Statistical Physics of Athermal Systems
Lecture 4

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Parts of lectures 4 and 5 will follow the **References:**
1. M. van Hecke, *Jamming of soft particles: geometry, mechanics, scaling and isostaticity*, Journal of Physics: Condensed Matter 22, 033101 (2010).
2. C. S. O'Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, *Jamming at zero temperature and zero applied stress: The epitome of disorder*, Physical Review E 68, 011306 (2003).
3. M. Wyart, *On the rigidity of amorphous solids*, Annales de Physique 30, 1–96 (2005).

1 Athermal dynamics

We begin with the dynamical point of view developed in Lecture 2. For overdamped particles,

$$\gamma \dot{\mathbf{r}}_i = -\nabla_i U + \mathbf{f}_i^{\mathrm{ext}}. \tag{1}$$

The rate of change of the potential energy is

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\gamma \sum_i |\dot{\mathbf{r}}_i|^2 + \sum_i \mathbf{f}_i^{\mathrm{ext}} \cdot \dot{\mathbf{r}}_i. \tag{2}$$

Thus the dissipation is fixed by the velocities, whereas the external protocol supplies mechanical work.

When the external driving is removed, the dynamics reduces to gradient descent,

$$\gamma \dot{\mathbf{R}} = -\nabla_{\mathbf{R}} U, \tag{3}$$

and therefore

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\frac{1}{\gamma} |\nabla_{\mathbf{R}} U|^2 \leq 0. \tag{4}$$

The dynamics terminates at a mechanically stable minimum. The important point for the statistical problem is that the minimum reached depends on the initial condition and on the preparation protocol.

1.1 From Microscopic Basins to Edwards Ensembles

As established in the previous lecture, an ensemble approach in athermal systems must be specified by the preparation protocol. Consider a common preparation route: particles are compressed from a dilute fluid phase to a specified packing fraction ϕ, followed by rapid energy minimization. The deterministic dissipative dynamics ensures that the final packing reached, $\mathbf{R}_i$, is a function of the prepared initial configuration $\mathbf{R}_0$.

We quantify this dynamic relationship through the basin of attraction $\mathcal{B}_i$ for state i, defined as:

$$\mathcal{B}_i = \left\{ \mathbf{R}_0 : \lim_{t\to\infty} \mathbf{R}(t|\mathbf{R}_0) = \mathbf{R}_i \right\}, \tag{5}$$

i.e. all the initial starting configurations $\mathbf{R}_0$ that end up at the final state $\mathbf{R}_i$ at late times belong to the basin of attraction of the jammed state i.

If the initial fluid state is uniform, such that any configuration $\mathbf{R}_0$ is equally likely, the minimization dynamics maps this uniformity to the packed state space. The probability P_i of selecting minimum i from this protocol is rigorously proportional to the volume of its basin:

$$P_i \propto \mathrm{Vol}(\mathcal{B}_i). \tag{6}$$

Crucially, in the overcompressed amorphous solid regime, the distribution of these basin volumes is broad and non-uniform, reflecting significant dynamic heterogeneity. The dynamic ensemble generated by realistic dissipation therefore carries an inherent bias toward packings with larger basins. We must therefore distinguish between this dynamically generated measure and the standard flat measure postulated by Sam Edwards, which assumes all distinct packings are equally likely ($P_i = 1/\Omega$).

To reach a truly biased statistical measure that validates Edwards' conjecture, the system must reach a state where the variance in the basin volume distribution vanishes. There is only one singular point where this unjamming occurs: the unjamming threshold $\phi \to \phi_J^+$. Recent simulations demonstrate that as the system decompresses to this marginal point, the basin volume distribution approaches a Dirac delta function. It is precisely at this marginal unjamming transition that the dynamically selected ensemble coincides with the equi-probable Edwards measure.

2 Frictionless Jamming

Consider N particles in d dimensions interacting through finite-range, purely repulsive potentials. For particles of effective contact diameter

$$\sigma_{ij} = \frac{\sigma_i + \sigma_j}{2}, \tag{7}$$

introduce the dimensionless overlap

$$\delta_{ij} = 1 - \frac{r_{ij}}{\sigma_{ij}}. \tag{8}$$

A contact is present only when $\delta_{ij} > 0$. A convenient family of interaction potentials is

$$V(\delta) = \frac{\epsilon}{\alpha}\delta^\alpha \Theta(\delta), \qquad \alpha > 1. \tag{9}$$

The harmonic interaction corresponds to $\alpha = 2$, and the Hertzian interaction to $\alpha = 5/2$.

The total energy is

$$U = \sum_{i<j} V(\delta_{ij}). \tag{10}$$

The jamming point is most naturally discussed in the zero-temperature, zero-rate limit. The transition is a loss of mechanical rigidity rather than a singularity in an equilibrium free energy. The packing fraction at which a particular protocol first produces a mechanically stable packing is denoted ϕ_J.

It is important to keep ϕ_J conceptually separate from an ordinary equilibrium critical point. In athermal systems the set of mechanically stable states, and the probabilities with which they are reached, depend on preparation. Consequently, the protocol is part of the specification of the state being studied.

2.1 Mechanical constraints and isostaticity

It is useful to derive the counting from the force-balance problem first, since this is the form in which isostaticity is most directly used for frictionless packings. A frictionless contact carries only one independent scalar force, its normal magnitude. Thus, if there are N_c contacts, the number of unknown contact-force amplitudes is

$$N_c = \frac{Nz}{2}. \tag{11}$$

Force balance on particle i gives a vector equation,

$$\sum_j f_{ij}\hat{\mathbf{n}}_{ij} = 0, \tag{12}$$

which represents d scalar equations. For N particles there are therefore dN force-balance equations, with the usual global redundancy associated with the total force on an isolated or periodic system. In the thermodynamic limit this distinction is subextensive, and a generic force network can exist only if

$$\frac{Nz}{2} \gtrsim Nd. \tag{13}$$

Thus

$$z \gtrsim 2d. \tag{14}$$

This is the force-balance side of the isostatic argument: there must be at least enough scalar contact forces to satisfy the vector equilibrium conditions.

There is a second condition at the jamming point. For a one-sided repulsive interaction, the contact forces vanish as the jamming point is approached from above, while the contacts approach the just-touching geometry. Each contact then supplies one scalar geometric constraint on the particle coordinates. After removing the global translations, the number of independent positional degrees of freedom is $Nd - d$ up to boundary effects. Thus, at a generic marginal point, the number of independent contact constraints and positional degrees of freedom meet at equality. In the thermodynamic limit this again gives

$$z_{\text{iso}} = 2d. \tag{15}$$

The two viewpoints are complementary: force balance determines how many contacts are needed to carry a force network, while the just-touching geometry limits how many independent contacts can be imposed at the zero-force jamming point. Special geometric degeneracies, boundaries and states of self stress modify the finite-size count, but not the generic large-system result.

We first count degrees of freedom and constraints. Consider frictionless particles and exclude rattlers. The N particles have Nd positional degrees of freedom. For a finite system the d global translations do not cost energy and should be removed from the count. Thus the number of nontrivial degrees of freedom is

$$N_{\mathrm{dof}} = Nd - d. \tag{16}$$

A frictionless contact supplies one scalar normal constraint. If there are N_c contacts,

$$N_c = \frac{Nz}{2}, \tag{17}$$

where z is the mean contact number.

At marginal stability, the number of independent constraints is comparable to the number of nontrivial positional degrees of freedom. For a large system this gives

$$\frac{Nz}{2} \simeq Nd, \tag{18}$$

and therefore

$$z_{\mathrm{iso}} = 2d. \tag{19}$$

Global translations, boundaries, and possible states of self stress give finite-size corrections whose precise form depends on the boundary conditions. They do not change the thermodynamic-limit value. Thus A more precise statement is obtained from the Maxwell–Calladine index theorem. If N_0 is the number of zero modes and N_{ss} is the number of independent states of self stress, then, for a periodic central-force network,

$$N_0 - N_{\mathrm{ss}} = Nd - N_c - d. \tag{20}$$

The d subtracted degrees of freedom are the global translations. Thus the finite-size contact count is not determined by z alone: zero modes and states of self stress also enter. For the generic frictionless jammed packings considered here, the thermodynamic-limit result is nevertheless $z_{\mathrm{iso}} = 2d$.

$$z_{\mathrm{iso}} = 4 \quad (d = 2), \qquad z_{\mathrm{iso}} = 6 \quad (d = 3). \tag{21}$$

Define the excess coordination by

$$\Delta z = z - z_{\mathrm{iso}}. \tag{22}$$

For $z < z_{\mathrm{iso}}$, Maxwell counting leaves an extensive number of floppy directions. Schematically,

$$N_{\mathrm{floppy}} \sim Nd - \frac{Nz}{2} \sim \frac{N}{2}(2d - z), \tag{23}$$

ignoring the subextensive zero modes and boundary corrections. At the isostatic point the extensive floppy subspace disappears.

2.2 Rattlers and finite-size counting

The counting refers to the load-bearing backbone. A rattler is a particle that is not part of this rigid backbone and has too few contacts to be constrained by it. Such particles contribute positional degrees of freedom but do not contribute contacts to the rigid network. They should therefore be removed before comparing the number of constraints with the number of degrees of freedom.

If N_b denotes the number of particles in the load-bearing backbone, rattlers should be excluded from the coordination number used in the isostatic count. The familiar relation is

$$z_{\text{iso}} = 2d + O(N_b^{-1}), \tag{24}$$

with the coefficient of the finite-size correction depending on the boundary conditions and on whether states of self stress are counted explicitly. For the thermodynamic-limit scaling discussed below, only $z_{\text{iso}} = 2d$ is needed. The packing fraction, however, is normally computed from all particles, including rattlers. Figure 1 is a schematic summary of this counting argument. The coordination below jamming is not meant to be a universal function of ϕ for every preparation protocol; the useful features are the loss of the load-bearing network below jamming, the approach to z_{iso} from the jammed side, and the growth of the excess coordination above it.

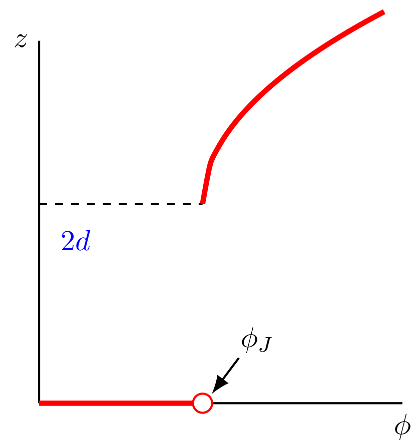

Figure 1: Coordination number z as a function of packing fraction ϕ, illustrating the jump from zero to the isostatic value $2d$ at the jamming transition ϕ_J.

The counting argument alone does not prove stability. A packing at the same coordination can be stable or unstable depending on the initial forces and the detailed contact geometry. The role of the forces enters through the prestressed Hessian.

3 Scaling immediately above jamming

Let

$$\Delta\phi = \phi - \phi_J > 0. \tag{25}$$

For small compression, the typical overlap scales as

$$\delta \sim \Delta\phi. \tag{26}$$

A contact then contributes an energy of order δ^α, so the energy density scales as The corresponding contact force and local stiffness scale as

$$f \sim \delta^{\alpha-1}, \qquad k \sim \delta^{\alpha-2}. \tag{27}$$

This distinction is useful throughout the lecture. Pressure is set by contact forces, whereas the small oscillation frequencies and elastic response involve the contact stiffness together with the geometry of the contact network.

$$\frac{U}{N} \sim \Delta\phi^{\alpha}. \tag{28}$$

The pressure is obtained from the derivative of the energy density with respect to volume, or equivalently from the scaling with packing fraction. Hence

$$p \sim \Delta\phi^{\alpha-1}. \tag{29}$$

For the two commonly used interaction laws,

$$\alpha = 2: \qquad p \sim \Delta\phi, \tag{30}$$

$$\alpha = \frac{5}{2}: \qquad p \sim \Delta\phi^{3/2}. \tag{31}$$

The contact number obeys the central scaling law

$$\Delta z \sim \Delta\phi^{1/2}. \tag{32}$$

The square-root behavior of Δz is geometric. Close to jamming the pair correlation function has a strong near-contact contribution of the form

$$g(h) \sim h^{-1/2}, \qquad h = r - \sigma_{ij} > 0. \tag{33}$$

A compression by an amount h_0 brings particles with $0 < h < h_0$ into contact. Hence the additional number of contacts scales as

$$\Delta z \sim \int_0^{h_0} g(h)\,\mathrm{d}h \sim h_0^{1/2}. \tag{34}$$

Since h_0 is proportional to the small isotropic strain, and hence to $\Delta\phi$, this gives $\Delta z \sim \Delta\phi^{1/2}$. The pressure exponent, by contrast, depends on the force law through α. The form $g(h) \sim h^{-1/2}$ is a statement about the near-contact structure of the marginal packing. It is not implied by the potential alone. The square-root law for Δz and the force-law-dependent pressure exponent therefore have different origins: one is geometric, the other follows from the interaction law.

For a power-law contact potential, the characteristic contact stiffness is

$$k \sim V'' \sim \delta^{\alpha-2} \sim \Delta\phi^{\alpha-2}. \tag{35}$$

It is useful to keep k separate from the geometric factor Δz, because different observables contain these two ingredients in different combinations.

4 Athermal ensembles

The mechanically stable states are not characterized by a Boltzmann factor of the form

$$P_\nu \propto \mathrm{e}^{-\beta E_\nu}. \tag{36}$$

Instead the probabilities result from the preparation dynamics or from an assumed ensemble over jammed states.

A simple statistical question is therefore: how many mechanically stable states exist at a given macroscopic volume? Let

$$\Omega(V) = \sum_{\nu \in \text{jammed}} \delta(V - V_\nu). \tag{37}$$

Define the configurational entropy

$$S(V) = k_{\mathrm{B}} \ln \Omega(V). \tag{38}$$

The quantity conjugate to volume is the compactivity X,

$$\frac{1}{X} = \frac{\partial S}{\partial V}. \tag{39}$$

As is conventional in much of the Edwards literature, the factor k_{B} can be absorbed into the definition of X. With this convention the exponential weight below is dimensionless. Nothing physical depends on where the constant k_{B} is placed, provided the convention is used consistently.

If the jammed states at fixed volume are sampled uniformly, the corresponding volume ensemble is

$$\mathcal{Z}(X) = \sum_{\nu \in \mathrm{jammed}} \mathrm{e}^{-V_\nu/X}, \tag{40}$$

and

$$P_\nu = \frac{1}{\mathcal{Z}(X)} \mathrm{e}^{-V_\nu/X}. \tag{41}$$

The formal analogy with a thermal canonical ensemble is

$$E \leftrightarrow V, \qquad T \leftrightarrow X. \tag{42}$$

The analogy should not be pushed further than the measure warrants. The Edwards weight is an additional assumption about the statistics of jammed states; it is not produced automatically by the dissipative equations of motion.

The basin-volume construction in the first part of the lecture makes this point concrete. A deterministic protocol generates

$$P_\nu = \int_{\mathcal{B}_\nu} \mathrm{d}\mathbf{R}_0 \, \rho_0(\mathbf{R}_0), \tag{43}$$

whereas the Edwards ensemble assigns a weight from a macroscopic state variable such as V_ν. There is no general reason for the two measures to agree.

5 Stress and the force-moment tensor

Volume is not the only extensive mechanical variable. A jammed packing also carries a force-moment tensor. For a collection of contacts,

$$\mathbf{\Sigma} = \sum_{\langle ij \rangle} \mathbf{r}_{ij} \otimes \mathbf{f}_{ij}, \tag{44}$$

up to the conventional geometric factors used in defining the macroscopic stress.

The natural state-counting function at fixed force moment is

$$\Omega(\mathbf{\Sigma}) = \sum_{\nu \in \mathrm{jammed}} \delta(\mathbf{\Sigma} - \mathbf{\Sigma}_\nu), \tag{45}$$

with entropy

$$S(\mathbf{\Sigma}) = k_{\mathrm{B}} \ln \Omega(\mathbf{\Sigma}). \tag{46}$$

The inverse-angoricity tensor is defined by

$$\boldsymbol{\alpha} = \frac{\partial S}{\partial \mathbf{\Sigma}}. \tag{47}$$

We use the same convention as for compactivity and absorb the entropy unit into the definition of $\boldsymbol{\alpha}$ when writing the statistical weight. Thus $\boldsymbol{\alpha}$ is the intensive quantity conjugate to the force moment in the exponent below. It should be noted that sometimes the angoricity tensor itself is defined as the inverse of this quantity.

A stress ensemble can then be written as

$$P_\nu \propto \exp\left[-\boldsymbol{\alpha} : \boldsymbol{\Sigma}_\nu\right], \tag{48}$$

where

$$\boldsymbol{\alpha} : \boldsymbol{\Sigma} = \sum_{ij} \alpha_{ij} \Sigma_{ij}. \tag{49}$$

A combined volume–stress ensemble takes the form

$$P_\nu \propto \exp\left[-\frac{V_\nu}{X} - \boldsymbol{\alpha} : \boldsymbol{\Sigma}_\nu\right]. \tag{50}$$

Angoricity is not a temperature. It is a parameter conjugate to a mechanical extensive variable in a proposed ensemble of mechanically stable states. Whether this ensemble accurately represents a particular preparation protocol is a separate physical question.

Mechanically stable states can then be viewed statistically through an athermal measure over jammed configurations. Volume gives the Edwards construction through compactivity, while force moments give the stress ensemble and angoricity.

6 Mechanical constraints in configuration space

The set of mechanically stable configurations is strongly constrained before any statistical weight is assigned. For frictionless particles with contact normals $\hat{\mathbf{n}}_{ij}$ and normal contact forces $f_{ij} \geq 0$, force balance on each particle requires

$$\sum_j f_{ij} \hat{\mathbf{n}}_{ij} = 0. \tag{51}$$

For frictional particles there is, in addition, torque balance,

$$\sum_j \mathbf{r}_{ij} \times \mathbf{f}_{ij} = 0, \tag{52}$$

and the Coulomb inequality,

$$|f_{t,ij}| \leq \mu_f f_{n,ij}. \tag{53}$$

These relations define the mechanically admissible region of force and configuration space. They are constraints, not probability weights. A statistical ensemble can only be defined after this admissible set has been specified.

The corresponding continuum statement is force balance for the stress field,

$$\partial_i \sigma_{ij} = 0, \tag{54}$$

possibly together with stress symmetry when torque balance is imposed in the usual way.

7 Stress correlations from a constrained field theory

The mechanical constraint can be incorporated directly into a coarse-grained field theory. In two dimensions, a symmetric divergence-free stress tensor can be written in terms of an Airy stress function ψ,

$$\sigma_{xx} = \partial_y^2 \psi, \tag{55}$$

$$\sigma_{yy} = \partial_x^2 \psi, \tag{56}$$

$$\sigma_{xy} = -\partial_x \partial_y \psi. \tag{57}$$

The force-balance equation is then satisfied identically.

Consider a Gaussian statistical weight

$$\mathcal{P}[\psi] = \frac{1}{\mathcal{Z}} \exp\left[-\frac{1}{2\mathcal{A}} \int \mathrm{d}^2 r \, (\nabla^2 \psi)^2\right], \tag{58}$$

where $\mathcal{A}$ measures the amplitude of fluctuations in the chosen mechanical ensemble.

It is useful to write the same statistical weight directly in terms of the stress tensor. From the Airy representation,

$$\nabla^2 \psi = \partial_x^2 \psi + \partial_y^2 \psi = \sigma_{xx} + \sigma_{yy} = \mathrm{Tr}\, \boldsymbol{\sigma}. \tag{59}$$

The Gaussian measure may therefore be written as

$$\mathcal{P}[\boldsymbol{\sigma}] \propto \exp\left[-\frac{1}{2\mathcal{A}} \int \mathrm{d}^2 r \, (\mathrm{Tr}\, \boldsymbol{\sigma})^2\right], \tag{60}$$

where the integration is restricted to symmetric stress fields satisfying

$$\partial_i \sigma_{ij} = 0. \tag{61}$$

The force-balance constraint and the statistical weight have different roles. Force balance specifies the allowed stress fields; the quadratic functional is the simplest local Gaussian choice for weighting those allowed fields. It is therefore an assumption about the coarse-grained ensemble, not a consequence of mechanical equilibrium alone.

Relating the Coarse-Grained Measure to Angoricity: The Gaussian measure for the stress field can be viewed as the quadratic (fluctuation) expansion of the macroscopic stress ensemble, where the angoricity tensor α_{ij} is the intensive variable conjugate to the total force-moment tensor

$$\Sigma_{ij} = \int d^2 r \, \sigma_{ij}(r). \tag{62}$$

The linear constraint $\alpha : \Sigma$ fixes only the saddle point, i.e., the uniform background stress $\langle \sigma_{ij} \rangle$ (the $k = 0$ component). Writing $\sigma_{ij}(r) = \langle \sigma_{ij} \rangle + \delta\sigma_{ij}(r)$ and, for an isotropic packing, retaining the leading scalar invariant dominated by local pressure fluctuations, we use Eq. (61) to identify $\mathrm{Tr}(\delta\sigma) = \delta\sigma_{xx} + \delta\sigma_{yy} = \nabla^2 \delta\psi$. The corresponding Gaussian weight is

$$P[\delta\sigma] \propto \exp\Big[-\tfrac{1}{2} \int d^2 r \, \chi^{-1} \big(\mathrm{Tr}\ \delta\sigma\big)^2\Big], \tag{63}$$

so the Airy-field fluctuation amplitude is the macroscopic susceptibility,

$$\mathcal{A} = \chi = -\Big(\frac{\partial \alpha_p}{\partial p}\Big)^{-1}, \qquad p = \tfrac{1}{2}\, \mathrm{Tr}\langle \sigma \rangle, \tag{64}$$

with α_p the isotropic trace of α_{ij}. Thus α_{ij} sets the mean stress background, while $\mathcal{A}$ controls the strength of allowed local fluctuations; together with force balance, this leads to the k^{-4} Airy correlations derived above.

For $\mathbf{k} \neq 0$ in two dimensions, symmetry and force balance leave only one independent stress amplitude. Defining the unit vector transverse to $\mathbf{k}$ by

$$\hat{\mathbf{t}} = \frac{1}{k}(-k_y, k_x), \qquad \mathbf{k} \cdot \hat{\mathbf{t}} = 0, \tag{65}$$

we may write

$$\sigma_{ij}(\mathbf{k}) = s(\mathbf{k})\, t_i t_j, \tag{66}$$

so that $\operatorname{Tr} \boldsymbol{\sigma} = s$ and $\sigma_{ij}\sigma_{ij} = s^2$. The Gaussian measure is therefore equivalently a Gaussian measure for the single scalar mode $s(\mathbf{k})$, and with the Airy convention used below,

$$s(\mathbf{k}) = -k^2 \psi_{\mathbf{k}}, \tag{67}$$

which makes the origin of the k^{-4} Airy correlation and the resulting stress correlations transparent.

Expand the field in Fourier modes,

$$\psi(\mathbf{r}) = \int_{\mathbf{k}} \psi_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \qquad \int_{\mathbf{k}} = \int \frac{\mathrm{d}^2 k}{(2\pi)^2}. \tag{68}$$

Then

$$\nabla^2 \psi = -\int_{\mathbf{k}} k^2 \psi_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \tag{69}$$

and orthogonality of Fourier modes gives

$$\int \mathrm{d}^2 r\, (\nabla^2 \psi)^2 = \int_{\mathbf{k}} k^4 \psi_{\mathbf{k}} \psi_{-\mathbf{k}}. \tag{70}$$

Hence

$$\mathcal{P}[\psi] \propto \exp\left[-\frac{1}{2\mathcal{A}} \int_{\mathbf{k}} k^4 \psi_{\mathbf{k}} \psi_{-\mathbf{k}}\right]. \tag{71}$$

Each Fourier mode is Gaussian. For a normalized one-dimensional Gaussian $P(x) \propto \exp[-ax^2/2]$, one has $\langle x^2 \rangle = 1/a$. Applying this mode by mode gives

$$\langle \psi_{\mathbf{k}} \psi_{-\mathbf{k}} \rangle = \frac{\mathcal{A}}{k^4}. \tag{72}$$

Equivalently, using $s(\mathbf{k}) = -k^2 \psi_{\mathbf{k}}$,

$$\langle s(\mathbf{k}) s(-\mathbf{k}) \rangle = \mathcal{A}. \tag{73}$$

Thus the wavevector-independent amplitude is what is being sampled by the Gaussian stress ensemble; the k-dependence of the stress correlations below comes entirely from the geometry of the force-balance constraint.

For a nonzero uniform background stress it is the fluctuating part,

$$\delta\sigma_{ij} = \sigma_{ij} - \langle \sigma_{ij} \rangle, \tag{74}$$

that should be correlated. The uniform contribution resides at $\mathbf{k} = 0$.

In Fourier space,

$$\sigma_{xx}(\mathbf{k}) = -k_y^2\psi_{\mathbf{k}}, \tag{75}$$
$$\sigma_{yy}(\mathbf{k}) = -k_x^2\psi_{\mathbf{k}}, \tag{76}$$
$$\sigma_{xy}(\mathbf{k}) = k_x k_y\psi_{\mathbf{k}}. \tag{77}$$

Therefore

$$C_{xyxy}(\mathbf{k}) = \langle\sigma_{xy}(\mathbf{k})\sigma_{xy}(-\mathbf{k})\rangle = \mathcal{A}\frac{k_x^2k_y^2}{k^4}, \tag{78}$$
$$C_{xxxx}(\mathbf{k}) = \mathcal{A}\frac{k_y^4}{k^4}, \tag{79}$$
$$C_{yyyy}(\mathbf{k}) = \mathcal{A}\frac{k_x^4}{k^4}. \tag{80}$$

Writing $k_x = k\cos\theta$ and $k_y = k\sin\theta$ gives

$$C_{xxxx}(\mathbf{k}) = \mathcal{A}\sin^4\theta, \tag{81}$$
$$C_{xyxy}(\mathbf{k}) = \mathcal{A}\sin^2\theta\cos^2\theta, \tag{82}$$
$$C_{yyyy}(\mathbf{k}) = \mathcal{A}\cos^4\theta. \tag{83}$$

The factor k^{-4} in the Airy-field correlation is cancelled by the four powers of k generated when two stress components are correlated. It is worth seeing where the cancellation occurs. The Airy field has $\langle\psi_{\mathbf{k}}\psi_{-\mathbf{k}}\rangle \sim k^{-4}$ because the Gaussian action contains $(\nabla^2\psi)^2$. Each stress component contains two derivatives of ψ, and hence contributes two powers of wavevector. A stress–stress correlator therefore contains four powers of k and removes the radial k^{-4} dependence, leaving only the angular tensor structure. The result is therefore independent of the magnitude of k at small wavevector, but it retains angular dependence. This is the Fourier-space signature of the long-range stress correlations generated by the mechanical constraint.

To make the real-space decay explicit, define the biharmonic Green function

$$G_\psi(\mathbf{r}) = \int_{\mathbf{k}} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{k^4}. \tag{84}$$

It obeys

$$\nabla^4 G_\psi(\mathbf{r}) = \delta^{(2)}(\mathbf{r}). \tag{85}$$

In two dimensions, up to cutoff-dependent local terms,

$$G_\psi(r) = \frac{r^2}{8\pi}\ln\frac{r}{r_0}. \tag{86}$$

Since $\sigma_{xy} = -\partial_x\partial_y\psi$, the two minus signs cancel in the product, giving

$$C_{xyxy}(\mathbf{r}) = \mathcal{A}\partial_x^2\partial_y^2 G_\psi(\mathbf{r}). \tag{87}$$

With $G_\psi(r) = r^2\ln(r/r_0)/(8\pi)$, four derivatives produce a factor of r^{-2} and an angular function. For $r \neq 0$, Writing $x = r\cos\theta$ and $y = r\sin\theta$ makes the origin of the fourth angular harmonic explicit. Direct differentiation gives

$$\partial_x^2\partial_y^2\left[r^2\ln(r/r_0)\right] = -\frac{2}{r^2}\cos 4\theta, \qquad r \neq 0, \tag{88}$$

so that

$$C_{xyxy}(\mathbf{r}) = -\frac{\mathcal{A}}{4\pi r^2}\cos 4\theta. \tag{89}$$

Thus the nonlocal part of the stress correlation is long ranged and decays as $1/r^2$ in two dimensions. This completes the mechanical progression developed in the previous lectures: the dissipative dynamics selects mechanically stable configurations; the Hessian determines the response of a selected packing; isostaticity controls the soft spectrum and elastic scaling; and mechanical equilibrium constrains the allowed coarse-grained stress fields.

More generally, the inverse Fourier transform of k^{-4} has a nonlocal contribution scaling as r^{4-d}, with logarithmic modifications at special dimensions. Since each stress tensor contains two derivatives of the Airy field, a stress–stress correlation contains four derivatives and therefore has the large-distance form

$$C_{ijkl}(\mathbf{r}) \sim \frac{1}{r^d}\mathcal{A}_{ijkl}(\hat{\mathbf{r}}), \qquad r \to \infty. \tag{90}$$

The radial decay is a consequence of the mechanical constraint and the chosen coarse-grained quadratic measure. It is not a consequence of equilibrium FDT. The amplitude $\mathcal{A}$ is not fixed by the equilibrium temperature of Lectures 1–2. In the athermal problem it is determined by the chosen statistical measure, or ultimately by the preparation protocol. The constraint fixes the spatial form of the correlations once that measure has been specified.