Crystal bases for reduced Imaginary Verma Modules of untwisted Quantum affine algebras.

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Let the index set $I = \{0, 1, \ldots, N\}$ and $A = (a_{ij})_{0 \le i,j \le N}$ be a generalized affine Cartan matrix for an untwisted affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ over the field of complex numbers $\mathbb{C}$. Let $D = diag(d_0, d_1, \ldots, d_N)$ be a diagonal matrix with relatively prime integer entries such that DA is symmetric. The numbers $d_0, \ldots, d_N$ are given as follows: Types ADE: $d_i = 1$ for all $i \in I$, type B: $d_i = 2$ for $i \in I \setminus \{N\}$ and $d_N = 1$, type C: $d_0 = d_N = 2$ and $d_i = 1$ for $i \neq 0, N$, Type F: $d_0 = d_1 = d_2 = 2$, $d_3 = d_4 = 1$ and type G: $d_0 = d_1 = 3$ and $d_2 = 1$.
The Chevalley-Serre presentation of $\hat{\mathfrak{g}}$ is given by generators e_i, f_i, h_i for $0 \le i \le N$ and d subject to the defining relations:

$$[h_i, h_j] = 0 \quad [d, h_i] = 0 \quad [h_i, e_j] = a_{ij} e_j \quad [h_i, f_j] = -a_{ij} f_j$$
$$[e_i, f_j] = \delta_{i,j} h_i \quad [d, e_i] = \delta_{0,i} e_i \quad [d, f_i] = -\delta_{0,i} f_i$$
$$(\mathrm{ad} e_i)^{1-a_{ij}}(e_j) = 0 \quad (\mathrm{ad} f_i)^{1-a_{ij}}(f_j) = 0.$$

The abelian subalgebra $\hat{\mathfrak{h}} = \mathrm{span}\{h_0, \ldots, h_N, d\}$ is the Cartan subalgebra of $\hat{\mathfrak{g}}$.

Untwisted affine Lie algebra $\hat{\mathfrak{g}}$

Let Δ be the set of roots of $\hat{\mathfrak{g}}$ with simple roots $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_N\}$ and let $\delta = \alpha_0 + \theta$ be the null root where θ is the longest root of the underlying simple Lie algebra $\mathfrak{g}$. Recall that $\Delta = \Delta^{re} \cup \Delta^{im}$, where Δ^{re} and Δ^{im} denotes the real and imaginary sets of roots. Let Q, P, $\check{Q}$ and $\check{P}$ denote the root lattice, the weight lattice, the coroot lattice and the coweight lattice respectively. The standard non-degenerate symmetric bilinear form $(.|.)$ on $\hat{\mathfrak{h}}^*$ satisfies $(\alpha_i|\alpha_j) = d_i a_{ij}$, $(\delta|\alpha_i) = (\delta|\delta) = 0$ for all $i, j \in I$.

A subset S of Δ is said to be closed if whenever $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in S$. We also say that S is a closed partition if S is closed, $\Delta = S \cup -S$ and $S \cap -S = \emptyset$. We consider the closed partition $S = \{\alpha + n\delta | \alpha \in \Delta_{0,+}, n \in \mathbb{Z}\} \cup \{k\delta | k > 0\}$ which is inequivalent to the standard partition of Δ.

Let $I_0 = \{1, 2, \dots, N\}$ and $\mathfrak{g}$ be the underlying finite dimensional simple Lie algebra with Cartan matrix $(a_{ij})_{1 \le i,j \le N}$. Let Δ_0 be the set of roots of $\mathfrak{g}$ and $\Pi_0 = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots. From now on, any subscript 0 will refer to the same sets for the simple Lie algebra $\mathfrak{g}$.

Quantum untwisted affine Lie algebra $U_q(\hat{\mathfrak{g}})$

The quantum (untwisted) affine algebra $U_q(\hat{\mathfrak{g}})$ is the associative and unital $\mathbb{C}(q^{1/2})$-algebra with generators $E_i, F_i, K_\alpha, \gamma^{\pm 1/2}, D^{\pm 1}$ for $0 \leq i \leq N$, $\alpha \in Q$ and defining relations:

$$DD^{-1} = D^{-1}D = K_\alpha K_{-\alpha} = K_{-\alpha}K_\alpha = \gamma^{1/2}\gamma^{-1/2} = \gamma^{-1/2}\gamma^{1/2} = 1$$

$$[\gamma^{\pm 1/2}, U_q(\hat{\mathfrak{g}})] = [D, K_{\pm\alpha}] = [K_\alpha, K_\beta] = 0$$

$$(\gamma^{\pm 1/2})^2 = K_{\pm\delta}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$K_\alpha E_i K_{-\alpha} = q^{(\alpha|\alpha_i)}E_i, \quad K_\alpha F_i K_{-\alpha} = q^{-(\alpha|\alpha_i)}F_i$$

$$DE_iD^{-1} = q^{\delta_{i,0}}E_i, \quad DF_iD^{-1} = q^{-\delta_{i,0}}F_i$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{ij}-s)}E_jE_i^{(s)} = \sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{ij}-s)}F_jF_i^{(s)} = 0, \quad i \neq j$$

where $q_i = q^{d_i}$, $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $[n]_i! = [n]_i[n-1]_i \cdots [2]_i[1]_i$, $K_i = K_{\alpha_i}$, $E_i^{(s)} = E_i^s/[s]_i!$ and $F_i^{(s)} = F_i^s/[s]_i!$.

Quantum untwisted affine Lie algebra $U_q(\hat{\mathfrak{g}})$

There is also another realization due to Drinfeld in the spirit of the loop space realization called the Drinfeld realization as follows. $U_q(\hat{\mathfrak{g}})$ is the associative unital $\mathbb{C}(q^{1/2})$-algebra generated by $x_{ir}^{\pm}, h_{is}, K_i^{\pm 1}, \gamma^{\pm 1/2}, D^{\pm 1}$ for $i \in I_0$, $r, s \in \mathbb{Z}$ and $s \neq 0$ subject to the relations:

$$D^{\pm 1} D^{\mp 1} = K_i^{\pm 1} K_i^{\mp 1} = \gamma^{\pm 1/2}\gamma^{\mp 1/2} = 1$$

$$[\gamma^{\pm 1/2}, U_q(\hat{\mathfrak{g}})] = [D, K_i^{\pm 1}] = [K_i, K_j] = [K_i, h_{js}] = 0$$

$$D h_{ir} D^{-1} = q^r h_{ir}, \qquad D x_{ir}^{\pm} D^{-1} = q^r x_{ir}^{\pm}$$

$$K_i x_{jr}^{\pm} K_i^{-1} = q^{\pm(\alpha_i|\alpha_j)} x_{jr}^{\pm}$$

$$[h_{ik}, h_{jl}] = \delta_{k,-l}\frac{1}{k}[ka_{ij}]_i \frac{\gamma^k - \gamma^{-k}}{q_j - q_j^{-1}}$$

$$[h_{ik}, x_{jl}^{\pm}] = \pm\frac{1}{k}[ka_{ij}]_i \gamma^{\mp|k|/2} x_{j,k+l}^{\pm}$$

$$x_{i,k+1}^{\pm} x_{jl}^{\pm} - q^{\pm(\alpha_i|\alpha_j)} x_{jl}^{\pm} x_{i,k+1}^{\pm} = q^{\pm(\alpha_i|\alpha_j)} x_{ik}^{\pm} x_{j,l+1}^{\pm} - x_{j,l+1}^{\pm} x_{ik}^{\pm}$$

$$[x_{ik}^{+}, x_{jl}^{-}] = \delta_{ij}\frac{1}{q_i - q_i^{-1}}(\gamma^{(k-l)/2}\psi_{i,k+l} - \gamma^{(l-k)/2}\phi_{i,k+l})$$

Quantum untwisted affine Lie algebra $U_q(\hat{\mathfrak{g}})$

where

$$\sum_{k=0}^{\infty} \psi_{ik} z^k = K_i \exp\Big((q_i - q_i^{-1}) \sum_{l>0} h_{il} z^l \Big)$$

$$\sum_{k=0}^{\infty} \phi_{i,-k} z^{-k} = K_i^{-1} \exp\Big(-(q_i - q_i^{-1}) \sum_{l>0} h_{i,-l} z^{-l} \Big)$$

and for $i \neq j$,

$$\mathrm{Sym}_{k_1,\ldots,k_{1-a_{ij}}} \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i x_{ik_1}^{\pm} \cdots x_{ik_r}^{\pm} x_{ij}^{\pm} x_{ik_{r+1}}^{\pm} \cdots x_{ik_{1-a_{ij}}}^{\pm} = 0.$$

Define the following generating functions:

$$\phi_i(u) = \sum_{p\in\mathbb{Z}} \phi_{ip} u^{-p}, \quad \psi_i(u) = \sum_{p\in\mathbb{Z}} \psi_{ip} u^{-p}, \quad x_i^{\pm}(u) = \sum_{p\in\mathbb{Z}} x_{ip}^{\pm} u^{-p}$$

Quantum untwisted affine Lie algebra $U_q(\hat{\mathfrak{g}})$

Then the defining relations of $U_q(\hat{\mathfrak{g}})$ become:

$$[\phi_i(u), \phi_j(v)] = [\psi_i(u), \psi_j(v)] = 0$$

$$\phi_i(u)\psi_j(v)\phi_i(u)^{-1}\psi_j(v)^{-1} = g_{ij}(uv^{-1}\gamma^1)/g_{ij}(uv^{-1}\gamma)$$

$$\phi_i(u)x_j^{\pm}(v)\phi_i(u)^{-1} = g_{ij}(uv^{-1}\gamma^{\mp 1/2})^{\pm 1}x_j^{\pm}(v)$$

$$\psi_i(u)x_j^{\pm}(v)\psi_i(u)^{-1} = g_{ji}(vu^{-1}\gamma^{\mp 1/2})^{\mp 1}x_j^{\pm}(v)$$

$$(u - q^{\pm(\alpha_i|\alpha_j)v})x_i^{\pm}(u)x_j^{\pm}(v) = (q^{\pm(\alpha_i|\alpha_j)u-v})x_j^{\pm}(v)x_i^{\pm}(u)$$

$$[x_i^+(u), x_j^-(v)] = \delta_{ij}(q_i - q_i^{-1})(\delta(u/v\gamma)\psi_i(v\gamma^{1/2}) - \delta(u\gamma/v)\phi_i(u\gamma^{1/2}))$$

where $g_{ij}(t) = g_{ij,q}(t)$ is the Taylor expansion at $t = 0$ of the function $(q^{(\alpha_i|\alpha_j)}t - 1)/(t - q^{(\alpha_i|\alpha_j)})$ and $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$.

(Quantum) Imaginary Verma modules

For the closed partition $\Delta = S \cup -S$ where $S = \{\alpha + n\delta | \alpha \in \Delta_{0,+}, n \in \mathbb{Z}\} \cup \{k\delta | k > 0\}$, we define the following subalgebras of $U_q(\hat{\mathfrak{g}})$.

- $U_q^+(S)$ generated by x_{ik}^+, h_{il} for $i \in I_0$, $k \in \mathbb{Z}$ and $l > 0$.
- $U_q^-(S)$ generated by $x_{ik}^-, h_{i,-l}$ for $i \in I_0$, $k \in \mathbb{Z}$ and $l > 0$.
- $U_q^0(S)$ generated by $K_i^{\pm 1}, \gamma^{\pm 1/2}, D^{\pm 1}$ for $i \in I_0$.

For $\lambda \in P$, a wight module V of $U_q(\hat{\mathfrak{g}})$ is called an S-weight module with highest weight λ if there is a non zero vector $v \in V$ of weight λ such that $u^+ v = 0$ for all $u^+ \in U_q^+(S) \setminus \mathbb{C}(q^{1/2})$ and $V = U_q(\hat{\mathfrak{g}})v$.
Let us consider the Borel subalgebra B_q of $U_q(\hat{\mathfrak{g}})$, which is generated by $U_q^+(S) \cup U_q^0(S)$.
Consider now the one dimensional module $\mathbb{C}(q^{1/2})_\lambda$, in which the B_q-module $U_q^+(S)$ acts trivially and if $\mathbf{1}$ is the generator of the module then $K_i^{\pm 1}\mathbf{1} = q^{\pm\lambda(h_i)}\mathbf{1}$, $i \in I_0$, $\gamma^{\pm 1/2}\mathbf{1} = q^{\pm\lambda(c)/2}\mathbf{1}$ and $D^{\pm 1}\mathbf{1} = q^{\pm\lambda(d)}\mathbf{1}$.
The *(quantum) imaginary Verma module* $M_q(\lambda)$ of weight $\lambda \in P$ is defined as
$M_q(\lambda) := U_q(\hat{\mathfrak{g}}) \otimes_{B_q} \mathbb{C}(q^{1/2})_\lambda$.

Reduced Imaginary Verma modules

The following theorem is known.

Theorem

$M_q(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.

Let us assume that $M_q(\lambda)$ is reducible, i.e. $\lambda(c) = 0$ and so $\gamma^{\pm 1/2}$ acts by 1. Denote by $J^q(\lambda)$ the left ideal of $U_q(\hat{\mathfrak{g}})$ generated by x_{ik}^{+}, h_{il} for $i \in I_0, k, l \in \mathbb{Z}, l \neq 0$ and $K_i^{\pm 1} - q^{\pm\lambda(h_i)}, \gamma^{\pm 1/2} - 1$ and $D^{\pm 1} - q^{\pm\lambda(d)}$. Define

$$\tilde{M}_q(\lambda) = U_q(\hat{\mathfrak{g}})/J^q(\lambda).$$

This quotient of $M_q(\lambda)$ is called the *reduced imaginary Verma module*. Then the following theorem is known.

Theorem

Let $\lambda \in P$ such that $\lambda(c) = 0$. Then $\tilde{M}_q(\lambda)$ is irreducible if and only if $\lambda(h_i) \neq 0$ for all $i \in I_0$.

Ω-operators

Observe that

$$\tilde{M}_q(\lambda) = \bigoplus_{\substack{i_1,\dots,i_r \\ k_1,\dots,k_r}} \mathbb{C}(q^{1/2}) x^-_{i_1 k_1} \cdots x^-_{i_r k_r} v_\lambda$$

where v_λ stands for the generator **1**. We will say that a monomial $x^-_{i_1 k_1} \cdots x^-_{i_r k_r}$ is ordered if and only if $i_1 + k_1 \geq i_2 + k_2 \geq \cdots \geq i_r + k_r$. Consider the subalgebra $\mathcal{N}_q^-$ of $U_q(\hat{\mathfrak{g}})$ generated by $\gamma^{\pm 1/2}$ and x^-_{il} for $l \in \mathbb{Z}$, $i \in I_0$ and the relations between them. The elements of $\mathcal{N}_q^-$ are sums of the elements of the form $P^{j_1,\dots,j_k}_{m_1,\dots,m_k} = \gamma^{l/2} x^-_{j_1 m_1} \cdots x^-_{j_k m_k}$, for $m_i, l \in \mathbb{Z}$, $k \geq 0$ and $1 \leq j_i \leq N$. Such an element is a summand of

$$P^{j_1,\dots,j_k} = P^{j_1,\dots,j_k}(v_1, \dots, v_k) := \gamma^{l/2} x^-_{j_1}(v_1) \cdots x^-_{j_k}(v_k).$$

We set $\overline{P}^{j_1,\dots,j_k} = x^-_{j_1}(v_1) \cdots x^-_{j_k}(v_k)$ and $\overline{P}^{j_1,\dots,j_k}_l = x^-_{j_1}(v_1) \cdots x^-_{j_{l-1}}(v_{j_{l-1}}) x^-_{j_{l+1}}(v_{j_l+1}) \cdots x^-_{j_k}(v_k)$.

Ω-operators

We denote

$$G_{il} = G_{il}^{1/q} = G_{il}^{1/q}(v_{j1}, \dots, v_{j_l}, v_l) := \delta_{i,j_l} \prod_{m=1}^{l-1} g_{i,j_m,q^{-1}}(v_{j_m}/v_l)$$

$$G_{il}^{q} = G_{il}^{q}(v_{j1}, \dots, v_{j_l}, v_l) := \delta_{i,j_l} \prod_{m=1}^{l-1} g_{i,j_m,q}(v_l/v_{j_m})$$

where $G_{i1} = \delta_{i,j_1}$. We define operators $\Omega_{\psi_i}(k), \Omega_{\phi_i}(k) : \mathcal{N}_q^- \to \mathcal{N}_q^-$ for $k \in \mathbb{Z}$ in terms of the generating functions

$$\Omega_{\psi_i}(u) = \sum_{l \in \mathbb{Z}} \Omega_{\psi_i}(l) u^{-l}, \qquad \Omega_{\phi_i}(u) = \sum_{l \in \mathbb{Z}} \Omega_{\phi_i}(l) u^{-l}$$

by

$$\Omega_{\psi_i}(u)(\overline{P}^{j_1,\dots,j_k}) = \sum_{l=1}^{k} G_{il} \overline{P}_l^{j_1,\dots,j_k} \delta(u/v_l\gamma)$$

$$\Omega_{\phi_i}(u)(\overline{P}^{j_1,\dots,j_k}) = \sum_{l=1}^{k} G_{il}^{q} \overline{P}_l^{j_1,\dots,j_k} \delta(u\gamma/v_l).$$

Ω-operators

Note that $\Omega_{\psi_i}(u)(1) = \Omega_{\phi_i}(u)(1) = 0$.
Recall that $x_i^-(v) = \sum_m x_{im}^- v^{-m}$. So we can consider left multiplication operators $x_{im}^- : \mathcal{N}_q^- \to \mathcal{N}_q^-$. There are several identities between the Ω-operators and the x^--operators, we just list a few:

$$q^{(\alpha_i|\alpha_j)}\gamma\Omega_{\psi_j}(m)x_{i,n+1}^- - \Omega_{\psi_j}(m+1)x_{in}^- = (q^{(\alpha_i|\alpha_j)}\gamma - 1)\delta_{ij}\delta_{m,-n-1} + \gamma x_{i,n+1}^-\Omega_{\psi_j}(m) - q^{(\alpha_1|\alpha_j)}x_{in}^-\Omega_{\psi_j}(m+1).$$

$$q^{(\alpha_i|\alpha_j)}\Omega_{\phi_j}(m)x_{i,n+1}^- - \gamma\Omega_{\phi_j}(m+1)x_{in}^- = (q^{(\alpha_i|\alpha_j)} - \gamma)\delta_{ij}\delta_{m,-n-1} + x_{i,n+1}^-\Omega_{\phi_j}(m) - q^{(\alpha_1|\alpha_j)}\gamma x_{in}^-\Omega_{\phi_j}(m+1).$$

$$\Omega_{\psi_j}(k)x_{im}^- = \delta_{ij}\delta_{k,-m}\gamma^k + \sum_{r\geq 0} g_{i,j,q^{-1}}(r)x_{i,m+r}^-\Omega_{\psi_j}(k-r)\gamma^r. \tag{5.1}$$

$$\Omega_{\psi_i}(k)\Omega_{\phi_j}(m) = \sum_{r\geq 0} g_{i,j}(r)\gamma^{2r}\Omega_{\phi_j}(r+m)\Omega_{\psi_i}(k-r). \tag{5.2}$$

Kashiwara Algebra

We define the **Kashiwara algebra** $\mathcal{K}_q$ as the $\mathbb{C}(q^{1/2})$-algebra with generators $\Omega_{\psi_j}(m), x_i^-(n), \gamma^{\pm 1/2}$ for $m, n \in \mathbb{Z}$, $1 \leq i, j \leq N$ where $\gamma^{\pm 1/2}$ are central and the defining relations are:

$$q^{(\alpha_i|\alpha_j)}\gamma\Omega_{\psi_j}(m)x_{i,n+1}^- - \Omega_{\psi_j}(m+1)x_{in}^- = (q^{(\alpha_i|\alpha_j)}\gamma - 1)\delta_{ij}\delta_{m,-n-1} + \gamma x_{i,n+1}^-\Omega_{\psi_j}(m) - q^{(\alpha_1|\alpha_j)}x_{in}^-\Omega_{\psi_j}(m+1)$$

$$q^{(\alpha_i|\alpha_j)}\Omega_{\psi_i}(k+1)\Omega_{\psi_j}(l) - \Omega_{\psi_j}(l)\Omega_{\psi_i}(k+1) = \Omega_{\psi_i}(k)\Omega_{\psi_j}(l+1) - q^{(\alpha_i|\alpha_j)}\Omega_{\psi_j}(l+1)\Omega_{\psi_i}(k)$$

$$x_{i,k+1}^- x_{jl}^- - q^{-(\alpha_i|\alpha_j)}x_{jl}^- x_{i,k+1}^- = q^{-(\alpha_i|\alpha_j)}x_{ik}^- x_{j,l+1}^- - x_{j,l+1}^- x_{ik}^- \qquad (5.3)$$

and

$$\gamma^{\pm 1/2}\gamma^{\mp 1/2} = 1.$$

Kashiwara Algebra

Some of the properties of the Kashiwara algebra $\mathcal{K}_q$ that holds for any quantum untwisted affine algebras are summarized in the following proposition.

Proposition (CFM2015)

For a quantum affine algebra associated to any untwisted affine Lie algebra, there exists a unique non-degenerate symmetric form $(-,-)$ defined on $\mathcal{N}_q^-$ satisfying $(x_{ij}^- a, b) = (a, \Omega_{\psi_i}(-j)b)$ and $(1,1) = 1$. Moreover, $\mathcal{N}_q^-$ is a left $\mathcal{K}_q$-module such that $\mathcal{N}_q \cong \mathcal{K}_q / \left(\sum_{i=1}^{N} \sum_{k\in\mathbb{Z}} \mathcal{K}_q \Omega_{\psi_i}(k) \right)$. Furthermore, $\mathcal{N}_q^-$ is simple as a left $\mathcal{K}_q$-module.

Operators $\tilde{\Omega}$ and $\tilde{x}$

On the algebra $\mathcal{N}_q^-$ we define the *"twisted concatenation product"* as follows: For elements $x_{im}^-, x_{jn}^- \in \mathcal{N}_q^-$, we define

$$x_{im}^- \star x_{jn}^- = \begin{cases} x_{im}^- x_{jn}^- & \text{if } i+m \geq j+n \text{ or } i+m < j+n \\ & \text{and } (\alpha_i|\alpha_j) > 0 \\ q^{-(\alpha_i|\alpha_j)} x_{im}^- x_{jn}^- & \text{if } j = i+1 \text{ and } n = m+1 \\ & \text{or } i = j+2 \text{ and } n = m+4 \\ q^{-(\alpha_i|\alpha_j)}(x_{im}^- x_{jn}^- - x_{j,n-1}^- x_{i,m+1}^-) & \text{if } i = j+1,\ m+1 < n,\ j+n-1 < i+m+1 \\ & \text{or } i = j+2 \text{ and } n > m+4 \\ q^{-(\alpha_i|\alpha_j)}(x_{im}^- x_{jn}^- - q^{(\alpha_i|\alpha_j)} x_{i,m+1}^- x_{j,n-1}^-) & \text{if } i = j+1,\ m+1 < n,\ i+m+1 < j+n-1 \\ & \text{or } i = j+2 \text{ and } m+2 < n < m+4 \\ & \text{or } j = i+2 \text{ and } m < n \\ q^{-(\alpha_i|\alpha_j)}(x_{im}^- x_{jn}^- - x_{i,m+1}^- x_{j,n-1}) & \text{if } j = i+1 \text{ and } m+1 < n \\ & \text{or } j = i+2 \text{ and } n = m+1 \\ x_{im}^- x_{jn}^- - x_{i,m-1}^- x_{j,n+1} & \text{if } j = i+2 \text{ and } m = n+1 \end{cases}$$

Operators $\tilde{\Omega}$ and $\tilde{x}$

We define the operator $\tilde{x}^-_{jm}$ over a $\star-$monomials $(x^-_{i_1k_1} \star (\cdots \star (x^-_{i_{l-1}k_{l-1}} \star x^-_{i_lk_l}) \cdots))$ as follows:

$$\tilde{x}^-_{jm}(x^-_{i_1k_1} \star (\cdots \star (x^-_{i_{l-1}k_{l-1}} \star x^-_{i_lk_l}) \cdots)) := x^-_{jm} \star (x^-_{i_1k_1} \star (\cdots \star (x^-_{i_{l-1}k_{l-1}} \star x^-_{i_lk_l}) \cdots)) = ((x^-_{jm} \star x^-_{i_1k_1}) \star (x^-_{i_2k_2} \star (\cdots \star (x^-_{i_{l-1}k_{l-1}} \star x^-_{i_lk_l}) \cdots)).$$

From the definition of the $\star-$product, we have that it coincides with the usual product on ordered monomials, and the $\star-$product is associative over ordered monomial.

We define the operator $\tilde{\Omega}_{\psi_i}(m)$ by induction on $\star-$monomials $(x^-_{i_1k_1} \star (\cdots \star (x_{i_{l-1}k_{l-1}} \star x^-_{i_lk_l}) \cdots))$ as follows:

$$\tilde{\Omega}_{\psi_i}(m)(x^-_{jk}) := \delta_{ij}\delta_{-m,k}.$$

and

Operators $\tilde{\Omega}$ and $\tilde{x}$

$$\tilde{\Omega}_{\psi_i}(m)((x^-_{i_1k_1}\star(\cdots\star(x^-_{i_{l-1}k_{l-1}}\star x^-_{i_lk_l})\cdots))) = \delta_{ii_1}\delta_{-m,k_1}(x^-_{i_2k_2}\star(\cdots\star(x^-_{i_{l-1}k_{l-1}}\star x^-_{i_lk_l})\cdots))$$

$$+\sum_{r\geq 0} q^{p^{mk_1}_{ii_1}} g_{i,i_1,q^{-1}}(r)(x^-_{i_1,m_1+r}\star\tilde{\Omega}_{\psi_i}(m-r)(x^-_{i_2k_2}\star(\cdots\star(x^-_{i_{l-1}k_{l-1}}\star x^-_{i_lk_l})))$$

where $p^{mk_1}_{i_1i} := p^{k_2,\dots,k_l}_{i_2,\dots,i_l}$ is defined as

$$p^{mk_1}_{i_1i} := \begin{cases} 0 & \text{if } (\alpha_i|\alpha_j) > 0 \\ -(\alpha_i|\alpha_j)\,\text{mín}\{\ell \mid \tilde{\Omega}_{\psi_i}(m-\ell)((x^-_{i_2k_2}\star(\cdots\star(x^-_{i_{l-1}k_{l-1}}\star x^-_{i_lk_l}))=0\}+1 & \text{if } (\alpha_i|\alpha_j) < 0 \\ 2 & \text{if } (\alpha_i|\alpha_j) = 0. \end{cases}$$

Recall that ordered monomials are the defining elements on $\tilde{M}_q(\lambda)$, so we define the operators $\tilde{x}^-_{jm}$ and $\tilde{\Omega}_{\psi_i}(m)$ on $\tilde{M}_q(\lambda)$ as follows: For an ordered monomial $x^-_{i_1k_1}\cdots x^-_{i_lk_l}v_\lambda$ we have:

Operators $\tilde{\Omega}$ and $\tilde{x}$

$$\tilde{x}_{jm}^-(x_{i_1k_1}^- \cdots x_{i_rk_r}^- v_\lambda) = \tilde{x}_{jm}^-(x_{i_1k_1}^- \star \cdots \star x_{i_rk_r}^- v_\lambda) := (x_{jm}^- \star x_{i_1k_1}^-) \star \cdots \star x_{i_rk_r}^- v_\lambda$$

$$\begin{aligned}
\tilde{\Omega}_{\psi_i}(m)(x_{i_1k_1}^- \cdots x_{i_lk_l}^- v_\lambda) &= \tilde{\Omega}_{\psi_i}(m)(x_{i_1k_1}^- \star \cdots \star x_{i_lk_l}^- v_\lambda) \\
&= \delta_{ii_1}\delta_{-m,k_1} x_{i_2k_2}^- \star \cdots \star x_{i_lk_l}^- v_\lambda \\
&+ \sum_{r\geq 0} q^{p_{ii_1}^{mk_1}} g_{i,i_1,q^{-1}}(r) x_{i_1,m_1+r} \star \tilde{\Omega}_{\psi_i}(m-r)(x_{i_2k_2}^- \star \cdots \star x_{i_lk_l}^-) v_\lambda \\
&= \delta_{ii_1}\delta_{-m,k_1} x_{i_2k_2}^- \cdots x_{i_lk_l}^- v_\lambda \\
&+ \sum_{r\geq 0} q^{p_{ii_1}^{mk_1}} g_{i,i_1,q^{-1}}(r) x_{i_1,m_1+r} \star \tilde{\Omega}_{\psi_i}(m-r)(x_{i_2k_2}^- \cdots x_{i_lk_l}^-) v_\lambda
\end{aligned}$$

Operators $\tilde{\Omega}$ and $\tilde{x}$

We have the following Propositions:

Proposition

$\tilde{x}^-_{jm}(x^-_{ik})$ could be written as a linear combination of ordered monomials of length 2 with coefficients in $\mathbb{Z}[q]$.

Proposition

If $x^-_{i_1k_1}\cdots x^-_{i_lk_l}$ is an ordered monomial then $\tilde{\Omega}_{\psi_i}(m)(x^-_{i_1k_1}\cdots x^-_{i_lk_l})$ is a linear combination of ordered monomials of length l-1 with coefficients in $\mathbb{Z}[q]$.

Next we define a new bilinear form on ordered monomials of $\mathcal{N}_q^-$. For its definition we consider the unique non-degenerate bilinear form $(-,-)$ defined on $\mathcal{N}_q^-$.
Let $x^-_{i_1m_1}\cdots x^-_{i_km_k}$, $x^-_{j_1n_1}\cdots x^-_{j_ln_l}$ be two ordered monomials of $\mathcal{N}_q^-$. We define by induction the following bilinear form on $\mathcal{N}_q^-$:

$$\langle x^-_{im}, x^-_{jn}\rangle = (1, \tilde{\Omega}_{\psi_i}(-m)(x^-_{jn}))$$

$$\langle x^-_{i_1m_1}\cdots x^-_{i_km_k}, x^-_{j_1n_1}\cdots x^-_{j_ln_l}\rangle := \langle x^-_{i_2m_2}\cdots x^-_{i_km_k}, \tilde{\Omega}_{\psi_{i_1}}(-m_1)(x^-_{j_1n_1}\cdots x^-_{j_ln_l})\rangle$$

Bilinear form

For $\mathbf{i} = (i_1, \ldots, i_k), \mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$ the monomial $x^-_{i_1 m_1} \cdots x^-_{i_k m_k}$ is denoted by $x^-_{\mathbf{im}}$.
We have following results.

Lemma

Let $\mathbf{i} = (i_1, \ldots, i_k) \in I_0^k, \mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, $\mathbf{j} = (j_1, \ldots, j_l) \in I_0^l$ and $\mathbf{n} = (n_1, \ldots, n_l) \in \mathbb{Z}^l$ such that $i_1 + m_1 \geq \cdots \geq i_k + m_k$, $j_1 + n_1 \geq \cdots \geq j_l + n_l$ and $k > l$, then $\langle x^-_{\mathbf{im}}, x^-_{\mathbf{jn}} \rangle = 0$.

Lemma

Let $\mathbf{i} = (i_1, \ldots, i_k), \mathbf{j} = (j_1, \ldots, j_k) \in I_0^k$ and $\mathbf{m} = (m_1, \ldots, m_k), \mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, and consider ordered monomials $x^-_{\mathbf{im}} = x^-_{i_1 m_1} \cdots x^-_{i_k m_k}$ and $x^-_{\mathbf{jn}} = x^-_{j_1 n_1} \cdots x^-_{j_k n_k}$. Then

$$\langle x^-_{\mathbf{im}}, x^-_{\mathbf{jn}} \rangle \in \mathbb{Z}[q]$$

Bilinear form

Proposition

Let $\mathbf{i} = (i_1, \dots, i_k) \in I_0^k, \mathbf{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$, $\mathbf{j} = (j_1, \dots, j_l) \in I_0^l$ *and* $\mathbf{n} = (n_1, \dots, n_l) \in \mathbb{Z}^l$, *and consider ordered monomials* $x_{\mathbf{im}}^-, x_{\mathbf{jn}}^-$ *such that* $\sum_{r=1}^k (i_r + m_r) = \sum_{r=1}^l (j_r + n_r)$, *then*

$$\langle x_{\mathbf{im}}^-, x_{\mathbf{jn}}^- \rangle = (\delta_{kl}\delta_{\mathbf{ij}}\delta_{\mathbf{mn}} + q\mathbb{Z}) \mod q^2\mathbb{Z}[q]$$

Let $\mathbb{A}_0 = \mathbb{C}[q^{1/2}]_{(q)}$ be the ring of rational functions in $q^{1/2}$ regular at 0 and $\pi = \{-\alpha + n\delta | \alpha \in \Delta_{0,+}, n \in \mathbb{Z}\} \cup \{0\}$.

Definition

Let M *be a* $U_q(\hat{\mathfrak{g}})$*-module. We call a free* $\mathbb{A}_0$*-submodule* $\mathcal{L}$ *of* M *an imaginary crystal lattice of* M *if the following holds:*

1. $\mathbb{C}(q^{1/2}) \otimes_{\mathbb{A}_0} \mathcal{L} \cong M$.
2. $\mathcal{L} \cong \oplus_{\lambda \in \pi} \mathcal{L}_\lambda$ *and* $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$.
3. $\tilde{\Omega}_{\psi_i}(m)\mathcal{L} \subseteq \mathcal{L}$ *and* $\tilde{x}_{im}^-\mathcal{L} \subseteq \mathcal{L}$, *for* $i \in I_0$ *and* $m \in \mathbb{Z}$.

Imaginary Crystal Basis

Let $\lambda \in P$ such that $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$, $i \in I_0$, then $\tilde{M}_q(\lambda)$ is a simple reduced imaginary Verma module. Consider the following $\mathbb{A}_0$-submodule:

$$\mathcal{L}(\lambda) := \bigoplus_{\substack{k \geq 0 \\ i_1+m_1 \geq \cdots \geq i_k+m_k \\ i_l, m_l \in \mathbb{Z}}} \mathbb{A}_0 x^-_{i_1 m_1} \cdots x^-_{i_k m_k} v_\lambda$$

Then $\mathcal{L}(\lambda)$ is an imaginary crystal lattice of $\tilde{M}_q(\lambda)$.

Definition

An imaginary crystal basis of a reduced imaginary Verma module $\tilde{M}_q(\lambda)$ is a pair $(\mathcal{L}, \mathcal{B})$ satisfying:

1. *$\mathcal{L}$ is an imaginary crystal lattice of $\tilde{M}_q(\lambda)$.*
2. *$\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{C} \otimes_{\mathbb{A}_0} \mathcal{L}$.*
3. *$\mathcal{B} = \cup_{\mu \in \pi} \mathcal{B}_\mu$ where $\mathcal{B}_\mu = \mathcal{B} \cap (\mathcal{L}_\mu / q\mathcal{L}_\mu)$.*
4. *$\tilde{x}^-_{im}\mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$ and $\tilde{\Omega}_{\psi_i}(m)\mathcal{B} \subset \pm\mathcal{B} \cup \{0\}$, for $i \in I_0$ and $m \in \mathbb{Z}$.*
5. *For $m \in \mathbb{Z}$ and $i \in I_0$ if $\tilde{\Omega}_{\psi_i}(-m)b \neq 0$ and $\tilde{x}^-_{im}b \neq 0$ for $b \in \mathcal{B}$, then $\tilde{x}^-_{im}\tilde{\Omega}_{\psi_i}(-m)b = \tilde{\Omega}_{\psi_i}(-m)\tilde{x}^-_{im}b$.*

Imaginary Crystal Basis

For $\lambda \in P$ such that $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$, $i \in I_0$ define

$$\mathcal{B}(\lambda) = \left\{ x^-_{i_1 m_1} \cdots x^-_{i_k m_k} v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \;\middle|\; \begin{array}{c} i_1 + m_1 \geq \cdots \geq i_k + m_k, \\ m_1, \ldots, m_k \in \mathbb{Z}, i_1, \ldots, i_k \in I_0 \end{array} \right.$$

Theorem

If $\lambda \in \mathfrak{h}^*$ such that $\lambda(c) = 0$ and $\lambda(h_i) \neq 0$ for all $i \in I_0$, then $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is an imaginary crystal bases for $\tilde{M}_q(\lambda)$.

Category $\mathcal{O}^q_{red,im}$

Let G_q be the quantized Heisenberg subalgebra generated by h_{in} and γ, for $i \in I_0$, $n \in \mathbb{Z} \setminus \{0\}$.
We will say that a $U_q(\hat{\mathfrak{g}})$-module V is G_q-compatible if:

(i) V has a decomposition $V = T(V) \oplus TF(V)$ where $T(V)$ and $TF(V)$ are non-zero G_q-modules, called, respectively, torsion and torsion free module associated to V.

(ii) h_{im} for $i \in I_0$, $m \in \mathbb{Z} \setminus \{0\}$ acts bijectively on $TF(V)$, i.e., they are bijections on $TF(V)$.

(iii) $TF(V)$ has no non-zero $U_q(\hat{\mathfrak{g}})$-submodules.

(iv) $G_q \cdot T(V) = 0$.

Consider the set

$$\mathfrak{h}^*_{q,red} = \{\lambda \in P \mid \lambda(c) = 0, \lambda(h_i) \neq 0, i \in I_0\}.$$

Category $\mathcal{O}^q_{red,im}$

The category $\mathcal{O}^q_{red,im}$ is defined as the category whose objects are $U_q(\hat{\mathfrak{g}})$-modules M such that

1. M is $\hat{\mathfrak{h}}^*_{q,red}$-diagonalizable, that means,

$$M = \bigoplus_{\nu \in \hat{\mathfrak{h}}^*_{q,red}} M_\nu, \text{ where } M_\nu = \{m \in M | K_i m = q^{\lambda(h_i)} m, Dm = q^{\lambda(d)} m, i \in I_0\}$$

2. For any $i \in I_0$ and any $n \in \mathbb{Z}$, x^+_{in} acts locally nilpotently.
3. M is G_q-compatible.
4. the morphisms in $\mathcal{O}^q_{red,im}$ are $U_q(\hat{\mathfrak{g}})$-homomorphisms.

Theorem

(1) *If $\lambda, \mu \in \hat{\mathfrak{h}}^*_{q,red}$ then $\mathrm{Ext}^1_{\mathcal{O}^q_{red,im}}(\tilde{M}_q(\lambda), \tilde{M}_q(\mu)) = 0$.*

(2) *If M is an irreducible module in the category $\mathcal{O}^q_{red,im}$, then $M \cong \tilde{M}_q(\lambda)$ for some $\lambda \in \hat{\mathfrak{h}}^*_{q,red}$. Moreover, if N is an arbitrary object of $\mathcal{O}^q_{red,im}$ then $N \cong \bigoplus_{\lambda_i \in \hat{\mathfrak{h}}^*_{q,red}} \tilde{M}(\lambda_i)$, for some λ_i's.*

Category $\mathcal{O}^q_{red,im}$

Now suppose $M \in \mathcal{O}^q_{red,im}$. Then there exists $\lambda_k \in \hat{\mathfrak{h}}^*_{q,red}$ for $k \in J$ (J an index set) such that $M \cong \bigoplus_{k \in J} \tilde{M}_q(\lambda_k)$. For $k \in J$, let $(\mathcal{L}(\lambda_k), \mathcal{B}(\lambda_k))$ be the imaginary crystal basis of $\tilde{M}_q(\lambda_k)$. Set $\mathcal{L} = \bigoplus_{k \in J} \mathcal{L}(\lambda_k)$ and $\mathcal{B} = \bigsqcup_{k \in J} \mathcal{B}(\lambda_k)$.

Theorem

Let $M \in \mathcal{O}^q_{red,im}$ such that $M \cong \bigoplus_{\lambda_k \in J} \tilde{M}_q(\lambda_k)$ as above. Then the pair $(\mathcal{L}, \mathcal{B})$ is an imaginary crystal basis for M.

THANKS!