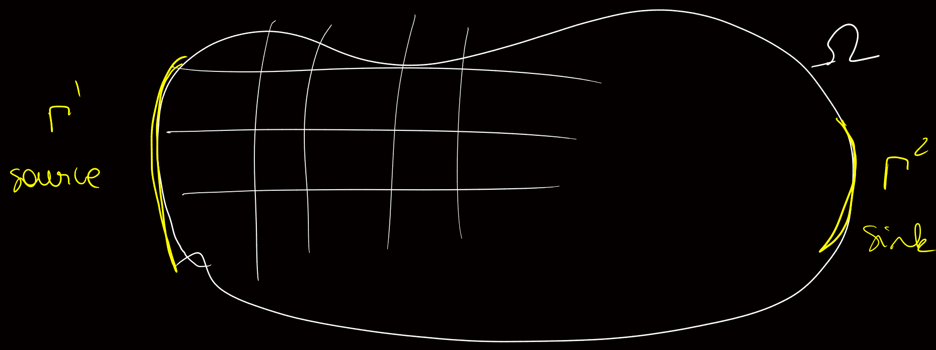
maximal flow and minimal cutset in FPP on $\mathbb{Z}^d$

I. Introduction of the model and definitions

$d \geq 2$ $\quad (\mathbb{Z}^d, \mathbb{E}^d)$ $\quad$ G distribution on $(0, M)$ $\quad M > 0$

$(c(e))_{e \in \mathbb{E}^d}$ i.i.d. family with law G



$c(e) =$ capacity of the edge e.
maximal amount of water that can flow through e / unit of time

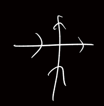
__definition:__ stream $f : \mathbb{E}^d \to \mathbb{R}^d$

$f(e)$ — $\|f(e)\|_2$: amount of water that flow through e

$\rightleftarrows$ $\quad \dfrac{f(e)}{\|f(e)\|_2}$; $\quad \underset{e}{\rightleftarrows}$

an admissible stream in $(\Omega, \Gamma^1, \Gamma^2)$:

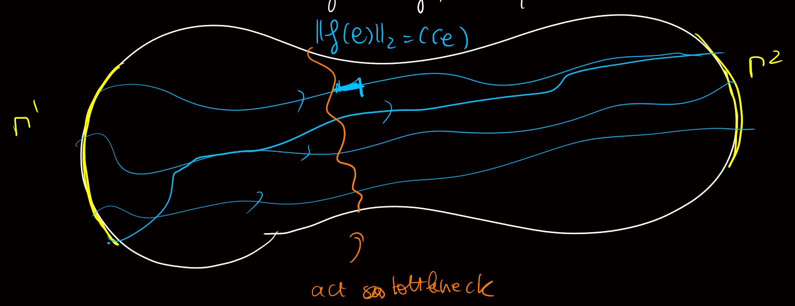
- $f = 0$ outside Ω
- $\forall e \in \mathbb{E}^d \quad \|f(e)\|_2 \leq c(e)$
- the node law is respected everywhere outside Γ^1 and Γ^2



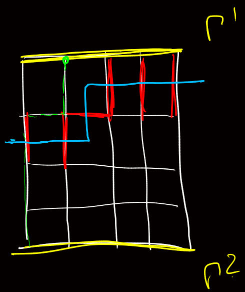
- the flow of a stream $\quad flow(f) = \sum_{\substack{e \in \Gamma^1 \\ \cancel{\mathbb{E}}\, \Gamma^1}} \pm \|f(e)\|_2$

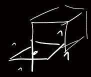
__maximal flow__ : $\Phi(\Omega, \Gamma^1, \Gamma^2) = \sup \{ flow(f) : f \text{ admissible stream} \}$

if f is such that $flow(f) = \Phi(\Omega, \Gamma^1, \Gamma^2)$ we call f a maximal stream.



Def: [cutset in Ω from Γ^1 to Γ^2] $E \subset \mathbb{E}^d$ s.t. $\forall$ path γ from Γ^1 to Γ^2 in Ω, we have $E \cap \gamma \neq \emptyset$





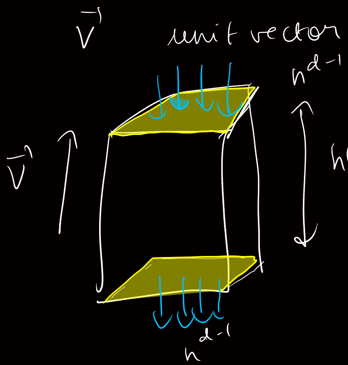
capacity of a cutset E: $\quad cap(E) = \sum_{e \in E} c(e)$.

<u>max-flow min. cut theorem:</u> $\quad \phi(\Omega, \Gamma^1, \Gamma^2) = \inf\{cap(E) : E \text{ cutset}\}$

admissible stream $\downarrow$ flow(f) $\qquad$ cutset $\downarrow$ cap(E)

II. <u>Maximal flow in a given direction</u>

$\vec{v}$ unit vector



$\phi_n = \phi(\square, \square, \square)$ $\quad n^{d-1}$

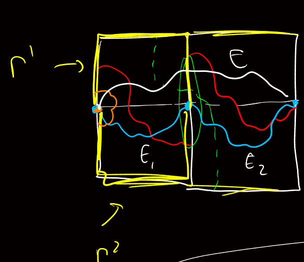
$\frac{\phi_n}{n^{d-1}}$

<u>Thm</u> (Kesten 87) $\quad \frac{\phi_n}{n^{d-1}} \to \nu_G(\vec{v})$ $\quad d = 3$, $\vec{v}$ // to axis

flow constant $\qquad h(n)$ polynomial n

<u>Thm</u> (Rossignol & Théret 10) $\quad \frac{\phi_n}{n^{d-1}} \to \nu_G(\vec{v})$ any d. $\quad \frac{h(n)}{n} \to 0$

any $\vec{v}$ $\qquad h(n) \to \infty$



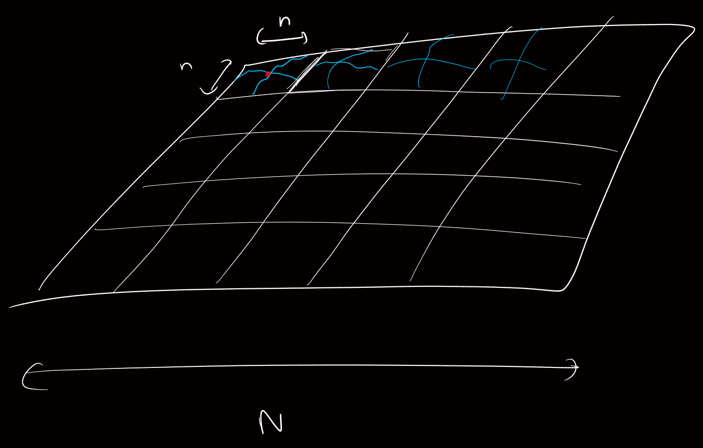
$\tau_n = \phi(\square \ h(n), \square, \square)$ $\quad n^{d-1}$

$$\boxed{cap(E) \leq cap(E_1 \cup E_2) = cap(E_1) + cap(E_2)}$$

$$\frac{\tau_n}{n^{d-1}}$$

$|I| \approx \left(\frac{N}{n}\right)^{d-1}$

$N \gg n$



$$\tau_N \le \sum_{i \in I} \tau_n(i)$$

$$\mathbb{E}(\tau_N) \le \left(\frac{N}{n}\right)^{d-1} \mathbb{E}(\tau_n)$$

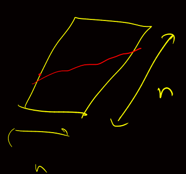
$$\frac{1}{N^{d-1}} \mathbb{E}(\tau_N) \le \frac{1}{n^{d-1}} \mathbb{E}(\tau_n)$$

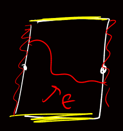
$$\limsup_{N \to \infty} \frac{1}{N^{d-1}} \mathbb{E}(\tau_N) \le \liminf_{n \to \infty} \frac{1}{n^{d-1}} \mathbb{E}(\tau_n)$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{1}{n^{d-1}} \mathbb{E}(\tau_n) = \lim_{n \to \infty} \frac{\Phi_n}{n^{d-1}} = \nu_G(\vec{v}) \quad a.s.$$

Concentration arguments.

$\nu_G(\vec{v})$: maximal amount of water that can flow in direction $\vec{v}$ asymptotically





$h(n) \ll n$

$h(n)\, n^{d-2}$

$E \cup F \quad F$

$$\Phi_n \le \tau_n$$

$$\tau_n \le \Phi_n + \underbrace{\{ \quad \}}_{h(n)\, n^{d-2}}$$

$$\lim \frac{\tau_n}{n^{d-1}} = \lim \frac{\Phi_n}{n^{d-1}}$$

μ distributed on $[0, M]$

$(X_i)_{i \le n}$ i.i.d distributed given μ

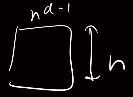
LLN $\quad \frac{1}{n} \sum_{i=1}^{n} X_i \to \mathbb{E}(X_1)$

$$\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} X_i \ge \mathbb{E}(X_1) + \varepsilon\right] \approx e^{-n \underbrace{I(\varepsilon)}}$$

speed of large deviation

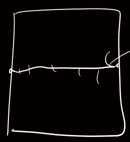
cost

Large deviations for τ_n



Lower large deviations $\quad \mathbb{P}\left(\frac{\tau_n}{n^{d-1}} \leq \nu_G(\vec{v}) - \varepsilon\right] \approx e^{-n^{d-1} J(\varepsilon)}$

Rossignol & Théret 2010



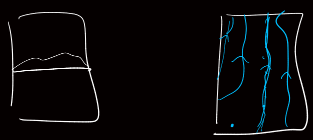
decrease all the capacity

$\varepsilon^{n^{d-1}}$

$\varepsilon(\nu_G(\vec{v}) - \varepsilon)\, n^{d-1}$

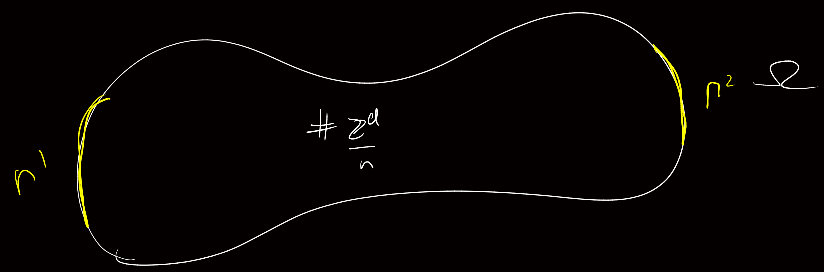
Upper large deviations: $\quad \mathbb{P}\left(\frac{\tau_n}{n^{d-1}} \geq \nu_G(\vec{v}) + \varepsilon\right]$ of volumic order

$\approx e^{-n^d}$.



$\limsup_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}\left(\quad \right) > 0$

III. maximal flow in a general domain



$\phi_n(\Omega, \Gamma^1, \Gamma^2)$

Thm: (Cerf & Théret 11] $\quad \phi_\Omega \geq 0 \qquad \frac{\phi_n(\Omega,\Gamma^1,\Gamma^2)}{n^{d-1}} \to \phi_\Omega$ a.s.

E_n^{min} minimal cut set $\qquad E_n^{min}$ "$\to$" continuous minimal cut set.

f_n^{max} maximal stream $\qquad f_n^{max}$ "$\to$" continuous maximal stream

we want to encode f_n into a measure

$$\vec{\mu}_n(f_n) = \sum_{e \in \mathbb{E}_n^d} f_n(e)\, \delta_{c(e)} \in \mathcal{M}(\mathbb{R}^d)^d$$

$c(e) \in \mathbb{R}^d$ dirac mass at the center of e.

$\vec{\sigma} \in \mathcal{L}^d$

vector field

$\vec{\sigma} = 0$ outside Ω.

$\operatorname{div} \vec{\sigma} = 0$ in Ω $\iff$ node law

ULD

$\lambda > \phi_\Omega \quad \mathbb{P}(\phi_n \geq \lambda n^{d-1})$

Thm (D, Théret 20) $\forall A$ Borelian set $\mathcal{B}(\mathcal{M}(\mathbb{R}^d)^d)$, $I \geq 0$

$$-\inf_{\mathring{A}} I \leq \liminf_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}(\;) \leq \limsup_{n\to\infty} \frac{1}{n^d} \log \mathbb{P}(\exists f_n \text{ st } \vec{\mu}_n(f_n) \in A) \leq \inf_{\bar{A}} I$$

$$\mathbb{P}(\exists f_n \text{ s.t. } \vec{\mu}_n(f_n) \approx \vec{\sigma}\mathcal{L}^d) \approx e^{-I(\vec{\sigma}) n^d}$$

$I(\vec{\sigma}) = \int_\Omega \vec{I}(\vec{\sigma}(x))\, d\mathcal{L}^d(x)$

$$J(\lambda) = \inf\{ I(\vec{\sigma}) : \text{flow}(\vec{\sigma}) \geq \lambda \}$$

$\frac{\text{flow}(f_n)}{n^{d-1}} \geq \lambda$

$\vec{\mu}(f_n) \approx \vec{\sigma}\mathcal{L}^d$

$\text{flow}(\vec{\sigma}) \geq \lambda$

$$\begin{aligned}
\mathbb{P}(\phi_n \geq \lambda n^{d-1}) &= \mathbb{P}(\exists f_n \text{ admissible s.t. } \text{flow}(f_n) \geq \lambda n^{d-1}) \\
&\approx \mathbb{P}(\exists f_n \text{ s.t. } \vec{\mu}_n(f_n) \in \{\vec{\sigma}\mathcal{L}^d : \text{flow}(\vec{\sigma}) \geq \lambda\}) \\
&\approx e^{-n^d J(\lambda)}
\end{aligned}$$

$\mathbb{P}(\phi_n \leq \lambda n^{d-1}) \qquad \lambda < \phi_\Omega$