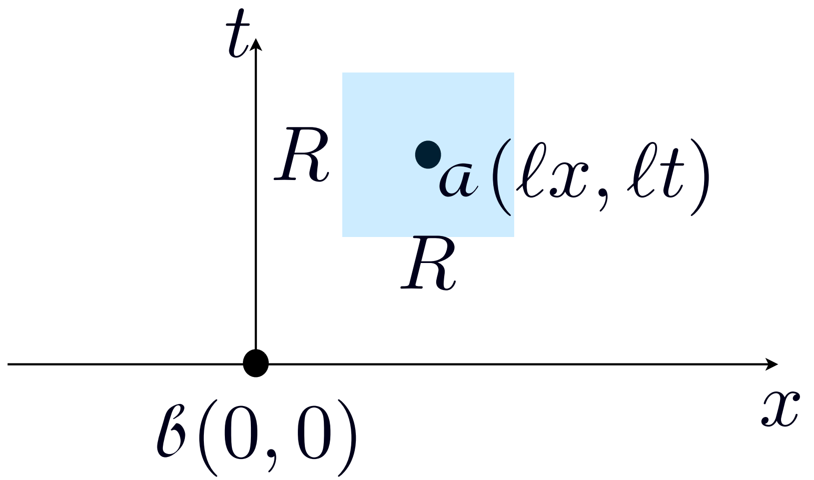
Theorem [BD 2020]: under an appropriate definition of “local” this holds for every local conserved densities $q_j,\ q_m$.

Boltzmann-Gibbs principle (conventional hydrodynamic projections)

A more physical way of re-writing this is using fluid-cell averaging

$$\langle \overline{a}(\ell x, \ell t) b(0,0) \rangle^{\rm c} \sim \ell^{-1} \sum_{i,j,n} \langle a\,, q_i \rangle_0 \, \mathsf{C}^{ij} \, \delta(x - \mathsf{A}t)_j^{\;n} \, \langle q_n, b \rangle_0$$

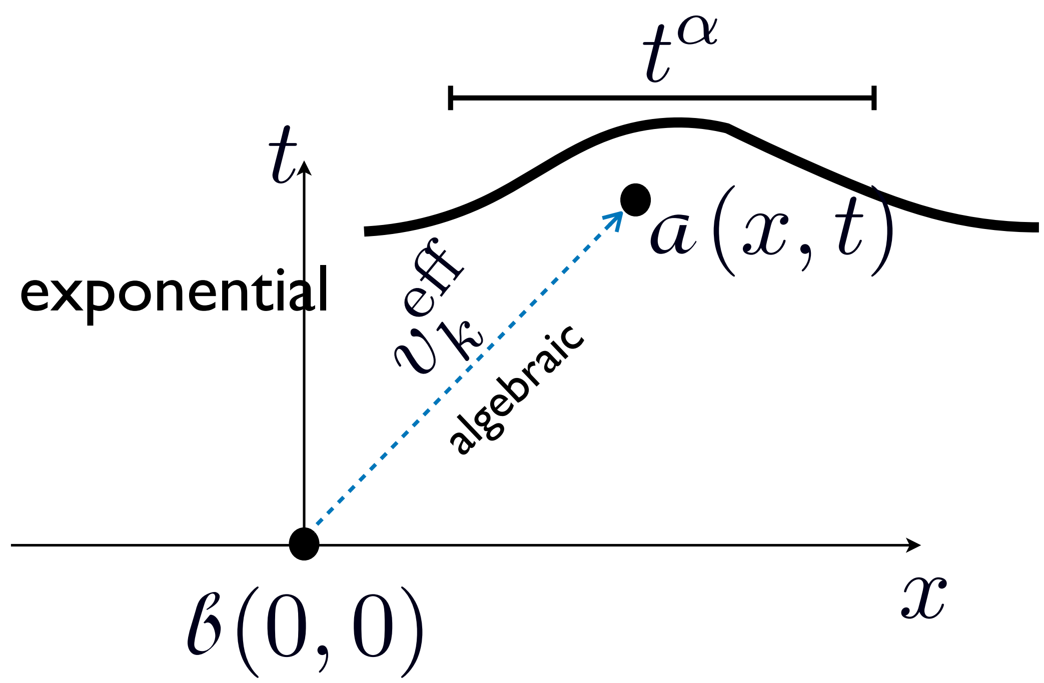
$$\overline{a}(x,t) = \frac{1}{R^2} \sum_{y=-R}^{R} \int_{-R}^{R} {\rm d}s \, a(x+y, t+s)$$



$$\ell \gg R \to \infty$$

Boltzmann-Gibbs principle (conventional hydrodynamic projections)

Typically algebraic decay along velocities $\mathrm{spectrum}(\mathsf{A}) = \{v_k^{\mathrm{eff}}\}$:



Application to XX model

Jordan-Wigner transformation

$$\sigma_x^+ = \frac{1}{2}(\sigma_x^1 + \mathrm{i}\sigma_x^2) = \exp\left(\mathrm{i}\pi \sum_{y=0}^{x-1} a_y^\dagger a_y\right) a_x^\dagger, \quad \sigma_x^- = (\sigma_x^+)^\dagger, \quad \sigma_x^3 = 2a_x^\dagger a_x - 1$$

where $a_x, a_x^\dagger$ are canonical fermions. Then H becomes bilinear: free fermions

$$H = -2\sum_{x=0}^{N-1}\left[a_x^\dagger a_{x+1} + a_{x+1}^\dagger a_x + h a_x^\dagger a_x\right] + hN$$

Diagonalised by Fourier transform (here $N \to \infty$)

$$a_x = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{\mathrm{i}kx} c(k)$$

where modes have energies

$$E(k) = 2h - 4\cos(k)$$

Application to XX model

The Hilbert space of extensive conserved charges has “continuous basis”

$$Q_k = c^\dagger(k)c(k) : k \in [-\pi, \pi)$$

The flux Jacobian A has continuous spectrum

$$\mathrm{spectrum}(\mathsf{A}) = \{v(k) = E'(k) = 4\sin(k)\}$$

One may evaluate all ingredients, e.g.

$$\langle \sigma_0^3, q_k \rangle_0 = 2\langle a_0^\dagger a_0 c^\dagger(k)c(k)\rangle - \langle 2a_0^\dagger a_0\rangle\langle c^\dagger(k)c(k)\rangle$$
$$\langle \sigma_0^\pm, q_k \rangle_0 = 0$$

σ_x^3 projects onto conserved charges, but $\sigma_x^\pm$ has zero projection!

Application to XX model

This gives

$$\langle \overline{\sigma}_x^3(t)\sigma_0^3(0)\rangle^{\rm c} \sim \frac{2}{\pi}\int_{-\pi}^{\pi} {\rm d}k\,\delta(x-v(k)t)n(k)(1-n(k))$$

$$= \frac{2}{\pi t\,\sqrt{16-\xi^2}}\sum_{a=\pm} n_a(1-n_a) \qquad (|\xi|<4)$$

$$\langle \overline{\sigma}_x^+(t)\sigma_0^-(0)\rangle^{\rm c} \sim 0$$

Standard Boltzmann-Gibbs principle only gives the monotonic (non-oscillating) algebraic part, because of fluid-cell averaging (non-oscillating) and projection onto “slowly decaying modes” (algebraic)!

Finite-frequency hydrodynamic projections

The theorems on hydrodynamic projections stay valid if we replace time evolution by

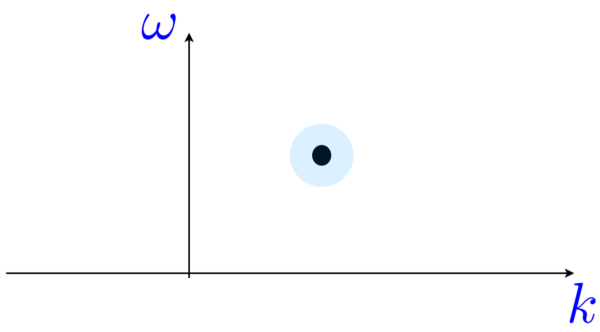
$$\tau_t = e^{\mathrm{i}t\,\mathrm{ad}H} \quad \to \quad \tilde\tau_t = e^{-\mathrm{i}\omega t} e^{\mathrm{i}t\,\mathrm{ad}H}$$

and if we replace space translation by

$$\iota_x : a(y) \mapsto a(y+x) \quad \to \quad \tilde\iota_x = e^{\mathrm{i}kx}\iota_x$$

That is, Hilbert space of "homogeneous" extensive observables $\tilde{\mathscr{H}}$ is based on $\tilde\iota_x$, and $\tilde\tau_t$ is valid time-evolution unitary operators it. [BD 2020]

Redo the same hydrodynamic projection construction, but based on oscillating time evolution and space translation: hydrodynamics near (ω, k) instead of $(0,0)$.



Finite-frequency hydrodynamic projections

New fluid-cell average that extracts the oscillating part:

$$\overline{a}(x,t) = \frac{1}{R^2}\sum_{y=-R}^{R}\int_{-R}^{R}\mathrm{d}s\, e^{\mathrm{i}ky-i\omega s}a\,(x+y,t+s)$$

New “susceptibility”

$$\langle a\,,b\rangle_0 = \sum_x e^{\mathrm{i}kx}\langle a^\dagger(x)b(0)\rangle^{\mathrm{c}}$$

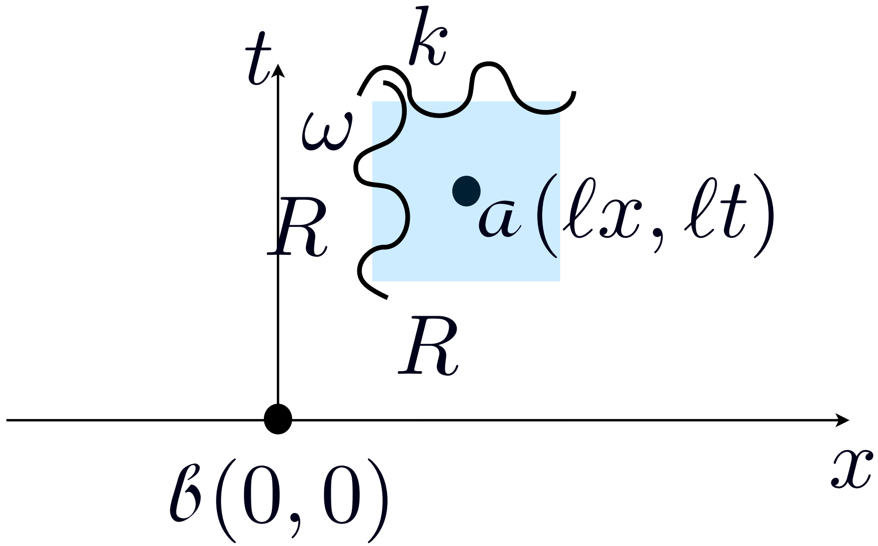
New condition of conservation

$$\mathrm{i}[H,Q_i] = \omega Q_i, \qquad Q_i = \sum_x e^{\mathrm{i}kx}q_i(x)$$

Finite-frequency hydrodynamic projections

Same formula

$$\langle \overline{a}(\ell x, \ell t) b(0,0)\rangle^{\rm c} \sim \ell^{-1} \sum_{i,j,n} \langle a\,, q_i\rangle_0\, \mathsf{C}^{ij}\, \delta(x - \mathsf{A}t)_j^{\ n}\, \langle q_n\,, b\rangle_0$$



Application to XX model

For every $\xi \in (-4, 4)$ there are two modes of velocity ξ

$$k_\pm : v(k_\pm) = 4\sin(k_\pm) = \xi$$

Then, there is an extensive (ω, k)-conserved quantity with $\omega = E(k_+) - E(k_-)$ and $k = k_+ - k_-$:

$$Q = c^\dagger(k_+)c(k_-)$$

This is extensive because both modes have the same velocity ξ.

We can again calculate all necessary ingredients of the projection formula, and obtain

$$\langle\overline{\sigma}_x^3(t)\sigma_0^3(0)\rangle^{\mathrm{c}} \overset{\omega,k}{\sim} \frac{2\mathrm{i}}{\pi t\sqrt{16-\xi^2}} \sum_{a=\pm} a n_a(1-n_{-a})(-1)^x e^{-2a\mathrm{i}(x\arcsin(\xi/4)+t\sqrt{16-\xi^2})}$$

Finite-frequency hydrodynamic projections give the correct oscillating algebraic decay! It is due to the presence of two modes of different energies but with the same velocity.

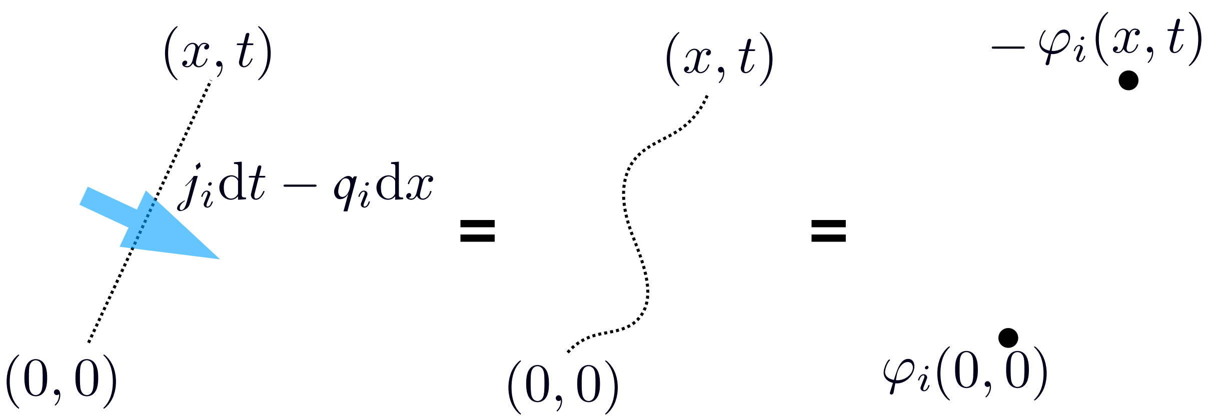
Correlation functions of twist fields and ballistic fluctuation theory

A “twist field” is a field $e^{\lambda\varphi_i(x,t)}$ where the “potential” $\varphi_i(x,t)$ is formally defined by solving the continuity equations:

$$q_i(x,t) = \partial_x \varphi_i(x,t), \qquad j_i(x,t) = -\partial_t \varphi_i(x,t)$$

The two-point function is an exponential of a path-independent line integral in space-time:

$$\langle e^{-\lambda(\varphi_i(x,t)-\varphi_i(0,0))}\rangle = \langle \exp\left[\lambda \int_{(0,0)}^{(x,t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right]\rangle$$



Correlation functions of twist fields and ballistic fluctuation theory

Recall that

$$\sigma_x^+ = \frac{1}{2}(\sigma_x^1 + \mathrm{i}\sigma_x^2) = a_x^\dagger \exp\left(\mathrm{i}\pi \sum_{y=0}^{x-1} a_y^\dagger a_y\right)$$

This suggests the identification

$$\sigma_x^+(t) = a_x^\dagger(t) e^{\mathrm{i}\pi\varphi_0(x,t)}$$

with $q_0(x,t) = a_x^\dagger(t) a_x(t)$ the fermion density.

Taking into account the even / odd boundary conditions at $x = 0$ coming from the JW transformation, and using the Lieb-Robinson bound and various other principles, it is possible to argue that indeed we get a "**space-time Jordan-Wigner string**"

$$\langle \sigma_x^+(t)\sigma_0^-(0)\rangle \asymp \langle a_x^\dagger(t) e^{\mathrm{i}\pi(\varphi_0(x,t)-\varphi_0(0,0))} a_0(0)\rangle$$

The presence of a "string" leads to a field that is semi-local, and this is one reason why $\sigma_x^\pm$ do not project onto extensive conserved charges.

Correlation functions of twist fields and ballistic fluctuation theory

To the quantity

$$\langle \exp\left[\lambda \int_{(0,0)}^{(x,t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right]\rangle$$

we apply **large-deviation theory**:

$$\langle \exp\left[\lambda \int_{(0,0)}^{(\ell x,\ell t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right]\rangle \asymp \exp\left[\ell F_i(\lambda; x, t)\right]$$

The **ballistic fluctuation theory** tells us how to evaluate $F_i(\lambda; x, t)$ using solely objects from Euler hydrodynamics. [Myers, Bhaseen, Harris, Doyon 2019; Doyon, Myers 2019]

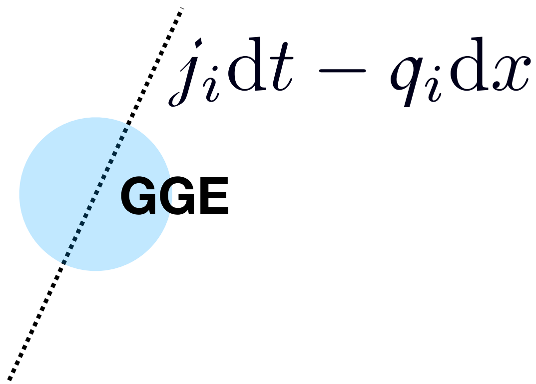
Correlation functions of twist fields and ballistic fluctuation theory

Its basis is the concept of “measure bias”

$$\lim_{\ell\to\infty} \frac{\langle \exp\left[\lambda \int_{(-\ell x,-\ell t)}^{(\ell x,\ell t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right] \cdots\rangle}{\langle \exp\left[\lambda \int_{(-\ell x,-\ell t)}^{(\ell x,\ell t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right]\rangle} = \langle\cdots\rangle_\lambda$$

By using path-invariance and (conventional) hydrodynamic projections, one can show, at least order by order in λ, that $\langle\cdots\rangle_\lambda$ must be a (G)GE, and that the λ-dependent GGE satisfies a **flow equation**

$$\partial_\lambda \beta^j(\lambda;\xi) = \mathrm{sgn}\big(x\,\mathbf{1} - t\,\mathsf{A}(\lambda;\xi)\big)_i^{\ j}, \qquad \beta^j(0;\xi) = \beta^j, \quad \xi = x/t.$$



Correlation functions of twist fields and ballistic fluctuation theory

The flow determines the large-deviation, with associated “specific free energy” – scaled cumulant generating function – given by

$$F_i(\lambda;x,t) = \int_0^\lambda \mathrm{d}\lambda' \big(t\,\mathsf{j}_i(\lambda';\xi) - x\,\mathsf{q}_i(\lambda';\xi)\big)$$

In the XX model for the fermion number ($i=0$), the GGE along the flow is described by the function

$$w(\lambda;\xi;k) = \beta E(k) + \lambda\,\mathrm{sgn}\big(x - t\,v(k)\big)$$

and the scaled cumulant generating function is

$$F_0(\lambda;x,t) = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\,|x - t\,v(k)|\,\log\Big(\frac{1+e^{-w(\lambda;\xi;k)}}{1+e^{-w(k)}}\Big).$$

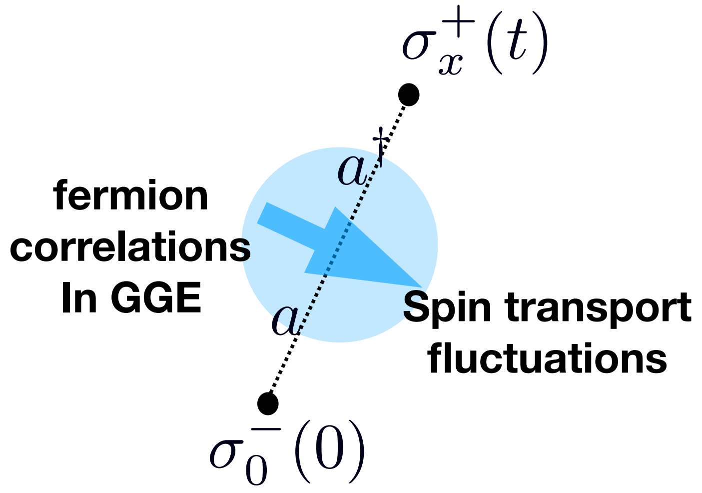
Correlation functions of twist fields and ballistic fluctuation theory

Using the flow on states and the SCGF, the required correlation function factorises into:

[BD, Del Vecchio Del Vecchio 2021]

- an exponential decay due to the interaction between the "boundary fermions" that occurs well within the region between them where λ-GGE is established,
- and a contribution from the large-deviation for fluctuations of total spin, or total spin transport:

$$\langle a^\dagger_{\ell x}(\ell t) e^{\lambda(\varphi_0(\ell x,\ell t)-\varphi_0(0,0))} a_0(0)\rangle \asymp \langle a^\dagger_{\ell x}(\ell t) a_0(0)\rangle_\lambda \exp\left[\ell F_0(\lambda; x,t\right]$$



Correlation functions of twist fields and ballistic fluctuation theory

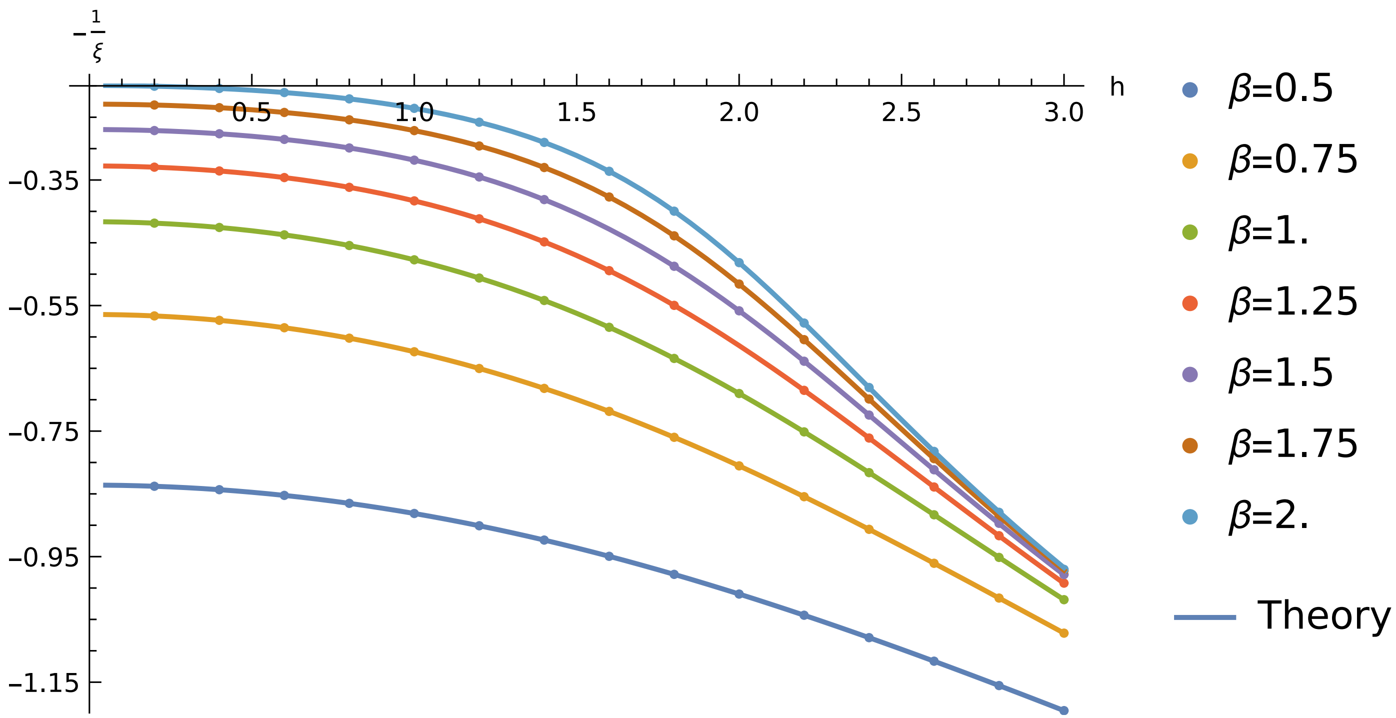
An analysis of both factors (saddle point, and ballistic fluctuation theory) gives the correct results

$$\langle \sigma_x^+(t)\sigma_0^-(0)\rangle \qquad\qquad (E(k) = 4h - 2\cos k,\ v(k) = 4\sin k)$$

$$\asymp \begin{cases} \exp\left[|t|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,|\xi - v(k)|\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi|\leq 4) \\ \exp\left[|x|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi|>4,\ |h|\leq 2) \\ e^{\mathrm{i}\Phi(x,t)}\exp\Big[-|x|\min\big(\operatorname{arccosh}(|h|/2), & \\ \qquad\qquad \operatorname{arccosh}(|\xi|/4)-\sqrt{1-\frac{16}{\xi^2}}\,\big)\Big]\times & \\ \qquad \times\exp\left[|x|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi|>4,\ |h|>2) \end{cases}$$

Correlation functions of twist fields and ballistic fluctuation theory

Comparison with numerics, e.g. in space-like region, $e^{-|x|/\xi}$



Conclusions

Finite-frequency hydro projection

- Related works in interacting models where “dynamical symmetries” are used to bound finite-frequency Drude weights [Buça, Tindall, Jaksch 2019; Medenjak, Prosen Zadnik 2020].
- Probably the principles used here can be extended to generic integrable models using the finite-density form factors (see reviews [De Nardis, BD, Medenjak, Panfil 2021; Cortés Cubero, Yoshimura, Spohn 2021]).
- Finite-frequency hydrodynamic equations? Higher-point functions?

Twist fields

- Immediately applicable to other fields of interest such as $e^{\mathrm{i}\alpha\phi}$ in the sine-Gordon model [in progress with del Vecchio del Vecchio, Kormos], potentially for Ψ field in Lieb-Liniger.
- Can be used to study non-equilibrium dynamics of entanglement entropy....
- Connected to macroscopic fluctuation theory for integrable systems (see talk T. Yoshimura).