Prospects in post-Newtonian theory and gravitational self force

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Tuesday, October 28, 2025

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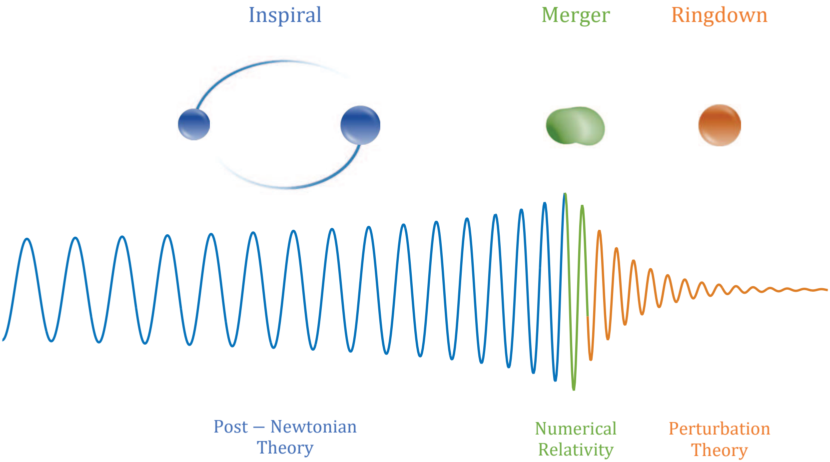
The three stages of a binary



[Antelis & Moreno (2017), arXiv:1610.03567]

Post-Newtonian theory in a nutshell

Two-body system obeys virial theorem:

$$\left(\frac{v_{12}}{c}\right)^2 \approx \frac{Gm_{\text{tot}}}{r_{12}c^2}$$

Newton's law of gravitation : $\mathbf{a}_1 = -\frac{Gm_2}{{r_{12}}^2}\mathbf{n}_{12}$.

The quadrupole formula predicts they emit GWs as:

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 R} \perp_{ij,ab}^{\text{TT}} \frac{\mathrm{d}^2 Q_{ab}}{\mathrm{d}t^2}\,,$$

where $Q_{ij} = m_1\left[y_1^i(t)y_1^j(t) - \frac{1}{3}\delta^{ij}|y_1|^2\right] + (1 \leftrightarrow 2)$.

Systems that orbit faster (but not too fast!) can be described with post-Newtonian (PN) corrections in powers of $(v/c)^2 \ll 1$

A correction of order $(v/c)^{2n}$ is said to be of order nPN.

Review of post-Newtonian results

Conservative vs dissipative dynamics

In the center of mass frame, define relative position $\boldsymbol{x} = r\boldsymbol{n} = \boldsymbol{y}_1 - \boldsymbol{y}_2$, relative velocity $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ and relative acceleration $\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2$

The (relative) equations of motion are a differential equation of the type

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \mathfrak{F}(\boldsymbol{x}, \boldsymbol{v}; m_1, m_2)$$

They contain all the information about the motion. They can be divided into conservative + dissipative piece, $\boldsymbol{a} = \boldsymbol{a}^{\text{cons}} + \boldsymbol{a}^{\text{diss}}$

Conservative dynamics

The conservative piece is time-even, ignores dissipative (radiation-reaction) effects.

Only even powers in $1/c$ (Newtonian, 1PN, 2PN, ...)

Derives from Hamiltonian $H(\boldsymbol{x}, \boldsymbol{p}) \Rightarrow$ exists conserved energy E^{cons}:

$$\left.\frac{\mathrm{d}E^{\text{cons}}}{\mathrm{d}t}\right|_{\boldsymbol{a}^{\text{cons}}} = 0$$

as well as conserved angular momentum $\boldsymbol{J}^{\text{cons}}$

At 4PN, the dynamics become nonlocal due to tail; equations of motion are integro-differential, acceleration at time t depends on all previous positions and velocities. $E^{\text{cons}} \neq H(\boldsymbol{x}, \boldsymbol{p})\big|_{\text{on shell}}$

Conservative dynamics (nonspinning)

For now, **ignore** spin contributions. Dynamics described either by the acceleration or the Hamiltonian

How? / Order	PN-MPM	ADM	EFT
3PN	[ItFu '03][It '04] [BcDaEs '04]	[DaJaSc '01]	[FoSt '11]
4PN	[BeBcBoFaMs '17]	[DaJaSc '14] [DaJaSc '15]	[FoPoRoSt '19] [BüMiMqSc '20]
5PN (disputed/partial)		[BiDaGe '20]	[BüMiMqSc '22] [AlMüFoSt '24] [PoRiYa '24]
6PN (partial)		[BiDaGe '20]	[BüMrMqSc '22]

Al= Almeida, Be = Bernard, Bi = Bini, Bc = Blanchet, Bo = Bohé, Bü = Blümlein, Da = Damour, Es = Esposito-Farèse, Fa = Faye, Fu = Futamase, Fo = Foffa, Iy = Itoh, Ja = Jaranowski, Mc = Marchand, Ms = Marsat, Mr = Maier, Mq = Marquard, Mü = Müller, Po = Porto, Ri = Riva, Ro = Rothstein, Sch = Schäfer, St= Sturani, Ya = Yang

Dissipative dynamics

The dissipative dynamics is time-odd, contains purely dissipation, drives radiation-reaction

Only odd powers in $1/c$ (2.5PN, 3.5PN) ... until we hit hereditary contributions (dissipative tail at 4PN)

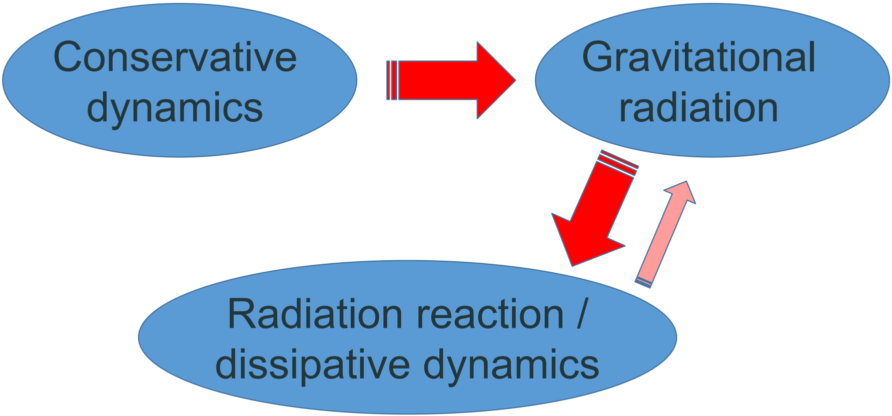
[At 4PN there is still a well-defined split cons + diss ... but does this carry on at 5PN?]

Radiation-**reaction** $\Rightarrow$ link to flux balance law:

$$\left.\frac{\mathrm{d}E^{\mathsf{bind}}}{\mathrm{d}t}\right|_{\boldsymbol{a}^{\mathsf{cons}}+\boldsymbol{a}^{\mathsf{diss}}} = -\mathcal{F}_{\mathsf{E}}^{\infty} - \mathcal{F}_{\mathsf{E}}^{\mathcal{H}_1} - \mathcal{F}_{\mathsf{E}}^{\mathcal{H}_2}$$

where $E^{\mathsf{bind}} - E^{\mathsf{cons}} = E^{\mathsf{Schott}} = \mathcal{O}\left(1/c^5\right)$

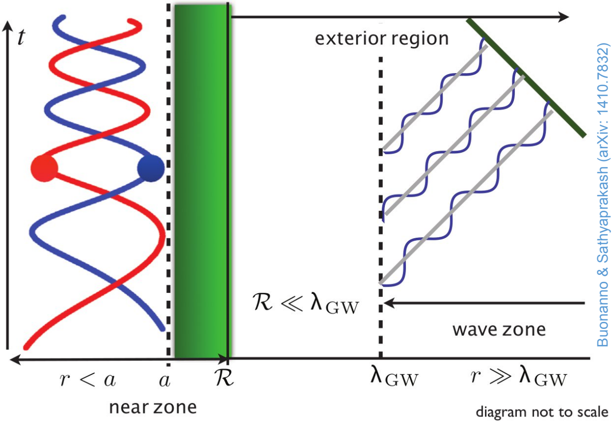
The three sectors of a PN computation



$$\boldsymbol{a} = \boldsymbol{a}_{\mathrm{N}} + \boldsymbol{a}_{1\mathrm{PN}} + \boldsymbol{a}_{2\mathrm{PN}} + \boldsymbol{a}_{2.5\mathrm{PN}} + \dots$$

$$F = F_{\mathrm{N}} + F_{1\mathrm{PN}} + F_{2\mathrm{PN}} + F_{2.5\mathrm{PN}} + \dots$$

Relating near-zone and exterior vacuum zone



- In NZ, obtain PN expansion of metric [up to homogeneous solution]
- In FZ, obtain PM expansion of metric [up to homogeneous solution]
- Both homogeneous solutions obtained by imposing asymptotic matching in buffer zone

Radiation: energy and angular momentum fluxes (nonspinning)

Order \ Type	Flux of energy	Flux of angular momentum
2PN	[BcDaIy '95] [WiWs '96] [LbMiYa '19]	[GoIy '97]
3PN	[AmYaPo '24]	[ArBcIyQu '08ab] [ArBcIySh '09]
3.5PN	[BcFaIyJo '05]	
4PN	[BcFaHeLaTr '23]	
4.5PN	[McBcFa '16]	

Waveform: depends on each mode, too complicated to present.

Highest known order: h_{22} mode at 4PN for circular orbits [BcFaHeLaTr '23]

Am = Amalberti, Bc = Blanchet, Da = Damour, Fa = Faye, Go = Gopakumar, He = Henry, Iy = Iyer, Jo = Joguet, La = Larrouturou, Lb = Leibovich, Ma = Maia, Mc = Marchand, Ms = Marsat, Po = Porto, Tr = Trestini, Wi = Will, Ws = Wiseman, Ya = Yang,

Radiation reaction: dissipative EOM (nonspinning)

Order \ Method	Balance-laws	Matching	
Coordinates	Parametrized	Harmonic	Burke-Thorne
3.5PN	[IyWi '95]	[PtWi '02] [NiBc '05]	[Bc '93][Bc '97] [IyWi '95]
4PN (tails)		[BlDa '88]	
4.5PN	[GoIyIy '98]	[LbPrYa '23]	[BcFaTr '24]

At 4.5PN, we found that the radiation-reaction force has a hereditary term in the center-of-mass frame

Reason: center-of-mass frame defined not only with respect to matter content, but also to radiation (gravitational recoil)

Bc = Blanchet, Fa = Faye, Go = Gopakumar, Iy = Iyer, Lb = Leibovich, Ni = Nissanke, Pd = Pardo, Pt = Pati, Tr = Trestini, Wi = Will, Ya = Yang,

What about spin ?

At 4PN, complete spin contributions are known:

SO	SS	SSS	SSSS
[Cho, Porto & Yang '22]		[Marsat '14]	[Siemonsen, Steinhoff & Vines '18]

For oscillatory waveform with spin, see [Henry '22] [Henry & Khalil '23]

For memory waveform with spin, see [Cunningham et al. '25]

For horizon fluxes with spin, see [Saketh, Steinhoff, Vines & Buonanno '22]

etc. etc.

Quasi-Keplerian parametrization

Once we know the equations of motion (and the spin-precession equations), in theory we know everything $\Rightarrow$ just solve numerically

But this is slow: need to resolve orbital timescales and radiation reaction timescales

Better approach when possible: find an analytical solution to the conservative equations of motion

Then evolve the "constants" of motion adiabatically

At leading order, Keplerian parametrization (acquires corrections):

$$\begin{aligned} r &= a_r(1 - e_r \cos u) \\ n(t - t_0) &= u - e \sin u \\ K(\phi - \phi_0) &= u + 2 \arctan\left[\frac{\beta(e) \sin u}{1 - \beta(e) \cos u}\right] \end{aligned}$$

where u in eccentric anomaly and $\beta(e) = e/(1 + \sqrt{1 - e^2})$

Solution to the conservative dynamics (nonspinning)

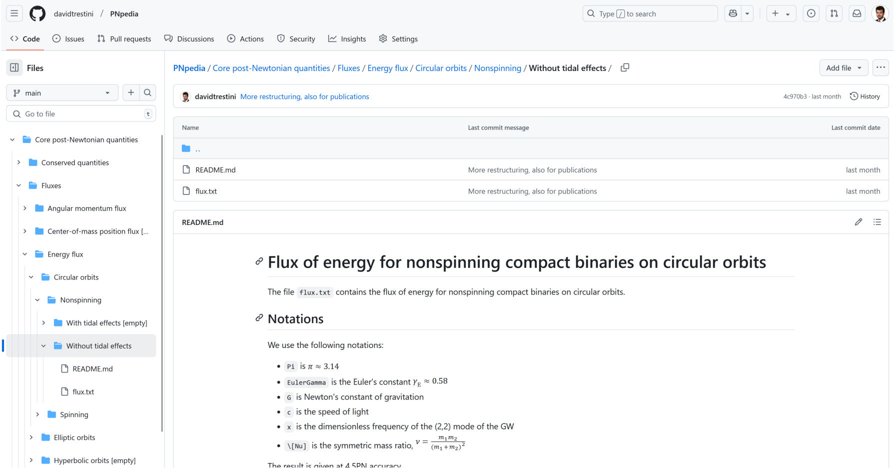
In the conservative section, the **quasi-Keplerian parametrization** yields a PN-accurate solution to the *conservative* EOM for quasielliptic orbits.

Equivalently, on can evolve the orbital parameters of the Newtonian elliptic solution using the "Lagrange planetary equations" [Tucker & Will '19]

Order	References
1PN	[Damour & Deruelle '85]
2PN	[Damour & Schäfer '88] [Schäfer & Wex '93]
3PN	[Memmesheimer, Gopakumar & Schäfer '04]
4PN (partial)	[Cho, Tanay, Gopakumar & Lee '22][Trestini '25b (in prep)]

These have be extended to hyperbolic orbits [Bini & Damour '12, '17] [Bini, Damour & Gericalo 20, '21, '23][Cho, Gopakumar, Haney & Lee 18'][Cho, Dandapat & Gopakumar 22'] [Cho '22] [De Vittori, Gopakumar, Gupta & Jetzer '14]

PNpedia



https://github.com/davidtrestini/PNpedia

Please use, verify, populate and cite!

Prospects

Potentials and super-potentials

In the near-zone PN computation, we need to solve for potentials, which parametrize the metric, e.g.

$$g_{00} = -1 + \frac{2}{c^2} V + \mathcal{O}\left(\frac{1}{c^4}\right)$$

At lowest order, easy:

$$\Box_\eta V = -4\pi G \times \frac{T^{00} + T^{ii}}{c^2} = \text{(compact terms)}$$

Next iteration harder

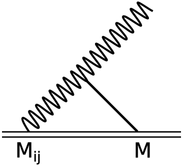
$$\Box_\eta \hat{W} = -4\pi G \times \text{(compact terms)} - \partial_i V \partial_j V$$

Non compact source $\Rightarrow$ regularize Green's integral

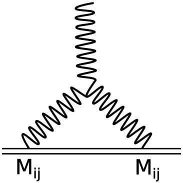
As we iterate harder and hard; "super-potential" techniques to treat them. But for how long?

Tails, memory and beyond

In MPM iteration, obtain nonlocal hereditary contributions that arise from nonlinearities of GR. At quadratic order we have tails and memory

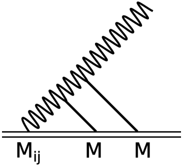


Tails

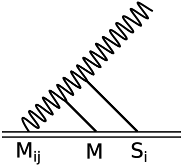


Memory

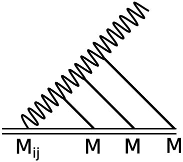
At higher order, one gets iterated tails



Tails of tails



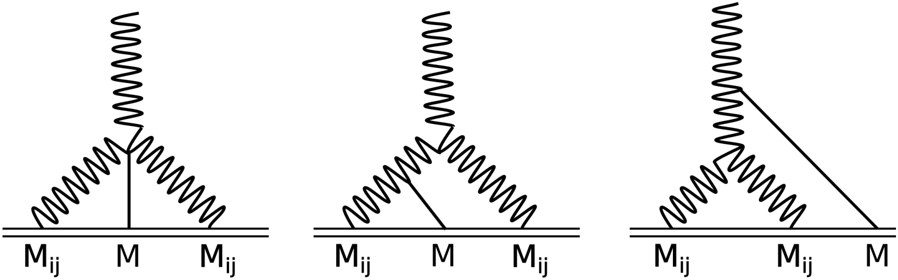
Spin-quadrupole tails



Quartic tails

Tails, memory and beyond

At cubic order, very complicated **tails of memory** [Trestini & Blanchet '22]



Obtained tractable result only after "magic" simplications

How long will we be able to iterate this way ?

Dimensional regularization

One is often led to evaluate the metric $g_{\mu\nu}$ at the location of the particle $A = 1, 2$, which we denote $(g_{\mu\nu})_A$

The metric is obviously formally divergent at the particle, so we need to regularize (remove self-field)

- Option 1 : match to a hydrodynamic body $\Rightarrow$ intractable [Will '25]
- Option 2 : Hadamard regularization : easy but ambiguous at higher orders [requires multiplying distributions, which is not possible]
- Option 3 : dimensional regularization : hard, looses Huygens principle in $d \in \mathbb{C}$ dimensions so everything nonlocal

In practice, compute (i) Hadamard regularization and (ii) what needs to be added to it in the $d \to 0$ limit (never full d-dim computation)

Big challenge for future computations: can we learn from EFT methods?

Precession

We know how to solve the equations of motion for planar motion, but what about precession (caused by misaligned spins) ? Especially in eccentric case

Some results only at 1.5PN [Samanta, Tanay & Stein ‘22] and [Tanay, Cho & Stein ‘21] and 2PN [Klein et al.] , but very convoluted

Problem: we loose separability (in the sense of action-angles).

Possible solution: obtain action-angles perturbatively (KAM)

Important to solve this: we want generic waveform for LISA, ET & CE! More important than high-PN

Conclusions

- PN has been making steady progress
- more PN orders have always improved accuracy
- focus now: expand parameter space
- challenges lie ahead, but hopefully not insurmountable

Schott term in binding energy [DT 2025]

For generic orbits, E^{Schott} is small, of order 2.5PN.

It was previously thought that E^{Schott} was a 5PN effect in the case of circular orbits.

In [DT 2025], I showed that it was it fact 4PN, and read over generic orbits

$$
\begin{aligned}
E^{\mathsf{tail}}_{\mathsf{Schott}} = \frac{2G^2\mathrm{M}}{5c^8}\Bigg\{ & -\mathrm{M}^{(3)}_{ij}\int_0^{\infty}\mathrm{d}\tau\,\ln\left(\frac{c\tau}{2b_0}\right)\left[\mathrm{M}^{(4)}_{ij}(t-\tau)-\mathrm{M}^{(4)}_{ij}(t+\tau)\right] \\
& -\mathrm{M}^{(2)}_{ij}\int_0^{\infty}\mathrm{d}\tau\,\ln\left(\frac{c\tau}{2b_0}\right)\left[\mathrm{M}^{(5)}_{ij}(t-\tau)+\mathrm{M}^{(5)}_{ij}(t+\tau)\right] \\
& +\mathrm{M}^{(1)}_{ij}\int_0^{\infty}\mathrm{d}\tau\,\ln\left(\frac{c\tau}{2b_0}\right)\left[\mathrm{M}^{(6)}_{ij}(t-\tau)+\mathrm{M}^{(6)}_{ij}(t+\tau)\right]-\frac{11}{12}\mathrm{M}^{(3)}_{ij}\mathrm{M}^{(3)}_{ij} \\
& +\int_0^{\infty}\mathrm{d}\rho\,\mathrm{M}^{(4)}_{ij}(t-\rho)\int_0^{\infty}\mathrm{d}\tau\,\ln\left(\frac{c\tau}{2b_0}\right)\left[\mathrm{M}^{(4)}_{ij}(t-\rho-\tau)-\mathrm{M}^{(4)}_{ij}(t-\rho+\tau)\right]\Bigg\}
\end{aligned}
$$

Non-vanishing because of hereditary terms!

Schott term for circular orbits

For circular orbits, the Schott term finally reads

$$E^{\mathsf{4PN}}_{\mathsf{Schott}} = -\frac{c^2 m \nu^2 x^5}{2}\left[\frac{128}{5}\ln\left(\frac{Gm}{c^2 b_0}\right) - \frac{192}{5}\ln(x) - \frac{128}{5}\gamma_{\mathrm{E}} - \frac{256}{5}\ln(2) - \frac{32}{15}\right]$$

Recall $E_{\mathsf{bind}}(x;b_0) = E_{\mathsf{cons}}(x) + E_{\mathsf{Schott}}(x;b_0)$

The 4PN flux in terms of the orbital frequency x depends on the arbitrary constant b_0 — denote this $\mathcal{F}(x;b_0)$. [Blanchet, Faye, Henry, Larrouturou & DT 2023]

This translates into the evolution equation for the chirp:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\mathcal{F}(y;b_0)}{\mathrm{d}E_{\mathsf{bind}}(y;b_0)/\mathrm{d}y}$$

We find that b_0 drops out from the right-hand side
— expected from gauge-invariance !