Statistical Physics of Athermal Systems
Lectures 1

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1 Athermal Systems

The systems considered in this course span a broad range of disordered materials, including molecular and colloidal glasses, foams, emulsions, granular materials, and dense suspensions. For thermal systems, the transition between a fluid and a glassy solid is usually discussed in the (T, ϕ) plane. In contrast, for athermal systems the temperature becomes dynamically irrelevant and the relevant control parameters are the density and the externally imposed stress. Typically, a jamming transition is encountered as the system crosses a critical packing fraction ϕ_J, or, more generally, when sufficient stress is applied to a system that is below the density at which it can jam at zero stress. In such athermal systems, rigidity arises from mechanical constraints and the existence of a mechanically stable contact network. A useful way to organize these systems

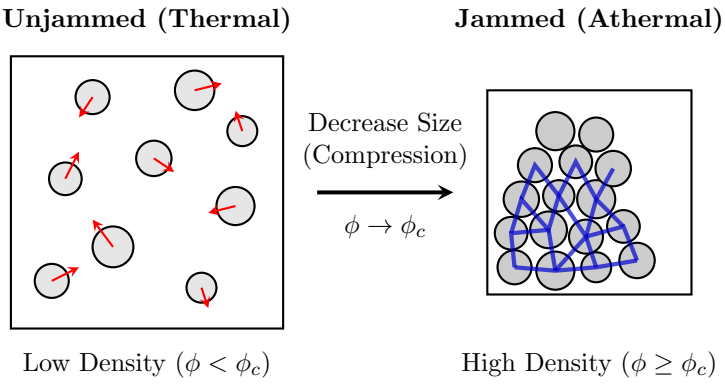

Figure 1: Schematic of the jamming transition under volume compression. (Left) At low packing fractions ($\phi < \phi_c$), the system is unjammed and governed by thermal, kinetic fluctuations. (Right) As the system size is decreased, the density crosses the critical jamming threshold. The dynamics become structurally arrested (athermal), and mechanical stability is maintained by a percolating network of force chains (blue lines) acting across grain contacts.

is through the jamming phase diagram. In its simplest form, the phase diagram is spanned by the packing fraction ϕ, the shear stress σ, and the temperature T. The central idea is that the distinction between a fluid and a solid cannot in general be specified by temperature alone: a sufficiently dense system can be rigid at zero temperature, while a system below its jamming density can be made mechanically rigid by applying an external stress. It is important that the jamming point is not simply a conventional thermodynamic critical point. In particular, the location of the transition can depend on the preparation protocol, particle interactions, friction, particle shape, and the manner in which the system is driven. Nevertheless, the vicinity of the

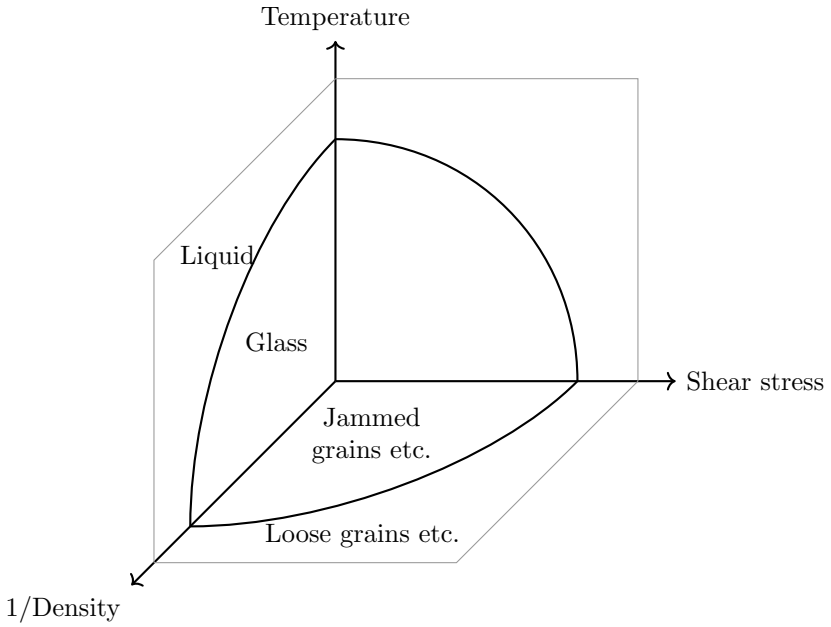

Figure 2: Generalized jamming phase diagram. The enclosed volume near the origin represents the structurally arrested, jammed state. A system can be unjammed and driven into a flowing or yielding state by increasing thermal fluctuations (temperature), decreasing the volume fraction (increasing 1/density), or applying a mechanical load exceeding the critical yield threshold (shear stress). Adapted from Liu and Nagel, Nature **396**, 21–22 (1998).

jamming point exhibits a number of remarkably robust features, most notably the emergence of an isostatic contact network and the appearance of characteristic diverging length scales.

Lectures 1 and 2 follow the **References:**

1. L. Berthier and J. Kurchan, Lectures on nonequilibrium active systems, arXiv:1906.04039 (2019).
2. A. J. Liu, S. R. Nagel, W. van Saarloos, and M. Wyart, The jamming scenario—an introduction and outlook, arXiv:1006.2365 (2011).
3. D. Bi, S. Henkes, K. E. Daniels, and B. Chakraborty, The Statistical Physics of Athermal Materials, Annu. Rev. Condens. Matter Phys. 6:63–83 (2015).
4. L. Radzihovsky course on Advanced Statistical Mechanics Lecture 4: Field Theory Primer

2 Interaction Potentials

It is useful to consider simple model systems with which we can study the nature of such athermal systems. We will focus primarily on two paradigmatic interaction potentials. Paradigmatic Interaction Potentials **1. Soft One-Sided Spheres:** This model isolates the purely athermal jamming transition. Particles interact only when geometrically overlapping ($\delta_{ij} = 1 - r_{ij}/\sigma_{ij} > 0$):

$$V(r_{ij}) = \frac{\epsilon}{\alpha}\left(1 - \frac{r_{ij}}{\sigma_{ij}}\right)^{\alpha} \Theta(\sigma_{ij} - r_{ij}) \tag{1}$$

Common variants include harmonic ($\alpha = 2$) and Hertzian ($\alpha = 5/2$) interactions. Above the jamming density ($\Delta\phi = \phi - \phi_{\mathrm{J}} > 0$), macroscopic properties scale algebraically:

$$\frac{U}{N} \sim \Delta\phi^{\alpha}, \qquad p \sim \Delta\phi^{\alpha-1} \tag{2}$$

2. The Weeks–Chandler–Andersen (WCA) Potential: The WCA model bridges thermal liquids and amorphous glasses using a purely repulsive, truncated Lennard-Jones potential acting at $r < 2^{1/6}\sigma$:

$$V_{\mathrm{WCA}}(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} + \frac{1}{4}\right]$$

Unlike soft spheres, it lacks a sharp geometric contact cutoff, allowing a continuous crossover to dynamical arrest as density increases or temperature drops.

2.1 When does a system become athermal?

A system is effectively "athermal" when thermal fluctuations are negligible on the energy scales relevant for microscopic rearrangements,

$$k_{\mathrm{B}}T \ll E_{\mathrm{barrier}}. \tag{3}$$

Thus, athermal behavior does not require $T = 0$. For soft particles with interaction scale ϵ, a necessary condition is $k_{\mathrm{B}}T/\epsilon \ll 1$, but the relevant comparison is with the barriers between mechanically distinct configurations. Near jamming these mechanical scales become small; for harmonic interactions, $E_{\mathrm{mech}} \sim \epsilon(\Delta\phi)^2$, so thermal effects can become important sufficiently close to ϕ_{J}. It is useful to distinguish this from glass formation. In a thermal glass, particles continue to undergo thermal fluctuations and the equilibrium distribution is

$$P(\{\mathbf{r}_i\}) \propto \exp\left[-\frac{U(\{\mathbf{r}_i\})}{k_{\mathrm{B}}T}\right], \tag{4}$$

with rigidity emerging through the slowing down of structural relaxation. In an athermal jammed solid, by contrast, thermal fluctuations are negligible and the configuration is selected by mechanical stability,

$$\frac{\partial U}{\partial \mathbf{r}_i} = 0, \tag{5}$$

together with the stability of the corresponding mechanically constrained state. Importantly, athermal does not mean fluctuation-free. In the absence of thermal equilibration, quenched disorder, mechanical constraints, and external driving can produce large spatial and temporal fluctuations, including heterogeneous contact forces and force chains. The statistical physics of athermal matter must therefore characterize these fluctuations without assuming that they originate from a thermal bath.

3 Equilibrium Statistical Mechanics

A microscopic classical system is specified by a point in a very high-dimensional phase space. For N particles, we may write

$$\Gamma = \{\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N\}. \tag{6}$$

The microscopic equations of motion determine a trajectory $\Gamma(t)$ once the initial condition is specified. In principle, therefore, the microscopic problem is deterministic. The statistical-mechanical problem is different. We are rarely interested in the exact position and momentum of every particle at every instant. Instead, we measure macroscopic quantities such as

$$E, \quad P, \quad \rho, \quad \sigma_{ij}, \quad J_i, \tag{7}$$

where E is the energy, P the pressure, ρ a density, σ_{ij} a stress tensor, and J_i a current. These observables depend on enormous numbers of microscopic degrees of freedom. We therefore replace the detailed trajectory by a probability measure over microscopic states. If $P(\Gamma, t)$ is the probability density, then for any observable $A(\Gamma)$,

$$\langle A \rangle_t = \int d\Gamma \, P(\Gamma, t) A(\Gamma). \tag{8}$$

The normalization condition is

$$\int d\Gamma \, P(\Gamma, t) = 1. \tag{9}$$

What determines $P(\Gamma, t)$? For Hamiltonian dynamics,

$$\dot{\mathbf{r}}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \qquad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{r}_i}. \tag{10}$$

The Hamiltonian flow preserves phase-space volume. More precisely, if $\rho(\Gamma, t)$ denotes the phase-space density, Liouville's equation is

$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0, \tag{11}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. Since the Hamiltonian itself is conserved,

$$\frac{dH}{dt} = 0, \tag{12}$$

the microcanonical density

$$P_E(\Gamma) \propto \delta(E - H(\Gamma)) \tag{13}$$

is stationary. For a system in contact with a thermal reservoir, the corresponding stationary measure becomes canonical,

$$P_{\mathrm{eq}}(\Gamma) = \frac{1}{Z} e^{-\beta H(\Gamma)}, \qquad \beta = \frac{1}{k_{\mathrm{B}} T}. \tag{14}$$

The key point is that the Gibbs measure is not postulated here as an arbitrary guess. In this lecture we will construct the reservoir explicitly and derive the stochastic dynamics that leaves this measure invariant. Equilibrium gives us much more than a stationary one-time distribution. Suppose A and B are observables. A one-time average is

$$\langle A \rangle_{\mathrm{eq}} = \int d\Gamma \, P_{\mathrm{eq}}(\Gamma) A(\Gamma), \tag{15}$$

while a two-time correlation is

$$C_{AB}(t,t') = \langle A(t)B(t')\rangle. \tag{16}$$

The latter cannot be obtained from the stationary distribution alone; it depends on the transition probabilities generated by the dynamics. Two systems can have exactly the same equilibrium distribution and yet have very different relaxation times, diffusion constants, or viscosities. Equilibrium removes the need to solve the dynamics for static averages, but it does not remove dynamics from transport or time-dependent response.

4 Constructing an explicit heat bath

We now seek a microscopic mechanism for the equilibrium measure by coupling the system to a large bath at temperature T and then eliminating the bath variables. This yields an effective stochastic dynamics for the system even though the underlying system+bath Hamiltonian dynamics is deterministic; the randomness reflects coarse-graining over bath coordinates. In the reduced Langevin equation, dissipation and fluctuations are linked in equilibrium by the fluctuation–dissipation relation, which we aim to derive rather than assume.

4.1 System, bath, and coupling

We now construct a heat bath explicitly from a large collection of harmonic oscillators. This construction is known as the Caldeira–Leggett bath. We begin with one system degree of freedom. We denote its generalized coordinate by $q(t)$ and its conjugate momentum by $p(t)$. For a particle-like coordinate of mass m, $p = m\dot{q}$. The isolated-system Hamiltonian is

$$H_{\text{sys}} = \frac{p^2}{2m} + V(q), \tag{17}$$

where $V(q)$ is the conservative potential, so that the corresponding force is $-V'(q)$. The bath is a collection of oscillators labelled by $a = 1, \dots, M$. For oscillator a, x_a is its coordinate, π_a is its conjugate momentum, and ω_a is its natural angular frequency, defined by $\ddot{x}_a + \omega_a^2 x_a = 0$ when the oscillator is uncoupled. We use mass-normalized bath coordinates, so

$$H_{\text{bath}} = \sum_{a=1}^{M} \left[\frac{\pi_a^2}{2} + \frac{\omega_a^2 x_a^2}{2} \right]. \tag{18}$$

At this point the complete microscopic state is

$$\Gamma_{\text{tot}} = \left(q, p; \{x_a, \pi_a\}_{a=1}^{M}\right). \tag{19}$$

The semicolon separates the system variables from the bath variables. We couple the system linearly to each bath coordinate. The coefficient c_a is the system–bath coupling strength; the interaction energy is $-c_a q x_a$. Hence the force $+c_a x_a$ acts on the system, while the force $+c_a q$ drives oscillator a. A convenient form of the full Hamiltonian, including the counterterm, is

$$H_{\text{tot}} = \frac{p^2}{2m} + V(q) + \sum_{a=1}^{M} \left[\frac{\pi_a^2}{2} + \frac{\omega_a^2}{2} \left(x_a - \frac{c_a}{\omega_a^2} q \right)^2 \right]. \tag{20}$$

Expanding the square,

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{bath}} - q \sum_a c_a x_a + \frac{q^2}{2} \sum_a \frac{c_a^2}{\omega_a^2}. \tag{21}$$

Every term in H_{tot} has dimensions of energy. Since $c_a q x_a$ is an energy, the same coupling c_a appears in the force $c_a x_a$ acting on the system and in the driving term $c_a q$ acting on oscillator

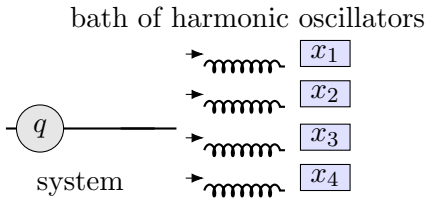

Figure 3: Schematic bath construction. The system coordinate q is coupled linearly to many harmonic bath modes. Eliminating the bath produces both a memory kernel and a fluctuating force.

a. The combination c_a^2/ω_a^2 therefore has precisely the dimensions needed by the counterterm multiplying q^2. At fixed system coordinate q, the bath contribution for oscillator a is

$$H_a = \frac{\pi_a^2}{2} + \frac{\omega_a^2 x_a^2}{2} - c_a q x_a + \frac{c_a^2}{2\omega_a^2} q^2. \tag{22}$$

Complete the square,

$$H_a = \frac{\pi_a^2}{2} + \frac{\omega_a^2}{2}\left(x_a - \frac{c_a}{\omega_a^2} q\right)^2. \tag{23}$$

The conditional bath partition function is therefore

$$Z_a(q) = \int_{-\infty}^{\infty} d\pi_a\, dx_a \exp\left[-\beta H_a\right] \tag{24}$$

$$= \left(\int d\pi_a\, e^{-\beta\pi_a^2/2}\right)\left(\int dx_a\, e^{-\beta\omega_a^2 (x_a - c_a q/\omega_a^2)^2/2}\right). \tag{25}$$

Shift the integration variable to

$$y_a = x_a - \frac{c_a}{\omega_a^2} q. \tag{26}$$

Because the integration limits are infinite,

$$\int dx_a\, e^{-\beta\omega_a^2 (x_a - c_a q/\omega_a^2)^2/2} = \int dy_a\, e^{-\beta\omega_a^2 y_a^2/2}, \tag{27}$$

which is independent of q. Hence

$$Z_{\rm tot} = Z_{\rm bath}^{(0)} \int dq\, dp\, e^{-\beta[p^2/(2m) + V(q)]}. \tag{28}$$

The counterterm guarantees that the equilibrium marginal of the system is exactly the desired Gibbs distribution rather than a bath-renormalized potential. This is also the explicit partition-function content of the counterterm construction emphasized in the source notes .

4.2 Initial statistical independence

For the standard derivation we prepare the shifted bath variables $y_a(0)$ and $\pi_a(0)$ thermally, conditional on the initial system coordinate $q(0)$. This means that the bath is Gaussian once the appropriate equilibrium shift is made. The system coordinate itself need not be independently sampled from the bath in an arbitrary nonequilibrium preparation; what matters for the fluctuation–dissipation relation is that the bath degrees of freedom are initially distributed according to their conditional equilibrium distribution. This allows us to integrate out the bath dynamically, which is an exact change of description. Next, assuming a statistical state for the bath, which is an additional physical assumption, turns the deterministic bath force into a statistical noise with a calculable covariance.

4.3 Equations of motion

Hamilton's equations give

$$\dot{q} = \frac{p}{m}, \tag{29}$$

$$\dot{p} = -V'(q) + \sum_a c_a x_a - q \sum_a \frac{c_a^2}{\omega_a^2}, \tag{30}$$

and

$$\ddot{x}_a + \omega_a^2 x_a = c_a q(t). \tag{31}$$

5 Eliminating the bath: the generalized Langevin equation

Before solving: what is the Green's function? The retarded Green's function $G_a(t)$ is defined as the solution of

$$\left(\frac{d^2}{dt^2} + \omega_a^2\right) G_a(t) = \delta(t), \qquad G_a(t<0) = 0. \tag{32}$$

The second condition is causality: the bath cannot respond before the system has driven it. For the oscillator,

$$G_a(t) = \Theta(t) \frac{\sin(\omega_a t)}{\omega_a}. \tag{33}$$

Here $\Theta(t)$ is the Heaviside step function, equal to one for $t > 0$ and zero for $t < 0$. The convolution with G_a automatically enforces causality. It is useful to perform a dimensional check. Because q and x_a are coordinates, the convolution term

$$c_a \int dt'\, G_a(t-t') q(t') \tag{34}$$

must have the dimensions of x_a. Since G_a has dimensions of time, this fixes the dimensions of the coupling coefficient consistently with the Hamiltonian.

5.1 Exact solution of one bath oscillator

For a fixed a, the equation

$$\ddot{x}_a + \omega_a^2 x_a = c_a q(t) \tag{35}$$

is a driven harmonic oscillator. The solution is the homogeneous solution plus a particular solution. The homogeneous solution is

$$x_a^{\mathrm{hom}}(t) = x_a(0)\cos(\omega_a t) + \frac{\pi_a(0)}{\omega_a}\sin(\omega_a t). \tag{36}$$

Using the Green's function of the oscillator,

$$G_a(t) = \frac{\sin(\omega_a t)}{\omega_a}\Theta(t), \tag{37}$$

we obtain

$$x_a(t) = x_a^{\mathrm{hom}}(t) + c_a \int_0^t ds\, G_a(t-s) q(s). \tag{38}$$

Thus,

$$x_a(t) = x_a(0)\cos(\omega_a t) + \frac{\pi_a(0)}{\omega_a}\sin(\omega_a t) + \frac{c_a}{\omega_a}\int_0^t ds\, \sin[\omega_a(t-s)] q(s). \tag{39}$$

5.2 Substitution back into the system equation

Insert Eq. (38) into Eq. (30). We obtain

$$m\ddot{q} = -V'(q) + \sum_a c_a x_a(0)\cos(\omega_a t) + \sum_a \frac{c_a \pi_a(0)}{\omega_a}\sin(\omega_a t) \tag{40}$$

$$+ \sum_a \frac{c_a^2}{\omega_a}\int_0^t ds\, \sin[\omega_a(t-s)]q(s) - q\sum_a \frac{c_a^2}{\omega_a^2}. \tag{41}$$

We now simplify the convolution term. The integral

$$I_a(t) = \int_0^t ds\, \sin[\omega_a(t-s)]q(s) \tag{42}$$

can be integrated by parts. Since

$$\frac{d}{ds}\cos[\omega_a(t-s)] = \omega_a \sin[\omega_a(t-s)], \tag{43}$$

we have

$$I_a(t) = \frac{1}{\omega_a}\int_0^t ds\, q(s)\frac{d}{ds}\cos[\omega_a(t-s)] \tag{44}$$

$$= \frac{1}{\omega_a}\left[q(s)\cos[\omega_a(t-s)]\right]_0^t - \frac{1}{\omega_a}\int_0^t ds\, \dot{q}(s)\cos[\omega_a(t-s)] \tag{45}$$

$$= \frac{q(t) - q(0)\cos(\omega_a t)}{\omega_a} - \frac{1}{\omega_a}\int_0^t ds\, \dot{q}(s)\cos[\omega_a(t-s)]. \tag{46}$$

Substituting this into Eq. (41), the term proportional to $q(t)$ cancels the counterterm exactly. The remaining equation is

$$m\ddot{q}(t) = -V'(q(t)) - \int_0^t ds\, \Gamma(t-s)\dot{q}(s) + \eta(t), \tag{47}$$

where the memory kernel is

$$\Gamma(t) = \sum_{a=1}^{M}\frac{c_a^2}{\omega_a^2}\cos(\omega_a t) \tag{48}$$

$\Gamma(t)$ has the dimensions of a friction coefficient divided by time, so that

$$\int_0^t ds\, \Gamma(t-s)\dot{q}(s) \tag{49}$$

has the dimensions of a force. The kernel tells us how strongly the bath remembers the system velocity at earlier times. A slowly decaying $\Gamma(t)$ gives non-Markovian dynamics; a sharply peaked kernel gives local friction. Why the kernel is a sum of cosines: Each independent harmonic bath mode remembers the past periodically. The total memory is the superposition of these oscillatory memories,

$$\Gamma(t) = \sum_a \Gamma_a(t), \qquad \Gamma_a(t) = \frac{c_a^2}{\omega_a^2}\cos(\omega_a t). \tag{50}$$

Thus a bath with many closely spaced frequencies can have a rapidly decaying collective memory even though each individual oscillator is perfectly coherent. and the fluctuating force is

$$\eta(t) = \sum_a c_a\left[x_a(0)\cos(\omega_a t) + \frac{\pi_a(0)}{\omega_a}\sin(\omega_a t) - \frac{c_a q(0)}{\omega_a^2}\cos(\omega_a t)\right]. \tag{51}$$

This is the generalized Langevin equation. The bath has not been replaced by an arbitrary friction coefficient. The friction kernel has been derived from a Hamiltonian model. The fluctuating force has also been derived. We have not yet assumed a Markov process and we have not yet postulated the fluctuation–dissipation theorem. It will emerge after specifying the statistical state of the bath.

6 The fluctuation–dissipation relation

6.1 Thermal preparation of the bath

We assume that at $t = 0$ the bath degrees of freedom are sampled from a Gibbs distribution at temperature T. Because the counterterm shifts the oscillator centers, it is convenient to define

$$y_a = x_a - \frac{c_a}{\omega_a^2} q(0). \tag{52}$$

For fixed $q(0)$, the bath Hamiltonian is

$$H_{\mathrm{bath}}^{\mathrm{shifted}} = \sum_a \left[\frac{\pi_a^2}{2} + \frac{\omega_a^2 y_a^2}{2} \right]. \tag{53}$$

Therefore the initial bath distribution factorizes into independent Gaussian variables:

$$\langle y_a(0) \rangle = 0, \tag{54}$$
$$\langle \pi_a(0) \rangle = 0, \tag{55}$$
$$\langle y_a(0) y_b(0) \rangle = \frac{k_{\mathrm{B}} T}{\omega_a^2} \delta_{ab}, \tag{56}$$
$$\langle \pi_a(0) \pi_b(0) \rangle = k_{\mathrm{B}} T \, \delta_{ab}, \tag{57}$$
$$\langle y_a(0) \pi_b(0) \rangle = 0. \tag{58}$$

6.2 Mean noise

Because every initial Gaussian variable has zero mean,

$$\langle \eta(t) \rangle = 0. \tag{59}$$

6.3 Noise correlation

We write the noise in terms of y_a:

$$\eta(t) = \sum_a c_a \left[y_a(0) \cos(\omega_a t) + \frac{\pi_a(0)}{\omega_a} \sin(\omega_a t) \right]. \tag{60}$$

Therefore

$$\big\langle \eta(t) \eta(t') \big\rangle = \sum_{a,b} c_a c_b \Bigg[\langle y_a y_b \rangle \cos(\omega_a t) \cos(\omega_b t') \tag{61}$$
$$+ \frac{\langle \pi_a \pi_b \rangle}{\omega_a \omega_b} \sin(\omega_a t) \sin(\omega_b t') \tag{62}$$
$$+ \frac{\langle y_a \pi_b \rangle}{\omega_b} \cos(\omega_a t) \sin(\omega_b t') \tag{63}$$
$$+ \frac{\langle \pi_a y_b \rangle}{\omega_a} \sin(\omega_a t) \cos(\omega_b t') \Bigg]. \tag{64}$$

The mixed averages vanish. Using Eq. (58), only $a = b$ terms survive,

$$\langle \eta(t)\eta(t') \rangle = k_{\mathrm{B}}T \sum_a \frac{c_a^2}{\omega_a^2} \left[\cos(\omega_a t)\cos(\omega_a t') + \sin(\omega_a t)\sin(\omega_a t') \right] \tag{65}$$

$$= k_{\mathrm{B}}T \sum_a \frac{c_a^2}{\omega_a^2} \cos[\omega_a (t - t')]. \tag{66}$$

Hence

$$\langle \eta(t)\eta(t') \rangle = k_{\mathrm{B}}T\, \Gamma(t - t') \tag{67}$$

with Γ from Eq. (48). This is the fluctuation–dissipation relation of the second kind.

7 The Markovian limit

A stochastic process is Markovian if, once the full state at time t is known, the conditional probability for future states does not depend on the earlier history. For the underdamped Langevin equation the natural Markov state is (q, p), not q alone. The coordinate $q(t)$ retains information about the past through the current momentum. After momentum relaxes on the timescale $\tau_m = m/\gamma$, the reduced coordinate can often be treated as an approximately Markovian variable. Let τ_B be the decay time of the bath correlation function and τ_S the timescale on which $\dot{q}$ changes appreciably. The small parameter controlling the Markov approximation is

$$\varepsilon_M = \frac{\tau_B}{\tau_S} \ll 1. \tag{68}$$

A local friction law is therefore not a definition of a Markov process; it is the leading approximation obtained when the bath forgets its past much faster than the system evolves. A process is Markovian if the conditional probability for its future depends on the past only through the present state. For a first-order stochastic equation driven by white noise, the current state contains all information needed to advance the process. Arovas emphasizes a useful subtlety: even if the random forcing has zero correlation time, the position process of an underdamped particle is not Markovian in q alone because the velocity stores memory. The pair (q, p) is the Markov state, while q alone becomes approximately Markovian only after the velocity relaxation time has been eliminated . Let τ_B be the decay time of the bath correlation function and let τ_S denote the timescale over which $\dot{q}$ changes appreciably. The Markov approximation requires

$$\varepsilon_M = \frac{\tau_B}{\tau_S} \ll 1. \tag{69}$$

Write the friction term as

$$I(t) = \int_0^t ds\, \Gamma(t - s)\dot{q}(s). \tag{70}$$

Set $u = t - s$, so that

$$I(t) = \int_0^t du\, \Gamma(u)\dot{q}(t - u). \tag{71}$$

For $u \lesssim \tau_B \ll \tau_S$, Taylor expand

$$\dot{q}(t - u) = \dot{q}(t) - u\ddot{q}(t) + \frac{u^2}{2}\dddot{q}(t) + \cdots. \tag{72}$$

Therefore

$$I(t) = \dot{q}(t) \int_0^t du\, \Gamma(u) - \ddot{q}(t) \int_0^t du\, u\Gamma(u) + \frac{\dddot{q}(t)}{2} \int_0^t du\, u^2\Gamma(u) + \cdots. \tag{73}$$

If $t \gg \tau_B$, the upper limit may be extended to infinity. Defining the leading friction coefficient

$$\gamma_0 = \int_0^\infty du\, \Gamma(u), \tag{74}$$

where γ_0 has dimensions of mass/time. We will identify $\gamma_0 \equiv \gamma$ in the Markovian equation. The first moment of the memory kernel

$$\gamma_1 = \int_0^\infty du\, u\Gamma(u), \tag{75}$$

we find

$$I(t) = \gamma_0 \dot{q}(t) - \gamma_1 \ddot{q}(t) + O(\varepsilon_M^2). \tag{76}$$

Thus the leading Markovian contribution is local friction. The next term is an "inertial renormalization" and is parametrically smaller when the timescale separation is good.