ICTS Bengaluru
Grothendieck-Teichmüller Theory
February 26-March 1, 2024

The elliptic Grothendieck-Teichmüller Lie algebra and the elliptic associator

Leila Schneps

Part 1: the elliptic Grothendieck-Teichmüller Lie algebra

Recall that the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ is a derivation algebra of the 4-strand braid Lie algebra $\mathrm{Lie}[x, y]$ acting by

$$D_f(x) = 0, \quad D_f(y) = [y, f]$$

for $f \in \mathfrak{grt} \subset \mathrm{Lie}[x, y]$, and that the pentagon equation

$$f(x_{12}, x_{23}) + f(x_{34}, x_{45}) + f(x_{51}, x_{12}) + f(x_{23}, x_{34}) + f(x_{45}, x_{51}) = 0$$

with $x_{ij} \in \mathrm{Lie}\, P_5$ ensures that $\mathfrak{grt} \hookrightarrow \mathrm{Der}\,\mathrm{Lie}\, P_5$, and that by the two-level principle (proven here by Ihara), this then ensures that $\mathfrak{grt} \hookrightarrow \mathrm{Der}\,\mathrm{Lie}\, P_n$ for all $n > 5$.

The elliptic Grothendieck-Teichmüller Lie algebra is defined on the same idea except that the 4-strand and 5-strand genus zero braid Lie algebras are replaced by the genus 1 braid Lie algebra $\mathfrak{t}_{1,2}$ on two strands and the genus 1 braid Lie algebra $\mathfrak{t}_{1,3}$ on three strands.

The Lie algebra $\mathfrak{t}_{1,n}$ is generated by elements $x_i^{\pm}$ for $i = 1, \ldots, n$ and relations

$$\begin{cases} \sum_{i=1}^{n} x_i^{\pm} = 0 & \\ [x_i^{\pm}, x_j^{\pm}] = 0 & \text{for } i \neq j \\ [x_i^{+}, x_j^{-}] = [x_j^{+}, x_i^{-}] & \text{for } i \neq j \\ [x_k^{\pm}, [x_i^{+}, x_j^{-}]] = 0 & \text{for } i, j, k \text{ distinct.} \end{cases}$$

Thus $\mathfrak{t}_{1,2}$ is generated by $x_1^{\pm}, x_2^{\pm}$ with $x_1^{\pm} + x_2^{\pm} = 0$, so it is in fact generated by $a := x_1^{+}$ and $b := x_1^{-}$ and it is just the free Lie algebra $\mathrm{Lie}[a, b]$. We set

$$t_{ij} := [x_i^{+}, x_j^{-}] \in \mathfrak{t}_{1,n}.$$

The elliptic Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_{ell}$ is defined to be the space of triples $(\psi, \alpha_+, \alpha_-)$ such that $\psi \in \mathfrak{grt}$, the triple induces a derivation on $\mathfrak{t}_{1,2} = \mathrm{Lie}[x_1^+, x_1^-]$ by

$$x_1^{\pm} \mapsto \alpha_{\pm}$$

such that $[x_1^+, x_1^-]$ is mapped to 0 and such that taking the following extension to $\mathfrak{t}_{1,3}$ by

$$\begin{cases} x_1^+ \mapsto \alpha_+(x_1^+, x_1^-) + [x_1^{\pm}, \psi(t_{12}, t_{23}] \\ x_2^{\pm} \mapsto \alpha_{\pm}(x_2^+, x_2^-) + [x_2^{\pm}, \psi(t_{12}, t_{13})] \\ x_3^{\pm} \mapsto \alpha_{\pm}(x_3^+, x_3^-) \end{cases}$$

induces a derivation of $\mathfrak{t}_{1,3}$.
This condition imposes strong relations on the elements α_+, α_-.

Explicitly, we have

$$
\begin{aligned}
\mathfrak{grt}_{ell} :=\Big\{ &(\psi, \alpha_+, \alpha_-) \in \mathfrak{grt} \times (\mathfrak{t}_{1,2})^2 \,\Big| \\
&\alpha_\pm(x_1^+, x_1^-) + \alpha_\pm(x_2^+, x_2^-) + \alpha_\pm(x_3^+, x_3^-) + [x_1^\pm, \psi(t_{12}, t_{23})] + [x_2^\pm, \psi(t_{12}, t_{13})] = 0, \\
&[x_1^\pm, \alpha_\pm(x_3^+, x_3^-)] + [\alpha_\pm(x_1^+, x_1^-), x_3^\pm] - [x_1^\pm, [x_3^\pm, \psi(t_{12}, t_{23})]] = 0, \\
&[x_1^+, \alpha_-(x_2^+, x_2^-)] - [x_2^-, \alpha_+(x_1^+, x_1^-)] = [x_2^-, [x_1^+, \psi(t_{12}, t_{23})]] - [x_1^+, [x_2^-, \psi(t_{12}, t_{13})]] \Big\}.
\end{aligned}
$$

Lemma 1. *Given an element* $(\psi, \alpha_+, \alpha_-)$ *in* $\mathfrak{grt}_{ell}$*, the term* α_+ *uniquely determines* α_-*.*

Proof. This is always the case for derivations of a free algebra Lie$[a,b]$ such that the bracket $[a,b]$ is mapped to zero. Indeed this condition means that

$$[\alpha_+, b] + [a, \alpha_-] = 0$$

and α_- can be recovered from α_+ by writing

$$\alpha_+ = (\alpha_+)_a a + (\alpha_+)_b b$$

and setting

$$\alpha_- = \sum_{i \geq 0} \frac{(-1)^i}{i!} a^i b \partial_a^i((\alpha_+)_a).$$

Lemma 2. *Elements* $(0, \alpha_+, \alpha_-)$ *exist in* $\mathfrak{grt}_{ell}$*, forming a Lie subalgebra* $\mathfrak{r}_{ell}$.

Proof. We prove this by displaying some elements of $\mathfrak{r}_{ell}$, namely the derivations known as ϵ_{2k} defined by

$$\epsilon_{2k}(a) = ad(a)^{2k}(b), \quad \epsilon_{2k}([a,b]) = 0.$$

By direct computation, the triple $\big(0, \epsilon_{2k}(a), \epsilon_{2k}(b)\big)$ lies in $\mathfrak{r}_{ell}$ for all $k \geq 0$. It is conjectured but not known that they generate $\mathfrak{r}_{ell}$.

Lemma 3. *If* $(\psi, \alpha_+, \alpha_-) \in \mathfrak{grt}_{ell}$ *then* α_+ *determines* ψ *uniquely.*

Proof. Suppose $(\psi_1, alpha_+, \alpha_-)$ and $(\psi_2, alpha_+, \alpha_-)$ both lie in $\mathfrak{grt}_{ell}$. Then there is an element $(\psi, 0, 0) \in \mathfrak{grt}_{ell}$. From the defining relations of $\mathfrak{grt}_{ell}$ we must then have

$$[x_1^+, [x_3^+, \psi(t_{12}, t_{23})]] = 0. \qquad (*)$$

Consider the morphism p_1 mapping $x_1^\pm \mapsto 0$. $Ker(p_1)$ is a free Lie algebra generated by $x_1^\pm, t_{12}, t_{13}$. Since $p_1(t_{12}) = 0$, we have $p_1(\psi(t_{12}, t_{23})) = 0$ because the kernel is a normal subalgebra, and we also have $p_1([x_3^+, \psi(t_{12}, t_{23})]) = 0$. This shows that

$$T := [x_3^+, \psi(t_{12}, t_{23})] \in ker(p_1) = \mathrm{Lie}[x_1^\pm, t_{12}, t_{13}].$$

Thus $[x_1^+, [x_3^+, \psi(t_{12}, t_{23})]] = [x_1^+, T] = 0$ inside the free Lie algebra $ker(p_1)$, so $T = [x_3^+, \psi(t_{12}, t_{23})] = 0$. But by the same argument, since $\psi(t_{12}, t_{13}) \in ker(p_3)$ and $T = 0$, we must have $\psi(t_{12}, t_{13}) = 0$, which proves that $\psi = 0$.

Lemma 4. *The map* $\mathfrak{grt}_{ell} \to \psi$ *given by* $\alpha_+ \mapsto \psi$ *is surjective, and there is a section map*

$$\mathfrak{grt} \to \mathfrak{grt}_{ell}.$$

Proof. The first assertion will follow from the section map, which we describe explicitly.

Let

$$Ber_x(y) = \ (y) = \sum_{n \geq 1} \frac{B_n}{n!} ad(x)^{n-1}(y)$$

be the Bernoulli function. Let $a := x_1^+$, $b := x_1^-$, and let

$$t_{01} = Ber_b(-a), \quad t_{02} = Ber_{-b}(a), \quad t_{12} = [a, b].$$

These elements lie in the completed free Lie algebra on two generators Lie$[a, b]$ and satisfy

$$t_{01} + t_{02} + t_{12} = 0.$$

Let

$$\mathrm{Lie}[x,y] \hookrightarrow \mathrm{Lie}[a,b]$$

be the injective Lie algebra morphism given by

$$x \mapsto t_{12}, \quad y \mapsto t_{01},$$

and let $T = \mathrm{Lie}[t_{12}, t_{01}]$ denote the image.

Proposition. *Let $f \in \mathfrak{grt}$, and let $d_f(x) = 0$, $d_f(y) = [y, f]$ denote the Ihara derivation of* $\mathrm{Lie}[x,y]$ *associated to f. Let*

$$D_f : t_{12} \mapsto 0, \quad t_{01} \mapsto [t_{01}, f(t_{12}, t_{01})]$$

be the transport of this derivation to $\mathrm{Lie}[t_{12}, t_{01}]$. *Then there exists a unique extension of D_f to a derivation α of all of* $\mathrm{Lie}[a,b]$. *Setting*

$$\alpha_+ = \alpha(a) = \alpha(x_1^+), \quad \alpha_- = \alpha(b) = \alpha(x_1^-)$$

gives the element $(\psi, \alpha_+, \alpha_-) \in \mathfrak{grt}_{ell}$.

The elliptic associator

Enriquez defined an automorphism g_τ of $\mathbb{Q}\langle\langle a, b\rangle\rangle$, which is a function of the elliptic parameter τ, running through the fundamental domain of the action of $SL_2(\mathbb{Z})$ on the upper half-plane. It is a solution of the differential equation

$$\frac{1}{2\pi i}\frac{\partial}{\partial\tau}g_\tau = -\Big(\epsilon_0 + \sum_{k\geq 1}\frac{2}{(2k-2)!}G_{2k}(\tau)\epsilon_{2k}\Big)g_\tau,$$

$$G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n\geq 1}\sigma_{2k-1}(n)q^n$$

is the Hecke-normalized Eisenstein series, with $q = e^{2\pi i\tau}$. Enriquez singles out the solution g_τ by specifying its asymptotic behavior as $\tau \to i\infty$.

Let ϵ_{2k}, $k \geq 0$, be the weight $2k+1$ derivations of Lie$[a,b]$ defined by

$$\epsilon_{2k}(a) = ad(a)^{2k}(b), \quad \epsilon_{2k}([a,b]) = 0$$

(where weight $2k+1$ means that $\epsilon_{2k}(a)$ and $\epsilon_{2k}(b)$ are Lie elements of homogeneous degree $2k+1$). This definition completely fixes the value of ϵ_{2k} on b.

We can write

$$g_\tau = id + \sum_{\mathbf{k}=(2k_1,\ldots,2k_r)} \mathcal{G}_{\mathbf{k}} \epsilon_{\mathbf{k}}$$

where

$$\epsilon_{\mathbf{k}} = \epsilon_{2k_1} \circ \cdots \circ \epsilon_{2k_r}$$

and $\mathcal{G}_{\mathbf{k}}$ is the iterated integral of the Eisenstein series $G_{2k_1}, \ldots, G_{2k_r}$.

Definition of the elliptic associator

Instead of considering the generators a and b of the completed Lie algebra $\mathrm{Lie}[a, b]$, Enriquez consider the pair of generators t_{01}, b for this Lie algebra.

The elliptic associator defined by Enriquez is a pair (A_τ, B_τ) of group-like power series in the free non-commutative variables a, b with coefficients that are functions of τ, so lie in $\mathcal{O}(\mathcal{H})$, and such that

$$\begin{cases} e^{t_{01}} \mapsto A_\tau \\ e^b \mapsto B_\tau \end{cases}$$

induces an automorphism of the group of group-like power series in $\mathbb{Q}\langle\langle a, b\rangle\rangle \otimes_{\mathbb{Q}} \mathcal{O}(\mathcal{H})$ such that $e^{[a,b]}$ is fixed.

The condition that the automorphism fixes $e^{[a,b]}$ implies that in fact knowing A_τ, we can reconstruct B_τ. Therefore we will generally refer to A_τ as “the elliptic associator”.

The power series A_τ can be defined as followed in two steps. First we define a power series $A \in \mathbb{Q}\langle\langle a, b\rangle\rangle \otimes_{\mathbb{Q}} \mathcal{Z}$, where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra of multizeta values:

$$A = \Phi_{KZ}(t_{01}, t_{12})^{-1} e^{2i\pi t_{01}} \Phi_{KZ}(t_{01}, t_{12}).$$

Then we set

$$A_\tau := g_\tau(A).$$

The power series $g_\tau(a)$, A_τ and B_τ are all group-like.

In what way does this deserve to be called "the elliptic associator"?

• As the Drinfel'd associator arises from the usual KZB equation, the pair (A_τ, B_τ) arise from an elliptic KZB equation based on the function F_τ;

• As Φ_{KZ} is the iterated integral of the KZB differential form $\frac{x}{v} + \frac{y}{1-v}\, dv$, A_τ is the iterated integral of the differential form $F_\tau(u,v)\, dv$;

• As the Drinfeld associator yields an isomorphism between

$$\pi_1^{pro-unip}(\mathbf{P}^1 - \{0,1,\infty\}) \simeq \langle e^X, e^Y, e^Z | e^X e^Y e^Z = 1\rangle$$

and the graded version

$$\exp(\mathrm{Lie}[x,y,z|x+y+z=0]) \simeq \langle e^x, e^y, e^z | e^{x+y+z} = 1\rangle,$$

so (A_τ, B_τ) gives an isomorphism between

$$\pi_1^{pro-unip}(T_1) \simeq \langle e^A, e^B, e^C | (e^A, e^B) e^C = 1\rangle$$

and the graded version

$$\exp(\mathrm{Lie}[a,b,c|[a,b]+c=0]) = \langle e^a, e^b, e^c | e^{[a,b]+c} = 1\rangle,$$

where T_1 is the once-punctured torus. The isomorphism is given by $e^A \mapsto A_\tau$, $e^B \mapsto B_\tau$.

• Viewing $\langle e^X, e^Y \rangle$ as the pro-unipotent π_1 of the thrice-punctured sphere identifies it with the pure sphere 4-strand braid group. The graded formality isomorphism on $\langle e^X, e^Y \rangle$ induced by Φ_{KZ} extends to one of the pure sphere 5-strand braid group [Kohno-Drinfeld]. Similarly, viewing $\langle e^A, e^B \rangle$ as the pro-unipotent π_1 of T_1 identifies it with the 1-strand torus braid group, and the graded formality isomorphism given by (A_τ, B_τ) extends to one one the the 2-strand torus braid group.

• The extension to the 5-strand braid group is ensured by the *associator relations* satisfied by Φ_{KZ}. Similarly, (A_τ, B_τ) satisfy *elliptic associator relations* arising from the fact that it induces a graded formality isomorphism of the 2-strand torus braid group.

The ring of elliptic multizeta values

The *ring of elliptic multiple zeta values* $\mathcal{EZ}$ is generated by the coefficients of the power series A_τ and B_τ (for reasons of convenience, we add the function $2\pi i\tau$ to $\mathcal{EZ}$). We consider the ring $\overline{\mathcal{EZ}}$ modulo the ideal generated by $\zeta(2)$.

Theorem 1. *(i) Let $\mathcal{U}$ denote the ring generated over $\mathbb{Q}$ by the coefficients of $g_\tau(a)$, which are all linear combinations of the iterated integrals $\mathcal{G}_{\mathbf{k}}$. Then $\mathcal{U}$ is the dual of the universal enveloping algebra of the Lie algebra of derivations generated by the derivations ϵ_{2k}.*

(ii) The ring $\overline{\mathcal{EZ}}$ generated by the elliptic multiple zeta values is a $\mathbb{Q}$-algebra under the shuffle multiplication.

(iii) The $\mathbb{Q}$-algebra $\overline{\mathcal{EZ}}$ decomposes into a tensor product

$$\overline{\mathcal{EZ}} \simeq \mathcal{U} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$$

where $\mathcal{Z}$ denotes the $\mathbb{Q}$-algebra of real multizeta values.

The elliptic generating series

For this part, we will work modulo $\zeta(2)$. Let $\overline{\Phi_{KZ}}$ denote the Drinfeld associator with coefficients reduced mod $\zeta(2)$, and set

$$\overline{A} = \overline{\Phi}_{KZ}(t_{01}, t_{12})^{-1} e^{t_{01}} \overline{\Phi}_{KZ}(t_{01}, t_{12})$$

and

$$\overline{A}_\tau = g_\tau(\overline{A}).$$

Let $\overline{\mathcal{Z}} = \mathcal{Z}/\langle\zeta(2)\rangle$ and let

$$\phi_{KZ} \in \mathfrak{grt} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$$

denote the *Lie Drinfeld associator*, which satisfies

$$exp_{\mathfrak{grt}}(\phi_{KZ}) = \overline{\Phi}_{KZ},$$

where $\overline{\Phi}_{KZ}$ is the Drinfeld associator with coefficients reduced mod $\zeta(2)$ and $exp_{\mathfrak{grt}}$ denotes the exponential with respect to the Poincaré-Birkhoff-Witt multiplication law * in $\mathcal{U}\mathfrak{grt}$:

$$exp_{\mathfrak{grt}}(\phi_{KZ}) = 1 + \sum_{n\geq 1} \frac{1}{n!}\phi_{KZ}^{*n} = \overline{\Phi}_{KZ} \in \widehat{\mathcal{U}\mathfrak{grt}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

To the Lie series ϕ_{KZ} we associate the Ihara derivation $d_{\phi_{KZ}}$ of $\mathrm{Lie}[x,y] \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$ (the completed Lie algebra).
Transporting this over to a derivation $D_{\phi_{KZ}}$ of

$$T = \mathrm{Lie}[t_{12}, t_{01}] \subset \mathrm{Lie}[a,b] \otimes_{\mathbb{Q}} \overline{\mathcal{Z}},$$

the above Proposition shows that it has a unique extension to a derivation $\tilde{D}_{\phi_{KZ}}$ of $\mathrm{Lie}[a,b] \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. The exponential $A_{\phi_{KZ}} := exp(\tilde{D}_{\phi_{KZ}})$ is then an automorphism of

$$\mathbb{Q}\langle\langle a,b\rangle\rangle \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

We set

$$\overline{e} = \tilde{D}_{\phi_{KZ}}(a) \quad \text{and} \quad \overline{E} = A_{\phi_{KZ}}(e^a).$$

Theorem. *The automorphism* $A_{\phi_{KZ}}$ *of* $\mathbb{Q}\langle\langle a,b\rangle\rangle \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$ *mapping* $e^a \mapsto \overline{E}$ *fixes* $e^{[a,b]}$*, which determines it completely. It satisfies*

$$A_{\phi_{KZ}}(e^{t_{01}}) = \overline{\Phi}_{KZ}(t_{01},t_{12})^{-1}\, e^{t_{01}}\, \overline{\Phi}_{KZ}(t_{01},t_{12}).$$

Let $\overline{E}_\tau := g_\tau(\overline{E})$*. The automorphism* $g_\tau \circ A_{\phi_{KZ}}$ *of* $\mathbb{Q}\langle\langle a,b\rangle\rangle \otimes_{\mathbb{Q}} \overline{\mathcal{EZ}}$ *maps*

$$\begin{cases} e^a \mapsto \overline{E}_\tau \\ e^{t_{01}} \mapsto \overline{A}_\tau \\ e^b \mapsto \overline{B}_\tau. \end{cases}$$

In this way the (reduced) elliptic associator $(\overline{A}_\tau, \overline{B}_\tau)$ *can be constructed directly from* ϕ_{KZ} *and* g_τ*.*

The power series

$$\overline{E}_\tau = g_\tau(\overline{E}) = g_\tau\big(A_{\phi_{KZ}}(e^a)\big)$$

is called the *elliptic generating series*. Its coefficients generate the same ring $\mathcal{EZ}$ as those of $\overline{A}_\tau$ and $\overline{B}_\tau$.

Elliptic double shuffle relations

The elliptic associator should satisfy double-shuffle type relations, both by analogy with the genus zero case and because it is constructed from the Drinfeld associator. In fact, they do satisfy a family of *elliptic double shuffle relations.*

To simplify, we will restrict our attention to the elliptic generating series

$$\overline{E}_\tau = g_\tau \circ A_{\phi_{KZ}}(e^a).$$

In fact we will consider the Lie elliptic double shuffle relations on the generating series given by

$$\overline{e}_\tau := \log\big(g_\tau \circ A_{\phi_{KZ}}\big)(a).$$

The difficulty is that the elliptic double shuffle relations can't be expressed in terms of power series. Let us give an approximative version.

Let D be the operator which acts on polynomials $f(x,y) \in \mathrm{Lie}[x,y]$ by

$$D(f) = [x, f(x, [x,y])].$$

The key point is that it is possible to give a meaning to the inverse operator D^{-1}, but $D^{-1}(f)$ will not generally be a polynomial (we need to move into a world with denominators...). Accepting this, the *elliptic double shuffle relations* satisfied by the Lie elliptic generating series $\overline{e}_\tau$ are the depth-linearizations of the usual double shuffle relations, namely the relations that hold not in $\mathfrak{ds}$ but in $gr\,\mathfrak{ds}$, the associated graded for the depth filtration: instead of $D^{-1}\big(\overline{e}_\tau\big)$ being Lie-like for the coproduct Δ and $D^{-1}\big((\overline{e}_\tau)_*\big)$ for the coproduct Δ_*, they are both Lie-like for the coproduct Δ:

$$\Delta\big(D^{-1}(\overline{e}_\tau)\big) = D^{-1}(\overline{e}_\tau) \otimes 1 + 1 \otimes D^{-1}(\overline{e}_\tau)$$

and

$$\Delta\big(D^{-1}(\overline{e}_\tau)_*\big) = D^{-1}(\overline{e}_\tau)_* \otimes 1 + 1 \otimes D^{-1}(\overline{e}_\tau)_*.$$

One can give explicit meanings to $D^{-1}(\overline{e}_\tau)$ even though $D^{-1}(\overline{e}_\tau)$ is not a power series by using Écalle's *mould theory*.

Other relations (such as the *Fay relations*) satisfied by the elliptic associator have been studied, but like the elliptic double shuffle, they can be explained as transformations via the construction of relations already satisfied by the Drinfeld associator.