The systolic geometry of arithmetic locally symmetric spaces

Benjamin Linowitz

OBERLIN COLLEGE & CONSERVATORY | Department of Mathematics

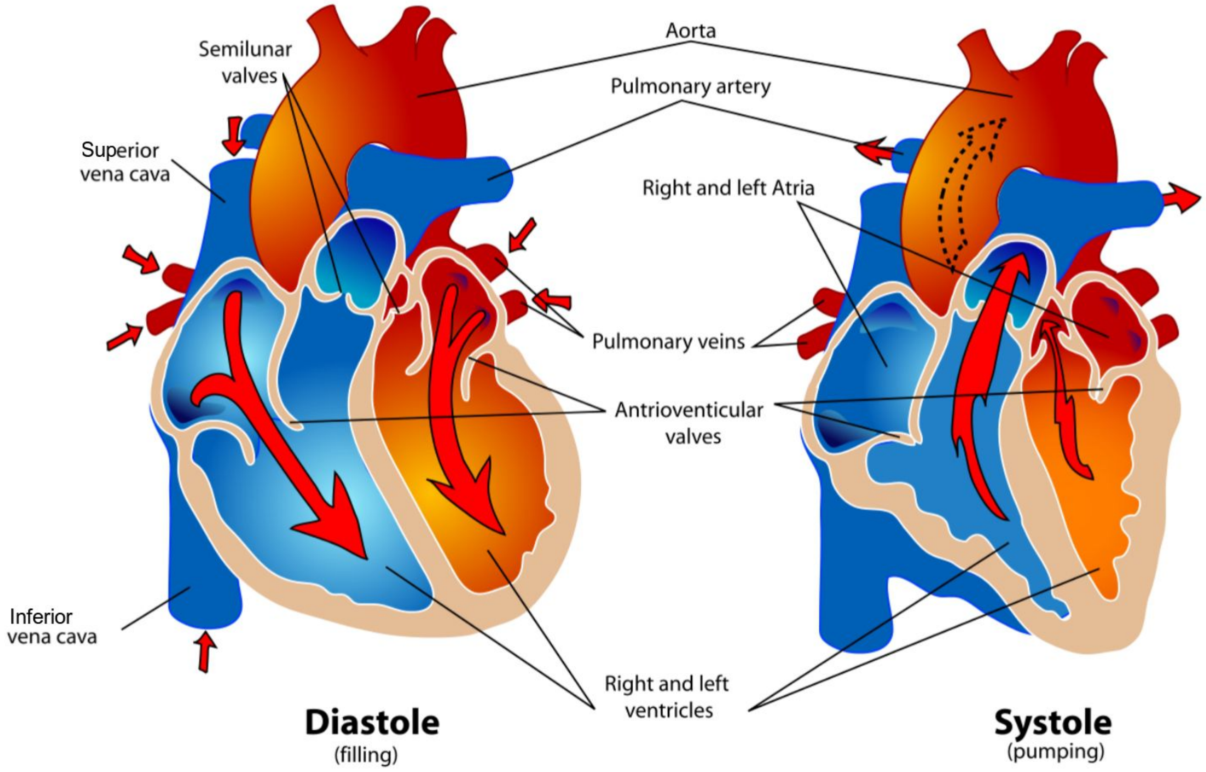
Systoles

OBERLIN
COLLEGE & CONSERVATORY

The **systole** of a compact Riemannian manifold is the least length of a non-contractible geodesic loop in the manifold.

- It is easy to construct manifolds with arbitrarily small systoles.
- It is also easy to construct manifolds with arbitrarily large systoles. *But naive constructions tend to yield manifolds with very large volumes.*

Why are they called “systoles”?



Why are they called "systoles"?

The medical term systole comes from the Greek word for "contraction". (If you have extra systolic beats in the medical and not geometrical sense, you had better consult your cardiologist.) The mathematical term systole was coined in 1980.

\- Marcel Berger, *What is... a systole*

Why are they called “systoles”?

OBERLIN
COLLEGE & CONSERVATORY

I was looking at the time for a word of the type ‘iso-???-ic’ both for the systoles and for the injectivity radius. I looked at Greek language dictionaries and found various wordings. Luckily, I was doing physical education together with a Greek literature colleague; he told me what I was proposing was ‘low Greek’. I explained to him, in ordinary words, what a systole and the injectivity radius are. At the next week’s physical education session, he came back with two proposals: ‘isoclysteric’ and ‘isosphincteric’. I was still young in the bad sense, say provocative, and I was [amused]; but I told these two wordings to the Besse seminar that week, they were horrified and told me ‘Marcel, you cannot use that’. So, at the next session I asked him to find less bad taste ‘ic’ words. The following week he came back with ‘isosystolic’ and ‘isoembolic’. The seminar people were happy; you understand that in short, I switched from proctology to cardiology.

Systole inequalities

Theorem (Loewner, 1949) – *If* (T^2, g) *is a* 2*-dimensional torus with a Riemannian metric, then the systole of* (T^2, g) *obeys the inequality*

$$systole(T^2, g) \leq C \cdot Area(T^2, g)^{\frac{1}{2}},$$

where $C = 2^{\frac{1}{2}} 3^{-\frac{1}{4}}$.

Systole inequalities

OBERLIN
COLLEGE & CONSERVATORY

Theorem (Besicovitch, 1952) – *If* (S, g) *is a closed oriented surface with* $genus(S, g) \geq 1$*, then*

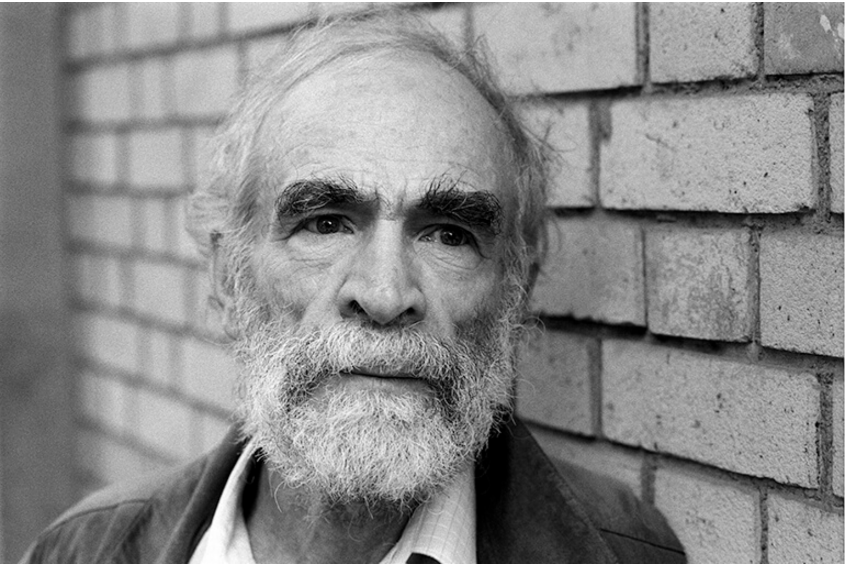
$$systole(S, g) \leq \sqrt{2} \cdot Area(S, g)^{1/2}.$$

Systole inequalities

Mais c'est fondamental!

- René Thom in 1961, referring to the aforementioned systole inequalities in conversation with Berger, who would go on to popularize systoles and systole inequalities.

Systole inequalities



Theorem (Gromov, 1983) – *If M is a closed aspherical n-manifold, then there exists a constant $c = c(n)$ such that $systole(M) \leq c \cdot vol(M)^{1/n}$.*

Systole inequalities

OBERLIN COLLEGE & CONSERVATORY

If S is a compact hyperbolic surface, then one can easily obtain a better estimate:

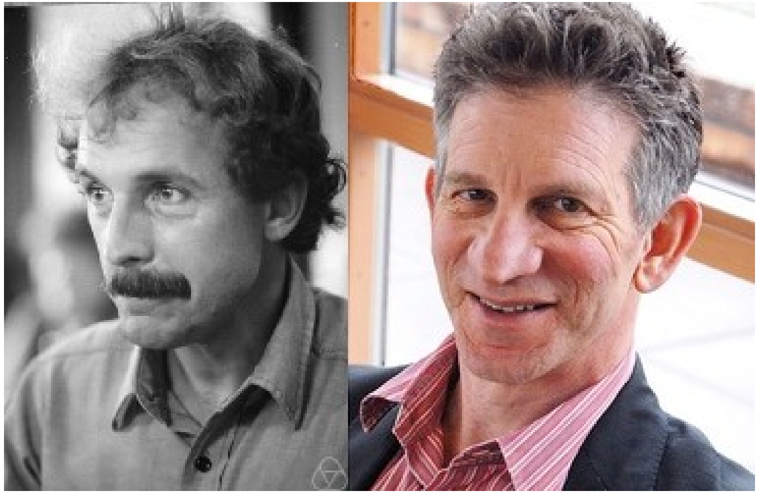
$$systole(S) \leq 2 \cdot \log(genus(S)).$$

The reason for the logarithmic character of this bound is that the area of a disk in the hyperbolic plane is exponential in the radius.

A virtually identical argument proves a logarithmic upper bound for the systole of a compact locally symmetric space of noncompact type.

Systole Growth Along Congruence Covers

OBERLIN
COLLEGE & CONSERVATORY



Theorem (Buser – Sarnak, 1994) – *There exist hyperbolic surfaces* S *such that for every prime* p *the congruence cover* S_p *of* S *satisfies*

$$systole(S_p) \geq \frac{4}{3} \cdot log\left(genus(S_p)\right) - c$$

for some constant c *that is independent of* p.

Systole Growth Along Congruence Covers

OBERLIN
COLLEGE & CONSERVATORY

The surfaces of Buser and Sarnak are among the strangest and most interesting examples in (Riemannian) geometry.

-Larry Guth, *Metaphors in Systolic Geometry,* Proceedings of the International Congress of Mathematicians, Hyderabad, India, 2010

Systole Growth Along Congruence Covers

OBERLIN
COLLEGE & CONSERVATORY

Arithmetic hyperbolic surfaces are remarkably hard to picture. When I meet a mathematician who studies the geometry of surfaces, I often ask them if they have any ideas about visualizing arithmetic hyperbolic surfaces. They just laugh. Part of the problem is that the systole of an arithmetic hyperbolic surface is only $\sim \log G$. *That means that to get interesting behavior, we need to look at huge values of* G. *Naturally, it is not easy to imagine a surface of genus* 10^6.

Systole Growth Along Congruence Covers

OBERLIN
COLLEGE & CONSERVATORY

Let us say that a family S_i of compact hyperbolic surfaces has a **large systole** if there are constants $\gamma > 0$ and c, which do not depend on i, such that

$$systole(S_i) \geq \gamma \cdot \log\big(genus(S_i)\big) - c.$$

The construction of Buser and Sarnak shows that principal congruence covers of certain fixed compact hyperbolic surfaces have large systoles, and that for these families of surfaces, the "best possible" associated value of γ is at least $\frac{4}{3}$.

On the other hand, our naive systole bound $systole(M) \leq 2 \cdot \log(genus(S))$ shows that this optimal value of γ is at most 2.

Systole Growth Along Congruence Covers

Theorem (Makisumi, 2012) – *For the family of principal congruence covers of a fixed compact hyperbolic surface, the best possible multiplicative constant is* $\gamma = 4/3$.

Systole Growth Along Congruence Covers

OBERLIN COLLEGE & CONSERVATORY

Theorem (Katz - Schaps - Vishne, 2007) – *If S is a compact arithmetic hyperbolic surface, then there is a constant $c = c(S)$ such that the congruence covers S_I of S satisfy*

$$systole(S_I) \geq \frac{4}{3} \cdot \log\big(genus(S_I)\big) - c.$$

Moreover, an analogous result holds for hyperbolic 3-manifolds, with the constant $\frac{4}{3}$ replaced by $\frac{2}{3}$ and genus replaced by simplicial volume.

Systole Growth Along Congruence Covers

OBERLIN
COLLEGE & CONSERVATORY

Theorem (Murillo, 2016) – *If* M *is Hilbert modular variety of real dimension* $2n$*, then there is a constant* $c = c(M)$ *such that the systole of the principal congruence coverings* M_I *satisfy*

$$systole(M_I) \geq \frac{4}{3\sqrt{n}} \cdot log\big(vol(M_I)\big) - c.$$

Systole Growth Along Congruence Covers

Theorem (Murillo, 2016) – *If* M *is an arithmetic hyperbolic* n*-manifold of the first type, then there is a constant* $c = c(M)$ *such that the systole of the principal congruence coverings* M_I *satisfy*

$$systole(M_I) \geq \frac{8}{n(n+1)} \cdot log\big(vol(M_I)\big) - c.$$

Systole Growth Along Congruence Covers

OBERLIN
COLLEGE & CONSERVATORY

Theorem (Murillo, 2016) – *If M is an arithmetic hyperbolic n-manifold of the first type, then there is a constant $c = c(M)$ such that the systole of the principal congruence coverings M_I satisfy*

$$systole(M_I) \geq \frac{8}{n(n+1)} \cdot log\big(vol(M_I)\big) - c.$$

In an appendix to the paper in which this result is proven, Murillo and Cayo Dória prove that the bound of $\frac{8}{n(n+1)}$ is best possible.

Systole Growth Along Congruence Covers

OBERLIN
COLLEGE & CONSERVATORY

Theorem (Lapan – L. - Meyer, 2021) – *The growth of the systoles of congruence covers of arithmetic simple locally symmetric manifolds is logarithmic in volume.*

In particular, the aforementioned results hold for finite-volume real, complex, and quaternionic arithmetic hyperbolic manifolds.

The Buser - Sarnak Construction

Let A be a quaternion division algebra over $\mathbf{Q}$ generated by $1, i, j, k,$ where

$$i^2 = a, \qquad j^2 = b, \qquad ij = k = -ji.$$

An arbitrary element of A is of the form $x = x_0 + x_1 i + x_2 j + x_3 k$ for $x_0, x_1, x_2, x_3 \in \boldsymbol{Q}$.

The reduced norm of A is the function $N(x) = x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2$.

Define $\Gamma = \{\, x \in A : x_0, x_1, x_2, x_3 \in \boldsymbol{Z},\ N(x) = 1\}$.

The Buser - Sarnak Construction

For an odd prime p, define the principal congruence subgroup $\Gamma(p)$ of Γ by

$$\Gamma(p) = \{x \in \Gamma : x \equiv 1 \ (mod\ p)\ \}.$$

Consider the embedding of A into $M_2(\boldsymbol{R})$ given by

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix}.$$

The image of Γ under this embedding is a lattice in $SL_2(\boldsymbol{R})$.

The Buser - Sarnak Construction

Because A is a division algebra and $\Gamma(p)$ is torsion-free, $\Gamma(p)\backslash \boldsymbol{H}^2$ is a compact Riemann surface. Its genus is $g_p = p(p-1)(p+1)v + 1$ for some $v > 0$ which does not depend on p.

But what is the systole of $\Gamma(p)\backslash \boldsymbol{H}^2$?

If $x = x_0 + x_1 i + x_2 j + x_3 k \in \Gamma(p)$, then $x \equiv 1 \ (mod\ p)$ implies that $p \mid x_1, x_2, x_3$.

Because $1 = N(x) = x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2$, it must be that $x_0 \equiv \pm 1 \ (mod\ p^2)$.

The Buser - Sarnak Construction

If $x \neq \pm 1$, then $x_0 \neq \pm 1$ (since Γ has no parabolic elements) , hence $|x_0| \geq p^2 - 1$ and $|Trace(x)| \geq 2(p^2 - 1)$.

Recall that the translation length ℓ_x of x satisfies the formula $\cosh\left(\frac{\ell_x}{2}\right) = \pm\frac{Trace(x)}{2}$.

It follows that $systole(\Gamma(p)\backslash \boldsymbol{H}^2) \geq 2 \cdot \log(p^2) \approx \frac{4}{3} \cdot \log(g_p)$.

A Generalization

How can this construction be generalized so as to prove logarithmic growth of the systoles of congruence covers of more general arithmetic locally symmetric manifolds?

The starting point lies with what we call *standard special linear manifolds*. These are manifolds of the form $\Gamma \backslash \mathrm{SL}_n(\boldsymbol{R})/SO(n)$, where $\Gamma \subset SL_n(\boldsymbol{R})$ is a torsion-free lattice arising from the elements of norm one in a maximal order in a central simple algebra.

Proving logarithmic systole growth of the systoles of congruence covers of standard special linear manifolds is similar to Buser and Sarnak's construction for surfaces.

A Generalization

OBERLIN
COLLEGE & CONSERVATORY

Let A be a central simple algebra over $\boldsymbol{Q}$ of dimension n^2 and O be a maximal order of A.

Given a natural number N, we have an ideal NO of O whose quotient O/NO is a finite ring.

We define *the level* N *principal congruence subgroup* $O^1(N)$ of O^1 to be the kernel of the homomorphism $O^1 \to \left(\frac{O}{NO}\right)^\times$ induced by the natural projection $O \to O/NO$.

Denote the images of O^1 and $O^1(N)$ in $SL_n(\boldsymbol{R})$ by Γ and $\Gamma(N)$.

A Generalization

Theorem – *Let* Γ *be as above and* p *be a prime which does not ramify in* A *and satisfies* $p > 2n$*. For every* $m \geq 1$ *and semisimple element* $x \in \Gamma(p^m), x \neq 1$*, there is an integer* q *such that* $|q| < \frac{n}{2}$ *and* $|Trace(x^q)| > p^m - n$*.*

A Generalization

OBERLIN
COLLEGE & CONSERVATORY

Theorem – *Let* Γ *be as above and* p *be a prime which does not ramify in* A *and satisfies* $p > 2n$. *For every* $m \geq 1$ *and semisimple element* $x \in \Gamma(p^m), x \neq 1$, *there is an integer* q *such that* $|q| < \frac{n}{2}$ *and* $|Trace(x^q)| > p^m - n$.

Proof:

Choose a basis of $A \otimes_{\mathbf{Q}} \mathbf{Q}_{\mathrm{p}}$ so that $A \otimes_{\mathbf{Q}} \mathbf{Q}_{\mathrm{p}} \cong M_n(\boldsymbol{Q}_p)$ and $O \otimes_{\mathbf{Z}} \mathbf{Z}_p \cong M_n(\mathbf{Z}_{\mathrm{p}})$.

Let φ_p denote the natural projection from $M_n(\mathbf{Z}_{\mathrm{p}})$ onto $M_n(\mathbf{Z}/p^m\mathbf{Z})$.

Then $\varphi_p(x) = Id_n$, which has trace n, hence $Trace(x) \equiv n \pmod{p^m}$.

This shows that if $x \in \Gamma(p^m)$, then $Trace(x) = p^m k + n$ for some integer k.

A Generalization

Theorem – *Let* Γ *be as above and* p *be a prime which does not ramify in* A *and satisfies* $p > 2n$*. For every* $m \geq 1$ *and semisimple element* $x \in \Gamma(p^m), x \neq 1$*, there is an integer* q *such that* $|q| < \frac{n}{2}$ *and* $|Trace(x^q)| > p^m - n$*.*

Proof:

If $Trace(x^q) = n$ for all integers q with $|q| < \frac{n}{2}$, then one can use Newton's Identities to show that the characteristic polynomial of x is $p_x(X) = (X - 1)^n$ and that $x = 1$.

This is a contradiction and proves the theorem.

A Generalization

Theorem (Trace-Length Bounds) – *For a semisimple element* $x \in SL_n(\boldsymbol{R})$, *the translation length* ℓ_x *of* x *satisfies*

$$\ell_x \geq \sqrt{2}\,\operatorname{arccosh}\left(\frac{Trace(x)}{n}\right).$$

A Generalization

OBERLIN
COLLEGE & CONSERVATORY

Theorem (Trace-Length Bounds) – *For a semisimple element* $x \in SL_n(\boldsymbol{R})$, *the translation length* ℓ_x *of* x *satisfies*

$$\ell_x \geq \sqrt{2}\,\mathrm{arccosh}\left(\frac{Trace(x)}{n}\right).$$

Combining the last two results allows us to deduce that

$$systole(\Gamma(p^m)\backslash \mathrm{SL}_n(\boldsymbol{R})/SO(n)) \geq \frac{2\sqrt{2}}{n}\,\mathrm{arccosh}\left(\frac{p^m - n}{n}\right).$$

A Generalization

OBERLIN
COLLEGE & CONSERVATORY

Employing a little algebra and a bound for the index of $\Gamma(p^m)$ in Γ yields the following systole bound for $M_{p^m} = \Gamma(p^m)\backslash \mathrm{SL}_n(\boldsymbol{R})/SO(n)$:

$$systole(M_{p^m}) \geq \frac{2\sqrt{2}}{n(n^2-1)} \cdot \log\left(vol(M_{p^m})\right) - c.$$

A Generalization

In this manner we can prove logarithmic growth for the systoles along congruence covers of standard special linear manifolds.

What about the systoles of more general locally symmetric spaces?

In this case we can make use of the fact that every arithmetic simple locally symmetric manifold is commensurable to an immersed totally geodesic submanifold of a standard special linear manifold of explicitly bounded degree.

Optimal Multiplicative Constants

We have shown that for a broad class of arithmetic locally symmetric spaces N, the congruence covers N_{p^m} (for all but finitely many primes p) satisfy

$$systole(N_{p^m}) \geq c_1 \log(vol(N_{p^m}) - c_2$$

for constants c_1, c_2 depending only on N.

We give explicit values for the constants c_1, c_2, and are even able to show the general dependence of c_1 on the volume of N.

For certain important special cases, we prove logarithmic systole inequalities for which the multiplicative constant c_1 depends only on the dimension. This is the case, for instance, for real, complex, and quaternionic hyperbolic orbifolds.

Optimal Multiplicative Constants

We did not, however, make any attempt to determine the value of the optimal multiplicative constants.

Shortly after our paper appeared on the arXiv, Inkang Kim determined the optimal multiplicative constant for several important classes of arithmetic lattices and gave much improved multiplicative constants in other cases.

Optimal Multiplicative Constants

OBERLIN COLLEGE & CONSERVATORY

Theorem (Kim, 2020) – *The multiplicative systole constant may be taken to be as follows:*

- *for lattices in* $SO(1,n)$, $c_1 = \frac{4}{n(n+1)}$,
- *for lattices in SU(n,1) of the first type,* $c_1 = \frac{4}{n(n+2)}$, *and for the other type,* $c_1 = \frac{2}{n(n+2)}$, *and*
- *for lattices in* $SL(n+1, \boldsymbol{R})$, $c_1 = \frac{\sqrt{2}}{n(n+2)}$.

Optimal Multiplicative Constants



Theorem (Emery-Kim-Murillo, 2022) – *The optimal multiplicative systole constant for compact quaternionic manifolds of dimension* $4n$ *is* $\frac{4}{(n+1)(2n+3)}$.

Optimal Multiplicative Constants

OBERLIN
COLLEGE & CONSERVATORY



Theorem (Emery-Kim-Murillo, 2022) – *The optimal multiplicative systole constant for compact quaternionic manifolds of dimension* $4n$ *is* $\frac{4}{(n+1)(2n+3)}$.

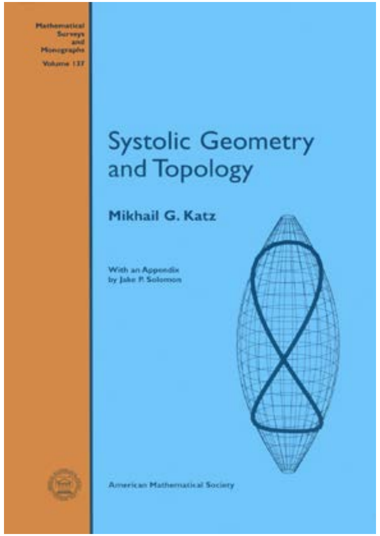
Correctness Check: Murillo showed that for arithmetic hyperbolic n-manifolds, the optimal constant is $\frac{8}{n(n+1)}$. Since $H^1_{\mathbb{H}}$ is isometric to four-dimensional real hyperbolic space, we need $\frac{4}{(1+1)(2+3)} = \frac{8}{4(4+1)}$.

Further Reading

OBERLIN
COLLEGE & CONSERVATORY

Our paper, *Systole inequalities for arithmetic locally symmetric spaces,* is to appear in *Communications in Analysis and Geometry* and is also on the arXiv.

The best place to learn about the field of systolic geometry more generally is undoubtedly Misha Katz's book *Systolic Geometry and Topology.*



Thank You!